Integrability and non-integrability of periodic non-autonomous Lyness recurrences*

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December 23, 2010

Abstract

This paper studies non-autonomous Lyness type recurrences of the form $x_{n+2} = (a_n + x_n)/x_{n+1}$, where $\{a_n\}_n$ is a k-periodic sequence of positive numbers with prime period k. We show that for the cases $k \in \{1, 2, 3, 6\}$ the behavior of the sequence $\{x_n\}_n$ is simple(integrable) while for the remaining cases satisfying $k \neq \dot{5}$ this behavior can be much more complicated(chaotic). The cases $k = \dot{5}$ are studied separately.

2000 Mathematics Subject Classification: 39A20, 39A11

Keywords: Periodic difference equations, integrability, non-integrability, meromorphic first integrals, chaos.

1 Introduction and main results

This paper deals with non-autonomous Lyness difference equations of the form

$$x_{n+2} = \frac{a_n + x_{n+1}}{x_n}, \tag{1}$$

* **Acknowledgements**. GSD-UAB and CoDALab Groups are supported by the Government of Catalonia through the SGR program. They are also supported by MCYT through grants MTM2008-03437 (first and second authors) and DPI2008-06699-C02-02 (third author).

where $\{a_n\}_n$ is a cycle of k positive numbers, i. e. $a_{n+k} = a_n$ for all $n \in \mathbb{N}$, being k the prime period and we consider positive initial conditions x_1 and x_2. As we will see, the behavior of the sequences $\{x_n\}_n$ can be essentially different according whether $k \in \{1, 2, 3, 6\}$, k is a multiple of 5 or it is not.

This section contains a first part with notations and definitions, a second one with a description of our main results on (1) in terms of k and a third one with the tools that we have developed that we believe that might be interesting by themselves.

1.1 Notations and definitions

Given a periodic sequence $\{a_n\}_n$ of prime period k we will say that its *rank* is m if

$$\mathrm{Card}\{a_1, a_2, \ldots, a_k\} = m \in \mathbb{N}.$$

The values $a_1, \ldots, a_k$ will be usually called *parameters*. In our context the recurrence (1) is called *persistent* if for any sequence $\{x_n\}_n$ there exist two real positive constants c and C, which depend on the initial conditions, such that for all n, $0 < c < x_n < C < \infty$.

For each k, the *composition maps* are

$$F_{a_k,\ldots,a_2,a_1} := F_{a_k} \circ \cdots \circ F_{a_2} \circ F_{a_1} \tag{2}$$

where each F_{a_i} is defined by

$$F_{a_i}(x, y) = \left(y, \frac{a_i + y}{x}\right)$$

and $a_1, a_2, \ldots, a_k$ are the k elements of the cycle. When there is no confusion, for the sake of shortness, we also will use the notation $F_{[k]} := F_{a_k,\ldots,a_2,a_1}$. Note that these maps are birational maps and are always well-defined in the open invariant set $Q^+ = \{(x, y) : x > 0, y > 0\} \subset \mathbb{R}^2$. Moreover

$$(x_1, x_2) \xrightarrow{F_{a_1}} (x_2, x_3) \xrightarrow{F_{a_2}} (x_3, x_4) \xrightarrow{F_{a_3}} (x_4, x_5) \xrightarrow{F_{a_4}} (x_5, x_6) \xrightarrow{F_{a_5}} \cdots$$

and in general,

$$F_{[k]}(x_1, x_2) = (x_{k+1}, x_{k+2}).$$

There are two concepts coexisting in this context, *the non-autonomous invariants* and *the first integrals*, that we will use in this paper. Given a difference equation of the form (1), a non-autonomous invariant is a function $V(x, y, n)$, such that

$$V\left(x_{n+1}, x_{n+2}, n + 1\right) = V(x_n, x_{n+1}, n),$$

for all initial conditions and all $n \in \mathbb{N}$. On the other hand, when the difference equation has k-periodic coefficients a first integral is a function H, which is a first integral for the discrete dynamical system generated by $F_{[k]}$, that is $H(F_{[k]}(x,y)) = H(x,y)$, for all points in an open set. In terms of the recurrence

$$H(x_{n+k}, x_{n+k+1}) = H(x_n, x_{n+1}),$$

for all initial conditions (x_n, x_{n+1}). We will relate both concepts in Section 3.

Two analytic functions $P, Q : \mathcal{U} \subset \mathbb{C}^2 \to \mathbb{C}$ are said to be coprime if the points of the set $\{(x,y) \in \mathcal{U} \,:\, P(x,y) = Q(x,y) = 0\}$ are isolated. A function $H = P/Q$, with P and Q coprime, will be called a *meromorphic function.* A *meromorphic first integral* of an analytic map $F : \mathcal{U} \to \mathbb{C}^2$ is a meromorphic function $H = P/Q$ such that

$$P(F(x,y))Q(x,y) = P(x,y)Q(F(x,y)) \quad \text{for all} \quad (x,y) \in \mathcal{U}.$$

Observe that from this definition $H(F(x,y)) = H(x,y)$ for all points of $\mathcal{U}$ for which both terms of this last equality are well-defined. When P and Q are polynomials then it is said that H is a *rational first integral.* Similarly we can talk about *meromorphic or rational invariants.*

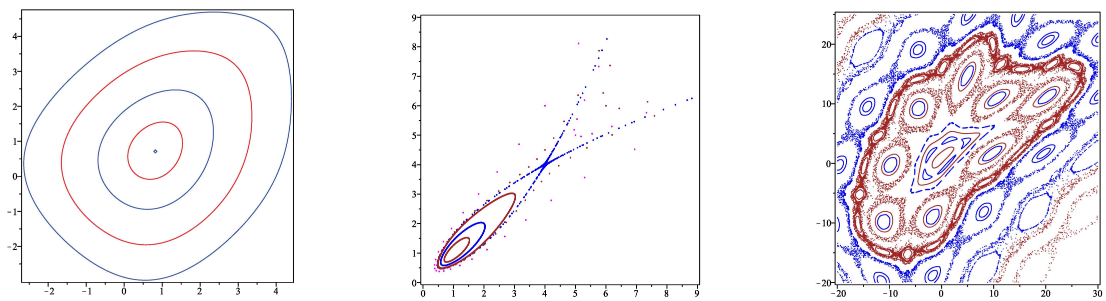
Finally, we will say that a planar map F has *structurally stable numerical chaos (SSNC)* when studying numerically several of its orbits, we observe that it presents all the features of a non-integrable perturbed twist map, that is: many invariant curves and, between them, couples of orbits of p-periodic points (for several values of p), half of them of elliptic type and the other half of hyperbolic saddle type. Moreover the separatrices of these hyperbolic saddles intersect transversally, see for instance see [1, Chapter 6].

1.2 Main results

This subsection collects the general outlines of all our results about the recurrence (1), in terms of k. Figure 1 shows some typical behaviors of the orbits of $F_{[k]}$. In fact we consider some maps $G_{[k]}$, which are conjugated to $F_{[k]}$, because the pictures are much more clear. See Lemma 9 for the definition of $G_{[k]}$.

Cases $\mathbf{k} \in \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{6}\}$ and other concrete integrable cases. For $k \in \{1,2,3\}$ it is already known that the recurrences (1) are persistent. Moreover, either each sequence $\{x_n\}_n$ is periodic, with period a multiple of k or it densely fills at most k disjoint intervals of $\mathbb{R}^+$, see [6]. A key point for the proof is the existence of a rational first integral for $F_{[k]}$. When $k = 6$ we can also prove the existence of a similar first integral (see Corollary 4) and the persistence of recurrence (1). Moreover we are confident that the same characterization of the sequences $\{x_n\}_n$ holds but we have only been able to prove the result when $F_{[k]}$ has a unique fixed point in the first quadrant, see Lemma 10 and Proposition 17.

It is also satisfied that for any $k \neq 5$ there are values $a_1, \ldots, a_k$, with prime period k and high rank, satisfying the property that all the sequences $\{x_n\}_n$ given by (1) are either periodic, with period a multiple of k, or they densely fill at most k disjoint intervals, see Theorem 19.



