

Witness Gabriel Graphs*

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August 6, 2010

Abstract

We consider a generalization of the Gabriel graph, the *witness Gabriel graph*. Given a set of vertices P and a set of witness points W in the plane, there is an edge ab between two points of P in the witness Gabriel graph $GG^-(P, W)$ if and only if the closed disk with diameter ab does not contain any witness point (besides possibly a and/or b). We study several properties of the witness Gabriel graph, both as a proximity graph and as a new tool in graph drawing.

1 Introduction

Originally defined to capture some concept of neighborliness, *proximity graphs* [9, 12, 15] can be intuitively defined as follows: given a set P of points in the plane, the vertices of the graph, there is an edge between a pair of vertices $p, q \in P$ if some specified region in which they interact contains no point from P , besides possibly p and q .

Proximity graphs have proved to be a very useful tool in shape analysis and in data mining [9, 16]. In graph drawing, a problem that has been attracting substantial research is to explore which classes of graphs admit a proximity drawing, for some notion of proximity, and when it is possible to efficiently decide, for a given graph, whether such a drawing exists [3, 12]. For all these reasons, several variations and extensions have been considered, from higher-order proximity graphs to the so-called weak proximity drawings [4, 9].

In the case of the *Gabriel graph*, $GG(P)$, the region of influence of a pair of vertices a, b is the closed disk with diameter ab , D_{ab} . An edge ab is in the Gabriel graph of a point set P if and only if $P \cap D_{ab} = \{a, b\}$ (see Figure 1(left)). Gabriel graphs were introduced by Gabriel and Sokal [8] in the context of geographic variation analysis.

*Part of this research was done while the third author was visiting Polytechnic, supported by Grant AGAUR-Generalitat de Catalunya 2007 BE-1 00033.

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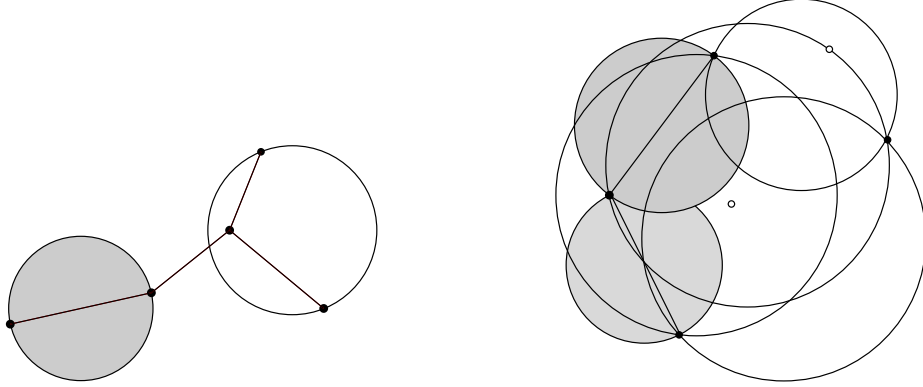


Figure 1: Left: Gabriel graph. The vertices defining the shaded disk are adjacent because their disk doesn't contain any other vertex, in contrast to the vertices defining the unshaded disk. Right: witness Gabriel graph. Black points are the vertices of the graph, white points are the witnesses. Each pair of vertices defining a shaded disk are adjacent and the pairs defining the remaining (unshaded) disks are not.

We consider in this work a generalization of the Gabriel graph, the *witness Gabriel graph*, $GG^-(P, W)$. It is defined by two sets of points P and W ; P is the set of vertices of the graph and W is the set of *witnesses*. There is an edge ab in $GG^-(P, W)$ if, and only if, there is no point of W in $D_{ab} \setminus \{a, b\}$ (see Figure 1(right)).

Notice that witness Gabriel graphs are a proper generalization of Gabriel graphs, because when the set W of witnesses coincides with the set P of vertices, we clearly obtain $GG^-(P, P) = GG(P)$. This was the main underlying idea of the generic concept of *witness graphs*, which were introduced as a general framework in [2] to provide a generalization of proximity graphs, allowing witnesses to play a negative role as in this paper or a positive one as well. Several examples were described in [1, 2], including in particular witness versions of Delaunay graphs and rectangle-of-influence drawings. A systematic study is developed in [6]. As already mentioned in this introduction, generalizing basic proximity graphs has attracted several research efforts. This is also our main motivation. On the other hand, a witness graph $W(P, Q)$ is an instrument for describing the position of P with respect to Q . We believe that once these graphs are well understood, by considering simultaneously $W(P, Q)$ and $W(Q, P)$ we would have useful tools for the description of the interaction/discrimination between the two sets; this is a main topic of our ongoing research.

In this paper we prove several fundamental properties of witness Gabriel graphs, describe algorithms for their computation, and present results on the realizability of some combinatorial graphs.

We assume throughout the paper that the points in $P \cup W$ are in general position, that is, that there are no three points of $P \cup W$ on a line and no four on a circle.

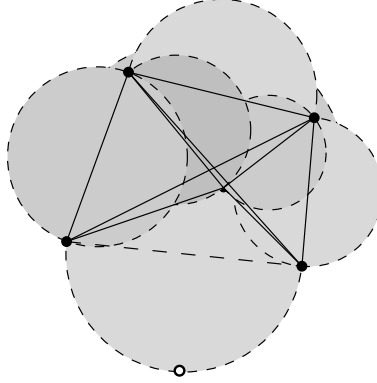


Figure 2: The white point is a witness that removes the dashed edge.

2 Some Properties of Witness Gabriel Graphs

It is known that $\text{MST}(P) \subseteq \text{GG}(P) \subseteq \text{DT}(P)$ [10], where $\text{MST}(P)$ is the minimum spanning tree and $\text{DT}(P)$ is the Delaunay triangulation. As a consequence, $|\text{MST}(P)| \leq |\text{GG}(P)| \leq |\text{DT}(P)|$, where we have used $|\cdot|$ to denote the number of edges in a graph. Expressing this in terms of $n = |P|$, we have that $n - 1 \leq |\text{GG}(P)| \leq 3n - 6$. In [14], a more detailed analysis gives a tighter upper bound of $3n - 8$.

For witness Gabriel graphs $\text{GG}^-(P, W)$, the situation is quite different, as for any fixed set P of n points, by varying the size of W and the location of the witnesses, the number of edges in $\text{GG}^-(P, W)$ can attain any value from 0 to $\binom{n}{2}$. For example, when $W = \emptyset$, we obviously obtain $\text{GG}^-(P, \emptyset) = K_n$.

Theorem 1. *For any set P of n points in the plane, a witness Gabriel graph $\text{GG}^-(P, W)$ can have any number of edges from 0 to $\binom{n}{2}$ edges, by a suitable choice of the set W of witnesses.*

Proof. Consider any given graph $\text{GG}^-(P, W)$ and take the union U of the diametral disks $D_{p_i p_j}$, $p_i, p_j \in P$, that do not contain a point $q \in W$. The boundary of the union consists of circular arcs $C_{p_i p_j}$ of disks $D_{p_i p_j}$, for some $p_i, p_j \in P$. Put a point $q \in W$ in the relative interior of one such arc $C_{p_i p_j} \setminus \{p_i, p_j\}$. Point q lies in the closed disk $D_{p_i p_j} \setminus \{p_i, p_j\}$. By construction, it lies outside all other disks. Therefore adding q to W would eliminate precisely one edge, namely (p_i, p_j) .¹ By iterating this procedure to remove the edges one by one from the witness Gabriel graph, one can see that any number of edges can be attained (see Figure 2). \square

The reverse problem is more interesting: as the witness points can be thought as *interferences* that prevent the points in P from being adjacent, one may wonder how many witnesses are required to completely eliminate all edges in $\text{GG}^-(P, W)$. Trivially, if there is a witness inside each disk D_{ab} , for all $a, b \in P$, then $\text{GG}^-(P, W)$ has no edge. This can be achieved, for instance, by putting a witness close to the midpoint of every pair a, b of points from P ,

¹This choice of q is in some sense degenerate, but q can be moved slightly into the interior of $D_{p_i p_j}$ without affecting the argument.

