Periodic solutions with nonconstant sign in Abel equations of the second kind

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Abstract

The study of periodic solutions with constant sign in the Abel equation of the second kind can be made through the equation of the first kind. This is because the situation is equivalent under the transformation $x \mapsto x^{-1}$, and there are many results available in the literature for the first kind equation. However, the equivalence breaks down when one seeks for solutions with nonconstant sign. This note is devoted to periodic solutions with nonconstant sign in Abel equations of the second kind. Specifically, we obtain sufficient conditions to ensure the existence of a periodic solution that shares the zeros of the leading coefficient of the Abel equation. Uniqueness and stability features of such solutions are also studied.

Keywords: Abel differential equations, periodic solutions

1. Introduction

Abel Ordinary Differential Equations (ODE) of the second kind [1],

$$[b_0(t) + b_1(t)x]\,\dot{x} = a_0(t) + a_1(t)x + a_2(t)x^2, \quad a_i(t), b_i(t) \in C([0,T]), \tag{1}$$

can be regarded as a generalization of Riccati's equation [1]. This family of equations deserves special interest in the applied mathematics field because it appears in different contexts, running from control problems [2] to mathematical physics and nonlinear mechanics issues [1, 3]. It is also remarkable that a class of Abel equations of the first kind [1] can be written as (1) with the change of variables $x \mapsto x^{-1}$.

Indeed, polynomial differential equations of the type

$$\dot{x} = \sum_{i=0}^{n} a_i(t)x^i, \quad a_i(t) \in C([0,T]), \quad i = 0,\dots,n, \tag{2}$$

are also known as Abel-like [4] or generalized Abel [5] equations because, when $n = 3$, (2) is an Abel ODE of the first kind. The existence of periodic solutions in (2), i.e. solutions verifying $x(0) = x(T)$, has attracted considerable research effort: see, for example, [4, 5, 6, 7, 8, 9, 10, 11] and references therein. This is mainly due to its relation with the number of limit cycles of planar polynomial systems and, therefore, with Hilbert's 16th problem [12]. Contrarily, few results regarding periodic solutions in Abel ODE of the second kind have been published [2, 13, 14].

Notice that the change of variables $x \mapsto x + b_0/b_1$ allows to recast (1) as

$$x\dot{x} = A(t) + B(t)x + C(t)x^2, \quad A, B, C \in C([0,T]). \tag{3}$$

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[1]Partially supported by the spanish Ministerio de Ciencia e Innovación (MICINN) under project DPI2010-15110 and also through the Programa Nacional de Movilidad de Recursos Humanos of the Plan Nacional de I+D+i 2008-2011.
[2]Partially supported by the spanish Ministerio de Educación (MEC) under project MTM2008-06349 C03 01.

The transformation being time-preserving, the study of periodic solutions in (1) and (3) whenever b_1 has constant sign, i.e. $b_1(t) \neq 0$, for all t, is equivalent.

The existence of nontrivial periodic solutions of constant sign in (3) may be carried out after transforming it into the Abel equation of the first kind

$$\dot{x} = A(t)x^3 + B(t)x^2 + C(t)x \tag{4}$$

using the aforementioned change $x \mapsto x^{-1}$, which keeps the equivalence between these class of solutions in (3) and (4). Different conditions are available in the literature yielding to upper and/or lower bounds on the number of periodic solutions of (4) or (2)-like Abel ODE with more or less generic coefficients. The results cover from the simplest case, $A(t) \neq 0$, for all t, in (4) [7, 13] or $a_n(t) \neq 0$, for all t, in (2) [5], to the most complex in which no sign condition is assumed on the coefficients [8, 11], going through situations in which some of the coefficients, often $A(t)$ or $a_n(t)$, are demanded not to change sign [4, 6, 7, 9, 10].

The study of periodic solutions of (3) with nonconstant sign is also a challenging problem that can not be tackled via the Abel ODE of the first kind and about which, as far as the authors know, no results have been yet reported. Notice also that if a solution of (3) has nonconstant sign, then its zeros are also zeros of $A(t)$. Hence, the search of periodic solutions in (3) with nonconstant sign only makes sense when $A(t)$ itself has nonconstant sign.

