A Discontinuous Galerkin Formulation for Solid Dynamics

J. Peraire[*], J. Bonet[**], A. Huerta[+], P.-O. Persson[*] and Y. Vidal[+*]

[*] Massachusetts Institute of Technology

[**] University College Swansea

[+] Polytechnic University of Catalonia

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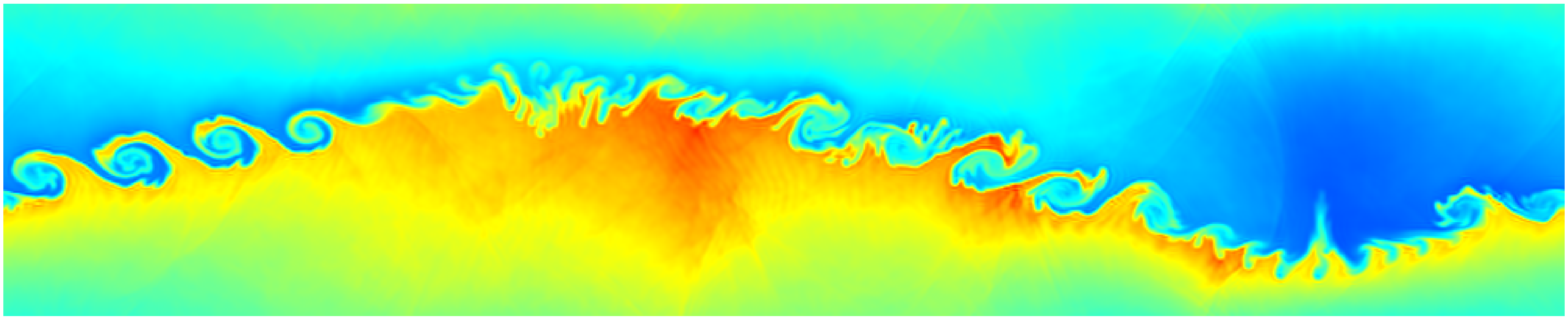
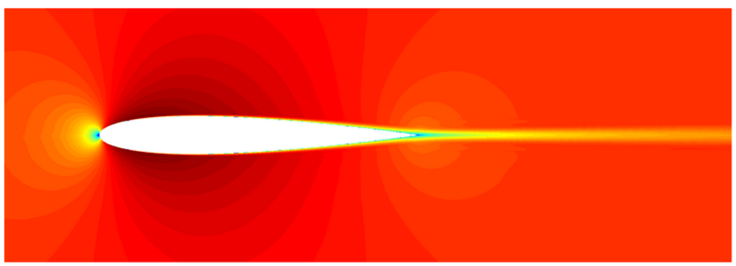
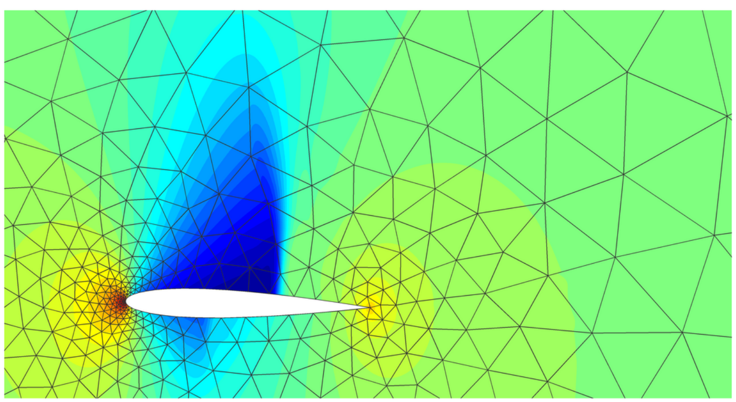
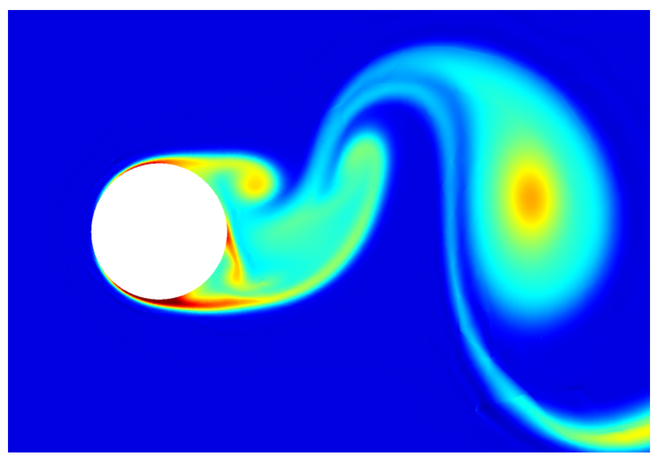
DG for Eulerian CFD

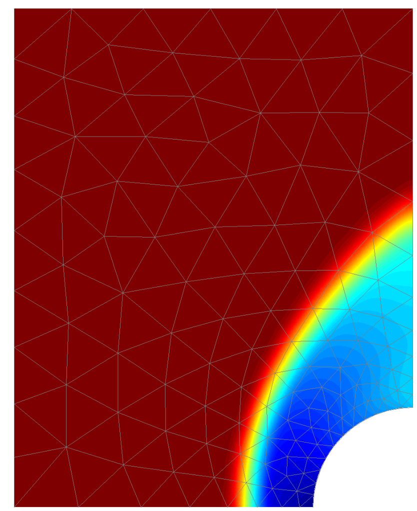
High Resolution, Complex Geometries/Physics, $0.001 \leq M \leq 10 - 20$











Objectives

- General Materials, e.g. Neo-Hookean, Visco-elasto-plastic
- Arbitrary Description (Lagarangian/Eulerian)
- High Energy/Large Deformation
- Shock waves incorporating correct entropy production
- Accurate resolution of isentropic shear/volumetric waves
- Low Numerical Dissipation/Dispersion
- Multi-material Interactions
- Accuracy Guarantees

Approach

- Cast Governing Equations as System of First Order Conservation Laws
- Account for involutions by appropriate selection of approximating spaces
- Use High-Order DG
- For Dissipative/Non-local Fluxes introduce local variables (LDG/CDG)
- Use “standard” shock capturing techniques

Here, we will present basic formulation for Lagrangian non-dissipative Neo-Hookean material

The Discontinuous Galerkin Method

- (Reed/Hill 1973, Cockburn/Shu 1998-2001, etc)
- Consider non-linear hyperbolic system in conservative form:

$$\boldsymbol{\mathcal{U}}_t + \nabla \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}}) = \boldsymbol{\mathcal{S}}(\boldsymbol{\mathcal{U}})$$

- Triangulate domain Ω into elements $\kappa \in T_h$
- Seek approximate solution $\boldsymbol{\mathcal{U}}_h$ in space of element-wise polynomials:

$$V_h^p = \{\boldsymbol{\mathcal{V}} \in L^2(\Omega) : \boldsymbol{\mathcal{V}}|_\kappa \in P^p(\kappa)\ \forall \kappa \in T_h\}$$