All k $\qquad$ $k = \dot{5}$ $\qquad$ $k \notin \{1, 2, 3, 6, \dot{5}\}$

Figure 1: Different possible behaviors of the orbits of $G_{[k]}$, according k. Other behaviors are possible for $k = \dot{5}$.

Cases $\mathbf{k = \dot{5}}$. When k is a multiple of 5, apart of the behaviors described above there appear others for an open set of values of $a_j, j = 1, \ldots, k$ and initial conditions. For instance we can find sequences $\{x_n\}_n$ such that

$$\liminf_{n\to\infty} x_n = 0 \quad \text{and} \quad \limsup_{n\to\infty} x_n = +\infty,$$

and others such that their adherence consists of k points, see Theorem 6. The existence for $k = 5$ of values of $a_j, j = 1, \ldots, 5$, for which the sequence $\{x_n\}_n$ has the first behavior has been already established in previous works, see [4, Example 5.43.1] or [7], but only for very concrete initial conditions and parameters $a_1, \ldots, a_5$.

Moreover in this case we can prove that for most values of the parameters the map $F_{[k]}$ has no meromorphic first integral (see Theorem 20). Nevertheless we want to comment that all our numerical approaches seem to show that the maps $F_{[k]}$ are always integrable. Furthermore the phase portrait of the map $F_{[k]}$ does not always coincide with the ones found in all the rational integrable cases, see for instance the second picture in Figure 1. In this case, apart of the celebrated Lyness map F_1 which satisfies $F_1^5 = F_{1,1,1,1,1} = \text{Id}$, there are values of the parameters $a_1, \ldots, a_k$ such that the number of fixed points of the maps is a 1-dimensional manifold, or $2, 1$, or 0 points, see Lemma 10.

Cases $\mathbf{k = 4}$ and $\mathbf{k \geq 7}$ satisfying $\mathbf{k \neq \dot{5}}$. When $k \in \{4, 7, 11\}$, for some values of the parameters, $a_1, \ldots, a_k$, we have numerically found SSNC, see Section 6. In fact we prove in Lemma 23 that based on these examples we can obtain values $a_1, \ldots, a_k$, all different, with a similar behavior for all the remaining values of k. So for all these values of k there are situations for which the sequence $\{x_n\}_n$ can have different behaviors to those given in the above situations. For instance there appear sequences which fill more than k intervals. Some concrete examples for $k = 4$ are shown in Section 6.

Finally note that the above results show that the only cases for which recurrence (1) can have a rational invariant for all values of the parameters $a_1, \ldots, a_k$ are $k \in \{1, 2, 3, 6\}$.

1.3 Main tools

In this subsection we present several results that we have obtained which we believe that are interesting by themselves. Other technical results will be given in Section 2.

The first result is a necessary condition for the meromorphic integrability of planar maps near a fixed point. Our approach follows the guidelines of Poincaré when he studied the same problem for ordinary differential equations, see [11] and the references there in for the approach to ordinary differential equations. In Section 5 we will apply the result below to study the case $k = \dot{5}$.

Theorem 1. *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ an analytic map defined in $\mathcal{U}$, an open neighborhood of the origin, such that $F(0, 0) = (0, 0)$ and $DF(0, 0)$ is diagonalizable with eigenvalues λ and μ. Assume that F has a meromorphic first integral H in $\mathcal{U}$.*

(i) If $\lambda\mu \neq 0$ then there exists $(p, q) \in \mathbb{Z}^2$, $(p, q) \neq (0, 0)$, such that $\lambda^p \mu^q = 1$.

(ii) If $\lambda \neq 0$ and $\mu = 0$ then there exists $n \in \mathbb{N}^+$ such that $\lambda^n = 1$.

When the map $F : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ is real valued and of class $\mathcal{C}^2(\mathcal{U})$ the proof of Theorem 1 can be adapted following the same steps. Taking into account that in this case, when $\lambda \in \mathbb{C}$ is an eigenvalue of $DF(0, 0)$, then $\bar{\lambda}$ it is so, and we have to deal with the resonant condition $\lambda^p \bar{\lambda}^q = 1$, we obtain the following result:

Corollary 2. *Let $F : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathcal{C}^2(\mathcal{U})$ map such that $F(0, 0) = (0, 0) \in \mathcal{U}$ and $DF(0, 0)$ is diagonalizable, with eigenvalues λ and μ. Assume that F has a meromorphic first integral H in $\mathcal{U}$.*

(i) If $\lambda, \mu \in \mathbb{R}$, $\lambda\mu \neq 0$, then there exists $(0, 0) \neq (p, q) \in \mathbb{Z}^2$ such that $\lambda^p \mu^q = 1$.

(ii) If $0 \neq \lambda \in \mathbb{C} \setminus \mathbb{R}$ (hence $\mu = \bar{\lambda}$), then either $|\lambda| = 1$ or $\lambda = |\lambda| e^{i\theta}$ and there exists $0 \neq n \in \mathbb{N}$ such that $(e^{i\theta})^{2n} = 1$.

(iii) If $\lambda \neq 0$ and $\mu = 0$ then there exists a $n \in \mathbb{N}^+$ such that $\lambda^n = 1$.

The above results will be applied to prove the meromorphic non-integrability of many cases when $k = \dot{5}$, see Theorem 20. On the contrary, next result will be the key point to prove the existence of rational integrable cases for all $k \neq 5$, see Theorem 19.

For the recurrence (1) we seek for non-autonomous invariants of the form

$$V(x, y, n) = \frac{\Phi_n(x, y)}{xy} \tag{3}$$

where

$$\begin{aligned}\Phi_n(x, y) = \quad & A_n + B_n x + C_n y + D_n x^2 + F_n y^2 + G_n x^3 + H_n x^2 y + I_n x y^2 \\ & + J_n y^3 + K_n x^4 + L_n x^3 y + M_n x^2 y^2 + N_n x y^3 + O_n y^4,\end{aligned}$$

with all the sequences of positive numbers. This method is introduced in [8] and the special form of V is inspired by this paper and the known invariant of the Lyness recurrences, see [2, 9, 10]. We prove:

Theorem 3. *If the recurrence* (1) *has an invariant of the form* (3) *then* $a_{n+6} = a_n$ *and*

$$\begin{aligned}\Phi_n(x, y) = {} & a_n F_{n+1} + (F_{n+2} + a_{n+1} F_{n+1})x + (F_{n+1} + a_n F_n)y + F_{n-3} x^2 \\ & + F_n y^2 + F_{n-2} x^2 y + F_{n-1} x y^2,\end{aligned} \tag{4}$$

where $\{F_n\}_n$ *satisfies that* $F_{n+6} = F_n$ *and* $a_{n+1} F_{n+2} - a_n F_{n-3} = 0$.

Corollary 4. *(i) The non-autonomous k-periodic recurrence* (1) *has invariants of the form* (3) *if and only if* $k \in \{1, 2, 3, 6\}$.

(ii) The first integrals of the maps $F_{[k]}$, for $k \in \{1, 2, 3, 6\}$, corresponding to the invariants given in Theorem 3 are:

$$\begin{aligned}
V_a(x, y) &= \frac{a + (a+1)x + (a+1)y + x^2 + y^2 + x^2 y + x y^2}{xy}, \\
V_{b,a}(x, y) &= \frac{ab + (a + b^2)x + (b + a^2)y + bx^2 + ay^2 + ax^2 y + bxy^2}{xy}, \\
V_{c,b,a}(x, y) &= \frac{ac + (a + bc)x + (c + ab)y + bx^2 + by^2 + cx^2 y + axy^2}{xy}, \\
V_{f,e,d,c,b,a}(x, y) &= \frac{af + (a + bf)x + (f + ae)y + bx^2 + ey^2 + cx^2 y + dxy^2}{xy}.
\end{aligned}$$

Remark 5. *Our proof of Theorem 3 does not use that the sequence of parameters $\{a_n\}_n$ is periodic.*

Observe that

$$V_a(x, y) + 2 + a = \frac{(x+1)(y+1)(a+x+y)}{xy}$$

is the usual first integral (invariant) of the map F_a associated to the classical Lyness recurrence, see for instance [2]. It is already known, see [6, 9, 10], that in the two and three periodic cases the functions $V_{b,a}$ and $V_{c,b,a}$ are first integrals of the maps $F_{b,a}$ and $F_{c,b,a}$, respectively. These first integrals play a crucial role for the understanding of the recurrence (1) when $k = 2, 3$, see again [6]. To the best of our knowledge the existence of a first integral for the general non-autonomous 6-periodic case was not known. In Section 4 we use it to describe the dynamics in this case.

