

# Very Accurate Computation of the Impedance Elements on the Discretization of the Magnetic Field Integral Equation with the Orthogonal Basis Functions

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**Abstract**—We show a novel integrating technique, the direct evaluation method, that provides maximum accuracy in the computation of the MFIE-interactions between neighboring non-coplanar basis functions sharing an edge or a vertex of the discretization. Unlike the previous techniques, this strategy requires no extraction of quasi-singular terms from the Kernel and accounts for both inner- and outer-integrals. We show that the recently proposed discretization of the MFIE with orthogonal facet-oriented basis functions provide best accuracy in the RCS computation of objects with small electrical dimensions when compared with other conventional basis functions sets.

## I. INTRODUCTION

The discretization in Method of Moments (MoM) of the Magnetic-Field Integral Equation (MFIE) with the RWG basis functions [1], a constant-normal linear-tangential (CN/LT) divergence-conforming set, shows some discrepancy in the RCS computation when compared with the Electric-Field Integral Equation (EFIE). This discrepancy is especially evident in the RCS computation of sharp-edged objects with moderately small electrical dimensions. The numerical methods group of AntennaLab at UPC has proposed the monopolar RWG basis functions [2] to mitigate such misbehavior. The Kernel of the MFIE shows a quasi-singular  $R^2$ -order dependence for adjacent non-coplanar facets that needs to be integrated carefully to establish a fair assesment of the behavior of the basis functions expanding the current. Conventional strategies either subtract these quasi-singular terms of the Kernel and apply closed analytical formulas of integration for the inner-integral of the impedance element [3]–[7] or apply some variable changes to cancel out the singularity [8]. The testing of the scattered fields in the impedance elements requires an outer-integral that is normally carried out numerically with a Gaussian Quadrature rule. For edge-adjacent non-coplanar interactions, though, the MFIE results in weakly singular logarithmic contributions in the field domain for which this numerical testing provides limited accuracy in reasonable times.

In this paper, we adopt the recently proposed zero-order and first-order orthogonal basis functions sets [9], [10] for the MoM-discretization of the MFIE and we compare their performance in the RCS computation with a traditional discretization, such as RWG. Up to now, it has been impossible the fast and very accurate computation of the interactions between adjacent non-coplanar basis functions. In this paper, we implement a recently developed integrating technique, so-called direct evaluation method [11]–[16], that computes with machine-precision accuracy the  $1/R^2$  contributions arising in the impedance elements of the MoM-MFIE discretization. This technique directly deals with the 4D singular integrals, and after removing the singularity with some proper variable changes and performing a re-ordering of the resultant integrals, it is possible to compute two integrals analytically. Furthermore, the last two integrals contain a smooth integrand, resulting in a very fast convergence with the number of integration points, even being held numerically. We then obtain a very fair assessment of the RCS improvement due to the discretization of the MFIE with the orthogonal basis functions, when compared with the RWG set, in the RCS computation for sharp-edged objects.

## II. ORTHOGONAL BASIS FUNCTIONS

A set of basis functions  $\{\mathbf{f}_n\}$  is orthogonal when

$$\langle \mathbf{f}_i, \mathbf{f}_j \rangle = \int \mathbf{f}_i(\mathbf{r}) \mathbf{f}_j(\mathbf{r}) ds = 0 \quad \text{if } i \neq j \quad (1)$$

The zero-order orthogonal basis functions are piecewise constant over the triangles arising from the discretization. Since we use planar facets, two basis functions per facet are defined with perpendicular directions (u-v) tangential to the surface [9]. The 0-order orthonormal basis functions (ORT-0),  $\mathbf{b}_{0,u}$

and  $\mathbf{b}_{0,v}$  stand for

$$\mathbf{b}_{0,u} = \frac{\hat{\mathbf{u}}}{\sqrt{A}} \quad \mathbf{b}_{0,v} = \frac{\hat{\mathbf{v}}}{\sqrt{A}} \quad (2)$$

where  $A$  stands for the area of the facet and  $\hat{\mathbf{u}}, \hat{\mathbf{v}}$  denote the unit vectors along the perpendicular directions. The number of unknowns is twice the number of facets.