- Multiply by test function $\boldsymbol{\mathcal{V}}_h \in \boldsymbol{\mathcal{V}}_h^p$ and integrate over element κ:

$$\int_\kappa \left[(\boldsymbol{\mathcal{U}}_h)_t + \nabla \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}}_h) - \boldsymbol{\mathcal{S}}(\boldsymbol{\mathcal{U}}_h)\right] \boldsymbol{\mathcal{V}}_h \, d\boldsymbol{x} = 0$$

The Discontinuous Galerkin Method

- Integrate by parts:

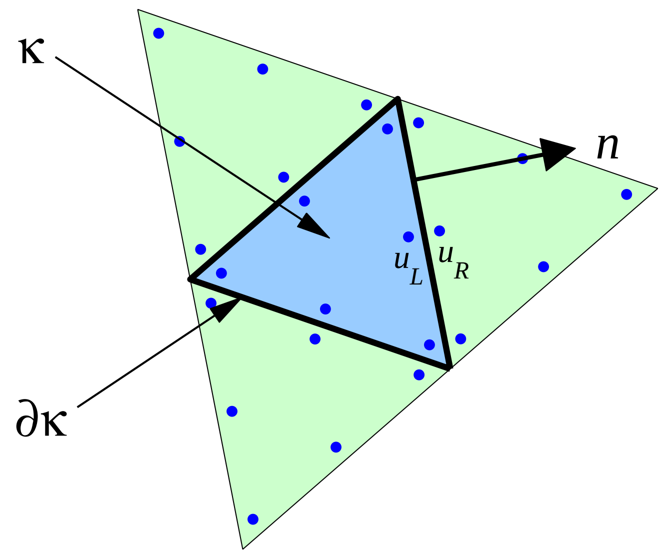
$$\int_\kappa [(\boldsymbol{\mathcal{U}}_h)_t]\, \boldsymbol{\mathcal{V}}_h \, d\boldsymbol{x} - \int_\kappa \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}}_h) \nabla \boldsymbol{\mathcal{V}}_h \, d\boldsymbol{x} + \int_{\partial\kappa} \mathcal{H}(\boldsymbol{\mathcal{U}}_h^+, \boldsymbol{\mathcal{U}}_h^-, \hat{\boldsymbol{n}}) \boldsymbol{\mathcal{V}}_h^+ \, ds = \int_\kappa \boldsymbol{\mathcal{S}}(\boldsymbol{\mathcal{U}}_h) \boldsymbol{\mathcal{V}}_h \, d\boldsymbol{x}$$

  with numerical flux function $\mathcal{H}(\boldsymbol{\mathcal{U}}_L, \boldsymbol{\mathcal{U}}_R, \hat{\boldsymbol{n}})$ for left/right states $\boldsymbol{\mathcal{U}}_L$,$\boldsymbol{\mathcal{U}}_R$ in direction $\hat{\boldsymbol{n}}$ (Godunov, Roe, Osher, Van Leer, **Lax-Friedrichs**, etc)

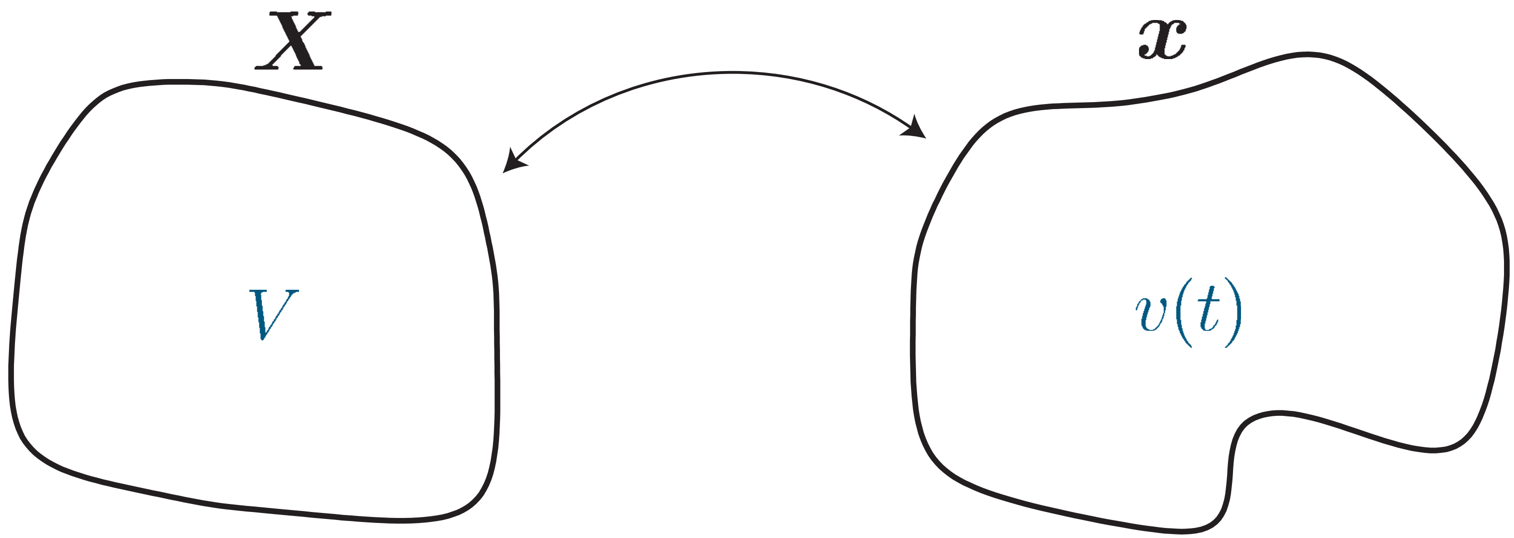
- Global view: Find $\boldsymbol{\mathcal{U}}_h \in \boldsymbol{\mathcal{V}}_h^p$ such that this weighted residual is zero for all $\boldsymbol{\mathcal{V}}_h \in \boldsymbol{\mathcal{V}}_h^p$

- Spatial Error $\mathcal{O}(h^{p+1})$ for smooth problems

- Explicit Time Integration Using RK Method



Lagrangian Formulation



$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$$

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\,, \qquad J = \det \boldsymbol{F}\,, \qquad \boldsymbol{v}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{x}}{\partial t}\,, \qquad \boldsymbol{p} = \rho_0 \boldsymbol{v}$$

Conservation Laws

- Momentum

$$\frac{d}{dt}\int_V \boldsymbol{p}\,dV = \int_V \rho_0 \boldsymbol{b}\,dV + \int_{\partial V} \boldsymbol{t}\,dA\,, \qquad \boldsymbol{t} = \boldsymbol{P}\boldsymbol{N}$$

$$\frac{\partial \boldsymbol{p}}{\partial t} = \boldsymbol{\nabla}\cdot\boldsymbol{P} + \rho_0\boldsymbol{b}$$

- Deformation Gradient

$$\frac{\partial \boldsymbol{F}}{\partial t} = \boldsymbol{\nabla}(\boldsymbol{p}/\rho_0) \qquad \text{or} \qquad \frac{\partial \boldsymbol{F}}{\partial t} = \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{I}\right)$$

$$\frac{d}{dt}\int_V \boldsymbol{F}\,dV = \int_{\partial V} \frac{1}{\rho_0}\boldsymbol{p}\otimes\boldsymbol{N}\,dA$$