As we have already commented, it is known that for very concrete values of $a_1, \ldots, a_5$ and suitable initial conditions, the behavior of $\{x_n\}_n$ is different to the ones appearing when $k \in \{1, 2, 3\}$ and in particular (1) is non-persistent, see [4, Example 5.43.1] or [7]. This behavior can also be seen considering

$$F_{a,1,1,1,1}(x, y) = \left(x, \frac{(x + a)\, y}{1 + x} \right).$$

Since $F_{a,1,1,1,1}(1, y) = (1, (1 + a)\, y/2)$ it is clear that for $a > 1$ the orbits of the points of the form $(1, y)$ with $y \neq 0$ are unbounded.

Our next result allows to establish, when $k = \dot{5}$, the non-persistence of the recurrence (1) for many values $a_1, \ldots, a_k$.

Theorem 6. *Consider recurrence* (1) *for* $k = \dot{5}$. *Set*

$$\phi_i := \prod_{\substack{n \,\equiv\, i \,(\mathrm{mod}\ 5) \\ n \,=\, 1, \ldots, k}} a_n, \quad \text{for} \quad i = 1, 2, \ldots, 5.$$

If for all $i = 1, \ldots, 5$, $\phi_i \neq 1$ *and*

$$\min_{i=1,\ldots,5} \{\phi_i\} < 1 < \max_{i=1,\ldots,5} \{\phi_i\}, \tag{5}$$

then (1) *is non-persistent. In fact, for an open set of initial conditions,* $\liminf_{n\to\infty}(x_n) = 0$ *and* $\limsup_{n\to\infty}(x_n) = +\infty$.

In the above result, the open set of initial conditions for which the result holds is sometimes the whole first quadrant Q^+. For instance, this is the case when $k = 5$ and $a_1 = a, a_2 = ac, a_3 = c, a_4 = 1/a$ and $a_5 = 1/(ac)$, when $a > 1$ and $ac > 1$, because

$$F_{\frac{1}{ac},\frac{1}{a},c,ac,a}(x, y) = \left(\frac{x}{a}, \frac{y}{ac} \right), \tag{6}$$

is a linear map with a stable node at the origin.

The rest of the paper is organized as follows. In Section 2 we introduce some preliminary results, while in Section 3 we prove the main tools described in Section 1.3. Section 4 is devoted to the cases for which we find rational integrability, while in Section 5 we prove the non-integrability results when k is a multiple of five. Finally, in Section 6 we present some numerical evidences of chaos.

2 Preliminary results

This section contains some technical preliminary results and other known results that we will use in the proofs given in next sections.

Lemma 7. *The map $F_{1/a,c,ac,a}$ is conjugated to the Lyness' map $F_{1/(ac^2)}$.*

Proof. Observe that $F_{\frac{1}{a},c,ac,a}(x,y) = \left(\frac{1+cx}{y}, \frac{x}{a}\right)$ and $F_{\frac{1}{ac^2}}(u,v) = \left(v, \frac{\frac{1}{ac^2}+v}{u}\right)$. If we consider the linear map

$$\varphi(x,y) = \left(\frac{y}{c}, \frac{x}{ac}\right),$$

it holds that $F_{\frac{1}{ac^2}} = \varphi \circ F_{\frac{1}{a},c,ac,a} \circ \varphi^{-1}$, as we wanted to prove. ∎

A nice consequence of the 5-global periodicity of the Lyness map $F_1(x,y) = (y,(1+y)/x)$ and the above lemma is the following result:

Corollary 8. *Recurrence* (1) *with* $k=4$ *and* $[a_1,a_2,\ldots] = [1/c^2, 1/c, c, c^2, 1/c^2, 1/c, \ldots]$ *is globally 20-periodic, i.e.* $F^5_{c^2,c,1/c,1/c^2}(x,y) = (x,y)$ *for all* $(x,y) \in Q^+$.

Next result will be very useful for our numerical simulations. As we will see, the new variables allows to "observe" much better the numerical non-integrability studied in Section 6. Its proof is straightforward.

Lemma 9. *The sequence* (1) *in the variables* $z_n := \log(x_n)$ *writes as*

$$z_{n+2} = -z_n + \log(a_n + \exp(z_{n+1})),$$

and the corresponding maps $F_{[k]}$ *are conjugated to* $G_{[k]}$*, where* $G_{[k]} = G_{a_k,a_{k-1}\ldots,2,1}$,

$$G_a(x,y) = (y, -x + \log(a + \exp(y))),$$

and each G_a *is defined on the whole plane,* $\mathbb{R}^2$*. The maps* $G_[k]$ *are area preserving.*

The following lemma studies the number of fixed points of $F_{[k]}$ for $k = 4,5,6$.

Lemma 10. *(i) There is a unique fixed point of* $F_{d,c,b,a}$ *in* Q^+ *and it satisfies*

$$\begin{cases} x = y^2 + (a-c)y - d, \\ y = x^2 + (d-b)x - a. \end{cases}$$

(ii) There are either 0,1,2 or a continuum of fixed points of $F_{e,d,c,b,a}$ *in* Q^+ *and they satisfy*

$$\begin{cases} (b-1)x + (1-d)y + (a-e) = 0, \\ cxy - (e+x)(a+y) + bx + y + a = 0. \end{cases}$$

(iii) Let $\mathcal{F}_{f,e,d,c,b,a} \subset Q^+$ be the set of fixed points of map $F_{f,e,d,c,b,a}$ and let $\mathcal{S}_{f,e,d,c,b,a} \subset Q^+$ be the set of singular points of its first integral $V_{f,e,d,c,b,a}$ given in Corollary 4. Then $\mathcal{F}_{f,e,d,c,b,a} = \mathcal{S}_{f,e,d,c,b,a}$ and both sets coincide with the set of points of Q^+ satisfying

$$\begin{cases} y^2 = \dfrac{(f+x)(a+bx)}{e+dx}, \\ x^2 = \dfrac{(a+y)(f+ey)}{b+cy}. \end{cases}$$

Moreover $\operatorname{card}(\mathcal{F}_{f,e,d,c,b,a}) \geq 1$.

Proof. (i) The set of fixed points of $F_{d,c,b,a}$ is exactly the set of points satisfying

$$F_d(F_c(F_b(F_a(x,y)))) = (x,y),$$

but it is not easy to handle these two equations. On the other hand, the equivalent condition

$$F_b(F_a(x,y)) = F_c^{-1}(F_d^{-1}(x,y)),$$

lead us to the system of the statement. Clearly both parabolas meet at a unique point in Q^+.

(ii) Studying the condition

$$F_c(F_b(F_a(x,y))) = F_d^{-1}(F_e^{-1}(x,y))$$

we obtain the system of the statement. The x-coordinate of a fixed point has to satisfy the quadratic equation

$$(c-1)(b-1)x^2 + (2e - 1 + bd + ac - ec - eb - ad)x + (e-1)(e-ad) = 0.$$

From this equation we easily obtain the result. Notice that simple cases having infinitely many fixed points appear for instance when $b = d = 1$ and $e = a$.