To justify the definition of our first-order orthogonal basis functions [10], we decompose the monopolar RWG [2] set of basis functions into its orthogonal components, of zero and first order. We adopt the monopolar RWG basis functions as reference because they provide, in our experience, remarkable accuracy improvement in the RCS computation of moderately electrically small sharp-edged objects. The monopolar RWG functions are defined as the RWG functions inside the facets but do not impose continuity of the normal-component of the current across the edge. Therefore, the monopolar RWG set is facet-oriented, just like the zero-order orthogonal basis functions ORT-0. The expansion of the current over a triangle arising from the discretization relies on three monopolar-RWG basis functions  $\mathbf{m}_i$  as

$$\mathbf{m}_i = \frac{(\mathbf{r} - \mathbf{r}_i)}{A} \quad i = 1, 2, 3 \quad (3)$$

where  $\mathbf{r}_i$  stands for the vertices defining the triangle. We can write the previous expression equivalently as

$$\mathbf{m}_i = \frac{(\mathbf{r}_c - \mathbf{r}_i)}{A} + \frac{(\mathbf{r} - \mathbf{r}_c)}{A} \quad i = 1, 2, 3 \quad (4)$$

where  $\mathbf{r}_c$  denotes the centroid or barycentre of the triangle. We see that the first term in (4) is uniform over the triangle and different for each monopolar-RWG basis function (see Fig. 1). In contrast, the second term is the same for each monopolar-RWG function  $\mathbf{m}_i$  and varies linearly over the triangle.

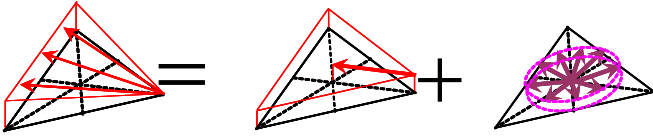


Fig. 1. Decomposition of the monopolar RWG basis functions in terms of its zero-order and first-order orthogonal components.

Therefore, being the first term piecewise constant, it can be expanded with generality by  $\mathbf{b}_{0,u}$  and  $\mathbf{b}_{0,v}$ , whereas the second linearly-varying term rules the definition of our first-order orthogonal basis function  $\mathbf{b}_{1,\rho}$

$$\mathbf{b}_{1,\rho} = \frac{(\mathbf{r} - \mathbf{r}_c)}{A} \quad (5)$$

Note now that the three basis functions  $\mathbf{b}_{0,u}$ ,  $\mathbf{b}_{0,v}$  and  $\mathbf{b}_{1,\rho}$  accomplish

$$\begin{aligned} \langle \mathbf{b}_{0,u}, \mathbf{b}_{0,v} \rangle &= \frac{1}{A} \int \hat{\mathbf{u}} \cdot \hat{\mathbf{v}} \, ds = 0 \\ \langle \mathbf{b}_{0,u}, \mathbf{b}_{1,\rho} \rangle &= \frac{1}{A^{3/2}} \int \hat{\mathbf{u}} \cdot (\mathbf{r} - \mathbf{r}_c) \, ds = 0 \\ \langle \mathbf{b}_{0,v}, \mathbf{b}_{1,\rho} \rangle &= \frac{1}{A^{3/2}} \int \hat{\mathbf{v}} \cdot (\mathbf{r} - \mathbf{r}_c) \, ds = 0 \end{aligned} \quad (6)$$

which is a proof for their orthogonality.

Since the ORT-1 discretization for triangular facets represents an orthogonal rearrangement of the monopolar-RWG set, they both lead to the same solution. Thanks to the property of orthogonality, the ORT-1 MoM-discretization of the MFIE leads to a Gram matrix that is diagonal.

