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A CONCEPT OF WEIGHTED CONNECTIVITY ON CONNECTED GRAPHS

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Abstract. The introduction of a $\{0, 1\}$-valued game associated to a connected graph allows us to assign to each node a value of weighted connectivity according to the different solutions that for the cooperative games are obtained by means of the semivalues. The marginal contributions of each node to the coalitions differentiate an active connectivity from another reactive connectivity, according to whether the node is essential to obtain the connection or it is the obstacle for the connection between the nodes in the coalition. Diverse properties of this concept of connectivity can be derived.

Key words and phrases. Graph; Connectivity; Cooperative game; Solution; Semivalue.

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1 Introduction

The graph concept is a mathematical model for communication networks constituted by nodes or vertices connected by means of connection channels, that generically we name edges. Each node is directly connected to other nodes by means of edges or indirectly through paths involving different edges and intermediate nodes. The geometry of the edges, in addition to their number, determines the importance of each node in the system of the graph, so that it seems reasonable to study the contribution of each node to the set of connections. In this work we propose to associate to each connected graph a cooperative game. A desirable characteristic for the game is that it is simple. For this reason, a two-valued game is constructed. Values 1 and 0 are respectively associated with the connected and not connected coalitions according to the graph.

2 Connected graphs and games

A graph $\Gamma = (N, E)$ is a pair where N is a finite set of nodes and $E = E(\Gamma)$ is a set of non-ordered pairs of different nodes called edges and denoted by $i : j$. From now on we suppose $N = \{1, 2, ..., n\}$ and we denote by $G(N)$ the set of all graphs with these n nodes. We say that two nodes related by an edge are adjacent nodes. A path between two nodes is a series of edges that link both nodes in such a way that each edge and its following edge have a common node. A graph is a connected graph when, for every pair of nodes, there exists a path between both nodes. We denote by $CG(N)$ the set of all connected graphs on N.

A cooperative game with transferable utility is a pair (N, v), where $N = \{1, 2, ..., n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is the so-called characteristic function, which assigns to every coalition $S \subseteq N$ a real number $v(S)$, the worth of coalition S, and satisfies the condition $v(\emptyset) = 0$. We denote by G_N the set of all cooperative games on N. Given a set of players N, we identify each game (N, v) with its characteristic function v.

Definition 2.1 *We define the cooperative game $v_\Gamma \in G_N$ [1] associated to the connected graph $\Gamma \in CG(N)$ by $v_\Gamma(S) = 1$ if $S \subseteq N$ is connected by graph Γ and $|S| > 1$, and $v_\Gamma(S) = 0$ otherwise.*

Value 1 is associated to success and value 0 to failure in the connection of all members of each coalition of nodes S. Γ is a connected graph, then $v_\Gamma(N) = 1$. Value $v_\Gamma(\{i\}) = 0, \forall i \in N$, responds to the consideration of which isolated elements do not enter in communication with any other node.

A function $\Psi : G_N \to \mathbb{R}^N$ is called a solution for the cooperative games in G_N and represents a method to measure the negotiation strength of the players in the game or a way to emphasize the importance of the role that each one of them carries out. The space $\mathbb{R}^N$ is the so-called allocation space or payoff vector space.

In order to calibrate the importance of each player i in the different coalitions S, we can look at its marginal contribution, i.e., $v(S) - v(S \setminus \{i\})$. If these marginal contributions are weighted by means of equal coefficients depending only on the size of the coalitions, we arrive at the solution concept known as *semivalue*, introduced by Dubey, Neyman and Weber in 1981. These solutions $\Psi : G_N \to \mathbb{R}^N$ are characterized [3] by means of four axioms:

A1. Linearity. $\Psi[\lambda u + \mu v] = \lambda\Psi[u] + \mu\Psi[v], \ \forall u, v \in G_N, \ \forall \lambda, \mu \in \mathbb{R}$.

A2. Symmetry. $\Psi_{\pi i}[\pi v] = \Psi_i[v], \ \forall v \in G_N, \ \forall i \in N, \ \forall \pi$ permutation of N, where game πv is defined by $(\pi v)(\pi S) = v(S), \ \forall S \subseteq N$.