This note deals with the existence of this type of periodic solutions in Abel equations of the second kind. The main result reads as follows:

Theorem 1. *Let $A(t), B(t), C(t)$ be C^1, T-periodic functions. If $A(t)$ has at least one zero in $[0, T]$ and*

$$\min |B(t)|^2 > -4 \min \dot{A}(t) \cdot [1 + T \max |C(t)|], \tag{5}$$

then (3) has a T-periodic solution that has the sign of $-A(t)B(t)$, and it is also C^1.

Furthermore, such solutions are shown to be the unique T-periodic solutions of (3) with nonconstant sign. In some cases, it is proved that there exists only one solution of this type. Also, a stability analysis reveals that these solutions are unstable. These results are later applied to the normal form of the Abel ODE of the second kind, which is obtained setting $B(t) = 1$ and $C(t) = 0$ in (3). For this case, restriction (5) is shown to be sharp.

The remainder of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1. Section 3 deals with the uniqueness and stability of the periodic solutions of (3) with nonconstant sign. Finally, Section 4 considers the application of the previous results to the normal form of the Abel ODE of the second kind.

2. Proof of Theorem 1

Firstly, let us establish a generic and rather straightforward result that will be used in subsequent demonstrations.

Lemma 1. *Consider the ODE*

$$\dot{x} = S(t, x), \quad S : \Omega \to \mathbb{R}, \tag{6}$$

where $\Omega := \mathbb{R} \times \mathbb{R}^*$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and S is a locally Lipschitz function. Assume that $m, n \in \mathbb{R}$ and let $r := \{(t, x) : x = mt + n\}$ be a straight line of slope m, which splits $\mathbb{R}^2$ into the half planes $\Omega_r^+ = \{(t, x) : x > mt + n\}$, $\Omega_r^- = \{(t, x) : x < mt + n\}$. Finally, let $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$.

(i) Assume that $S(t, x) > m$ for all $(t, x) \in [t_1, t_2) \times \mathbb{R}^* \cap r$. Then, any maximal solution $x(t)$ of (6) defined for all $t \in I_\omega \subseteq \mathbb{R}$ with $(t_1, x(t_1)) \in \overline{\Omega_r^+}$ is such that $(t, x(t)) \in \Omega_r^+$, for all $t \in (t_1, t_2) \cap I_\omega$.

(ii) Assume that $S(t, x) < m$ for all $(t, x) \in [t_1, t_2) \times \mathbb{R}^* \cap r$. Then, any maximal solution $x(t)$ of (6) defined for all $t \in I_\omega \subseteq \mathbb{R}$ with $(t_1, x(t_1)) \in \overline{\Omega_r^-}$ is such that $(t, x(t)) \in \Omega_r^-$, for all $t \in (t_1, t_2) \cap I_\omega$.

Proof. (i) Notice that if $(t_1, x(t_1)) \in \overline{\Omega_r^+}$ then, by continuity, for t close enough to t_1 it happens that $(t, x(t)) \in \Omega_r^+$: for $(t_1, x(t_1)) \in \Omega_r^+$ it is rather immediate, while for $(t_1, x(t_1)) \in r$ the claim follows because $\dot{x}(t_1) > m$.

Assume that $x(t)$ contacts r for the first time in (t_1, t_2) at $t = c > t_1$, i.e. that $(c, x(c)) \in r$. Then, $x(c) = mc + n$, and $x(t) > mt + n$ for $t_1 < t < c$, so

$$\dot{x}(c) = \lim_{t \to c^-} \frac{x(t) - x(c)}{t - c} \leq \lim_{t \to c^-} \frac{mt + n - (mc + n)}{t - c} = m,$$

which contradicts the hypothesis $\dot{x}(c) = S(c, x(c)) > m$. Hence, $x(t)$ can not contact again r and, therefore, it remains in Ω_r^+ for all $t \in (t_1, t_2) \cap I_\omega$.

The proof of (ii) is analogous. □

The hypotheses of Theorem 1 are assumed to be fulfilled throughout the rest of the section. Furthermore, the T-periodicity and C^1 character of $A(t)$ implies that $\min \dot{A}(t) \leq 0$. Hence, it is immediate from (5) that $B(t) \neq 0$ for all t. Thus, using the change of variables $x \mapsto -x$ if necessary, it is no loss of generality to assume $B(t) > 0$, for all t, for the remainder of the section.

Theorem 1 considers a case in which $A(t)$ has nonconstant sign. The next Lemmas study the behavior of the solutions of (3) in an open interval (a, b) where $A(t) > 0$, a, b being two consecutive zeros of $A(t)$.