(iii) The two conditions given by

$$F_c(F_b(F_a(x,y))) = F_d^{-1}(F_e^{-1}(F_f^{-1}(x,y))),$$

directly lead to the system of the statement. The set of singular points of $V_{f,e,d,c,b,a}$ is formed by the points satisfying

$$\{(x,y) \in Q^+ \,:\, \frac{\partial}{\partial x}\left(V(x,y)\right) = \frac{\partial}{\partial y}\left(V(x,y)\right) = 0\}.$$

The two equations describing the above set exactly coincide again with the two equations given in the statement. So $\mathcal{F}_{f,e,d,c,b,a} = \mathcal{S}_{f,e,d,c,b,a}$. That $\operatorname{card}(\mathcal{F}_{f,e,d,c,b,a}) \geq 1$ can be seen studying the behavior of the functions, $\frac{(f+x)(a+bx)}{e+dx}$ and $\frac{(a+y)(f+ey)}{b+cy}$, near 0 and $+\infty$. ∎

Next result proves that all the fixed points $\mathbf{p}$ of some map $F_{[k]}$, in $\mathbb{R}^2$ such that $F_{a_j,a_{j-1},\ldots,a_1}(\mathbf{p})$, for all $j \leq k$, is well defined are resonant. As we will see in Proposition 21, the hypothesis on the maps $F_{a_j,a_{j-1},\ldots,a_1}(\mathbf{p})$ is unavoidable because for $j = \dot{5}$ there appear some cancelations that make that the maps $F_{[5j]}$ have as fixed point $\mathbf{p} = (0,0)$ and this point can be of saddle type with arbitrary eigenvalues. In fact this property will be the key point for our proof of non-existence of meromorphic first integrals for most $F_{[k]}, k = \dot{5}$, see Theorem 20.

Proposition 11. *Let $\mathbf{p} \in \mathbb{R}^2$ be a fixed point of a composition map $F_{[k]}$ and such that $F_{a_j,a_{j-1},\ldots,a_1}(\mathbf{p})$ is well defined for all $j \leq k$. Then*

$$\det\left(DF_{[k]}(\mathbf{p})\right) = 1. \tag{7}$$

Proof. Fixed $\{a_n\}_n$ with $a_n \in \mathbb{R}$ we introduce the following notation

$$F_{[m]}(x,y) = \left(\psi_{m-1}(x,y), \psi_m(x,y)\right), \quad m \in \mathbb{N}$$

where $\psi_{-1}(x,y) = x$ and $\psi_0(x,y) = y$. Clearly

$$\psi_{\ell+1}(x,y) = \frac{a_{\ell+1} + \psi_\ell(x,y)}{\psi_{\ell-1}(x,y)}.$$

Let us show first that

$$\det\big(DF_{[m]}(x,y)\big) = \prod_{\ell=1}^{m} \frac{\psi_\ell(x,y)}{\psi_{\ell-2}(x,y)} = \frac{\psi_m(x,y)\psi_{m-1}(x,y)}{xy}. \tag{8}$$

We prove (8) by induction. Clearly,

$$\det\big(DF_{[1]}(x,y)\big) = \det\left(DF_{a_1}(x,y)\right) = \frac{a_1 + y}{x^2} = \frac{\psi_1(x,y)}{\psi_{-1}(x,y)}.$$

Assume now that (8) holds for $m = \ell$, then

$$\det\big(DF_{[\ell+1]}(x,y)\big) = \det\big(D(F_{a_{\ell+1}}(F_{[\ell]}(x,y)))\big) = \det\big(DF_{a_{\ell+1}}(F_{[\ell]}(x,y)))\big) \det\big(DF_{[\ell]}(x,y)\big).$$

Hence, to prove (8) for $m = \ell + 1$ it suffices to prove that

$$\det\big(DF_{a_{\ell+1}}(F_{[\ell]}(x,y)))\big) = \frac{\psi_{\ell+1}(x,y)}{\psi_{\ell-1}(x,y)}.$$

Notice that the left hand side of the above equality is

$$\left(\frac{a_{\ell+1} + \bar{y}}{\bar{x}^2}\right)\Bigg|_{(\bar{x},\bar{y})=F_{[\ell]}(x,y)} = \frac{a_{\ell+1} + \psi_\ell(x,y)}{\psi_{\ell-1}^2(x,y)} = \frac{\psi_{\ell+1}(x,y)}{\psi_{\ell-1}(x,y)},$$

as we wanted to prove.

Since $\mathbf{p} = (x_0, y_0)$ is a fixed point of $F_{[k]}$, $(\psi_{k-1}(x_0, y_0), \psi_k(x_0, y_0)) = (x_0, y_0)$, and using (8) we get that

$$\det\left(DF_{[k]}(x_0, y_0)\right) = \frac{x_0 y_0}{x_0 y_0} = 1,$$

as we wanted to prove. ∎

As a consequence of the above Proposition we have:

Corollary 12. *Let $\mathbf{p} \in \mathbb{R}^2$ be a m-periodic point of a composition map $F_{[k]}$ such that $F_{a_j, a_{j-1}, \ldots, a_1}(\mathbf{p})$ is well defined for all $j \leq km$. Then*

$$\det\left(DF_{[k]}^m(\mathbf{p})\right) = 1.$$

Next result allows to know the dynamics of each $F_{[k]}$ when the map has a smooth first integral.

Theorem 13 ([5])**.** *Let $\mathcal{U} \subset \mathbb{R}^2$ be an open set and let $F : \mathcal{U} \to \mathcal{U}$ be a diffeomorphism such that it has a smooth regular first integral $V : \mathcal{U} \to \mathbb{R}$ and there exists a smooth function $\mu : \mathcal{U} \to \mathbb{R}^+$ such that for any $(x, y) \in \mathcal{U}$, $\mu(F(x, y)) = \det(DF(x, y))\, \mu(x, y)$. Then the following holds:*

(i) If a level set $\Gamma_h := \{(x, y) \in \mathcal{U} : V(x, y) = h\}$ is a simple closed curve invariant under F, then the map F restricted to Γ_h is conjugated to a rotation.

(ii) If Γ_h is diffeomorphic to an open interval curve and invariant under F, then the map F restricted to Γ_h is conjugated to a translation.

The applicability of the above result to any integrable $F_{[k]}$ is guaranteed by the following lemma.

Lemma 14. *For every choice of positive numbers $a_1, \ldots, a_k$,*

(i) The map $F_{[k]}$ preserves the measure $\boldsymbol{m}(B) = \int_B \frac{1}{xy}\, dx\, dy$.

(ii) It holds that $\mu(F_{[k]}(x, y)) = \det(DF_{[k]}(x, y))\, \mu(x, y)$, where $\mu(x, y) = xy$.

(iii) The map $G_{[k]}$ preserves the Lebesgue measure $\boldsymbol{n}(B) = \int_B dx\, dy$.

(iv) It holds that $\mu(G_{[k]}(x, y)) = \det(DG_{[k]}(x, y))\, \mu(x, y)$, where $\mu(x, y) \equiv 1$.

3 Proof of the main tools

This section is devoted to prove Theorems 1, 3 and 6.

Proof of Theorem 1. Write

$$H(x,y)=\frac{P(x,y)}{Q(x,y)}=\frac{P_{\tilde n}(x,y)+O(\tilde n+1)}{Q_{\tilde m}(x,y)+O(\tilde m+1)},$$

where $P_{\tilde n}$ and $Q_{\tilde m}$ are homogeneous polynomials with degrees $\tilde n\geq 0$ and $\tilde m\geq 0$, respectively, and $O(k)$ denotes terms of order at least k. Firstly we prove that it is not restrictive to assume that $\tilde n\geq \tilde m$ and that if $\tilde n=\tilde m$ then $P_{\tilde n}(x,y)/Q_{\tilde m}(x,y)$ is not constant. Notice that if H is a first integral then $1/H$ it is so. Hence we can assume that $\tilde n\geq\tilde m$. If $\tilde n=\tilde m$ and $P_{\tilde n}=\eta Q_{\tilde n}$ for some $0\neq\eta\in\mathbb{C}$, take $\tilde H=H-\eta$. Clearly $\tilde H$ is a new first integral of the form $\tilde H(x,y)=(O(\tilde n+1))/(Q_{\tilde m}(x,y)+O(\tilde m+1))$, as we wanted to see.

It is also clear that in a neighborhood of the origin we can assume that $F(x,y)=(\lambda x+O(2),\mu y+O(2))$.