A3. Monotonicity. v monotonic $\Rightarrow \ \Psi_i[v] \geq 0, \ \forall i \in N$.

A4. Projection. $\Psi_i[v] = v(\{i\}), \ \forall v \in A_N$, where A_N denotes the set of additive games within G_N: games v such that $v(S \cup T) = v(S) + v(T)$ for $S \cap T = \emptyset$ and $S, T \subseteq N$.

It has been proved in [3] that there exists a one-to-one map between the semivalues on G_N and the weighting vectors (p_s), $s = 1, \ldots, n$, that verify conditions

$$\sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1 \quad \text{and} \quad p_s \geq 0 \ \text{ for } \ 1 \leq s \leq n.$$

The number of coalitions of size s that contain each player $i \in N$ is $\binom{n-1}{s-1}$. The above conditions give a probability distribution on the different coalitions that contain each player, assuming equal weight for equal size.

This way, the payoff to each player i in a game $v \in G_N$ by a semivalue ψ with weighting coefficients (p_s), $s = 1, \ldots, n$, is a weighted sum of its marginal contributions

$$\psi_i[v] = \sum_{S \subseteq N} p_s[v(S) - v(S \setminus \{i\})], \quad \text{where } s = |S|.$$

We denote the set of all semivalues defined on G_N by $Sem(G_N)$. Well known solutions for cooperative games as the Shapley value [6] and the Banzhaf value [2, 5] are semivalues.

3 Weighted connectivity

Definition 3.1 *We call weighted connectivity [1], according to semivalue $\psi \in Sem(G_N)$, of the node i in the connected graph Γ to the payoff that according to the semivalue ψ corresponds to the player i in the associated game v_Γ. We denote it by $\psi_i[v_\Gamma]$, $\forall i \in N$.*

Theorem 3.2 *The weighted connectivity $\psi_i[v_\Gamma]$ of a node i in a connected graph Γ can be decomposed in an active connectivity $\psi_i[v_\Gamma]^+$ and a reactive connectivity $\psi_i[v_\Gamma]^-$:*

$$\psi_i[v_\Gamma] = \psi_i[v_\Gamma]^+ + \psi_i[v_\Gamma]^- \quad \textit{where}$$

$$\psi_i[v_\Gamma]^+ = \sum_{s=2}^{n} \gamma_s p_s \textit{ with } \gamma_s = |\{S \subseteq N \mid i \in S, |S| = s, v_\Gamma(S) = 1 \textit{ and } v_\Gamma(S \setminus \{i\}) = 0\}| \textit{ and}$$

$$\psi_i[v_\Gamma]^- = -\sum_{s=3}^{n-1} \delta_s p_s \textit{ with } \delta_s = |\{S \subset N \mid i \in S, |S| = s, v_\Gamma(S) = 0 \textit{ and } v_\Gamma(S \setminus \{i\}) = 1\}|.$$

To prove the Theorem it suffices to consider that in cases $v_\Gamma(S) = v_\Gamma(S \setminus \{i\}) = 1$ and $v_\Gamma(S) = v_\Gamma(S \setminus \{i\}) = 0$ the marginal contributions vanish. In the first situation both coalitions are connected by Γ, whereas in the second situation the nodes are connected neither in S nor in $S \setminus \{i\}$. Case $v_\Gamma(S) = 1$, $v_\Gamma(S \setminus \{i\}) = 0$ supposes the loss of connection in S when we remove node i. The node i is essential in the connection of the nodes in S. Finally, $v_\Gamma(S) = 0$ with $v_\Gamma(S \setminus \{i\}) = 1$ shows the position of node i as an obstacle for the connection of the nodes in S, since coalition $S \setminus \{i\}$ is connected by Γ.