Remark 1. *Notice that there is no loss of generality in assuming $A(t) > 0$ in (a, b) because, otherwise, the change of variables $(t, x) \mapsto (-t, -x)$ reduces (3) to*

$$x\dot{x} = \hat{A}(t) + \hat{B}(t)x + \hat{C}(t)x^2,$$

with $\hat{A}(t) = -A(-t)$, $\hat{B}(t) = B(-t)$ and $\hat{C}(t) = -C(-t)$, for all $t \in (a, b)$.

Lemma 2. *If (5) is satisfied, then*

$$\min |B(t)|^2 > 2 \max |A(t)| \cdot \max |C(t)|, \tag{7}$$

Proof. Recalling that $\min \dot{A}(t) \leq 0$, (5) yields

$$\min |B(t)|^2 > -4 \min \dot{A}(t) \cdot [1 + T \max |C(t)|] \geq -2T \min \dot{A}(t) \max |C(t)|.$$

Then, it is sufficient to prove that

$$-T \min \dot{A}(t) \geq \max |A(t)|. \tag{8}$$

For, let $t_0 \in \mathbb{R}$ be such that $|A(t_0)| = \max |A(t)|$, and let t_1, t_2 be zeros of $A(t)$ such that $t_0 - T < t_1 \leq t_0 \leq t_2 < t_0 + T$. Then, applying the Mean Value Theorem,

$$\begin{aligned} A(t_0) &= A(t_0) - A(t_1) = (t_0 - t_1)\dot{A}(\xi_1) \geq T \min \dot{A}(t) \\ -A(t_0) &= A(t_2) - A(t_0) = (t_2 - t_0)\dot{A}(\xi_2) \geq T \min \dot{A}(t), \end{aligned}$$

from which we deduce (8) and therefore (7). □

Lemma 3. *Let $a, b \in \mathbb{R}$ be such that $A(a) = A(b) = 0$, with $A(t) > 0$, for all $t \in (a, b)$. Then, any negative solution $x(t)$ of (3) defined on $[t_1, t_2)$, $t_1 \geq a$, can be extended to $[t_1, b)$.*

Proof. The ODE (3) can be written as

$$\dot{x} = S(t, x) = \frac{A(t)}{x} + B(t) + C(t)x \tag{9}$$

in the domain $\Omega^- := \mathbb{R} \times \mathbb{R}^-$. Let $x(t)$ be a solution with $x(t_1) < 0$ and maximal interval of definition $I_\omega = (\omega_-, \omega_+)$, with $\omega_- < t_1$. Let us assume that $\omega_+ < b$ and proceed by contradiction.

As $\omega_+ < b$, $t \to \omega_+$ implies that either $x(t) \to -\infty$ or $x(t) \to 0$. Let us first see that it is not possible to have $x(t) \to -\infty$. For, let us select $M \in \mathbb{R}^+$ and let us define the straight line $r_M := \{(t, x) : \ x + M = 0\}$. If $C \not\equiv 0$, the selection

$$M = \sqrt{\frac{\max |A(t)|}{\max |C(t)|}}$$

and relation (7) in Lemma 2 indicate that

$$S(t, x) = -\frac{A(t)}{M} + B(t) + C(t)M > 0, \quad \forall\, (t, x) \in I_\omega \times \mathbb{R}^- \cap r_M. \tag{10}$$

Alternatively, if $C \equiv 0$, the selection of a sufficiently large M also yields (10). In any case, by Lemma 1.i, $x(t)+M > 0$, for all $t \in (t_1, \omega_+)$, so $x(t) \nrightarrow -\infty$ for $t \to \omega_+ < b$.

Let us now see that it cannot be $x(t) \to 0$ for $t \to \omega_+ < b$. For, let us take $c \in \mathbb{R}$, $t_1 < c < \omega_+$ and select $N \in \mathbb{R}^+$ small enough, in such a way that

$$S(t,x) = -\frac{A(t)}{N} + B(t) + C(t)N < 0, \quad \forall t \in [c, \omega_+].$$

Then, defining $r_N := \{(t,x) : \ x + N = 0\}$, it is immediate that $S(t,x) < 0$, for all $(t,x) \in [c,\omega_+] \times \mathbb{R}^- \cap r_N$ and, by Lemma 1.ii, $x(t) + N < 0$ for $t \in (c, \omega_+)$, i.e. $x(t) \nrightarrow 0$.