(i) We start studying the case $\lambda\mu\neq 0$. By imposing that H is a first integral of F in $\mathcal{U}$ we have that

$$P(F(x,y))Q(x,y)=P(x,y)Q(F(x,y)).$$

By taking the lower order terms of the above equality we obtain

$$P_{\tilde n}(\lambda x,\mu y)Q_{\tilde m}(x,y)=P_{\tilde n}(x,y)Q_{\tilde m}(\lambda x,\mu y). \tag{9}$$

Define

$$P_n(x,y)=\frac{P_{\tilde n}(x,y)}{\gcd(P_{\tilde n}(x,y),Q_{\tilde m}(x,y))},\quad Q_m(x,y)=\frac{Q_{\tilde m}(x,y)}{\gcd(P_{\tilde n}(x,y),Q_{\tilde m}(x,y))},$$

where $n\geq m$ are suitable non-negative integers. By using the homogeneity of P_n and Q_m, equation (9) writes as

$$\mu^n y^n P_n(\lambda x/(\mu y),1)y^m Q_m(x/y,1)=y^nP_n(x/y,1)\mu^m y^m Q_m(\lambda x/(\mu y),1), \tag{10}$$

where notice that we have canceled the common factor of $P_{\tilde n}$ and $Q_{\tilde m}$. By introducing the polynomials in one variable $p_n(w)=P_n(w,1)$, $q_m(w)=Q_m(w,1)$, with respective maximum degrees n and m, and $\rho=\lambda/\mu$, $w=x/y$, equation (10) writes as

$$\mu^{n-m}p_n(\rho w)q_m(w)=p_n(w)q_m(\rho w), \tag{11}$$

where we know that p_n and q_m are not identically zero and have no common root.

Notice that equality (11) implies that if $w=w^*$ is a root of p_n then ρw^* it is so, and hence $\rho^l w^*$ for any $l\in\mathbb{N}$, is a root of p_n. Since p_n has at most n roots, if $w^*\neq 0$ we have that $\rho^k = 1$ for some $k \le n$, proving the theorem, because $\lambda^k\mu^{-k} = 1$. A similar reasoning can be done for q_m. Hence it only remains to study the cases

$$p_n(w) = aw^{\hat{n}}, \quad 0 \le \hat{n} \le n, \quad \text{and} \quad q_m(w) = bw^{\hat{m}}, \quad 0 \le \hat{m} \le m,$$

for some complex numbers a and b, $ab \neq 0$. Remember that we know that both polynomials have no common roots. So, at least one of the two numbers $\hat{n}$ or $\hat{m}$ has to be zero. In any case, the equation (11) writes as

$$\mu^{n-m} a\rho^{\hat{n}} w^{\hat{n}} b w^{\hat{m}} = a w^{\hat{n}} b \rho^{\hat{m}} w^{\hat{m}},$$

giving $\mu^{n-m}(\lambda/\mu)^{\hat{n}-\hat{m}} = \lambda^{\hat{n}-\hat{m}}\mu^{n+\hat{m}-m-\hat{n}} = 1$, as we wanted to prove.

(ii) When $\lambda \neq 0$ and $\mu = 0$ equation (9), afer dropping the common factor of $P_{\tilde{n}}$ and $Q_{\tilde{m}}$, writes as

$$P_n(\lambda x, 0) Q_m(x, y) = P_n(x, y) Q_m(\lambda x, 0).$$

Notice that $P_n(x, 0) = ax^n$, $Q_m(x, 0) = bx^m$, and $(a, b) \neq (0, 0)$, because, otherwise P_n and Q_m would have y as a common factor. Hence

$$a\lambda^n x^n Q_m(x, y) = b\lambda^m x^m P_n(x, y),$$

and so $ab \neq 0$. Therefore $P_n(x, y) = a\lambda^{n-m} x^{n-m} Q_m(x, y)/b$. Since P_n and Q_m have no common factor, we get that $m = 0$ and so $Q_m = b$. Hence this last equality writes as $P_n(x, y) = a\lambda^n x^n$. Then $ax^n = P(x, 0) = a\lambda^n x^n$, giving $\lambda^n = 1$, as we wanted to prove. ∎

To illustrate the above result in next remark we present some examples of maps having (or not) meromorphic first integrals.

Remark 15. *(i) The linear maps $F(x, y) = (\lambda x, \mu y)$, with λ and μ satisfying the resonant condition $\lambda^p \mu^q = 1$ are the simplest maps with meromorphic first integrals $H(x, y) = x^p y^q$.*

(ii) The map $F(x, y) = (x + y(x - y), 0)$, with an eigenvalue 0, has the first integral $H(x, y) = (x - y + 1)(y + 1)$.

(iii) For maps with identically zero linear part we can have existence or not of meromorphic first integrals. For instance the map $F(x, y) = (x^2, xy)$ has the first integral $H(x, y) = x/y$. On the other hand, by using the same tools that in our proof of Theorem 1, we can prove that the map $F(x, y) = (x^2, y^2)$ has no meromorphic first integral.

Proof of Theorem 3. The condition that a function $V(x, y, n)$ of the form (3) is a non–autonomous invariant of the recurrence (1) writes as

$$V\left(y, \frac{a_n + y}{x}, n + 1\right) - V(x, y, n) = 0,$$

for all $(x,y)\in Q^+$ and all $n\in\mathbb{N}$. Imposing that the each one of the coefficients of the 31 monomials x^iy^j vanishes identically and playing a little bit with these conditions we obtain that (4) holds and moreover that

$$\begin{aligned} a_{n+1}F_{n+2}-a_nF_{n-3} &= 0, \\ F_{n+3}-F_{n-3}+a_{n+2}F_{n+2}-a_nF_{n-2} &= 0. \end{aligned} \tag{12}$$

From the first equation we get that

$$a_n=\frac{F_{n+2}}{F_{n-3}}\,a_{n+1}=\frac{F_{n+3}}{F_{n-2}}\frac{F_{n+2}}{F_{n-3}}\,a_{n+2}.$$

Plugging this equation in the second one we obtain that

$$\frac{F_{n+3}-F_{n-3}}{F_{n-3}}\,(F_{n-3}-a_{n+2}F_{n+2})=0.$$

Clearly the above equation holds if either $\{F_n\}_n$ is a 6-periodic sequence or $F_{n-3}=a_{n+2}F_{n+2}$.

In the first situation let us prove that if $\{F_n\}_n$ is a p-periodic sequence, $p\in\{1,2,3,6\}$ then a_n also has to be p-periodic. Assume for instance that $p=3$, then using (12) we have

$$\frac{a_{n+3}}{a_n}=\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}}\frac{a_{n+1}}{a_n}=\frac{F_{n-1}}{F_{n+4}}\frac{F_{n-2}}{F_{n+3}}\frac{F_{n-3}}{F_{n+2}}=\frac{F_{n+2}}{F_{n+4}}\frac{F_{n+4}}{F_{n+3}}\frac{F_{n+3}}{F_{n+2}}=1,$$

as we wanted to see. The other cases follow similarly.

In the second situation we have that $F_{n-3}=a_{n+2}F_{n+2}$. Using this equation and equality (12) we have that

$$a_{n+2}=\frac{F_{n-3}}{F_{n+2}}=\frac{a_{n+1}}{a_n}.$$

which is a well-known 6-periodic recurrence, as we wanted to see. Finally, using (12) six times we get that $F_{n+6}=F_n$. ■

Before proving Corollary 4, we explain here how non-autonomous invariants and first integrals are related. Consider a recurrence with k-periodic coefficients and having a non-autonomous invariant $V(x,y,n)$ that satisfies $V(x,y,n)=V(x,y,n+k)$. Then $H(x,y):=V(x,y,1)$ is a first integral of $F_{[k]}$. Conversely, if $H(x,y)$ is a first integral then

$$V(x,y,n):=H(F_{a_k,a_{k-1},\ldots,a_\ell}(x,y)),\quad \text{where}\quad 1\le\ell\le k,\quad n-\ell=\dot{k}$$

is a non-autonomous periodic invariant of the recurrence. These relations are used in next corollary for constructing the first integrals of $F_{[k]}$, $k=1,2,3,6$ using the invariant found in the above theorem.