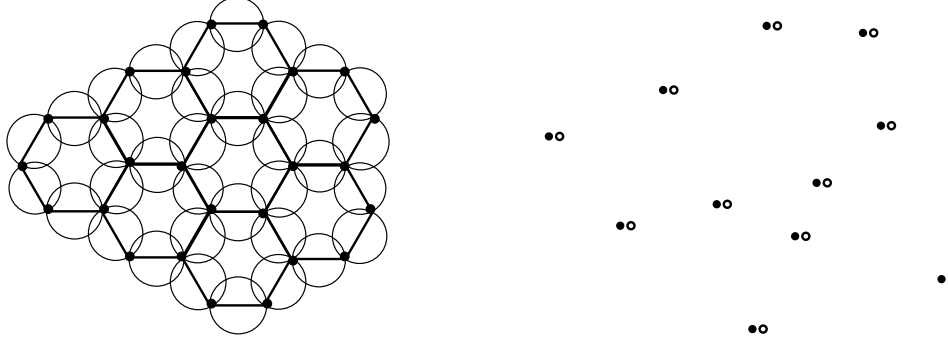


Figure 3: Left: Lower bound—any point in $W \setminus P$ can intersect at most 2 disks. Right: Upper bound—points in P are black and witnesses are white.

which would give $|W| \leq \binom{n}{2}$. In the following theorem we present a much better bound for the number of witnesses necessary to eliminate all edges of $\text{GG}^-(P, W)$.

Theorem 2. *$n - 1$ witnesses are always sufficient to eliminate all edges of an n -vertex witness Gabriel graph, while $\frac{3}{4}n - o(n)$ witnesses are sometimes necessary.*

Proof. We start with a lower bound construction. Place the points of P at the vertices of a hexagonal tiling (see Figure 3) and move them slightly so they are in general position. For each point p in P consider the three edges connecting it to the three closest points, and the closed disks that have these edges as diameter. By definition of the witness Gabriel graph, no point $q \in W$ at the intersection of three disks eliminates one of the considered edges as q would be at a point $p \in P$ defining the three disks. For any other position of $q \in W$, q never intersects more than 2 disks. Hence since we have $\frac{3}{2}n - o(n)$ disks, at least $\frac{3}{4}n - o(n)$ points in W are necessary to stab all the disks and eliminate the corresponding edges in the witness Gabriel graph.

Now we argue the upper bound. Without loss of generality, assume no two points of P lie on the same vertical lines—this can be achieved by an appropriate rotation of the coordinate system. Put a witness slightly to the right of each point of P , except for the rightmost one (see Figure 3). Every disk with diameter determined by two points of P will contain a witness. □

According to the preceding result, $n - 1$ suitable witnesses can always eliminate *all* the edges. Interestingly, realizing some witness Gabriel graphs that are not empty of edges may require strictly more witnesses.

Theorem 3. *For arbitrarily large n , there exist witness Gabriel graphs on n vertices that are not empty of edges, for which at least $\frac{3}{2}n - 8$ witnesses are necessary.*

Proof. Put the vertices on concentric circles, sixteen vertices evenly spaced per circle, the first vertex on each circle always on top of the disk (see Figure 4). The ratio between two consecutive circle radii must be between 1:1.82 and 1:1.92. On each circle, number the vertices clockwise, starting at the top. Number circles starting at the innermost. The edges of the

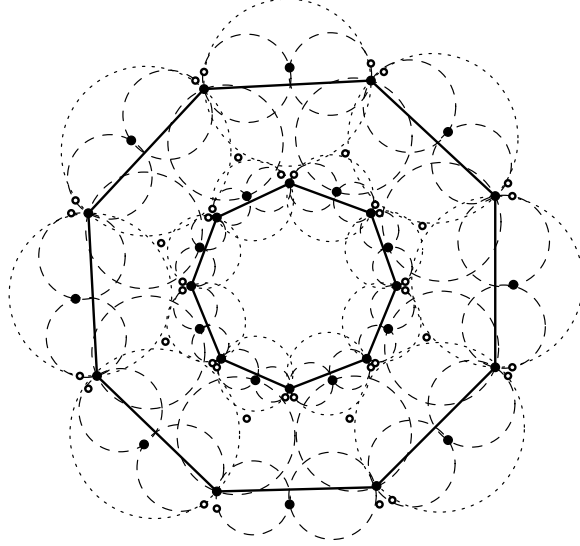


Figure 4: At least $\frac{3}{2}n - 8$ witnesses (the white points) are sometimes necessary. The black points are the n vertices of the graph.

geometric graph are the ones between consecutive even vertices on even circles and the ones between consecutive odd vertices on odd circles (see Figure 4).

Exactly n witnesses are necessary to remove the edges between every pair of consecutive vertices on each circle, without removing the edges between even (or odd) vertices. In addition, $\frac{n}{2} - 8$ witnesses are necessary to remove the edges between corresponding vertices on two consecutive circles. Summing up, we see that $\frac{3}{2}n - 8$ witnesses are required, as claimed. \square

3 Witness Gabriel Drawings

Given a combinatorial graph $G = (V, E)$, there is a *witness Gabriel drawing* of G if there is a point set P and a set of witnesses W such that the witness Gabriel graph of P and W is isomorphic to G . This witness Gabriel graph of P and W is a witness Gabriel drawing of G .

The fundamental question concerning witness Gabriel drawability is the following: Given a graph G , is it possible to construct a witness Gabriel drawing of G ? That question has been studied for (standard) Gabriel graphs. In [5] Bose et al. present a complete characterization of those trees that are drawable as a Gabriel graph. They proved that such trees cannot have vertices of degree greater than four and cannot have two adjacent vertices of degree four. They also characterized Gabriel-drawable trees by exhibiting families of forbidden subtrees and by showing that they don't contain members of these families.

Lubiw and Sleumer [13] showed that all maximal outerplanar graphs admit a Gabriel drawing in the plane. They also conjectured that any biconnected outerplanar graph has a Gabriel drawing. This was settled in the affirmative by Lenhart and Liotta [11].

As every Gabriel graph is also a witness Gabriel graph, since $GG(P) = GG^-(P, P)$, one may expect the witness Gabriel graphs to be a more powerful tool for representing graphs,

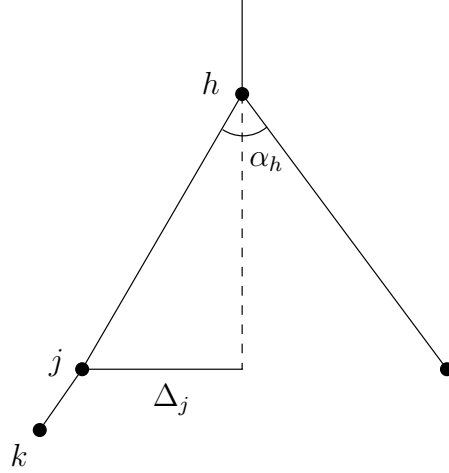


Figure 5: vertices h, j, k and the value of Δ_j .

compared to classical Gabriel graphs. This is indeed the case for trees.

Theorem 4. *Any tree can be drawn as a witness Gabriel graph.*

Proof. We construct a witness Gabriel drawing of a given tree T as follows. We assume, without loss of generality, that the tree is rooted. Draw the root of T as an arbitrary point. Number the nodes of T arbitrarily with 1 as root. Order the children of every node arbitrarily.

With each node j , we associate two values : α_j is the measure of the angles $\angle kjl$ with k, l being two consecutive children of j , and d_j is the number of children of the node j . Whenever it causes no confusion, in the remainder of the proof we do not distinguish between a vertex of T and the point representing it.

For the special case of the root (node 1), α_1 is set to $360^\circ/(d_1 + 1)$. For every other node j we define $\alpha_j = \alpha_h/d_j$, with h being the parent of j .

Recursively, for each node j , beginning with the root 1, draw the d_j children such that the angles between two edges incident to two consecutive children of j are α_j and the angles between the edges incident to an extremal child and the parent of j are $\frac{360^\circ - \alpha_j \times (d_j - 1)}{2}$.