Consequently, it has to be $\omega_+ \geq b$ and the solution $x(t)$ is defined in $[t_1, b)$. □

Lemma 4. *Let $a, b \in \mathbb{R}$ be such that $A(a) = A(b) = 0$, with $A(t) > 0$, for all $t \in (a,b)$. Then, there exists a C^1 solution $x^*(t)$ of (3) in (a,b), which is negative and such that*

$$x^*(t) \to 0 \quad \textit{and} \quad \dot{x}^*(t) \to \frac{B(a)}{2} - \sqrt{\frac{B(a)^2}{4} + \dot{A}(a)} \quad \textit{when} \quad t \to a^+.$$

Proof. The introduction of a new variable s allows to transform (3) in the following planar, generalized Liénard system [17]:

$$\frac{dt}{ds} = x, \tag{11}$$

$$\frac{dx}{ds} = A(t) + B(t)x + C(t)x^2. \tag{12}$$

Notice that, when $x \neq 0$, the portrait of the integral curves of (3) and the phase plane of (11)-(12) are coincident, preserving the orientation if $x > 0$ and reversing it if $x < 0$.

Since $A(a) = 0$ and $A > 0$ in (a,b), it results that $\dot{A}(a) \geq 0$. Let us study these two cases:

(i) If $\dot{A}(a) > 0$, then $(t,x) = (a,0)$ is a hyperbolic critical point, indeed a saddle, of (11)-(12). The eigenvalues are:

$$\lambda^a_{\pm} = \frac{B(a)}{2} \pm \sqrt{\frac{B(a)^2}{4} + \dot{A}(a)},$$

with $\lambda^a_{\pm} \in \mathbb{R}$ because of (5), the associated invariant subspaces of the linearized system being

$$\mathbb{E}^a_+ = \mathbb{E}^a_u = \text{span}\{(1, \lambda^a_+)\}, \quad \mathbb{E}^a_- = \mathbb{E}^a_s = \text{span}\{(1, \lambda^a_-)\}.$$

Hence, by the Stable Manifold Theorem [15], there exists a unique C^1 invariant stable manifold, tangent to $\mathbb{E}^a_s$ at $(a,0)$, with slope λ^a_-, i.e. lying on the subsets $\mathcal{A}^+ := \{(t,x) : \ t < a, \ x > 0\}$ and $\mathcal{A}^- := \{(t,x) : \ t > a, \ x < 0\}$ when $t \neq a$. The branch of the manifold that lies in $\mathcal{A}^+$ is a positive, C^1 solution $x^*(t)$ of (3) in $(a-\epsilon, a)$, $\epsilon > 0$, that satisfies $x^*(t) \to 0$ and $\dot{x}^*(t) \to \lambda^a_-$ when $t \to a^-$. Equivalently, the branch in $\mathcal{A}^-$ is a negative, C^1 solution $x^*(t)$ of (3) in $(a, a+\epsilon)$, $\epsilon > 0$, that satisfies $x^*(t) \to 0$ and $\dot{x}^*(t) \to \lambda^a_-$ when $t \to a^+$.

(ii) If $\dot{A}(a) = 0$, then $(t,x) = (a,0)$ is a non-hyperbolic critical point with eigenvalues

$$\lambda^a_u = B(a), \quad \lambda^a_c = 0,$$

the associated invariant subspaces of the linearized system being

$$\mathbb{E}^a_u = \text{span}\{(1, B(a))\}, \quad \mathbb{E}^a_c = \text{span}\{(1,0)\}.$$

Hence, by the Center Manifold Theorem [16], there exists a (not necessarily unique) C^1, invariant center manifold, tangent to $\mathbb{E}_c$ at $(a,0)$.

Let us finally see that this orbit lies on $\mathcal{A}^-$, which means that it matches a negative, C^1 solution $x^*(t)$ of (3) that satisfies $x^*(t) \to 0$ and $\dot{x}^*(t) \to 0$ when $t \to a^+$. Let us denote this orbit as $x = h(t)$, with $h(a) = \dot{h}(a) = 0$ and satisfying

$$h(t)(C(t)h(t) + B(t) - \dot{h}(t)) = -A(t).$$

As $C(a)h(a) + B(a) - \dot{h}(a) = B(a) > 0$, then $C(t)h(t) + B(t) - \dot{h}(t) > 0$ for $t - a$ small enough; consequently, h and $-A$ have the same sign in a neighborhood of $(a, 0)$, so $h(t) < 0$ for $0 < t - a << 1$.