Example 3.3 *In the set of nodes $N = \{1, 2, 3, 4, 5, 6\}$, we consider the graph Γ whose set of edges is $E(\Gamma) = \{1:2,\ 1:4,\ 2:3,\ 2:6,\ 3:4,\ 3:5,\ 4:5,\ 5:6\}$.*

For node 1, coalitions S that compute in active connectivity are $\{1, 2\}$, $\{1, 4\}$, $\{1, 2, 4\}$, $\{1, 2, 4, 5\}$ and $\{1, 2, 4, 6\}$, whereas for the reactive one they are $\{1, 3, 5\}$, $\{1, 5, 6\}$ and $\{1, 3, 5, 6\}$. Then: $\psi_1[v_\Gamma]^+ = 2p_2 + p_3 + 2p_4$, $\psi_1[v_\Gamma]^- = -2p_3 - p_4$ and $\psi_1[v_\Gamma] = 2p_2 - p_3 + p_4$.

Similar computations allow us to determine the weighted connectivity for the remaining nodes so that

$$\psi[v_\Gamma] = (2p_2 - p_3 + p_4,\ 3p_2 + 2p_3 + 5p_4 + 2p_5,\ 3p_2 + 2p_3 + 3p_4,\ 3p_2 + p_3 + 3p_4 + p_5,\\ 3p_2 + p_3 + 3p_4 + p_5,\ 2p_2 - p_3 + p_4).$$

The symmetrical role of nodes 1 and 6, and 4 and 5, respectively, gives rise to equal allocations by every semivalue. Now, by using diverse semivalues, we can offer relative measures of the importance of the nodes in the connection scheme of the graph. If we choose the Shapley value Sh on games with six players (weighting coefficients $p_1 = p_6 = 1/6$, $p_2 = p_5 = 1/30$ and $p_3 = p_4 = 1/60$):

$$Sh[v_\Gamma] = (0.0667,\ 0.2833,\ 0.1833,\ 0.2000,\ 0.2000,\ 0.0667).$$

According to the Banzhaf value β (weighting coefficients $p_s = 1/32$, $s = 1, \ldots, 6$), we have:

$$\beta[v_\Gamma] = (0.0625,\ 0.3750,\ 0.2500,\ 0.2500,\ 0.2500,\ 0.0625).$$

Finally, if we denote by ψ^* the semivalue with coefficients $p_1 = 32/243$, $p_2 = 16/243$, $p_3 = 8/243$, $p_4 = 4/243$, $p_5 = 2/243$ and $p_6 = 1/243$, the weighted connectivity of the nodes in graph Γ is

$$\psi^*[v_\Gamma] = (0.1152,\ 0.3621,\ 0.3128,\ 0.2881,\ 0.2881,\ 0.1152).$$

The weighting coefficients of semivalue ψ^* are in geometric progression with ratio 1/2: semivalue ψ^* belongs to the family of *binomial* semivalues, as they were introduced in [4].

4 General properties

Several properties of this concept of weighted connectivity can be derived. The following theorems introduce some of them, which can be easily proved from the definitions.

Given a node i in a graph Γ, the *degree* of i, denoted as $deg(i)$, is the cardinality of the set of adjacent nodes to i. The coefficient p_2 of any semivalue $\psi \in Sem(G_N)$ weights the marginal contributions to two-person coalitions. In the associated game to graph Γ, the two-person coalitions are formed by the own node i and all its adjacent nodes. Thus, the weight p_2 appears in the weighted connectivity of node i so many times as $deg(i)$.

We say that two graphs Γ_1 and Γ_2 are *isomorphic* iff there exists a one-to-one map between the respective sets of nodes, $f : N_1 \rightarrow N_2$, that preserves adjacencies, i.e., $i : j \in E(\Gamma_1) \Leftrightarrow f(i) : f(j) \in E(\Gamma_2)$. The map f is an *isomorphism* between the graphs Γ_1 and Γ_2.

Theorem 4.1 *If Γ_1 and Γ_2 are isomorphic graphs by map f, then*

$$\psi_{f(i)}(v_{\Gamma_2}) = \psi_i(v_{\Gamma_1}),\ \forall i \in N_1,\ \forall \psi \in Sem(G_N).$$

In particular, if two nodes play a symmetrical role in a given graph Γ, we can define a one-to-one map between the own nodes, so that both nodes obtain the same weighted connectivity by every semivalue.