The length of the edges is defined as follows. All the edges incident to the root have length 1. Consider a node j at depth i , its parent h and its child k . We set the length $|jk|$ of the edge jk to $\frac{1}{2}\Delta_j$, with $\Delta_j = |hj| \sin \frac{\alpha_h}{2}$ (see Figure 5).

The way the angles between consecutive children and the length of the edges are defined ensure that two edges never cross. Indeed, by construction, the Euclidean length of any path connecting j to its descendant in the tree is shorter than Δ_j and the same is true of any of its siblings j' . Since disks of radius $\Delta_j = \Delta_{j'}$ centered at j and j' do not overlap, the two paths cannot cross.

Now we shall place the witnesses. For every edge jh of the new tree connecting a non-root node j to its parent h , draw a Gabriel disk D_{jh} with diameter jh . Place two witnesses w_1 and w_2 on both sides of j such that they make an angle $\angle jhw_i = \frac{\alpha_h}{2}$, $i = \{1, 2\}$ and such

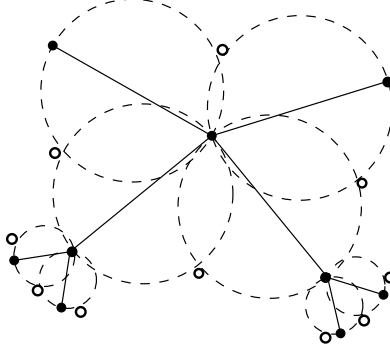


Figure 6: The vertices of the graph are black and the witnesses are white.

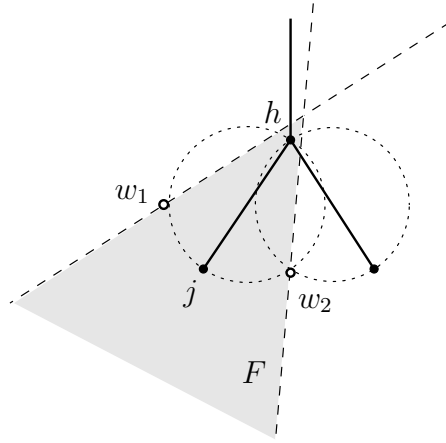


Figure 7: The wedge F around j ensures that j will be connected only to its parent and its children.

that they are very close but outside the Gabriel disk D_{jh} (and outside the Gabriel disks of h and its other children) (see Figure 6).

Let F be the intersection of two half planes H_1, H_2 defined as follows: H_i is the half plane containing j bounded by a line through w_i and perpendicular to fw_i . By construction j and all of its descendants lie in F since they are contained in a disk of radius strictly smaller than Δ_j centered at j . The placement of w_1, w_2 guarantees that no edge exists between any vertex in the subtree of j and the vertices outside of F (see Figure 7). Any point $p \in P$ outside F would make one of the angles $\angle pw_i j, i = \{1, 2\}$, larger than 90° and therefore would not be connected to j . Applying this reasoning to every node $j \neq 1$ in T , we conclude that each node is connected only to its parent and its children. Therefore $GG^-(P, W)$ is isomorphic to T . \square

We also prove that one can draw any complete bipartite graph as a witness Gabriel graph.

Theorem 5. *Every complete bipartite graph can be drawn as a witness Gabriel graph.*

Proof. We construct a drawing of $K_{m,n}$ with $m \geq n$. To avoid trivialities, we assume that

$m > 1$. Draw an axis-aligned rectangle $acdb$ such that the diametral disk $D = D_{ad} = D_{cb}$ is a unit disk, i.e., $|ad| = |cb| = 1$. Let p be any point on the segment ab and let q be any point on the segment cd (see Figure 8(left)). Let S be the horizontal strip bounded by the lines ab and cd . The diametral disk D_{pq} (dashed in the figure) is interior to S but for two circular segments that are contained in the circular segments determined by the chord ab and the chord cd on D , and its center lies on the line parallel to ab and cd through the center of the rectangle. Therefore, if we place no witness in $D \cup S$, then p and q would necessarily be adjacent in a witness Gabriel graph that included p and q as vertices.

Now, put m points $a = p_1, p_2, \dots, p_m = b$ equally spaced on ab , and n points $c = q_1, p_2, \dots, q_n = d$ equally spaced on cd , and let P be the set of these $m + n$ points.

Consider now a disk D' having as diameter two consecutive points on a horizontal edge of the rectangle, say p_i and p_{i+1} on segment ab . If the radius of D' is bigger than the height of the circular segment defined by the chord ab on D , then D' will stick out of the disk D (see Figure 8(right)).

The radius of D' is $|ab|/2(m-1)$, hence the preceding condition translates to $|ab|/(m-1) > |cb| - |ac|$. Equivalently, if we denote $x = |ab|$, taking into account that $|cb| = 1$, the condition becomes

$$\frac{1}{m-1} > \frac{1 - \sqrt{1 - x^2}}{x}. \quad (1)$$

Since the right-hand side of Eq. (1) tends to zero as x tends to zero, no matter how large m is, we can always select a value of x such that condition (1) is satisfied. Therefore taking this value of x all the disks having as diametral pair two of the p_i 's (or two of the q_j 's), whether consecutive or not, will stick out of the region $D \cup S$ and can be pierced by a set of witnesses W , all of them outside the region $D \cup S$. Therefore the witness Gabriel graph $GG^-(P, W)$ is isomorphic to $K_{m,n}$, and the claim is proved. Note that an infinitesimal perturbation of the points in $P \cup W$ would remove collinearities and still $GG^-(P, W) \simeq K_{m,n}$. \square

In the following, we observe some properties of witness Gabriel drawings, before deducing that some complete k -partite graphs, $k > 2$, have no such drawings.

Lemma 6. *In a witness Gabriel graph $GG^-(P, W)$, for any pair of incident edges ab and bc , all the points $p \in P$ in the triangle $\triangle abc$ are connected to the common vertex b , and there is no witness in $\triangle abc$ (except possibly for the vertices a, b, c , if they belong to $P \cap W$).*

Proof. Let d be the foot of the perpendicular from b to ac (see Figure 9). This point is at the intersection of the boundaries of the disks D_{ab} and D_{bc} of diameters ab and bc , respectively. For any edge be with $e \in ac$, $\angle bde = \angle bda = \angle bdc = 90^\circ$. For any $e \in ac$, $D_{be} \subset D_{ab} \cup D_{bc}$. As D_{ab} and D_{bc} don't contain any witnesses, neither does D_{be} .

Consider now a point f inside the triangle $\triangle abc$. Extend the segment bf until it meets ac at a point e . Now $D_{bf} \subseteq D_{be} \subseteq D_{ba} \cup D_{bc}$. Therefore D_{bf} is empty of witnesses and bf is an edge of $GG^-(P, W)$. This proves the first part of the Lemma.

For the second part, we use the fact that the two disks D_{ab} and D_{bc} with diameters ab and bc cover the interior of the triangle $\triangle abc$; therefore, there is no witness in $\triangle abc$ as any such witness inside the triangle would remove one or both edges ab and cd . \square

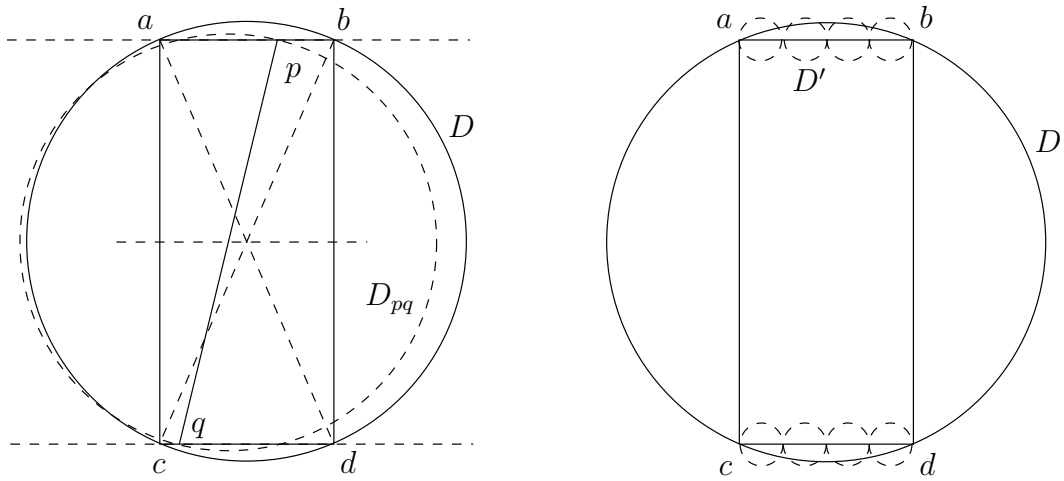


Figure 8: Left: Making sure that no witness eliminates any edge pq for $p \in ab$ and $q \in cd$. Right: Ensuring that small disks D' are not completely covered by the large disk D .