"generalized" form of continuity

Constitutive Model

- Strain Energy Potential

$$\psi(\boldsymbol{F}) = \psi_{iso}(J^{-1/3}\boldsymbol{F}) + \psi_{vol}(J)$$

$$\psi_{iso} = \frac{\mu}{2}[tr(J^{-2/3}\boldsymbol{F} : \boldsymbol{F}) - 3] , \qquad \psi_{vol} = \frac{1}{2}\kappa(J-1)^2$$

- First Piola-Kirchhoff Stress Tensor

$$\boldsymbol{P} = \boldsymbol{P}_{dev} + \boldsymbol{P}_{vol} \ ; \ \boldsymbol{P}_{dev} = \frac{\partial \psi_{iso}}{\partial \boldsymbol{F}} \ , \ \boldsymbol{P}_{vol} = \frac{\partial \psi_{vol}}{\partial \boldsymbol{F}}$$

$$\boldsymbol{P}_{vol} = \frac{d\psi_{vol}(J)}{dJ}\frac{\partial J}{\partial \boldsymbol{F}} = pJ\boldsymbol{F}^{-T} \ ; \ p = \frac{d\psi_{vol}(J)}{dJ} = \kappa(J-1)$$

$$\boldsymbol{P}_{dev} = \mu J^{-2/3}[\boldsymbol{F} - \frac{1}{3}(\boldsymbol{F} : \boldsymbol{F})\boldsymbol{F}^{-T}]$$

Conservative Form of the Governing Equations

$$\frac{\partial \boldsymbol{p}}{\partial t} = \boldsymbol{\nabla} \cdot \boldsymbol{P}(\boldsymbol{F}) + \rho_0 \boldsymbol{b}$$
$$\frac{\partial \boldsymbol{F}}{\partial t} = \boldsymbol{\nabla} \cdot \left(\frac{1}{\rho_0} \boldsymbol{p} \otimes \boldsymbol{I} \right) \qquad \Rightarrow \qquad \boldsymbol{\mathcal{U}}_t + \nabla \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}}) = \boldsymbol{\mathcal{S}}(\boldsymbol{\mathcal{U}})$$

$$\boldsymbol{\mathcal{U}} = \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{F} \end{pmatrix}, \quad \boldsymbol{\mathcal{F}} = \begin{pmatrix} \boldsymbol{P} \\ \frac{1}{\rho_0} \boldsymbol{p} \otimes \boldsymbol{I} \end{pmatrix}, \quad \boldsymbol{\mathcal{S}} = \begin{pmatrix} \rho_0 \boldsymbol{b} \\ \boldsymbol{0} \end{pmatrix}$$

If $\boldsymbol{x}$ is required we can solve (independently)

$$\frac{\partial \boldsymbol{x}}{\partial t} = \boldsymbol{v} = \frac{1}{\rho_0} \boldsymbol{p}$$

Conservative Form of the Governing Equations

$$\mathcal{U}_t + \nabla \cdot \mathcal{F}(\mathcal{U}) = \mathcal{S}(\mathcal{U})$$

$$\mathcal{U} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ F_{11} \\ F_{12} \\ F_{13} \\ F_{21} \\ F_{22} \\ F_{23} \\ F_{31} \\ F_{32} \\ F_{33} \end{pmatrix} ; \quad \mathcal{F}_I = \begin{pmatrix} -P_{1I}(\boldsymbol{F}) \\ -P_{2I}(\boldsymbol{F}) \\ -P_{3I}(\boldsymbol{F}) \\ -\delta_{I1} p_1/\rho_0 \\ -\delta_{I2} p_1/\rho_0 \\ -\delta_{I3} p_1/\rho_0 \\ -\delta_{I1} p_2/\rho_0 \\ -\delta_{I2} p_2/\rho_0 \\ -\delta_{I3} p_2/\rho_0 \\ -\delta_{I1} p_3/\rho_0 \\ -\delta_{I2} p_3/\rho_0 \\ -\delta_{I3} p_3/\rho_0 \end{pmatrix} \quad I = 1, 2, 3 ; \quad \mathcal{S} = \begin{pmatrix} \rho_0 b_1 \\ \rho_0 b_2 \\ \rho_0 b_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Eigenvalue Structure

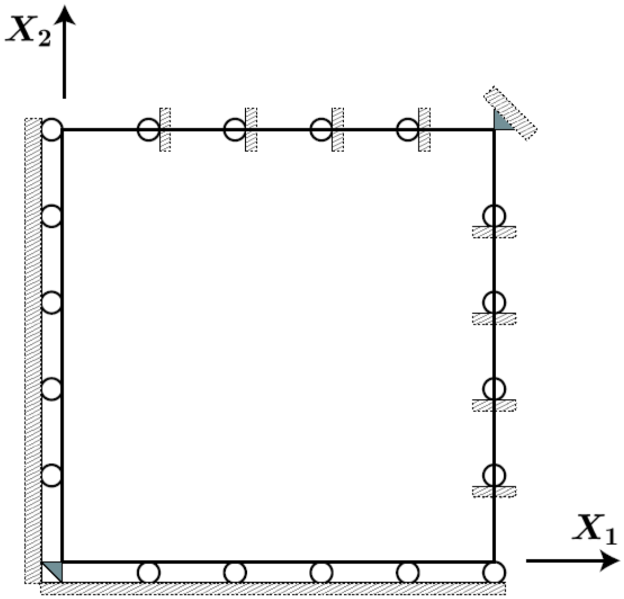
$$\boldsymbol{\mathcal{A}}_N = \frac{\partial \boldsymbol{\mathcal{F}}_N}{\partial \boldsymbol{\mathcal{U}}} = N_I \boldsymbol{\mathcal{A}}_I \ , \qquad \boldsymbol{\mathcal{A}}_I = \frac{\partial \boldsymbol{\mathcal{F}}_I}{\partial \boldsymbol{\mathcal{U}}} \qquad I = 1, 2, 3$$

- Two Acoustic Waves
- Four Shear Waves
- Six zero eigenmodes ($\delta \boldsymbol{F}^T = -\delta \boldsymbol{F}$ + “Plane Stress States”)

$$\lambda_{max} = \sqrt{\frac{\beta + \left(\frac{\alpha}{\varepsilon^2} + 2\gamma\right)}{\rho_0}} \ ,$$

$$\alpha = \kappa J^2 + \frac{5}{9}\mu J^{-2/3}(\boldsymbol{F} : \boldsymbol{F}) \ , \ \beta \ = \mu J^{-2/3} \ , \ \gamma = -\frac{2\mu}{3} J^{-2/3}$$