### III. INTEGRATION: DIRECT EVALUATION METHOD

#### A. Hyper-singular integrals

The hyper-singular integrals appearing in MFIE, and therefore in CFIE formulations, can be reduced to integrals of the form

$$I := \int_{E_P} \mathbf{g}(\mathbf{r}) \cdot \int_{E_Q} \nabla G(\mathbf{r}, \mathbf{r}') \times \mathbf{f}(\mathbf{r}') \, dS' \, dS \quad (7)$$

where the functions  $\mathbf{f}(\mathbf{r})$  and  $\mathbf{g}(\mathbf{r})$  are any of the previously defined basis functions or their  $\hat{\mathbf{n}} \times$  versions defined over the triangles  $E_Q$  and  $E_P$ , respectively.  $G(\mathbf{r}, \mathbf{r}')$  corresponds to the free space Green's function  $G(R) = e^{-jkR}/R$ , depending on the distance  $R = |\mathbf{r} - \mathbf{r}'|$ .

In particular, the case where the two triangles have a common edge (see Fig. 2 on the left), which is actually the most challenging one, is in order.

#### B. Theoretical description

The main idea of the direct evaluation method is to cancel out the singularity  $1/R^2$  of the integrand, performing a set of parameter transformations. Following the procedure described in [15] and references therein, we start by moving to equilateral parameter spaces  $\{\eta, \xi\}$  and  $\{\eta', \xi'\}$  in each triangle (see Fig. 2):

$$\begin{aligned} \mathbf{r} &= \begin{bmatrix} \frac{x_2+x_1}{2} \\ \frac{y_2+y_1}{2} \\ \frac{z_2+z_1}{2} \end{bmatrix} + \begin{bmatrix} \frac{x_2-x_1}{2} & \frac{2x_3-x_1-x_2}{2\sqrt{3}} \\ \frac{y_2-y_1}{2} & \frac{2y_3-y_1-y_2}{2\sqrt{3}} \\ \frac{z_2-z_1}{2} & \frac{2z_3-z_1-z_2}{2\sqrt{3}} \end{bmatrix} \begin{bmatrix} \eta \\ \xi \end{bmatrix} \\ \mathbf{r}' &= \begin{bmatrix} \frac{x_2+x_1}{2} \\ \frac{y_2+y_1}{2} \\ \frac{z_2+z_1}{2} \end{bmatrix} + \begin{bmatrix} \frac{x_1-x_2}{2} & \frac{2x_4-x_1-x_2}{2\sqrt{3}} \\ \frac{y_1-y_2}{2} & \frac{2y_4-y_1-y_2}{2\sqrt{3}} \\ \frac{z_1-z_2}{2} & \frac{2z_4-z_1-z_2}{2\sqrt{3}} \end{bmatrix} \begin{bmatrix} \eta' \\ \xi' \end{bmatrix} \end{aligned} \quad (8)$$

Subsequently, two further consecutive polar coordinates changes are carried out:

$$\begin{aligned} \eta' &= \rho \cos(\theta) - \eta & \text{and} & & \rho &= \Lambda \cos(\Psi) \\ \xi' &= \rho \sin(\theta) & & & \xi &= \Lambda \sin(\Psi). \end{aligned} \quad (9)$$