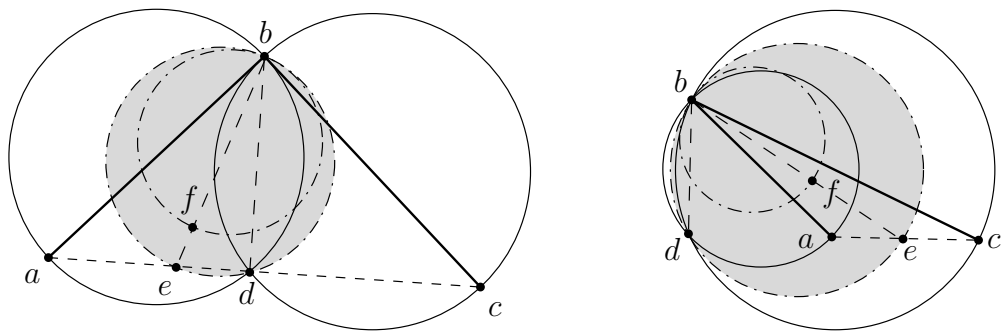


Figure 9: D_{be} is included in $D_{ab} \cup D_{bc}$.

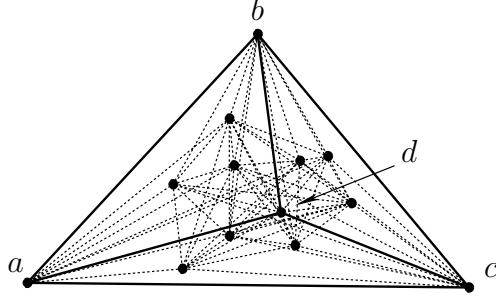


Figure 10: All the vertices inside the triangle $\triangle abc$ will be connected to d .

Proposition 1. *If a witness Gabriel graph has as a subgraph a triangle $\triangle abc$, the vertices a , b , c and the vertices inside this triangle form a complete subgraph (see Figure 10).*

Proof. Considering the three pairs of edges ab and bc , bc and ca , and ca and ab , by Lemma 6, all the vertices inside the triangle are connected to a , b , and c .

To complete the proof, consider a vertex d inside the triangle (see Figure 10). We have already seen that it will be connected to the vertices a , b and c . The edges ad , bd and cd together with the triangle edges define three new triangles $\triangle abd$, $\triangle acd$ and $\triangle bcd$. All the other points inside the triangle $\triangle abc$ are inside one of these three triangles. Therefore they will be connected to d , by another application of Lemma 6. \square

We denote by $K_{i,i,i}$ the $i \times i \times i$ complete 3-partite graph; we associate a color to each part; similarly, $K_{i,i,i,i}$ denotes the $i \times i \times i \times i$ complete 4-partite graph. In the following, we consider only complete k -partite graphs with $k \geq 3$ and at least two vertices of each color, as we have seen that $K_{m,n}$ can be drawn as a witness Gabriel graph for any m and n .

Lemma 7. *In every witness Gabriel drawing of a complete k -partite graph, there are at least two vertices of each color on the convex hull H of the set of vertices.*

Proof. First suppose that for some color, say black, there are no black vertices on H . Therefore there are at least two black vertices b_1, b_2 inside H . Draw all the edges between b_1 and the vertices of the hull (see Figure 11(left)). These are edges of the graph, so they must be in the witness Gabriel drawing. As b_2 is inside H , it is in a triangle $\triangle b_1mg$ defined by two edges incident to b_1 . By Lemma 6, b_2 is adjacent to b_1 , a contradiction.

Now suppose there is exactly one black vertex b_1 on H . Therefore there is at least one other black vertex b_2 inside H . Draw all edges between b_1 and other vertices of H (see Figure 11(right)). These edges have to be in the witness Gabriel drawing. As b_2 is inside H , it is in the triangle defined by two edges incident to b_1 . By Lemma 6, b_2 is adjacent to b_1 , a contradiction. \square

Lemma 8. *It is impossible, in a witness Gabriel drawing of a complete k -partite graph, for a line containing two vertices of one color, to divide the plane in two half-planes, each of them containing two points of a second and third color respectively (see Figure 12(left)). The second and the third colors may be the same.*

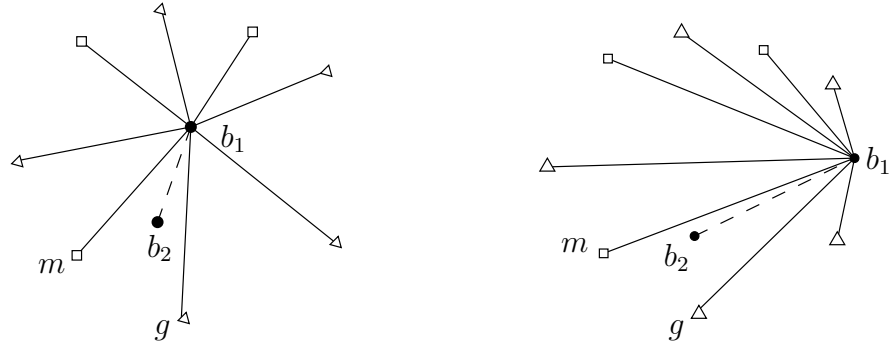


Figure 11: Left: No black vertex on H . Right: One black vertex on H .

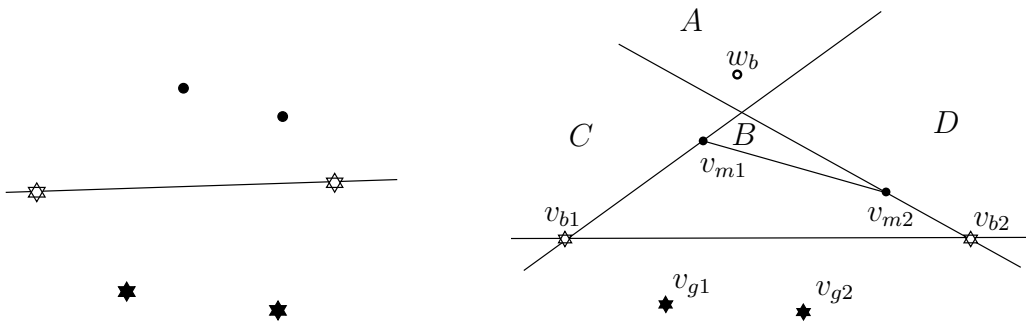


Figure 12: Left: The line containing the white-star points separates two black-dot points from two black-star points. Right: Illustration to the proof of Lemma 8.

Proof. Consider two white-star vertices v_{b1}, v_{b2} , two black-point vertices v_{m1}, v_{m2} , and two black-star vertices v_{g1}, v_{g2} . We will show that it is not possible to place the witness(es) to remove the white-star monochromatic edge, the black-point monochromatic edge, and the black-star monochromatic edge without removing a bi-chromatic edge. Let the witnesses removing the edges $v_{g1}v_{g2}, v_{b1}v_{b2}$, and $v_{m1}v_{m2}$ be w_g, w_b , and w_m , respectively. They need not be distinct.