Theorem 4.2 *Let i be a node in a graph $\Gamma \in CG(N)$. If $deg(i) = n-1, n-2$, then the reactive connectivity of vertex i vanish.*

Theorem 4.3 *Given a semivalue $\psi \in Sem(G_N)$ with weighting coefficients (p_s), $s = 1, \ldots, n$, we consider the set $CG(N)$ of all connected graphs with nodes in N. Then, for every node $i \in N$,*

$$\max_{\Gamma \in CG(N)} \psi_i[v_\Gamma] = 1 - p_1.$$

We say that a node i in a graph with $deg(i) = 1$ is an *antenna*. The unique node adjacent to an antenna is the *base* of the antenna.

The *star graph* with $n-1$ points S_{n-1} is formed by $n-1$ antennas and a unique node that is the base of all antennas or star center. We denote the set of nodes of S_{n-1} by $N = \{1\} \cup \{2, ..., n\}$, where the distinguished node 1 is its center.

Corollary 4.4 *Given a semivalue $\psi \in Sem(G_N)$ with weighting coefficients (p_s), $s = 1, \ldots, n$, the maximum value of weighted connectivity of the nodes in graphs of $CG(N)$ is attained in the center node of S_{n-1}.*

Theorem 4.5 *Given a semivalue $\psi \in Sem(G_N)$ with weighting coefficients (p_s), $s = 1, \ldots, n$, we consider the set $CG(N)$ of all connected graphs with nodes in N. Then, for every node $i \in N$,*

$$\min_{\Gamma \in CG(N)} \psi_i[v_\Gamma] = p_2 - \sum_{s=3}^{n-1} \binom{n-2}{s-1} p_s.$$

In the set of nodes N, we call *complete graph* K_N to the graph that is formed by the $\binom{n}{2}$ possible edges.

Corollary 4.6 *Given a semivalue $\psi \in Sem(G_N)$ with weighting coefficients (p_s), $s = 1, \ldots, n$, the minimum value of weighted connectivity of the nodes in graphs of $CG(N)$ is attained in an antenna node i with base in any node of $K_{N \setminus \{i\}}$.*

Corollary 4.7 *Given a semivalue $\psi \in Sem(G_N)$ with weighting coefficients (p_s), $s = 1, \ldots, n$, and a graph $\Gamma \in CG(N)$, the weighted connectivity of an antenna node i with base in a node b is a value independent from the number of edges between the base b and the remaining nodes.*

5 Connectivity of graph families

In this section we consider the weighted connectivity of some families of graphs whose edges have a disposition with geometric regularity. The graph families whose connectivity we want to evaluate have an element on each set of nodes with different cardinality: complete graphs K_N, star graphs S_{n-1} with $n-1$ points or *cycle graphs* formed by a unique closed way or cycle going through the n nodes.

Theorem 5.1 *For every semivalue $\psi \in Sem(G_N)$, $|N| \geq 3$, with weighting coefficients (p_s), $s = 1, \ldots, n$, the weighted connectivity has the following values.*

(a) Complete graph: $\psi_i[v_{K_N}] = (n-1)p_2$, $\forall i \in N$.

(b) Star graph:

$$\psi_1[v_{S_{n-1}}] = 1 - p_1, \quad 1 \text{ star center;}$$
$$\psi_i[v_{S_{n-1}}] = p_2, \qquad i = 2, ..., n \text{ star points.}$$

(c) Cycle graph:

$$\psi_i[v_{C_4}] = 2p_2 + p_3, \quad i = 1, 2, 3, 4;$$

$$\psi_i[v_{C_n}] = 2p_2 + \textstyle\sum_{s=3}^{n-2}(2s - n - 1)p_s + (n-3)p_{n-1}, \ \forall i \in N, \ n \geq 5.$$

6 Conclusion

Semivalues form a wide family of solutions for cooperative games whose allocations to the players are frequently used in Game Theory. The introduction of a cooperative game associated to each connected graph has turned out to be a useful tool to compute weighted connectivity of the nodes by means of the family of semivalues. Several properties of this concept of connectivity have been obtained.

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