Considering that the distance function  $R$  becomes proportional to  $\Lambda$ , the singularity, now placed at  $\Lambda = 0$ , is removed thanks to the global Jacobian

$$J = \frac{A_p A_q}{3} \Lambda^2 \cos(\Psi) \quad (10)$$

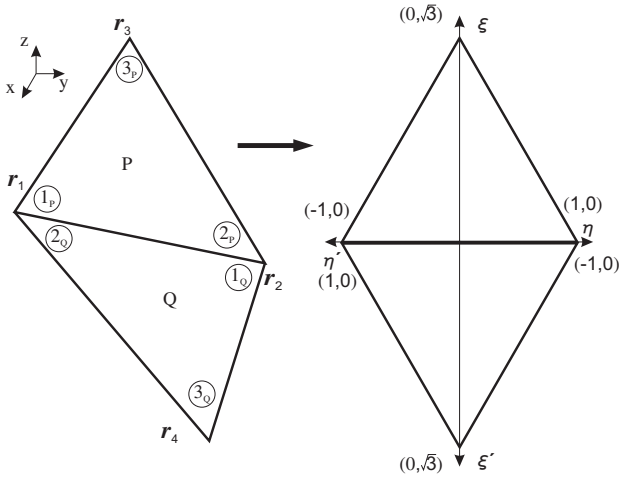


Fig. 2. Orientation of the triangular elements both in the original and the equilateral triangle parameter space.

yielding after some manipulations:

$$\begin{aligned}
 I = & \int_0^1 d\eta \int_0^{\Theta_1} d\theta \int_0^{\Psi_C} d\Psi \int_0^{\Lambda_{L_3}} F d\Lambda \\
 & + \int_0^1 d\eta \int_0^{\Theta_1} d\theta \int_{\Psi_C}^{\pi/2} d\Psi \int_0^{\Lambda_{L_2}} F d\Lambda \\
 & + \int_0^1 d\eta \int_{\Theta_1}^{\pi} d\theta \int_0^{\Psi_A} d\Psi \int_0^{\Lambda_{L_1}} F d\Lambda \\
 & + \int_0^1 d\eta \int_{\Theta_1}^{\pi} d\theta \int_{\Psi_A}^{\pi/2} d\Psi \int_0^{\Lambda_{L_2}} F d\Lambda
 \end{aligned} \quad (11)$$

where the transformed integrand  $F$  is smooth, with no singu-

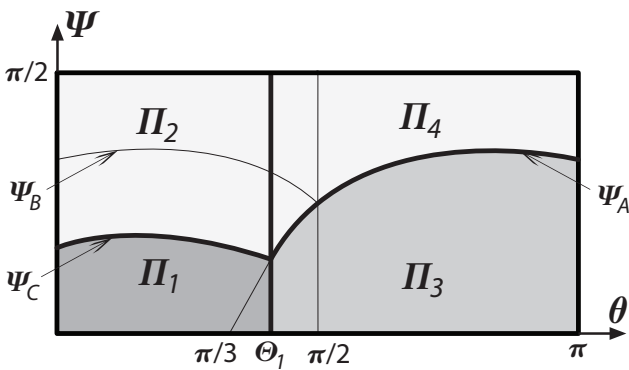


Fig. 3. Integration domains in eq. (11).

larities, and can be computed, in view of a symmetry at  $\eta = 0$

in the equilateral parameter space, as

$$\begin{aligned}
 F(\Lambda, \eta, \Psi, \theta) &= F^+(\Lambda, \eta, \Psi, \theta) + F^+(\Lambda, -\eta, \Psi, \pi - \theta) \\
 F^+(\Lambda, \eta, \Psi, \theta) &= \frac{A_p A_q}{3} \cos(\Psi) \\
 & \quad \left( \frac{e^{-jk\Lambda B(\theta, \Psi)}}{B(\theta, \Psi)^3} (1 + jk\Lambda B(\theta, \Psi)) \right) \\
 & \quad \mathbf{g}(\Lambda, \eta, \theta, \Psi) \cdot (\mathbf{f}(\Lambda, \eta, \theta, \Psi) \times \mathbf{B}(\theta, \Psi))
 \end{aligned} \quad (12)$$

being  $\mathbf{B}$  the proportionality constant, in terms of  $\Lambda$ , of the distance vector function  $\mathbf{R} = \Lambda \mathbf{B}(\theta, \Psi)$ :

$$\mathbf{B} = \alpha_{e_2} \sin(\Psi) + \alpha_{e_1} \cos(\Psi) \cos(\theta) + \alpha_{e_3} \cos(\Psi) \sin(\theta). \quad (13)$$