Without loss of generality, assume w_b lies above the line $v_{b1}v_{b2}$ (see Figure 12(right)), i.e., on the side containing v_{m1} and v_{m2} . Note that the vertices $v_{b1}, v_{b2}, v_{m1}, v_{m2}$ must be in convex position, as otherwise their convex hull would be a triangle, for example $v_{b1}v_{b2}v_{m1}$ with v_{m2} in its interior, implying by Lemma 6 v_{m1} and v_{m2} be adjacent. Therefore, the four vertices v_{b1}, v_{m1}, v_{m2} , and v_{b2} are in convex position and, without loss of generality, in this order around their convex hull. Moreover, applying Lemma 6 to the bi-chromatic edges $v_{b1}v_{m1}$ and $v_{b1}v_{m2}$ (and to $v_{m2}v_{b1}, v_{m2}v_{b2}$), we conclude that there is no witness in the convex hull of the four vertices.

The witness w_b cannot lie to the left of the line $v_{b2}v_{m2}$ as it would remove the edge $v_{b1}v_{m2}$. Indeed, as $\angle v_{b1}w_bv_{b2} \geq 90^\circ$, for w_b to the left of $v_{b1}v_{m2}$, $\angle v_{b1}w_bv_{m2} > \angle v_{b1}w_bv_{b2} \geq 90^\circ$, and the edge $v_{b1}v_{m2}$ would be removed. Symmetrically, w_b cannot lie to the right of the line $v_{b1}v_{m1}$ as it would remove the edge $v_{b2}v_{m1}$. Therefore, the lines $v_{b1}v_{m1}$ and $v_{b2}v_{m2}$ must cross and w_b lies in region A to the right of the line $v_{b2}v_{m2}$ and to the left of the line $v_{b1}v_{m1}$ (see Figure 12(right)).

Now consider the four regions A, B, C, D , external to the quadrilateral $v_{b1}v_{b2}v_{m2}v_{m1}$, as in Figure 12(right). Witness w_m cannot be in region C or D —irrespective of whether it lies above or below the line $v_{m1}v_{m2}$, its presence in C or D would eliminate at least one of the edges $v_{b1}v_{m2}, v_{b2}v_{m1}$. Indeed, $\angle v_{m1}w_mv_{m2} < \max\{\angle v_{b1}w_mv_{m2}, \angle v_{m1}w_mv_{b2}\}$ for all placements of w_m in C or D . Witness w_m can't be in region A either as it would remove both edges $v_{b1}v_{m2}$ and $v_{b2}v_{m1}$. Indeed, $\angle v_{m1}w_mv_{m2}$ is smaller than $\angle v_{b1}w_mv_{m2}$ and $\angle v_{m1}w_mv_{b2}$. Therefore w_m must be in region $B \subset \triangle v_{b1}w_bv_{b2}$, implying $\angle v_{b1}w_mv_{b2} > \angle v_{b1}w_bv_{b2} \geq 90^\circ$. But then w_m eliminates both $v_{b1}v_{m2}$ and $v_{b2}v_{m1}$, a contradiction. \square

Lemma 9. *Consider a set of points, colored by two or more colors, in convex position, such that there are no two consecutive black points. There is a triangulation of this set of points such that every triangle has exactly one black vertex.*

Proof. Find three consecutive points a, b, c , on the convex hull of the set of points such that exactly one of them is black. Add the triangle $\triangle abc$ to the triangulation and remove b from the set of points. Repeat this procedure until no three consecutive points a, b, c with exactly one black are found. At this moment, there must be either only two points left (in which case we have constructed the desired triangulation) or more than two points but no black.

Indeed, notice that if there were more than two points left, with at least one black one, this black point would have as neighbors points of a different color, and we could repeat the procedure described above at least once more. This is true as during the procedure above in which we add a triangle to the triangulation and remove one point, two cases may occur. In the first case, one black point is removed, in which case the neighborhood of all other black points does not change (as the black point removed had colored points as neighbors), and the

original condition that there are no two consecutive black points is maintained. In the other case, one point b of a different color than black is removed and the triangle abc is added to the triangulation. Either a or c might be black, but not both as all triangles are incident to exactly one black point. Suppose without loss of generality that a is black. Then a gets as a neighbor, instead of b , a new point c of a different color than black, and once again the original condition of no two consecutive black points is maintained.

If there are more than two points left, none black, remove the last triangle added and put back the point b that was removed last; b must be black. If b, c_1, c_2, \dots, c_m are the remaining points, in order, add the triangles $\triangle bc_1c_2, \triangle bc_2c_3, \dots, \triangle bc_{m-1}c_m$. The set of points is triangulated such that each triangle contains exactly one black vertex. \square

Lemma 10. *Consider the convex hull H of the vertices of a witness Gabriel drawing of a complete k -partite graph. Consider a subset of vertices of H , with at least one black point and no two consecutive black points. The interior of the convex hull of this subset is empty of black points.*

Proof. By Lemma 9, this subset of points can be triangulated such that each triangle has exactly one black vertex. In every triangle, the two edges incident to the black vertex are in the complete k -partite witness Gabriel drawing. Any black vertex b_2 inside the convex hull would be inside a triangle defined by two edges incident to a black vertex b_1 . By Lemma 6, b_2 would be incident to b_1 , a contradiction. \square

Consider a finite set of points P colored with k colors. A (*quasi-convex circular*) *quasi-ordering* \prec of P is a partition of P into $s \geq k$ subsets P_1, \dots, P_s , cyclically ordered as $P_1 \prec P_2 \prec \dots \prec P_s \prec P_1$, such that (a) every P_i contains only points of one color and consecutive sets have different colors, and (b) any subset $S \subset P$ with at most two elements from each P_i is in convex position and their cyclic ordering along $\text{CH}(S)$ is consistent with \prec ; we make no assumption on the relative order of the points coming from the same set P_i (hence the choice of the term *quasi-convex quasi-ordering*). Refer to Figure 13(left).

Lemma 11. *The set of vertices P of a witness Gabriel drawing of a complete k -partite graph can be quasi-ordered.*

Proof. Consider the convex hull $H := \text{CH}(P)$ of the vertices. Let $E \subset P$ contain the vertices that appear on H . Traversing H in counterclockwise order, group consecutive vertices of the same color together, forming a cyclically ordered partition E_1, \dots, E_s of E . Since E is in convex position, this partition is trivially a quasi-ordering, as define above. We now show how to extend it to the entire set P . Specifically, for each point in $P' := P \setminus E$, we assign it to one of the groups E_i . We say a group E_i is *trivial* if it contain just one point. We will see that no other point is ever added to such a group.

Let K_{CW} be the convex hull of the set consisting of the most *clockwise* point from each E_i . Analogously define K_{CCW} for most counterclockwise points.

Lemma 10 implies that no point of P' is contained inside $K_{\text{CW}} \cup K_{\text{CCW}}$. The remaining points of P lie in $H \setminus (K_{\text{CW}} \cup K_{\text{CCW}})$. This set naturally splits into at most $2s$ convex subsets: *pockets* Π_i are connected components of this set whose boundary contains one (and only one)

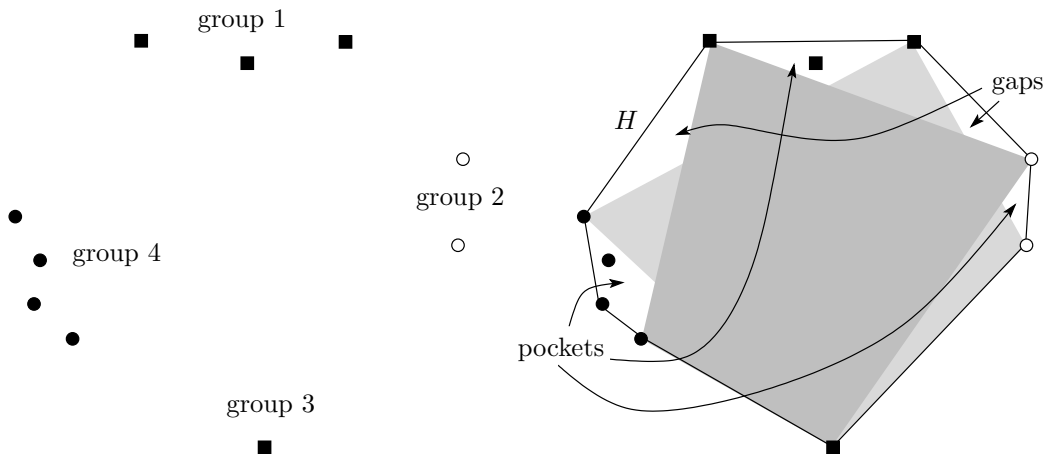


Figure 13: Left: An example of a quasi-ordered set. Right: Quasi-ordering of the vertex set of a witness Gabriel drawing of a complete k -partite graph. Sets K_{CW} and K_{CCW} are shaded. Pockets and gaps form the complement of $K_{\text{CW}} \cup K_{\text{CCW}}$ in H .

non-trivial group E_i , while *gaps* are components adjacent to two consecutive groups (refer to Figure 13(right)).