Finally notice that, by Lemma 3, this C^1 solution $x^*(t)$ is defined in (a, b), which completes the proof. □

Lemma 5. *Let $a, b \in \mathbb{R}$ be such that $A(a) = A(b) = 0$, with $A(t) > 0$, for all $t \in (a, b)$. Then, there exists a C^1 solution $x^*(t)$ of (3) in (a, b), which has the sign of $-A(t)$, and is such that*

$$x^*(t) \to 0 \quad \textit{and} \quad \dot{x}^*(t) \to \frac{B(a)}{2} - \sqrt{\frac{B(a)^2}{4} + \dot{A}(a)} \quad \textit{when} \quad t \to a^+,$$

$$x^*(t) \to 0 \quad \textit{and} \quad \dot{x}^*(t) \to \frac{B(b)}{2} - \sqrt{\frac{B(b)^2}{4} + \dot{A}(b)} \quad \textit{when} \quad t \to b^-.$$

Proof. Let $x^*(t)$ be the solution of (3) featured in Lemma 4. Then, it remains to be proved the behavior for $t \to b^-$.

Firstly, for all $t \in (a, b)$, the Mean Value Theorem ensures that there exists $\xi \in (t, b)$ such that $A(t) = \dot{A}(\xi)(t - b)$. Let now r_b be the straight line $r_b := \{(t, x) : \ x = \alpha(t - b)\}$, where

$$\alpha = \sqrt{\frac{-\min \dot{A}(t)}{1 + T \max |C(t)|}}.$$

Then, for all $t \in (a, b)$ such that $(t, x) \in r_b$,

$$\begin{aligned} S(t, x) &= \frac{A(t)}{x} + B(t) + C(t)x \\ &\geq \frac{\dot{A}(\xi)(t - b)}{\alpha(t - b)} + \min |B(t)| - \alpha T \max |C(t)| \\ &\geq \alpha + \min |B(t)| - \alpha(1 + T \max |C(t)|) + \frac{\min \dot{A}(t)}{\alpha} \\ &= \alpha + \min |B(t)| - 2\sqrt{-\min \dot{A}(t)(1 + T \max |C(t)|)} > \alpha, \end{aligned}$$

where in the last inequality we have used condition (5). As, by Lemma 4, $x^*(t) \to 0$ when $t \to a^+$, Lemma 1.i guarantees that $x^*(t) > \alpha(t - b)$ for all $t \in (a, b)$. But since $x^*(t)$ is negative in (a, b), then $\alpha(t - b) < x^*(t) < 0$ and, taking limits for $t \to b^-$, it is immediate that $x^*(t) \to 0$.

Secondly, consider the equivalent expression of (3) in terms of the planar, autonomous system (11)-(12). Since $A(b) = 0$ and $A(t) > 0$ in (a, b), it results that $\dot{A}(b) \leq 0$. Let us then split the study in two cases:

(i) If $\dot{A}(b) < 0$, then $(t, x) = (b, 0)$ is a hyperbolic critical point of (11)-(12). The eigenvalues are:

$$\lambda^b_\pm = \frac{B(b)}{2} \pm \sqrt{\frac{B(b)^2}{4} + \dot{A}(b)},$$

with $\lambda^b_\pm \in \mathbb{R}^+$ because of (5), the associated invariant subspaces of the linearized system being

$$\mathbb{R}^2 = \mathbb{E}^b_u = \text{span}\left\{\left(1, \lambda^b_-\right), \left(1, \lambda^b_+\right)\right\}.$$

Hence, it is an unstable node and, in $t = b$, all the orbits are tangent to one of the two eigenvectors that span $\mathbb{E}^b_u$. It is then immediate that the solution $x^*(t)$ of (3), which is known to satisfy $x^*(t) > \alpha(t - b)$, matches one of these orbits. Therefore, for t close enough to b it has to be $\dot{x}^*(t) < \alpha$ and, as it can be easily proved that $\lambda^b_- < \alpha < \lambda^b_+$, it results that

$$\dot{x}^*(t) \to \lambda^b_- = \frac{B(b)}{2} - \sqrt{\frac{B(b)^2}{4} + \dot{A}(b)} \quad \text{when} \ \ t \to b^-.$$