In the last expression,  $\alpha_{e_i}$  are constant vectors which only depend on the triangles' vertices:

$$\begin{aligned}
 \alpha_{e_1} &= \frac{\mathbf{r}_2 - \mathbf{r}_1}{2} \\
 \alpha_{e_2} &= \frac{2\mathbf{r}_3 - \mathbf{r}_1 - \mathbf{r}_2}{2\sqrt{3}} \\
 \alpha_{e_3} &= -\frac{2\mathbf{r}_4 - \mathbf{r}_1 - \mathbf{r}_2}{2\sqrt{3}}.
 \end{aligned} \quad (14)$$

The set of integration limits in expression (11) equal

$$\begin{aligned}
 \Theta_1(\eta) &= \frac{\pi}{2} - \tan^{-1} \left( \frac{\eta}{\sqrt{3}} \right) \\
 \Psi_A &= \tan^{-1} \left( \sin(\theta) - \sqrt{3} \cos(\theta) \right) \\
 \Psi_B &= \tan^{-1} \left( \sin(\theta) + \sqrt{3} \cos(\theta) \right) \\
 \Psi_C &= \tan^{-1} \left( \frac{1-\eta}{1+\eta} \left( \sin(\theta) + \sqrt{3} \cos(\theta) \right) \right) \\
 \Lambda_{L_1} &= \frac{\sqrt{3}(1-\eta)}{\cos(\Psi)(\sin(\theta) - \sqrt{3} \cos(\theta))} \\
 \Lambda_{L_2} &= \frac{\xi(\eta)}{\sin(\Psi)} = \frac{\sqrt{3}(1-\eta)}{\sin(\Psi)} \\
 \Lambda_{L_3} &= \frac{\sqrt{3}(1+\eta)}{\cos(\Psi)(\sin(\theta) + \sqrt{3} \cos(\theta))}.
 \end{aligned} \quad (15)$$

Bringing the integration with respect to  $\eta$  right after the integration with respect to  $\Lambda$ , properly re-ordering the integrals, it happens that both integrations,  $\Lambda$  first and  $\eta$  then, can be carried out analytically, whilst the remaining integrations with respect to  $\theta$  and  $\Psi$  are performed numerically, considering a smooth integrand. Results have proven the excellent convergence of this technique, leading to machine precision in less than one second in a common PC.

#### IV. RESULTS

We present computed RCS results for a cube with side of 0.1 m. The impinging planewave is x-polarized with +z-propagation and the wavelength is 1m. In Fig. 4, we show that the conventional strategy (cnv) to extract the  $1/R^2$  quasi-singular contributions in the impedance elements of the MoM-discretization of the MFIE [3], [5] with the ORTO basis

functions, MFIE[ORT0], does not reach stable results even after computing the outer integral and the inner integral of the lower order contributions of the Kernel with 12 points. In contrast, our new integrating technique (new) is perfectly stable already with 6 points, which is a consequence of its accuracy. Indeed, the conventional strategy requires more integrating points to compute numerically the testing-integral of the weak logarithmic field-contributions coming from the analytical source-integral of the terms  $(\rho' - \rho)/R^3$  between touching non-coplanar triangles.