We complete our quasi-ordering now: We claim that $P' \subset \bigcup \Pi_i$, i.e., there are no points of P' in the gaps. And a point in pocket Π_i is simply assigned to E_i .

It remains to prove (i) the emptiness of gaps and (ii) the fact that the resulting partition of P is a quasi-ordering. We start with (i), for which it is sufficient to argue that a gap between consecutive groups, say, white E_1 and black E_2 is empty of points of P' . Let w and b be the points of E_1 and E_2 , respectively, adjacent to the gap. For a contradiction, consider a point p in the gap. Suppose first that it is not black or white, say green. By Lemma 7 a green point g appears on H . By construction, the gap lies in $\triangle bwg$ and therefore so does p , contradicting Lemma 6, as it forces the existence of a green-green edge pg . Thus p must be white or black. We assume it is black, without loss of generality. Then we again take a green point g on H , forcing $\triangle bwg$ to contain p and ensuring the existence of a black-black edge bp , by Lemma 6—a contradiction. Therefore, indeed, the gaps are empty.

We now argue (ii): the resulting partition of points is indeed a quasi-ordering. We start by proving part (a) of the definition, namely that all points in the pocket Π_i have the color of E_i (consecutive groups have different colors by construction). For a contradiction, suppose a, say, blue point $b \in \Pi_i \cap P'$ lies in the pocket of purple group E_i . Let b' be a blue point on H . By definition of a pocket, b lies in a triangle formed by b' and two purple points of E_i , once again contradicting Lemma 6 and hence (a) is proved.

It remains to check that any subset of P formed by taking at most two points from each pocket Π_i (including E_i) is in convex position. If at most one point is used from each group, the assertion holds by construction. To finish the argument, it is sufficient to show that, for any two, say, cyan points c and c' , the line cc' leaves all the other pockets on the same side; indeed it is sufficient to prove this for the two pockets adjacent to the cyan pocket of c and c'

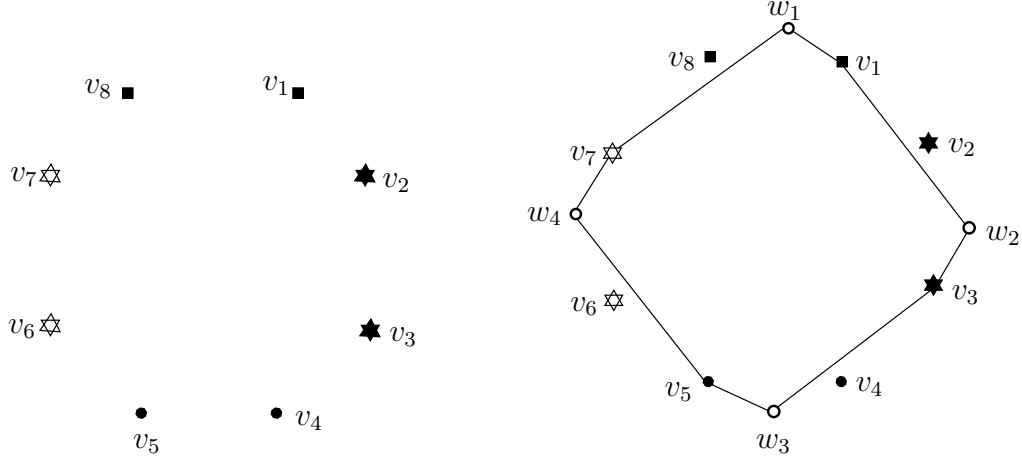


Figure 14: Left: A drawing of $K_{2,2,2,2}$ in which all the vertices of the same kind are consecutive. Right: Octagon in a 4-partite witness Gabriel drawing.

and thus of colors other than cyan. If the line cc' did not have both pockets entirely to one side of it, there would be two points p and q coming from these two pockets (one from each, or both from the same one) of color other than cyan, on opposite sides of the line cc' . Since c and c' lie in the same pocket and therefore on the same side of pq , this would force either $c \in \triangle pqc'$ or $c' \in \triangle pqc$, contradicting once again Lemma 6 and thereby completing the proof of the Lemma. \square

Lemma 12. *Any witness Gabriel drawing of $K_{2,2,2,2}$ must have vertices in convex position.*

Proof. By Lemma 7, there are at least 2 vertices of each color on the convex hull. \square

Lemma 13. *In a witness Gabriel drawing of a $K_{2,2,2,2}$, there is no witness inside the convex hull of the set of vertices.*

Proof. By Lemma 12, all the vertices are in convex position. Take any triangulation of the set of vertices. Each triangle will be incident to at most two vertices of the same color; therefore, for each triangle, at least two edges will be present in the witness Gabriel drawing. By Lemma 6, there can't be any witness in any of the triangles. \square

Lemma 14. *There is no witness Gabriel drawing of $K_{2,2,2,2}$ in which all the vertices of the same color are consecutive (see Figure 14(left)).*

Proof. By Lemma 12, all the vertices are in convex position, so the ordering of vertices is well defined. Name the vertices v_1, \dots, v_8 clockwise as in Figure 14(right). The witnesses w_1, \dots, w_4 eliminate the edges $v_8v_1, v_2v_3, v_4v_5, v_6v_7$, respectively. By Lemma 13, the witnesses are outside the convex hull of all the vertices. As we will see below, the four witnesses are distinct and necessary. For a contradiction, suppose one witness $w = w_2 = w_3$ removes two monochromatic edges, say v_2v_3 and v_4v_5 . The witness w sees v_2v_3 and v_4v_5 , respectively, with an angle larger than 90° , i.e. $\angle v_2ww_3 > 90^\circ$ and $\angle v_4ww_5 > 90^\circ$. As we already saw, w is

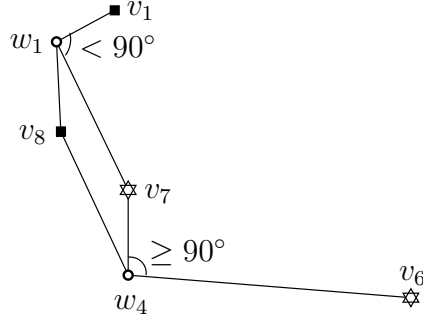


Figure 15: Detail of the proof of Theorem 14.

outside $\text{CH}(P)$, and therefore sees it with a maximum view angle smaller than 180° . Hence two cases are possible:

1. w sees the two edges overlapping, and without loss of generality, it sees the vertices from left to right in the following order: v_2, v_4, v_3, v_5 . But then w removes v_2v_5 , since $\angle v_2wv_5 > \angle v_2wv_3$.
2. w sees the two edges nested, and without loss of generality, it sees the vertices from left to right in that order: v_2, v_4, v_5, v_3 . But then w removes v_2v_5 and v_3v_4 , by similar reasoning.

Hence we can conclude that each witness w removes exactly one monochromatic edge, and four distinct witnesses are necessary.

Consider the octagon $w_1v_1w_2v_3w_3v_5w_4v_7$ (see Figure 14(right)). The interior angles at w_i measure less than 90° each; otherwise a witness would be inside a diametral disk of two vertices of different colors.