Proof of Corollary 4. (i) This result is proved along the proof of Theorem 3, above.

(ii) We only give the details for $k = 6$. The other cases follow similarly. We introduce the following notations for the non-autonomous 6-periodic recurrences:

$$\{a_n\}_n = a_1, a_2, \ldots = a, b, c, d, e, f, a, b, c, d, e, f, a, \ldots$$
$$\{F_n\}_n = F_1, F_2, \ldots = 1, \ell, m, n, o, p, 1, \ell, m, n, o, p, 1, \ldots$$

From the relations $a_{n+1}F_{n+2} - a_nF_{n-3} = 0$, we obtain that

$$\ell = \frac{f}{e}, \quad m = \frac{a}{e}, \quad n = \frac{b}{e}, \quad o = \frac{c}{e} \quad \text{and} \quad p = \frac{d}{e}.$$

Hence, with the notations of Theorem 3 we get that

$$\begin{aligned}\Phi_1(x,y) &= a_1F_2 + (F_3 + a_2F_2)x + (F_2 + a_1F_1)y + F_{-2}x^2 + F_1y^2 + F_{-1}x^2y + F_0xy^2 \\ &= a\ell + (m + b\ell)x + (\ell + a)y + nx^2 + y^2 + ox^2y + pxy^2 \\ &= \frac{af + (a + bf)x + (f + ae)y + bx^2 + ey^2 + cx^2y + dxy^2}{e}.\end{aligned}$$

Hence $V_{f,e,d,c,b,a}(x,y) = e\Phi_1(x,y)/(xy)$ is a first integral of $F_{[6]}$, as we wanted to prove. ∎

We first prove Theorem 6 for $k = 5$.

Proposition 16. *Consider recurrence* (1) *with* $k = 5$ *and* $a_i \neq 1$, *for* $i = 1, 2, \ldots, 5$ *and satisfying*

$$\min\{a_1, a_2, a_3, a_4, a_5\} < 1 < \max\{a_1, a_2, a_3, a_4, a_5\}. \tag{13}$$

Then the recurrence (1) *is non-persistent. Moreover for an open set of initial conditions* $\liminf_{n\to\infty} x_n = 0$ *and* $\limsup_{n\to\infty} x_n = +\infty$.

Proof. A computation shows that $F_{[5]}(x,y) = (P_1(x,y), P_2(x,y))$ where

$$P_1(x,y) = \frac{x\left(a_3xy + a_4y^2 + a_2x + (a_1a_4 + 1)y + a_1\right)}{(a_1 + y)(a_1 + a_2x + y)},$$
$$P_2(x,y) = \frac{yN(x,y)}{(a_1 + a_2x + y + a_3xy)(a_1 + a_2x + y)}$$

and

$$\begin{aligned}N(x,y) =& a_3x^2y + a_4xy^2 + a_2x^2 + (a_1a_4 + a_2a_5 + 1)xy + a_5y^2 + \\ & a_1(1 + a_2a_5)x + 2\,a_1a_5y + {a_1}^2a_5.\end{aligned}$$

First observe that, contrary to what it happens for $F_{[k]}, k < 4$, $F_{[5]}$ can be extended to a neighborhood of Q^+. Note also that $(0,0)$ is a fixed point and

$$DF_{[5]}(0,0) = \begin{pmatrix} \frac{1}{a_1} & 0 \\ 0 & a_5 \end{pmatrix}.$$

So, under our hypotheses, the origin is a hyperbolic fixed point of $F_{[5]}$. Arguing similarly with the shifted maps F_{a_1,a_5,a_4,a_3,a_2}, F_{a_2,a_1,a_5,a_4,a_3}, F_{a_3,a_2,a_1,a_5,a_4}, and F_{a_4,a_3,a_2,a_1,a_5}, we obtain that the origin is also a hyperbolic fixed point for these maps, with differential matrices

$$\begin{pmatrix} \frac{1}{a_2} & 0 \\ 0 & a_1 \end{pmatrix}, \begin{pmatrix} \frac{1}{a_3} & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} \frac{1}{a_4} & 0 \\ 0 & a_3 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{a_5} & 0 \\ 0 & a_4 \end{pmatrix},$$

respectively.

Condition (13) implies that at least there exist a parameter with a value less than one, and other grater that one. Since there are no parameters with value equal to one and the sequence is cyclic we can choose two contiguous parameters and such that $a_i < 1$ and $a_{i+1} > 1$. This implies that the origin is an attractive fixed point for some of the five shifted maps. For example, suppose that $a_2 < 1$ and $a_3 > 1$, then the origin is an stable node for F_{a_2,a_1,a_5,a_4,a_3}.

Taking an initial condition (x_0, y_0), with positive coordinates and in the basin of attraction of the origin for the corresponding shifted map we obtain that $\liminf_{n\to\infty} x_n = 0$ for the solution of equation (1) with initial condition $x_1 = x_0$ and $x_2 = y_0$.

Recall $F_{a_j}(x, y) = (y, (a_j + y)/x)$, $a_j \neq 0$. Thus the fact that $\limsup_{n\to\infty} x_n = +\infty$ follows because if some $\{(x_{n_s}, y_{n_s})\}_{n_s}$ tends to $(0, 0)$ then the second component of $\{F_{a_j}(x_{n_s}, y_{n_s})\}_{n_s}$ tends to $+\infty$. ∎

Proof of Theorem 6. Set $F_{[k]}$ for $k = 5m$. We can write $F_{[k]} = F_{a_{5m},\ldots,a_{5m-4}} \circ \ldots \circ F_{a_5,\ldots,a_1}$, so $F_{[k]}$ can be extended to a neighborhood of Q^+. Furthermore, observe that

$$DF_{[k]}(0,0) = DF_{a_{5m},\ldots,a_{5m-4}}(0,0) \circ \ldots \circ DF_{a_5,\ldots,a_1}(0,0) = \begin{pmatrix} \frac{1}{\phi_2} & 0 \\ 0 & \phi_1 \end{pmatrix}.$$

Similarly the differential matrices of the shifted maps have the form

$$\begin{pmatrix} \frac{1}{\phi_{i+1}} & 0 \\ 0 & \phi_i \end{pmatrix}.$$

Arguing as in Proposition 16, the relation (5) implies that at least there exists a couple (ϕ_i, ϕ_{i+1}) such that one of the values is greater than one and the other less than one. So the origin is an attractive fixed point for some of the m shifted maps. Now the proof follows again as in Proposition 16. ∎

4 Rational integrability and associated dynamics

As we have already explained in Subsection 1.2 the cases $k = 1, 2, 3$ are very similar and totally understood. From Corollary 4 we can prove in next proposition a similar result when

$k=6$. Before stating the result we introduce the following notation:

$$\mathbf{P}_{f,e,d,c,b,a} := \{(a,b,c,d,e,f) \in (\mathbb{R}^+)^6 \ : \ \text{system (14) has a unique solution in } Q^+\}$$

$$\begin{cases} y^2 = \dfrac{(f+x)(a+bx)}{dx+e}, \\ x^2 = \dfrac{(a+y)(f+ey)}{b+cy}. \end{cases} \tag{14}$$

Proposition 17. *For $k=6$ the recurrence (1) is persistent. Moreover if $(f,e,d,c,b,a) \in \mathbf{P}_{f,e,d,c,b,a}$, any sequence $\{x_n\}_n$ generated by (1) is either periodic, with period a multiple of 6, or it densely fills at most 6 disjoint intervals of $\mathbb{R}^+$.*

Proof. We follow the same steps that the proof of [6, Thm. 1]. To prove the persistence of (1), it suffices to show that each level curve $\{(x,y) \ : \ V_{f,e,d,c,b,a}(x,y) = h\} \cap Q^+$ is bounded. Since

$$\frac{af}{xy} + \frac{a+bf}{y} + \frac{f+ae}{x} + \frac{bx}{y} + \frac{ey}{x} + cx + dy = h,$$

we know that

$$\frac{f+ae}{h} \le x \le \frac{h}{c} \quad \text{and} \quad \frac{a+bf}{h} \le y \le \frac{h}{d}$$

and the persistence follows.