Example - Plane Strain



Unit Square

Linear Solution

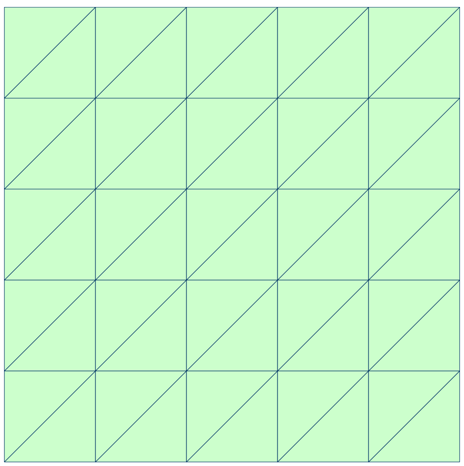
$$\boldsymbol{u}(t) = U_0 \cos(\frac{c_d \pi t}{\sqrt{2}}) \begin{bmatrix} \sin\left(\frac{\pi X_1}{2}\right) \cos\left(\frac{\pi X_2}{2}\right) \\ -\cos\left(\frac{\pi X_1}{2}\right) \sin\left(\frac{\pi X_2}{2}\right) \end{bmatrix} .$$

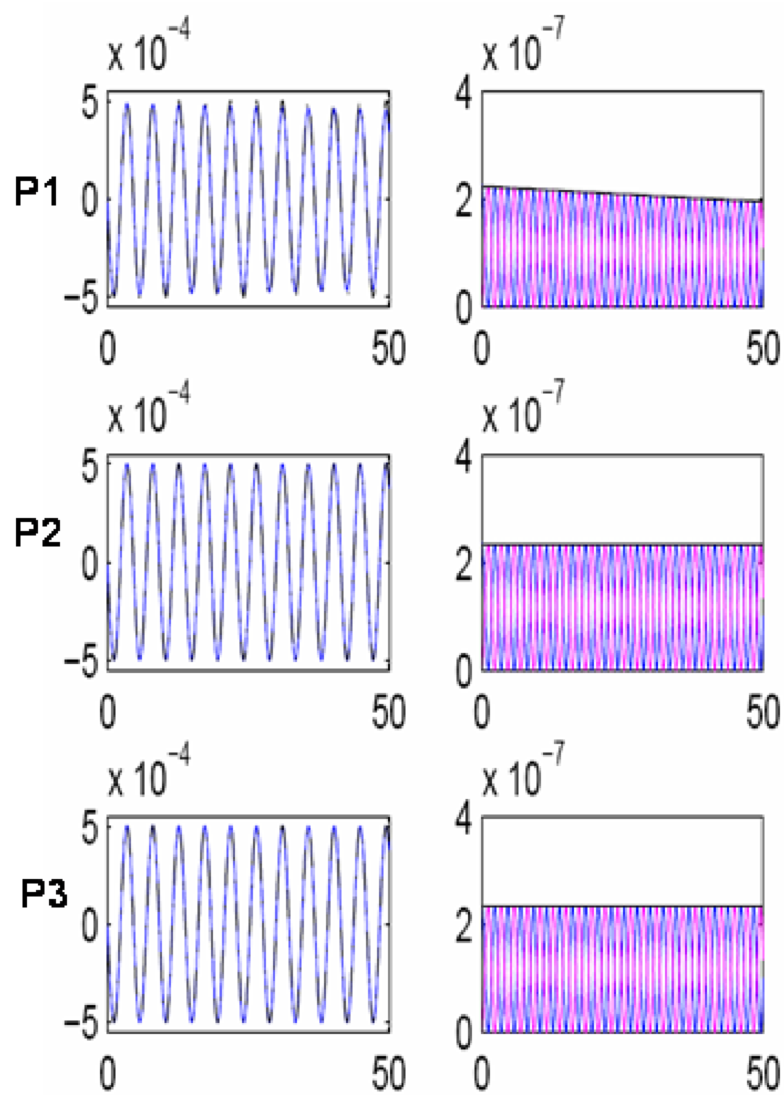
$$c_d = \sqrt{\frac{\mu}{\rho}}$$

Example - Plane Strain Solution

$U_0 = 5e - 4$ (linear)

$(5 \times 5 \times 2)$ mesh





Displacement

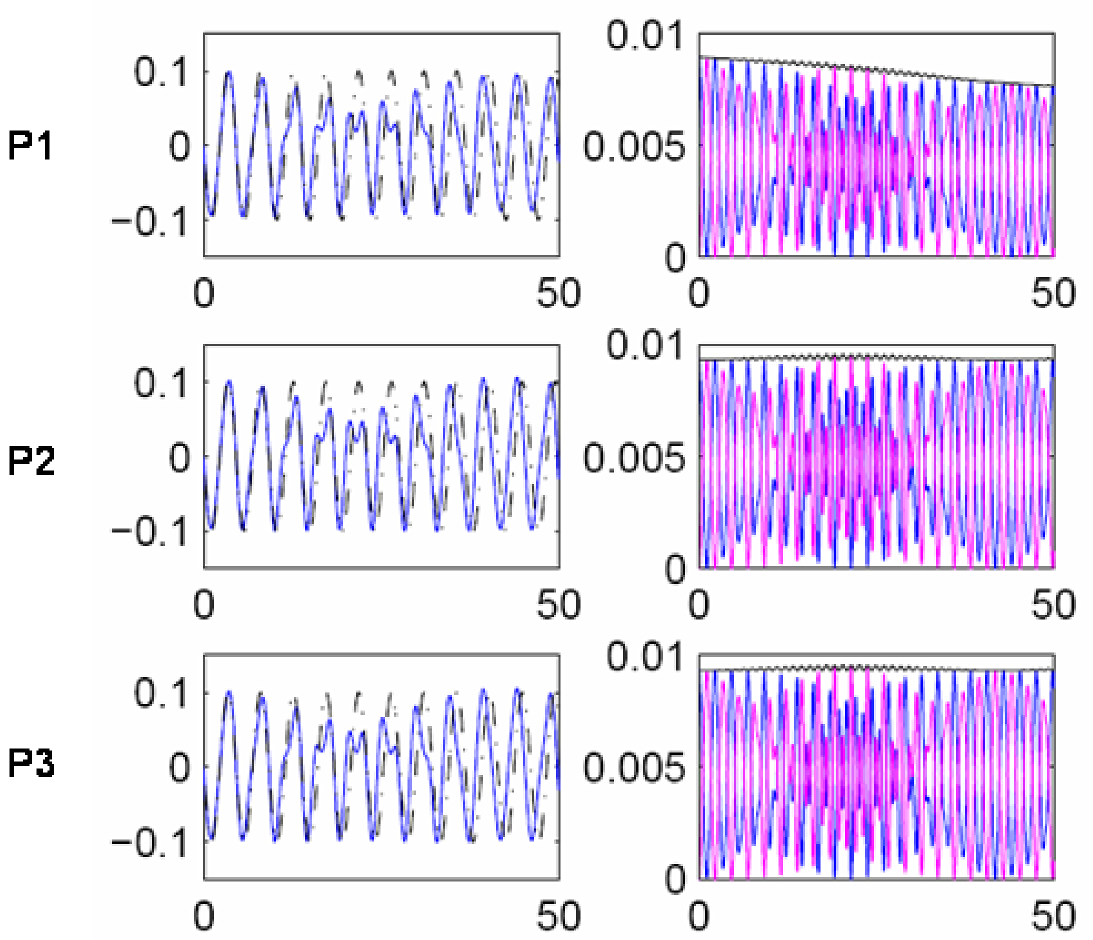
(Bottom Right Corner)