The interior angles at v_1, v_3, v_5, v_7 measure strictly less than 180° each. Indeed if one of these angles were equal to 180° , we would have three points on a line; this contradicts our assumption of general position. Now suppose that one of these angles, say, $\angle w_1v_7w_4 > 180^\circ$. By definition of the witness Gabriel drawing, we have $\angle v_1w_1v_7 < 90^\circ$ and $\angle v_1w_1v_8 \geq 90^\circ$ (see Figure 15). As $\angle v_7w_4v_6 \geq 90^\circ$, we would have $\angle v_8w_4v_6 > 90^\circ$, a contradiction.

Therefore the sum of the interior angles of this octagon is less than 1080° , which is impossible. \square

The constraints described in the preceding results lead to a graph that is not drawable:

Theorem 15. *There is no witness Gabriel drawing of $K_{3,3,3,3}$.*

Proof. Assume such a drawing exists. We consider the ordering of the colors of vertices of $K_{3,3,3,3}$, in the sense of Lemma 11. In the case analysis below, we argue that the only ordering of the vertices consistent with Lemmas 13 and 14 is such that all the vertices are in convex position and between every pair of consecutive vertices of one color, there is exactly one vertex of every other color (see Figure 16).

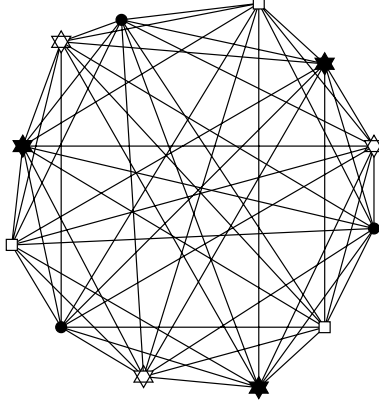


Figure 16: A tentative witness Gabriel drawing of $K_{3,3,3,3}$.

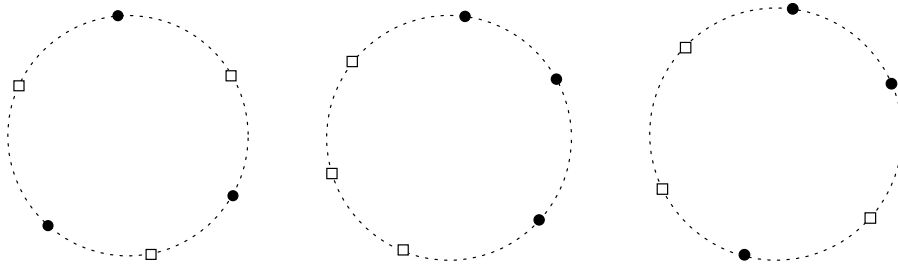


Figure 17: All possible ways to order the vertices of a witness Gabriel drawing of $K_{3,3}$.

All the possible ways to order three points of two different colors using the ordering defined in Lemma 11 are in Figure 17; notice in two of the three cases, the points must be in convex position by Lemma 11, and in the remaining (middle) case we must have the colors separated by a line and situated so that any choice of two points of each color is in convex position. All the ways to add three points of a third color to the cases in Figure 17 without violating Lemma 8 are in Figure 18. We draw the points on a circle for ease of visualization. Again, they must be in convex position unless there is a group of three consecutive points of the same color (second and third figures in the top row), in which case these points need not all appear on the convex hull of the entire set; see Lemma 11.

There is only one way to add three points of a fourth color to the set of points of Figure 18 without violating Lemma 8 and Lemma 14 (see Figure 19). Notice that in this Gabriel drawing of $K_{3,3,3,3}$, by Lemma 7, all the points are in convex position.

Now we will show that the tentative witness Gabriel drawing of $K_{3,3,3,3}$, depicted in Figure 16, where vertices are in convex position and such that between every pair of consecutive vertices of one color there is exactly one vertex of each other color, cannot be realized.

Consider the hexagon formed by the three black-star vertices b_1 , b_2 , and b_3 , and the three witnesses w_1 , w_2 , and w_3 that remove the edges b_1b_2 , b_2b_3 , and b_3b_1 , respectively (see Figure 20). The three witnesses are distinct as otherwise they would remove some bichromatic edges. The measure of each of the three interior angles $\angle b_1w_1b_2$, $\angle b_2w_2b_3$, and $\angle b_3w_3b_1$ are

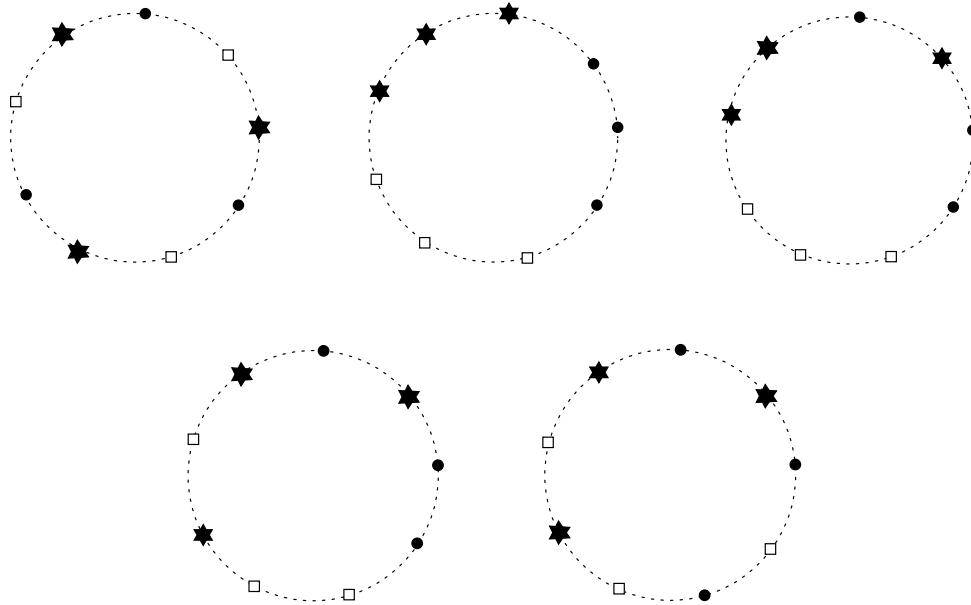


Figure 18: All possible ways to order the vertices of a witness Gabriel drawing of $K_{3,3,3}$.

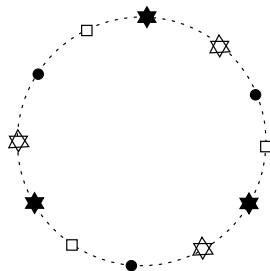


Figure 19: The only way to order the vertices of a witness Gabriel drawing of a $K_{3,3,3,3}$.

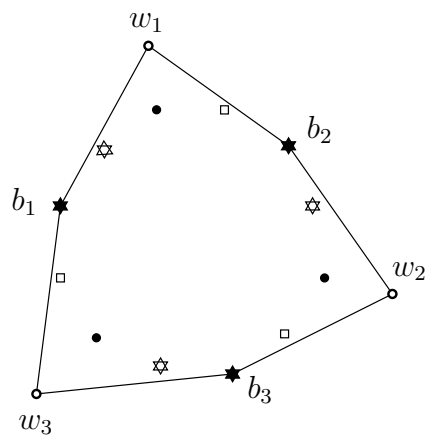


Figure 20: tentative Witness Gabriel drawing of $K_{3,3,3,3}$.

at least 90° . The sum of the measures of interior angles in a hexagon is 720° . Therefore, $\angle w_3b_1w_1$, $\angle w_1b_2w_2$, and $\angle w_2b_3w_3$ sum up to at most 450° .