By Lemma 10.(iii), under our hypotheses, the set of fixed points of $F_{[6]}$ and the set of singular points of $V_{[6]}$ coincide and consists of a single point. Following again the same guidelines of the proof of [6, Thm. 1], which in turn is based on [3, Prop. 2.1], we prove that all the level curves of $V_{[6]}$ in Q^+, but the fixed point, are diffeomorphic to circles. Hence by using Lemma 14 and Theorem 13.(i) the proposition follows. ∎

Remark 18. *We believe that $\mathbf{P}_{f,e,d,c,b,a}$ is the whole $(\mathbb{R}^+)^6$ but we have not been able to prove this equality. In any case it is easy to find sufficient conditions to ensure that some (a,b,c,d,e,f) belongs to $\mathbf{P}_{f,e,d,c,b,a}$. For instance, since*

$$\frac{\partial}{\partial x}\left(\frac{(f+x)(a+bx)}{e+dx}\right) = \frac{bdx^2+2bex+ae+bef-adf}{(e+dx)^2},$$
$$\frac{\partial}{\partial y}\left(\frac{(a+y)(f+ey)}{b+cy}\right) = \frac{cey^2+2bey+bf+abe-acf}{(b+cy)^2},$$

when both numerators have no positive real roots the point (a,b,c,d,e,f) is in the set, because the functions that we have derived are both increasing and so the curves defined by system (14) cut in a single point.

Next result collects our integrability results for any $k \neq 5$.

Theorem 19. *(i) For any $k \geq 15$, there exist sequences $\{a_n\}_n$ of prime period k and rank k such that $F_{[k]} = F_{a_k,\dots,a_2,a_1}$ is rationally integrable and the corresponding recurrence (1) is persistent.*

(ii) For any $k < 15, k \neq 5$, there exist sequences $\{a_n\}_n$ of prime period k with the ranks as in Table 1, such that $F_{[k]}$ is rationally integrable and the corresponding recurrence (1) is persistent.

(iii) Moreover it is possible to take in all the above cases parameters $a_1, a_2, \dots a_k$ such that each sequence $\{x_n\}_n$ is either periodic, with period a multiple of k, or it densely fills at most k disjoint intervals of $\mathbb{R}^+$.

k	1	2	3	4	5	6	7	8	9	$10 \leq k \leq 14$	$k \geq 15$
Rank	1	2	3	4	-	6	3	4	5	$k-5$	k

Table 1. Possible ranks for integrable $F_{[k]}$.

Proof. We start by introducing some notation. Given a_i and c_i positive, we consider the sets $S_i := \left\{\frac{1}{a_i c_i}, \frac{1}{a_i}, c_i, a_i c_i, a_i\right\}$. Assume that a_i and c_i are such that $\mathrm{Card}(S_i) = 5$ and consider

$$\Phi_i(x,y) = F_{\frac{1}{a_i c_i}, \frac{1}{a_i}, c_i, a_i c_i, a_i}(x,y) = \left(\frac{x}{a_i}, \frac{y}{a_i c_i}\right),$$

where we have used expression (6). Notice that $1 \notin S_i$. Given any natural number $m \geq 1$, we also consider m sets $S_1, S_2, \dots, S_m$, and define $\Phi^{[m]} = \Phi_m \circ \Phi_{m-1} \cdots \circ \Phi_1$. Then

$$\Phi^{[m]}(x,y) = \left(\frac{x}{\prod_{i=1}^{m} a_i}, \frac{y}{\prod_{i=1}^{m} a_i c_i}\right).$$

When $m = 0$ we consider $\Phi^{[0]}(x,y) = (x,y)$. Finally, choosing the values of the parameters such that $\prod_{i=1}^{m} a_i = 1$ and $\prod_{i=1}^{m} c_i = 1$, we obtain that for all $m \geq 3$, $\Phi^{[m]}(x,y) = (x,y)$. Moreover, for these values of m, the parameters a_i and c_i can be chosen such that $\mathrm{Card}(\cup_{i=0}^{m} S_i) = 5m$. Observe also that when $m = 1$ it is not possible to choose $a_1 = 1$ and $c_1 = 1$. When $m = 2$ it is again possible but with $\mathrm{Card}(\cup_{i=0}^{2} S_i) = 5$.

Now we can start the proof of the theorem. First we study the cases $k \leq 4$ and $k \geq 15$. Consider $k = 5m + \ell$ with $\ell \in \{0,1,2,3,4\}$ and $m = 0$ or $m \geq 3$.

Now, taking

$$\Psi(x,y) := \begin{cases} (x,y) & \text{for } \ell = 0, \\ F_a(x,y) & \text{for } \ell = 1, \\ F_{b,a}(x,y) & \text{for } \ell = 2, \\ F_{c,b,a}(x,y) & \text{for } \ell = 3, \\ F_{\frac{1}{a},c,ac,a}(x,y) & \text{for } \ell = 4, \end{cases} \tag{15}$$

with suitable values of a, b and c, we obtain that the orbits of

$$F_{[k]}(x,y) := \Psi \circ \Phi^{[m]}(x,y) = \Psi(x,y)$$

are like the ones of Ψ and the rank$(\{a_n\}) = k$. Then by using the known results for $k = 1, 2, 3$ and Lemma 7 the result follows for $k \geq 15$ and $k \leq 4$.

When $k = 6$ the result is proved in Proposition 17. Finally for $7 \leq k \leq 14$ we consider

$$\begin{array}{lll} k = 7, & F_{b,a,1,1,1,1,1} & \text{with Rank } 3, \\ k = 8, & F_{c,b,a,1,1,1,1,1} & \text{with Rank } 4, \\ k = 9, & F_{1/c,c,ac,a,1,1,1,1,1} & \text{with Rank } 5, \end{array}$$

and $\Psi \circ F_{\bar{a}\bar{c},\bar{a},1/\bar{c},1/(\bar{a}\bar{c}),1/\bar{a},1/(\bar{a}\bar{c}),1/\bar{a},\bar{c},\bar{a}\bar{c},\bar{a}} = \Psi$ for $10 \leq k \leq 14$, with $\bar{a}$ and $\bar{c}$ suitable chosen. All these $F_{[k]}$ have rank $k-5$, as we wanted to prove. ∎

5 Meromorphic non-integrability for the case $k = \dot{5}$

Our main result is the following theorem:

Theorem 20. *For $k = \dot{5}$ and most values of $\{a_n\}_n$ the map $F_{[k]}$ has no meromorphic first integral.*

Its proof is a consequence of the following result:

Proposition 21. *For $k = \dot{5}$ let ϕ_i, $i = 1, \ldots, 5$ be as in Theorem 6,*

$$\phi_i = \prod_{\substack{n \,\equiv\, i \,(\mathrm{mod}\ 5) \\ n\, =\, 1, \ldots, k}} a_n.$$

Then if $\{\phi_2, \phi_3, \phi_4, \phi_5\} \not\subset \{\phi_1^r,\, r \in \mathbb{Q}\}$ the map $F_{[k]}$ has no meromorphic first integral.