and

Energy

Example - Plane Strain Solution

$U_0 = 0.1$ (non-linear), $(5 \times 5 \times 2)$ mesh



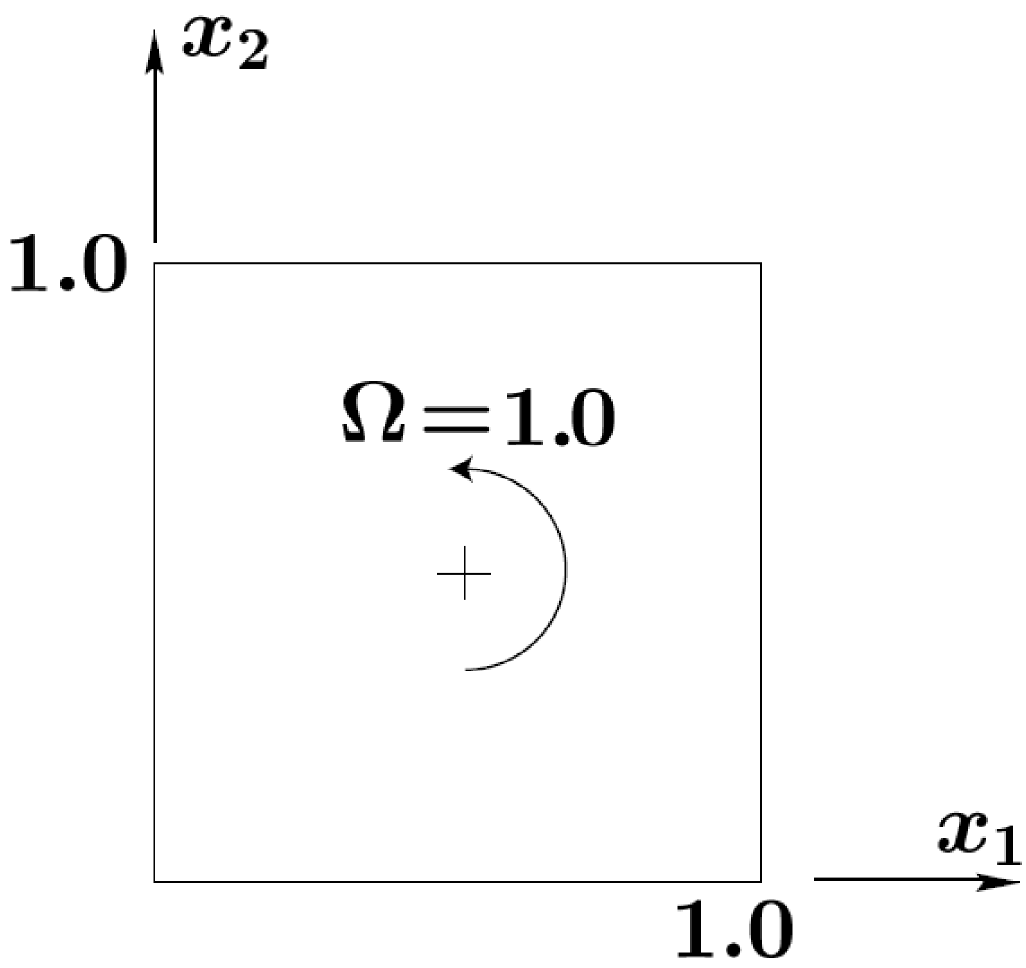
Displacement

(Bottom Right Corner)

and

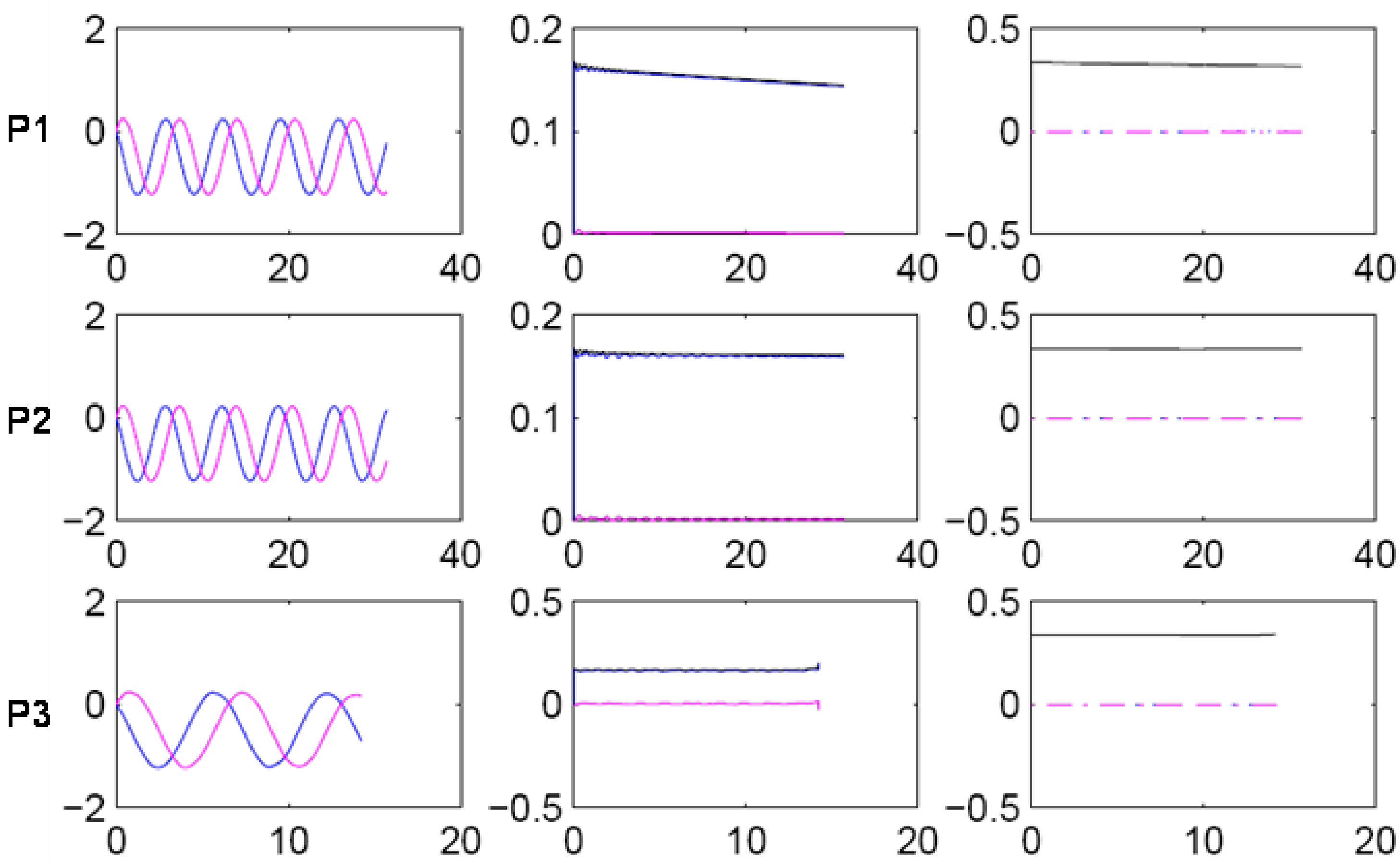
Energy

Example - Rotating Plate



"Plane Strain" Rotating Plate

Example - Rotating Plate



Displacement of point (1,1), Energy, Linear and Angular Momentum

The Problem

We solve for the evolution of $\boldsymbol{F}$

$$\frac{\partial \boldsymbol{F}}{\partial t} = \boldsymbol{\nabla} \cdot \left(\frac{1}{\rho_0} \boldsymbol{p} \otimes \boldsymbol{I} \right)$$