If one repeats the argument for each of the four colors and their corresponding witnesses, one obtains that the sum of the interior angles such that the vertex of the angle is a vertex of the graph adjacent to two witnesses, is at most 1800° . However, the sum of the interior angles of a 12-gon that is the convex hull of the vertices equals 1800° . Therefore for at least one color, say black-star without loss of generality, and because of our assumption of general position, at least one point of another color will be outside of the hexagon $b_1w_1b_2w_2b_3w_3$, and a bi-chromatic edge will be eliminated. \square

From the preceding result we immediately obtain the following:

Corollary 16. *No graph containing $K_{3,3,3,3}$ as an induced subgraph can be drawn as a witness Gabriel graph. In particular, there is no witness Gabriel drawing of $K_{p,q,r,s}$ for $p, q, r, s \geq 3$.*

4 Construction Algorithms

In this section we describe two algorithms to compute the witness Gabriel graph $GG^-(P, W)$ from two given sets of points P and W .

Theorem 17. *Given two point sets P, W with $|P| + |W| = n$, the graph $GG^-(P, W)$ can be computed in $\Theta(n^2)$ time.*

It is clear that in the worst case $\Omega(n^2)$ time is required, since the graph may have $\Theta(n^2)$ edges.

First algorithm: For each point $p \in P$, do the following: For each point $q \in W$, draw the line l_q through q , perpendicular to pq . Consider the interior of the intersection I_p of the half-planes containing p bounded by the lines l_q , $\forall q \in W$. Then, an edge pr , $r \in P \setminus \{p\}$, is in $GG^-(P, W)$ if and only if $r \in I_p$ (see Figure 21). Indeed any point $r \in P \setminus \{p\}$ in the interior of I_p will make an angle $\angle rqp < 90^\circ$ with any $q \in W$. On the other hand, any point $r \in P \setminus \{p\}$ on the boundary or outside I_p , will make an angle $\angle rqp \geq 90^\circ$ for at least one $q \in W$.

Once we have computed the circular ordering of points in $P \cup W$ around p , we can compute I_p and identify all edges pr in linear time, for a fixed p . The circular ordering, for all p , can be computed in quadratic time by standard methods using the dual arrangement of $P \cup W$ [7].

Second algorithm: Build the Voronoi diagram $\text{Vor}(W)$ of the points in W . For each $p \in P$, add the point p to $\text{Vor}(W)$ and consider all segments of the form pr with $r \in P \setminus \{p\}$. For each edge pr take the midpoint $m(p, r)$. Observe that $m(p, r)$ is in the Voronoi cell of p in $\text{Vor}(W \cup \{p\})$ if and only if the edge pr is in $GG^-(P, W)$. The algorithm can be implemented to run in quadratic time using standard tools. Again, it is useful to have the circular ordering of all points in $P \cup W$ around each point in P .

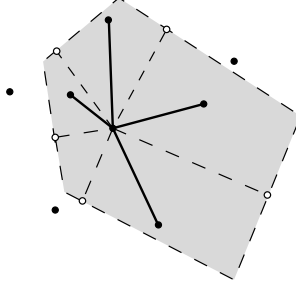


Figure 21: The first algorithm to build $GG^-(P, W)$. Black points are in P and white points in W .

As a final observation, it is worth mentioning an algorithm that would be more efficient in some cases, but not in the worst case. The *witness Delaunay graph* of a point set P in the plane, with respect to point set W of witnesses, denoted $DG^-(P, W)$, is the graph with vertex set P in which two points $x, y \in P$ are adjacent when there is a disk whose boundary passes through x and y and whose interior does not contain any witness $q \in W$. This graph was introduced in [2], and an algorithm for its computation with running time $O(e \log n + n \log^2 n)$, where e is the number of edges in the graph, was also described there.

Now, as $GG^-(P, W)$ is a subgraph of $DG^-(P, W)$, once the latter graph has been computed we can easily check in $O(\log n)$ time whether one of its edges, say pq belongs to $GG^-(P, W)$: if m is the midpoint of pq , we only have to find the point z from W which is closest to m , which can be achieved by standard point location in $\text{Vor}(W)$. Once z has been obtained, $pq \in GG^-(P, W)$ if and only if $d(m, z) > d(m, p)$. Therefore $GG^-(P, W)$ can be computed in additional $O(e \log n)$ time, once $DG^-(P, W)$ is available. To summarize, we can compute $GG^-(P, W)$ in time $O(e \log n + n \log^2 n)$, where $e \geq k$ is the number of edges in $DG^-(P, W)$ and k is the number of edges in $GG^-(P, W)$.

5 Verification Algorithm

In this section we present an algorithm to verify whether a graph $G = (V, E)$ embedded in the plane can be a witness Gabriel graph $GG^-(V, W)$, for some suitable set of witnesses W .

Theorem 18. *Given a straight-line graph $G = (V, E)$ embedded in the plane, checking if there exists a set of witnesses W so that G coincides with $GG^-(V, W)$ can be done in $O(|V|^2 \log |E|)$ time; if the answer is positive, such a set of witnesses W can be computed within the same time bounds.*

Algorithm: For each edge pq in G , draw a disk D_{pq} with diameter pq . Take the union $U = \bigcup_{pq \in E(G)} D_{pq}$ of these disks. Compute the Voronoi Diagram of the arcs and vertices of the boundary of U [17]. For each pair of vertices r and s such that there is no edge between them in G , draw a disk D_{rs} with diameter rs . Check if the center c of D_{rs} lies in U . If it does not, c (or any point sufficiently close to it) is a valid witness for rs . If it does, find which

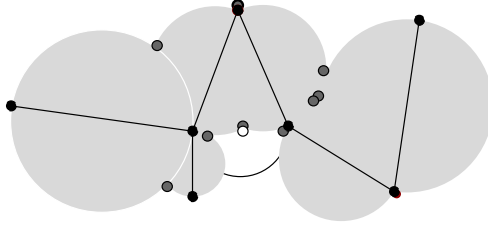


Figure 22: The algorithm to check if a geometric graph is a witness Gabriel drawing. Black points are in P .

cell C of the Voronoi diagram contains c and check if the site of C intersects D_{rs} . If the site of C does not intersect D_{rs} , $D_{rs} \subset U$, and it is impossible to place a witness to eliminate rs without also eliminating a legitimate edge of G . Therefore G is not a witness Gabriel graph $\text{GG}^-(V, W)$, for any W (see Figure 22). Otherwise, a suitable witness in $D_{rs} \setminus U$ is easy to identify. We continue to the next non-edge rs .

If none of the tests fail, we have produced a set W of witnesses such that $G = \text{GG}^-(P, W)$.

The algorithm can be implemented to run in time $O(|V|^2 \log |E|)$ using standard tools.

6 Final Remarks

We have described in this paper several properties of the witness Gabriel graph, as well as algorithms for its computation and verification. However, we have omitted the description of some extensions. For example, as the standard Gabriel graph can be extended to higher order, this can be done for the witness generalization: In a *witness k -Gabriel graph*, an edge ab , $a, b \in P$, is in the graph if there are fewer than k witnesses in $D_{ab} \setminus \{a, b\}$. Most of the preceding results can be easily modified to provide the corresponding conclusions about witness k -Gabriel graphs.

There are some obvious open problems left in this paper, such as closing the gaps between some bounds. In particular, it would be interesting to tighten the bounds in Theorem 2 on the maximum number of witnesses needed to eliminate all edges in a witness Gabriel graph. Perhaps more embarrassingly, we have no linear (nor, in fact *any subquadratic*) upper bound on the number of witnesses that are sufficient to realize an arbitrary witness Gabriel graph (Theorem 3).

On the algorithmic side, designing an *output-sensitive* algorithm for constructing a witness Gabriel graph, given its set of vertices and witnesses, i.e., one whose running time depends on the number of edges in the graph, is still an open problem. An ideal algorithm would pay a small, say, polylogarithmic, cost per additional graph edge. Another issue is whether finding the *minimum* number of witnesses required to realize a given geometric graph in the plane as a witness Gabriel graph (as in Theorem 18) is NP-hard, or whether it can be solved in polynomial time.

Finally, we also mention that some natural long-term goals, such as a complete characterization of the class of witness Gabriel graphs or the design of efficient algorithms testing

graphs for membership, remain elusive to date, which, on the other hand, is a common situation for most classes of standard proximity graphs.

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