Proof. For simplicity, we prove result for the case $k = 5$, being the proof in the general case similar. Note that the above condition reads as

$$\{b, c, d, e\} \not\subset \{a^r,\, r \in \mathbb{Q}\}. \tag{16}$$

If $F_{[5]} = F_{e,d,c,b,a}$ has a meromorphic first integral, the same holds for all the other maps $F_{a,e,d,c,b}$, $F_{b,a,e,d,c}$, $F_{c,b,a,e,d}$, and $F_{d,c,b,a,e}$. These five maps have the origin $(0,0)$ as a fixed point and are analytic in its neighborhood, see the proof of Proposition 16. Moreover the corresponding couples of eigenvalues of their linear parts at zero are $1/a, e$; $1/b, a$; $1/c, b$; $1/d, c$ and $1/e, d$, respectively. Hence, applying Theorem 1, we obtain the following necessary conditions for the existence of a meromorphic first integral

$$a^{n_1} e^{m_1} = a^{n_2} b^{m_2} = b^{n_3} c^{m_3} = c^{n_4} d^{m_4} = d^{n_5} e^{m_5} = 1,$$

for some $n_i, m_i \in \mathbb{Z}$, $i = 1, \ldots, 5$. From these equalities we get that $\{b, c, d, e\} \subset \{a^r,\, r \in \mathbb{Q}\}$. So the result follows. ∎

A simple corollary of the above result is:

Corollary 22. *The map $F_{e,d,c,b,1}$ has a meromorphic first integral if and only if $b = c = d = e = 1$.*

Of course, all the known rationally integrable cases, like $F_{[5]} = F_a^5$, $F_{[10]} = F_{ba}^5$, $F_{[15]} = F_{c,b,a}^3$ and $F_{[20]} = F_{1/a,c,ac,a}^5$ satisfy $\{\phi_2, \phi_3, \phi_4, \phi_5\} \subset \{\phi_1^r,\, r \in \mathbb{Q}\}$.

6 Numerical evidences of chaos

Our simulations show that there are examples of maps $F_{[k]}$ exhibiting SSNC when $k \in \{4, 7, 8, 11\}$. These behaviors can be seen by plotting some orbits of the corresponding conjugated maps $G_{[k]}$ for the k-periodic sequences of parameters $2, 2, \ldots, 2, 3$. More visual examples can be obtained by studying the maps $G_{\delta,7,4,2}$, $G_{1,\delta,1,\delta,7,4,2}$, $G_{\delta/2,1,\delta,1,\delta,7,4,2}$ and $G_{1,\delta,2,\delta/2,1,\delta,1,\delta,7,4,2}$, with $\delta = 0.001$. See some pictures in Figures 1 and 2.

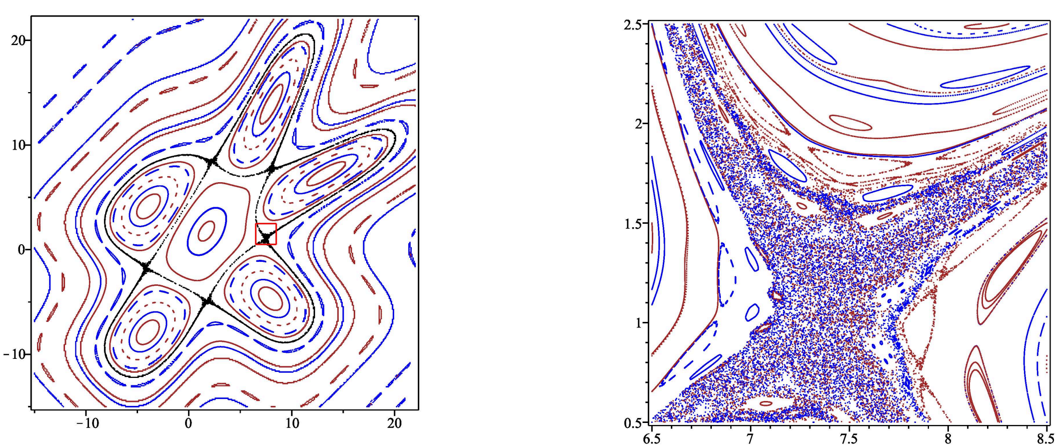


Figure 2: Some orbits of $G_{0.001,7,4,2}$ and a zoom with much more orbits.

In fact we prove:

Lemma 23. *If there exist maps of the form $F_{[4]}$,$F_{[7]}$ and $F_{[11]}$ exhibiting SSNC then, for any $k \neq \dot{5}$, $k \geq 7$, there exist maps of the form $F_{[k]}$ with* $\operatorname{rank}(\{a_n\}_n) = k$ *having also SSNC.*

Proof. For $m \geq 0$, the maps

$$\begin{aligned} F_{[5m+11]} &:= F_{1,1,1,1,1}^{m} \circ F_{[11]} = F_{[11]}, \\ F_{[5m+7]} &:= F_{1,1,1,1,1}^{m} \circ F_{[7]} = F_{[7]}, \\ F_{[5m+8]} &:= F_{1,1,1,1,1}^{m} \circ F_{[4]} \circ F_{[4]} = F_{[4]} \circ F_{[4]}, \\ F_{[5m+4]} &:= F_{1,1,1,1,1}^{m} \circ F_{[4]} = F_{[4]}, \end{aligned}$$

will also have SSNC. Note that, since $11 \equiv 1$, $7 \equiv 2$, $8 \equiv 3$ and $4 \equiv 4 \pmod 5$, the above maps cover all the values of k given in the statement. Since one of the features of these maps is the existence of transversal homoclinic points, which is a structural stable property, we can perturb each of the corresponding a_j by $a_j + \varepsilon_j$, with all the ε_j suitable and small enough, to obtain k-periodic sequences of parameters having SSNC and $\text{rank}(\{a_n\}_n) = k$, as we wanted to prove. ∎

We have (only numerically) shown the existence of SSNC for $k \in \{4, 7, 11\}$, but note that the above lemma allows to reduce all the other cases to these three ones.

Although the complicated behavior of the maps, see again Figure 2, induces to believe that even there is no upper bound for the number of intervals given by the adherence of a sequence, we only present here some simple examples. Concretely, for $k = 4$ we give a map $F_{[k]}$ and two sets of initial conditions such that the adherence of the sequences $\{x_n\}_n$ generated by (1) is formed by more than k intervals.

We have that for $a = 2, b = 4, c = 7$ and $d = 0.001$,

- the sequence starting at $13.35, 7.27$ is formed by 20 intervals;
- the sequence starting at $14.8, 8.25$ is formed by 7 intervals.

For instance, this last assertion can be seen by making the phase portraits of the orbit of $G_{0.01,7,4,2}$ starting at $(14.8, 8.25)$, which is formed by 5 islands, together with their images trough G_2, $G_{4,2}$ and $G_{7,4,2}$ and their projections in the x-axis, see Figure 3. The property of the existence of sequences $\{x_n\}_n$ generated by (1) such that their adherence consists of more that k intervals should be true for all the values of k given in Lemma 23. We also want to comment that, for these values of k, the initial conditions lying on the stable manifolds of the q-periodic saddle points of $G_{[k]}$, also have a curious behavior: the adherence of $\{x_n\}_n$ is the sequence itself together with q more points, corresponding to the saddle points. In general q also is greater than k. On the other hand, the most complicated orbits, that is the ones between two big invariant curves, which adherence seems to fill a region of positive measure give rise to a single interval when we consider their projections given by the sequence $\{x_n\}_n$.

Acknowledgements

We thank Joan Carles Tatjer for his suggestion to introduce the maps $G_{[k]}$ for a better approach to the numerical study of the maps $F_{[k]}$.

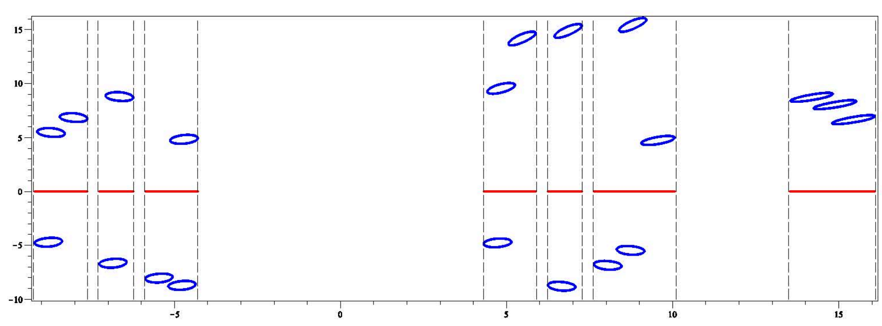


Figure 3. Case $k = 4$. An orbit of $G_{[4]}$, their images trough $G_{a_i,a_{i-1},\ldots,a_1}$, $i = 1, 2, 3$, and the projection corresponding to $\{x_n\}_n$.

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