but $\boldsymbol{F}$ is a gradient

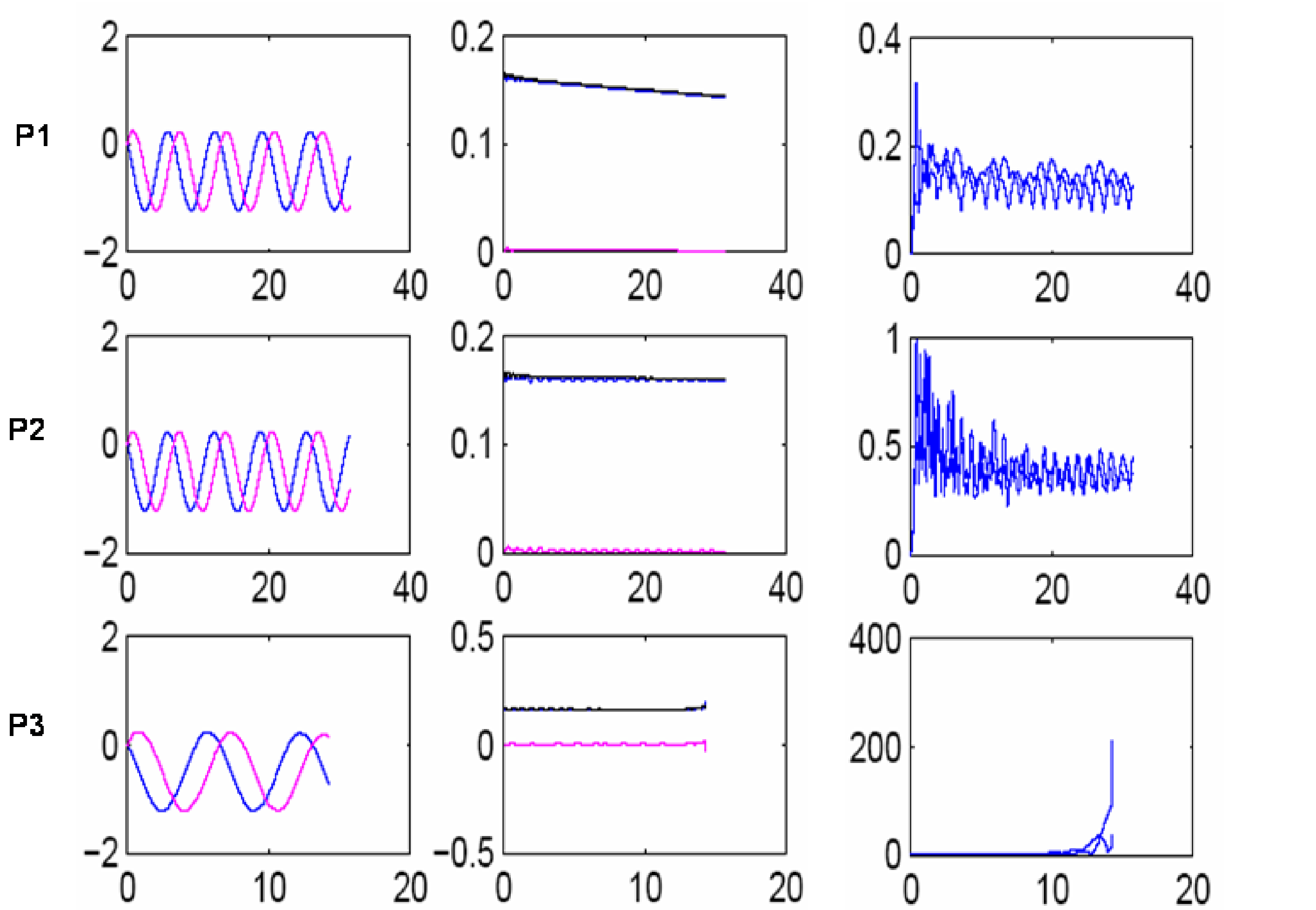
$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}^1 \\ \boldsymbol{F}^2 \\ \boldsymbol{F}^3 \end{pmatrix}$$

Therefore, we have

$$\nabla \times \boldsymbol{F}^i = \boldsymbol{0} \, , \qquad i = 1, 2, 3 \qquad \forall t$$

. . . this is called an **INVOLUTION**

Example - Rotating Plate



Displacement of point (1,1), Energy, $F_{11,2} - F_{12,1}$ and $F_{21,2} - F_{22,1}$

Dealing with the Involutions ($\nabla \times \boldsymbol{F}^i = 0$)

- Calculate a curl-free vector basis $\{\boldsymbol{\phi}_k\}$ within each element (SVD)