(ii) If $\dot{A}(b) = 0$, the situation is equivalent to the case $\dot{A}(a) = 0$ discussed in the proof of Lemma 4. Namely, $(t, x) = (b, 0)$ is a non-hyperbolic critical point with eigenvalues

$$\lambda_u^b = B(b), \quad \lambda_c^b = 0,$$

the associated invariant subspaces of the linearized system being

$$\mathbb{E}_u^b = \text{span}\{(1, B(b))\}, \quad \mathbb{E}_c^b = \text{span}\{(1, 0)\}.$$

A technique similar to the one followed in the preceding item shows that $\dot{x}^*(t) \to 0$ when $t \to b^-$. □

Let us now proceed with the proof of Theorem 1. As $A(t)$ has, at least, one zero in $[0, T]$ by hypothesis, let $t_0 \in \mathbb{R}$ be such that $A(t_0) = 0$. Then, we define

$$Z := \{t \in [t_0, t_0 + T] : \ A(t) = 0\}.$$

Let $\mathcal{P}$ and $\mathcal{N}$ denote the sets of maximal intervals in $[t_0, t_0 + T]$ where $A(t)$ is positive and negative, respectively. Let then $I_i = (a_i, b_i)$, $a_i, b_i \in Z$, denote an interval of $\mathcal{P} \cup \mathcal{N}$. Lemma 5 and Remark 1 ensure that, for every I_i, there exists a C^1 solution $x_i^*(t)$ of (3) on I_i, which has the sign of $A(t)$ in I_i, and is such that $x_i^*(t) \to 0$ when $t \to a_i^+$ and also when $t \to b_i^-$. Hence, the function constructed as

$$x^*(t) = \begin{cases} x_i^*(t) & \text{if} \quad t \in I_i \\ 0 & \text{if} \quad t \in Z, \end{cases} \tag{13}$$

is indeed a continuous solution of (3) in $\mathbb{R}$ which is also C^1 in every open interval I_i. Let us finally prove that $x^*(t)$ is C^1 for all $t_i \in Z$. Three different situations need to be considered:

(i) If $\dot{A}(t_i) > 0$ then the graph of $x^*(t)$ in a neighborhood of t_i is the orbit of the (unique) stable manifold of (11)-(12), so $x^*(t)$ is C^1 in t_i (see the discussion in the proof of Lemma 4 for the case $\dot{A}(a) > 0$).

(ii) If $\dot{A}(t_i) = 0$, then the graph of $x^*(t)$ in a neighborhood of t_i is the orbit of a center manifold of (11)-(12), so $x^*(t)$ is C^1 in t_i (see the discussion in the proof of Lemma 4 for the case $\dot{A}(a) = 0$).

(iii) If $\dot{A}(t_i) < 0$, then the graph of $x^*(t)$ in a neighborhood of t_i matches an orbit of (11)-(12) with slope

$$\dot{x}^*(t_i) = \frac{B(t_i)}{2} - \sqrt{\frac{B(t_i)^2}{4} + \dot{A}(t_i)},$$

so $x^*(t)$ is C^1 in t_i (see the discussion in the proof of Lemma 5 for the case $\dot{A}(b) < 0$).

Finally, the T-periodic extension of x^* is a C^1 solution of (3) defined in $\mathbb{R}$, which completes the proof. □

Remark 2. *Notice that if $A(t)$ has degenerate zeros, then the construction of the solution $x^*(t)$ requires the use of the Center Manifold Theorem. This means that, in such a case, there may exist a family of periodic solutions of (3) with the same sign as $-A(t)B(t)$.*

3. Uniqueness and stability

The next result reveals that the T-periodic solution(s) with non constant sign that arise from Theorem 1 are unique, in the sense that there do not exist other T-periodic solutions with nonconstant sign in (3) sharing only some of the zeros of $A(t)$.

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled. Then, all T-periodic solutions of (3) with nonconstant sign have the sign of $-A(t)B(t)$. Moreover, if all the zeros of $A(t)$ are simple, then (3) has a unique C^1, T-periodic solution with nonconstant sign.*

Proof. Let $x(t)$ be a solution of (3) with nonconstant sign and non identically 0 (otherwise the result is trivial). As stated in Section 1, the zeros of $x(t)$ are also zeros of $A(t)$. Then, let $I = (a, b)$ be an interval where $x(t) \neq 0$, for all $t \in I$, and such that $x(a) = x(b) = 0$. It is no loss of generality to assume that $x(t) < 0$ in I (see Remark 1), which yields $\dot{x}(a) \leq 0$. Let us consider three cases:

(i) If $\dot{A}(a) > 0$, it has been noticed in the proof of Lemma 4 that (11)-(12) posesses two invariant manifolds in $(a, 0)$, a stable one and an unstable one, with negative and positive slopes, respectively. As $x(t)$ has non positive derivative in $t = a$, it has to coincide with the solution curve corresponding to the stable manifold orbit and, consequently, with the periodic solution with nonconstant $x^*(t)$ guaranteed by Theorem 1, from $t = a$ to the next zero of $A(t)$, and this zero has to be in $t = b$.