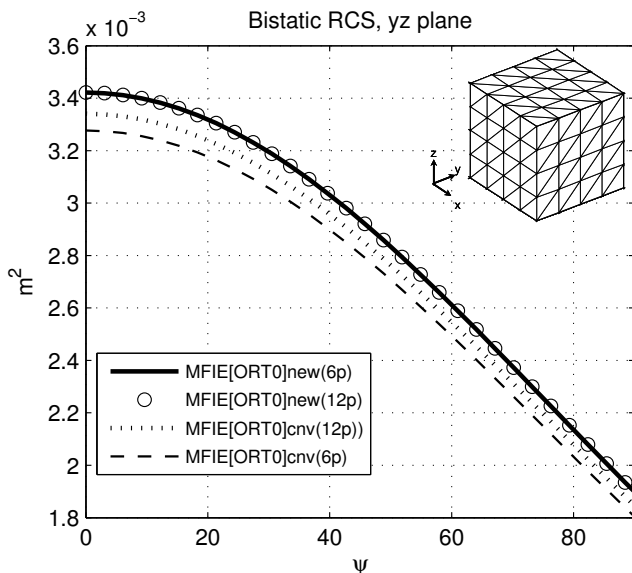


Fig. 4. yz-plane cut of the RCS for MoM-discretizations of the MFIE with ORT0 basis functions for a cube with side 0.1m meshed with 192 triangles and  $\lambda = 1m$  for an impinging x-polarized +z-propagating wave. Conventional and new integrating strategies of the  $1/R^2$  impedance contributions are applied.

In Fig. 5, we show the backward bistatic RCS of the MoM-discretizations of the MFIE with the zero- and first-order orthogonal basis functions, MFIE[ORT0] and MFIE[ORT1] with our new integrating technique and 6 quadrature points for the numerical computation of the remaining integrals (testing-integral and source-integral of the low-order Kernel contributions). We compare it with the conventional RWG-discretizations of the EFIE and MFIE; that is, EFIE[RWG] and MFIE[RWG]. In view of the figure, it is clear that MFIE[ORT0] and MFIE[ORT1] show a faster convergence against the number of unknowns than MFIE[RWG]. Therefore, the RCS-discrepancy between EFIE and MFIE with a RWG discretization gets drastically reduced with the adoption of the zero- and first-order orthogonal sets.

## V. CONCLUSIONS

In this paper, the use of the zero-order and first-order orthogonal basis functions in the MoM-MFIE computation has been presented, along with a recently developed very accurate integration technique, the direct evaluation method. In this manner, we are able to completely remove the errors

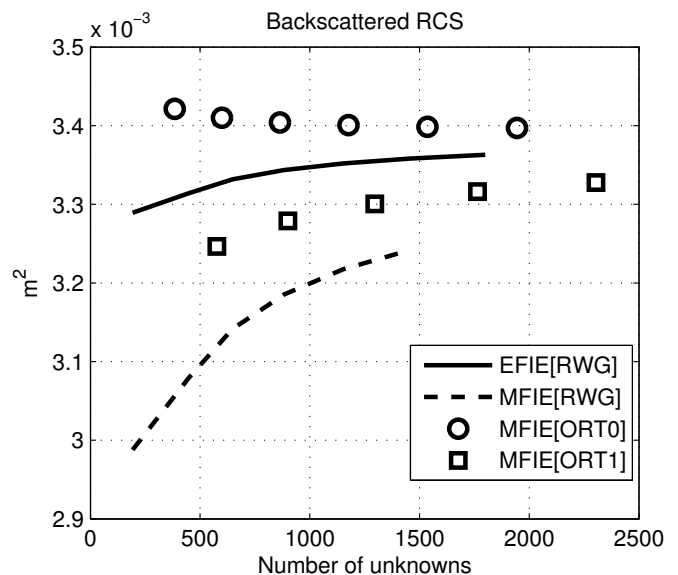


Fig. 5. Backscattered RCS for a cube with side 0.1m and  $\lambda = 1m$  against the number of unknowns for an impinging x-polarized +z-propagating wave.

coming from the integration of the singular kernels. We compare the RCS results with traditional CN/LT divergence-conforming discretizations of the MFIE and of the EFIE. We test several electrically small sharp-edged objects, where the traditional MoM-MFIE RCS-discrepancy is most evident and the proposed set of basis functions present a much better convergence.

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