$$\nabla \times \boldsymbol{\phi}_k = \mathbf{0}, \quad \forall k$$

- Approximate each row of $\boldsymbol{F}$ using the curl-free basis

$$\boldsymbol{F}^i = \sum_k \boldsymbol{F}^i_k \boldsymbol{\phi}_k$$

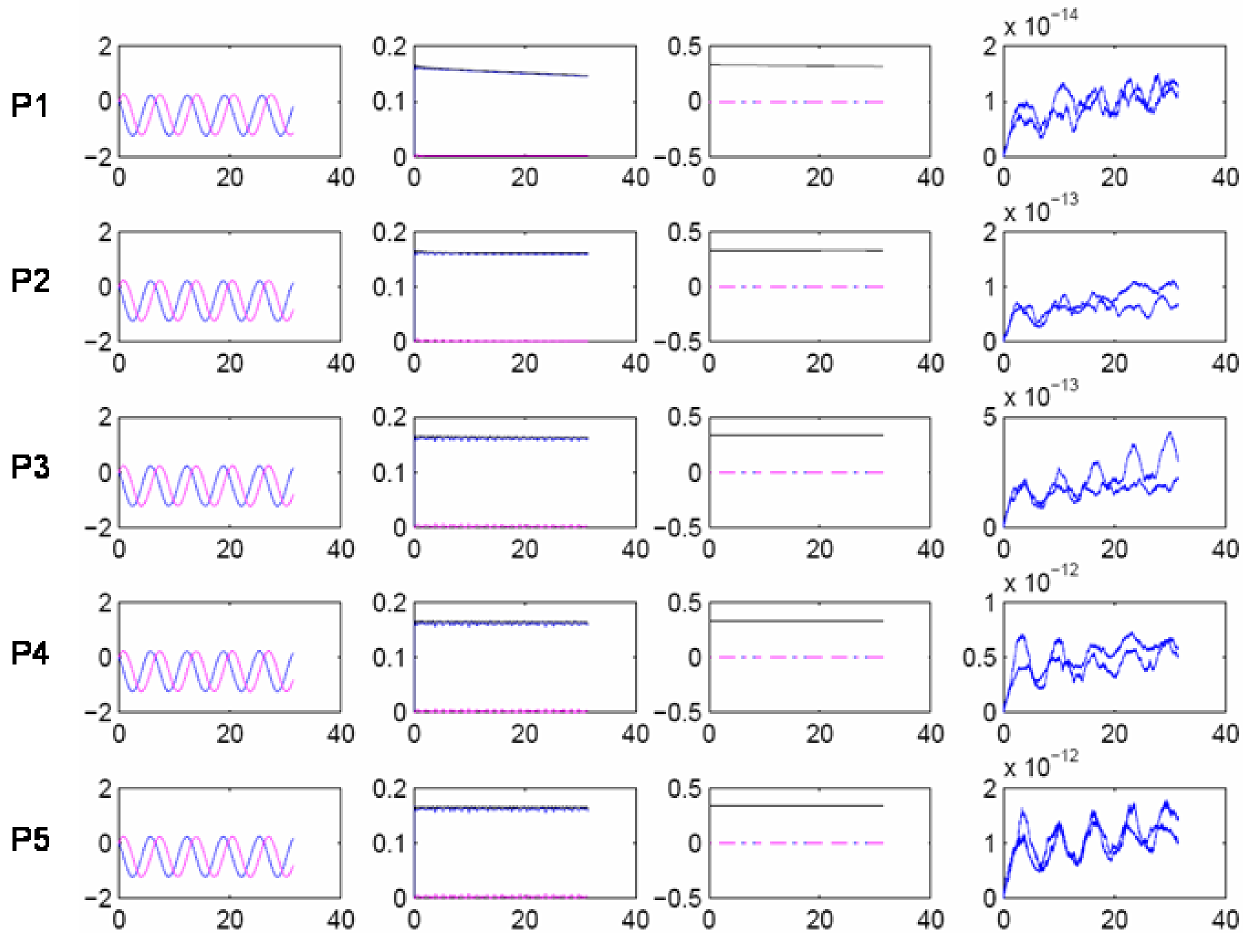
- Use standard DG approach with modified basis functions

- In practice, it can be implemented very efficiently by pre-calculating a modified inverse mass matrix $\boldsymbol{M}_*^{-1}$ for each element

$$\boldsymbol{M}^{-1} \;\Rightarrow\; \boldsymbol{M}_*^{-1} = \boldsymbol{W}(\boldsymbol{W}^T\boldsymbol{M}\boldsymbol{W})^{-1}\boldsymbol{W}^T$$

$\boldsymbol{W}$ is a rectangular matrix which expresses the curl-free basis in terms of the standard basis