(ii) If $\dot{A}(a) = 0$, it has been noticed in the proof of Lemma 4 that (11)-(12) possesses two invariant manifolds in $(a, 0)$, a (non necessarily unique) center one and an unstable one, with null and positive slopes, respectively. As $x(t)$ has non positive derivative in $t = a$, it has to coincide with one of the solution curves corresponding to the center manifold and, consequently, with the periodic solution with nonconstant $x^*(t)$ guaranteed by Theorem 1, from $t = a$ to the next zero of $A(t)$, and this zero has to be in $t = b$.

(iii) If $\dot{A}(a) < 0$, it has been noticed in the proof of Lemma 4 that (11)-(12) possesses an unstable node in $(a, 0)$, and also that all the solutions tending to this point have positive slope. This is in contradiction with $\dot{x}(a) \leq 0$, which means that there cannot exist a solution of (3) verifying $\dot{x}(a) \leq 0$ and $\dot{A}(a) < 0$.

It is therefore proved that $x(t) = x^*(t)$, for all t such that $x(t) \neq 0$. Furthermore, when $x(t) = 0$, then $A(t) = 0$ and $x^*(t) = 0$, which implies that $x(t) = x^*(t)$, for all $t \in \mathbb{R}$.

Finally, when all the zeros of $A(t)$ are simple, the uniqueness of the Stable Manifold (see the proof of Lemma 4) yields the existence of a unique T-periodic solution, $x^*(t)$, with nonconstant sign. □

The instability of the T-periodic solutions of (3) with nonconstant sign is claimed in next Theorem:

Theorem 3. *Let the assumptions of Theorem 1 be fulfilled. Then, any C^1, T-periodic solution of (3) with nonconstant sign is unstable.*

Proof. Assume that $B(t) > 0$, for all t, which is no loss of generality. Let $I = (a, b) \in \mathcal{P}$, i.e. an interval such that $A(t) > 0$, for all t (in case that $A(t) \leq 0$, for all t, use the change of variables $(t, x) \mapsto (-t, -x)$ suggested in Remark 1). Theorems 1 and 2 ensure that any T-periodic solution of (3) with nonconstant sign has the sign of $-A(t)$. Hence, let $x^*(t)$ denote one of these solutions, which satisfies $x^*(t) < 0$, for all $t \in I$.

Let $x_a \in \mathbb{R}^+$ and consider the straight line $r_{x_a} := \{(t, x) : x - x_a = 0\}$. It is rather immediate that there exists an open interval $J \subseteq \mathbb{R}^+$ such that, for all $x_a \in J$,

$$S(t, x) = \frac{A(t)}{x_a} + B(t) + C(t)x_a > 0, \quad \forall (t, x) \in [a, b) \times \mathbb{R}^+ \cap r_{x_a}.$$

Therefore, by Lemma 1.i, any solution $x(t, x_a)$ of (3) with $x(a) = x_a \in J$ is strictly positive for all $t \in I$. As a consequence, $|x(t, x_a) - x^*(t)| > |x^*(t)|$ in I, which yields the instability of $x^*(t)$. □

Remark 3. *When $t = a$ is a simple zero of $A(t)$, the behavior of the solutions brought out in Theorem 3 arises immediately from the fact that, in such a case, $(a, 0)$ is a saddle point of the corresponding two-dimensional system (11)-(12).*

4. An example case: the normal form

The normal form of Abel equations of the second kind is of the form

$$x\dot{x} = A(t) + x. \tag{14}$$

This type of ODE is specially important because the generic Abel equation of the second kind (3) is readily transformable to (14) with a change of variables that, however, does not preserve time [1].