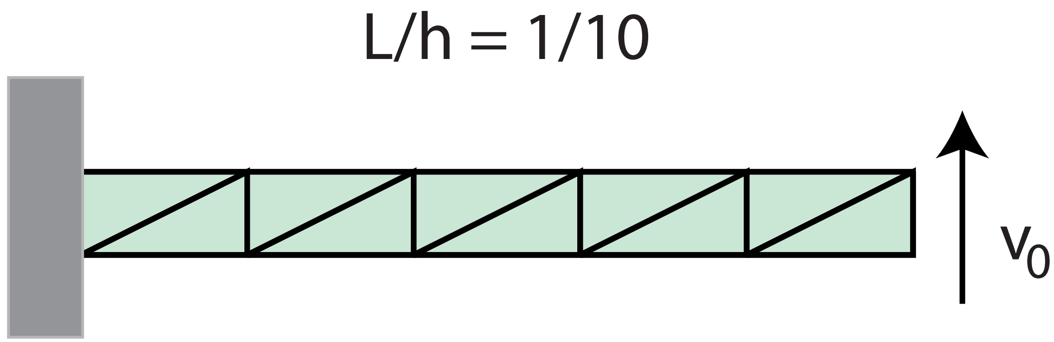
Example - Rotating Plate

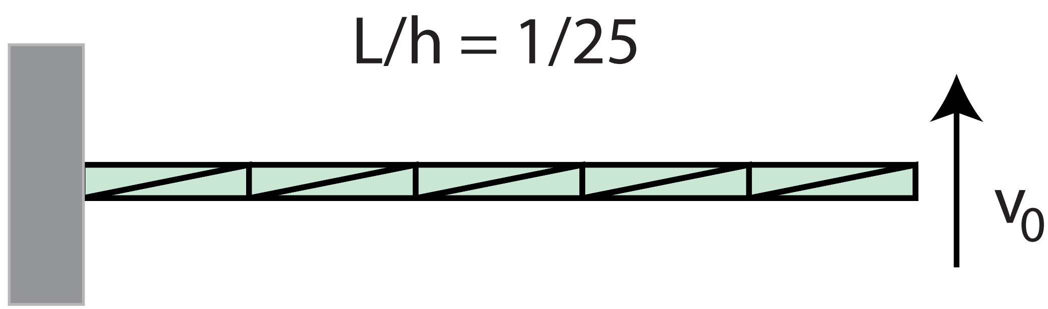


Displacement, Energy, Momentum, $\nabla \times \boldsymbol{F}^i$

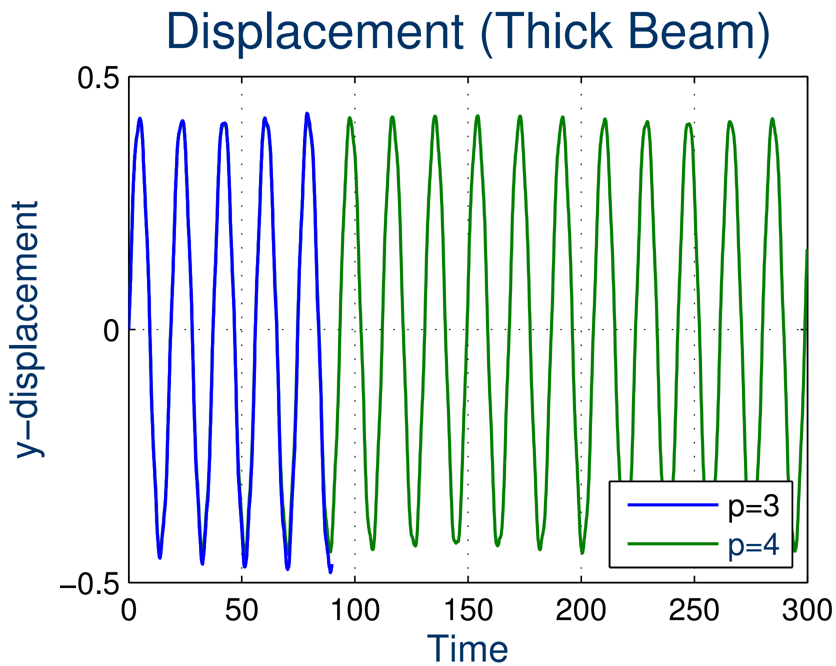
Bending Beam

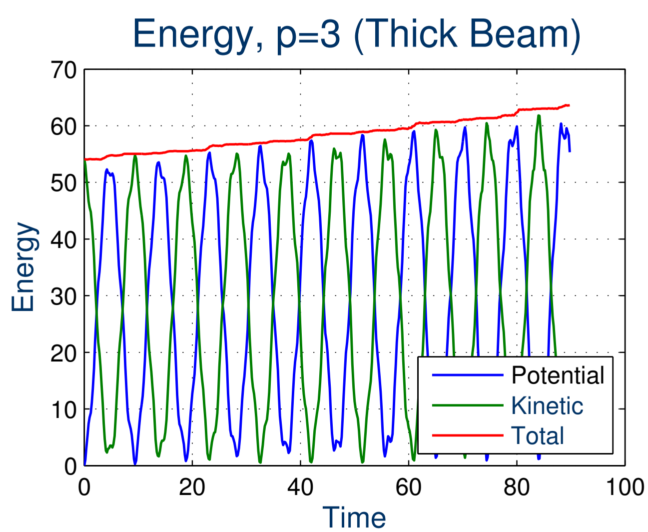
Initial velocity given by first eigenmode

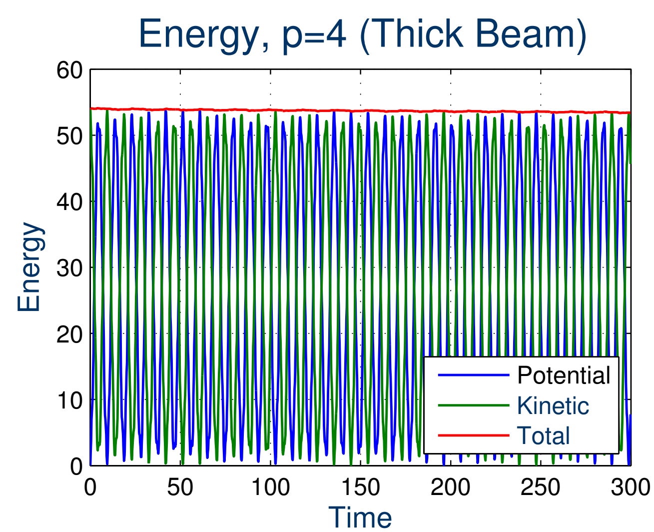




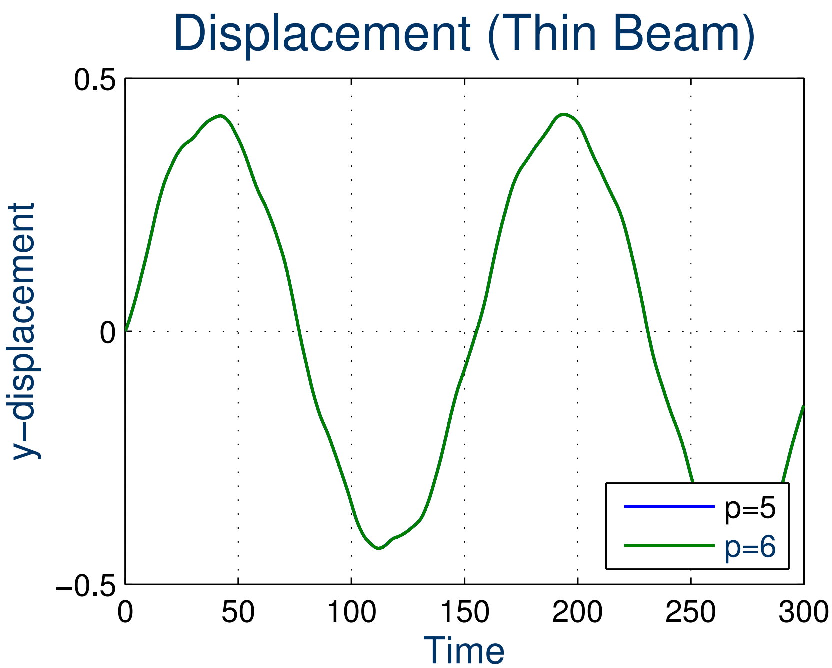
Thick Beam (1/10)

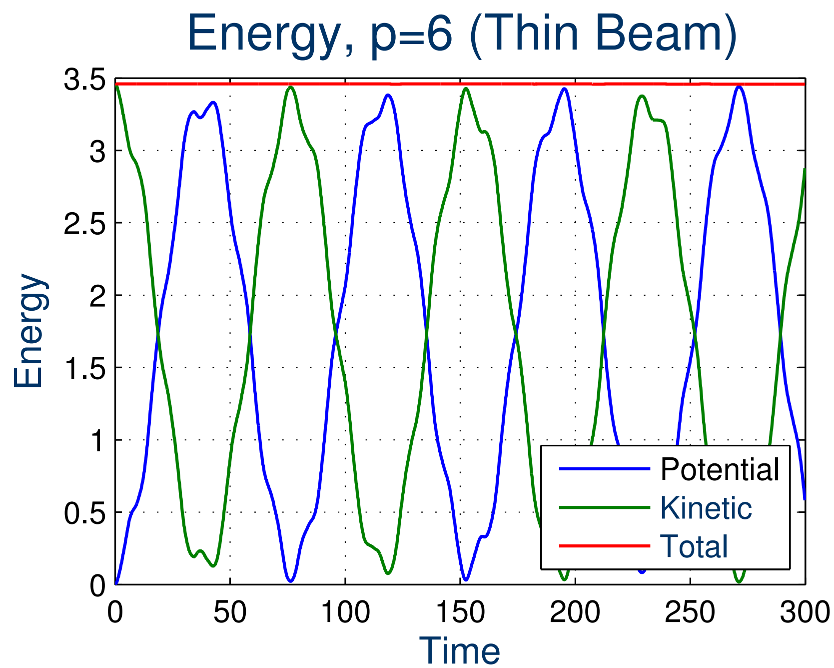






Thin Beam (1/25)





Current/Future Work

- More General Constitutive Relations $\psi(\boldsymbol{F}, \theta)$ e.g. Mie-Gruneisen
- Solve for Energy equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{q} - \frac{1}{\rho_0} \boldsymbol{P}^T \boldsymbol{p}) = 0$$

...plus additional constitutive internal variables

- Dissipation models (LDG, CDG)

$$\boldsymbol{\mathcal{U}}_t + \nabla \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{Q}}) = \boldsymbol{\mathcal{S}}(\boldsymbol{\mathcal{U}})$$
$$\boldsymbol{\mathcal{Q}} - \nabla \boldsymbol{\mathcal{U}} = \mathbf{0}$$

- Shock Capturing
- Arbitrary Lagarangian/Eulaerian Description
- Fluid Structure Interaction