Proposition 1. *Let $A(t)$ be a C^1, T-periodic function. If $A(t)$ has at least one zero in $[0, T]$ and*

$$\min \dot{A}(t) > -\frac{1}{4}, \tag{15}$$

then (14) *has, at least, one T-periodic solution, $x^*(t)$, that has the sign of $-A(t)$, and it is also C^1 and unstable. Furthermore, if all the zeros of $A(t)$ are simple, then $x^*(t)$ is the unique T-periodic solution of (14) with nonconstant sign. Additionally, if*

$$\int_0^T A(t)dt = 0, \tag{16}$$

then (14) has no other periodic solutions but $x^*(t)$.

Proof. The first part is straightforward from Theorems 1, 2 and 3.

The last part of the statement follows if one can ensure that, when (16) is assumed, (14) has no T-periodic solutions with constant sign. For, the change of variables $x \mapsto x^{-1}$ and Theorem 2.1 in [9] guarantee the nonexistence of positive periodic solutions in (14). An equivalent conclusion for negative periodic solutions follows using $x \mapsto -x^{-1}$. □

Remark 4. *It is worth mentioning that restriction (15) is sharp. For, recall from Section 2 that t_i denotes an element of Z, the set of time instants where $A(t)$ vanishes. Then, notice that if $\dot{A}(t_i) < -1/4$, the phase plane point $(t_i, 0)$ becomes a focus of (11)-(12), which implies that there can not exist any solution $x(t)$ of (14) such that $x(t_i) = 0$.*

References

[1] A.D. Polyanin, Handbook of exact solutions for ordinary differential equations (2nd ed.), Boca Raton: Chapman & Hall/CRC, 2003.
[2] E. Fossas, J.M. Olm, H. Sira-Ramírez, Iterative approximation of limit cycles for a class of Abel equations, Physica D 237 (2008) 3159–3164.
[3] D.E. Panayotounakos, N. Sotiropoulos, Exact analytic solutions of unsolvable classes of first and second-order nonlinear ODEs (Part II: Emden-Fowler and related equations), Appl. Math. Lett. 18 (2005) 367–374.
[4] P.J. Torres, Existence of closed solutions for a polynomial first order differential equation, J. Math. Anal. Applic. 328 (2007) 1108-1116.
[5] M.A.M. Alwash, Periodic solutions of Abel differential equations, J. Math. Anal. Applic. 329 (2007) 1161–1169.
[6] A. Gasull, J. Llibre, Limit cycles for a class of Abel equations, SIAM J. Math. Anal. 21 (1990) 12351244.
[7] K.M. Andersen, A. Sandqvist, Existence of closed solutions of an equation $\dot{x} = f(t, x)$ where $f'_x(t, x)$ is weakly convex or concave in x, J. Math. Anal. Appl. 229 (1999) 480-500.
[8] M.J. Álvarez, A. Gasull, H. Giacomini, A new uniqueness criterion for the number of periodic orbits of Abel equations, J. Differential Equations 234 (2007) 161–176.
[9] J.L. Bravo, J. Torregrosa, Abel-like differential equations with no periodic solutions, J. Math. Anal. Applic. 342 (2008) 931—942.
[10] M.J. Alvarez, J.L. Bravo, M. Fernandez. Uniqueness of limit cycles for polynomial first-order differential equations, J. Math. Anal. Applic. 360 (2009) 168-189.
[11] J.L. Bravo, M. Fernández and A. Gasull, Limit cycles for some Abel equations having coefficients without fixed signs, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 19 (2009) 3869–3876.
[12] S. Smale, Mathematical problems for the next century, Math. Intelligencer 20 (1998) 7–15.
[13] E. Fossas-Colet, J.M. Olm-Miras, Asymptotic tracking in DC-to-DC nonlinear power converters, Discrete Cont. Dynam. Syst. - B 2 (2002) 295–307.
[14] J.M. Olm, X. Ros-Oton, Y.B. Shtessel, Stable inversion of Abel equations: Application to tracking control in DC-DC nonminimum phase boost converters, Automatica (2010), in press.
[15] J. Guckenheimer, P. Holmes, Nonlinear oscillations, dynamical systems and bifurcations of vector fields, 2nd ed., Springer-Verlag, NY, 1985.
[16] A. Kelly, The stable, center-stable, center, center-unstable and unstable manifolds, J. Differential Equations 3 (1967) 546–570.
[17] A. Gasull, H. Giacomini, A New Criterion for Controlling the Number of Limit Cycles of Some Generalized Liénard Equations, J. Differential Equations 185 (2002) 54–73.