THE ENERGY EFFICIENCY OF THE ERGODIC FADING RELAY CHANNEL

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ABSTRACT

In this paper, we address the energy efficiency analysis of the relay channel under ergodic fading. The study considers full duplex and half duplex terminals. Since the capacity of general relay channels is unknown, we investigate achievable rates with decode and forward and capacity upper bounds with the cut-set bound. The maximum rate per energy and the slope of the spectral efficiency with the energy per bit are computed to asses the impact of the duplexing capabilities, the resource allocation and the channel fading distribution.

1. INTRODUCTION

In wireless networks, cooperation among users brings high data rates [1] and energy savings [2]. We consider the single relay channel (RC) under ergodic fading and additive white Gaussian noise (AWGN). For this channel, we study the spectral efficiency in the energy efficient regime. If the energy efficient regime is the low power regime any communication scheme is well characterized by computing its maximum rate per energy (RPE) or, equivalently, the minimum energy per bit ($\frac{E_b}{N_0}|_{\min}$) and the slope (S) of the spectral efficiency with respect to the $\frac{E_b}{N_0}$ [3].

The low power analysis of several communication schemes over ergodic fading can be found in the literature. For instance, the direct transmission with single or multiple antennas was studied in [3]. The multiple access, broadcast and interference channels were conducted in [4]. For the AWGN RC, the capacity remains still unknown and thus, only the known bounds on the capacity can be investigated. Over non-fading channels, the maximum RPE was first studied in [5]. There the *cut-set bound* (CB) and the *decode and forward* (DF) lower bound [6] on the capacity were used to derive upper and lower bounds on the maximum RPE. In particular, authors considered two scenarios: *i)* a relay in full duplex (FD) mode with coherent transmissions and *ii)* a relay in half duplex (HD) mode with orthogonal transmissions (OT) from the source and the relay. The FD capability at the relay allows to receive and transmit simultaneously in the same band. If FD operation is not possible, the relay works in HD mode. Then, the transmission and reception channels are orthogonal. These bounds were extended in [7] to include *side-information* and *linear relaying strategies*. As argued in [8], although in some scenarios these techniques can offer a better maximum RPE, it is not reached in the low power regime. In fact, in the low power regime these strategies can never improve DF. In these works, the computation of the maximum RPE is found useful to compare different relaying strategies and to demonstrate the suboptimality of orthogonal transmissions against simultaneous and coherent transmissions, see also [9]. The relay channel under ergodic fading was first considered in [10]. There, the ergodic rate bounds with DF and the CB, with FD terminals or with orthogonal transmission (OT) were studied. The results were extended in [11] to also include *amplify and forward*.

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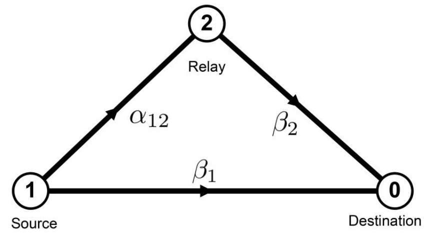

Figure 1: The relay channel model

In this work, we provide more insight into those results by studying the HD scenario. In the HD scenario, contrary to the OT scenario, the source is allowed to transmit during the relay transmission interval. Furthermore, we allow the resource allocation functions to be any differentiable function of the total power. In all these previous works, the power allocated to each node was restricted to be a linear function of the total power. Whereas, in the low power regime, we show that the HD mode does not provide any gain with respect to OT, the linear power allocation assumption results on an inefficient use of the bandwidth. We also discuss the bandwidth inefficiency incurred by other simpler but suboptimal resource allocation solutions.

The reminder of the paper is organized as follows. First, the channel model is given in Section 2. Then, ergodic rate lower bounds with DF and upper bounds with the CB for relays in HD or FD mode are presented in Section 3. An introduction to the low power analysis particularized to these channels is given in Section 4. Sections 5 and 6 are devoted to compute the maximum RPE and the slope, respectively. In Section 7 numerical results are presented and finally, in Section 8 conclusions are drawn.

2. CHANNEL MODEL

We consider the AWGN relay channel model depicted in Fig. 1. For the HD mode, the channel is divided into two intervals (orthogonal channels) $j \in \{1,2\}$. The source transmits to the relay and destination in the first interval $j=1$ and, in the second interval $j=2$, both nodes transmit to the destination. In FD mode, there is only one channel and the index j is dropped. The received signals during the interval j at the relay Y_{2j} and at the destination Y_{0j} are given by

$$\begin{aligned} Y_{2j} &= \sqrt{\alpha_{12}} X_{1j} + Z_{2j}, \\ Y_{0j} &= \sqrt{\beta_1} X_{1j} + \sqrt{\beta_2} X_{2j} + Z_{0j}. \end{aligned} \tag{1}$$

We assume, without claim of optimality, that the signals transmitted by the source X_{1j} and the relay X_{2j} are Gaussian with power P_{1j}, P_{2j}, respectively. The noise processes at the relay Z_{2j} and at the destination Z_{0j} are complex independent white Gaussian, each with unit variance $E[|Z_{ij}|^2] = N_0 = 1$. The channel gain from the source to the relay is denoted by α_{12}, from the source to destination by β_1, and from the relay to destination by β_2. Each orthogonal channel uses a fraction τ_j out of the total channel available. The power allocated to node $l \in \{1,2\}$ during the interval j is denoted as $E_{lj} = \tau_j P_{lj}$.

Let us consider that the channel coefficients $\alpha_{1,2}$, β_1, β_2 are independent random variables. Each time slot is assumed to be large enough so that the channel fading processes are ergodic. The receiver has perfect channel information while transmitters (source and relay) only have statistical information of the channel; namely, the mean $\mathbb{E}[\alpha_{1,2}], \mathbb{E}[\beta_1], \mathbb{E}[\beta_2]$ and the variance $\mathbb{E}\left[(\alpha_{1,2})^2\right], \mathbb{E}\left[(\beta_1)^2\right], \mathbb{E}\left[(\beta_2)^2\right]$.

3. ERGODIC RATE BOUNDS

The optimal relaying strategy is still unknown and only rate lower and upper bounds on the capacity exist. The energy efficiency analysis conducted in this work considers DF for rate lower bounds and the CB for rate upper bounds.

The rate bounds for HD and FD modes under some fixed resource allocation vectors $\boldsymbol{\tau}, \mathbf{E}$ can be found in [6] and [12], respectively. We write them in a unified manner as

$$R_F(\boldsymbol{\tau}, \mathbf{E}) = \min_{i \in \{1,2\}} C_i(\boldsymbol{\tau}, \mathbf{E}) \tag{2}$$

with

$$C_i(\boldsymbol{\tau}, \mathbf{E}) = \mathbb{E}\left[\sum_{j=1}^{2} \tau_j \log\left(1 + \frac{g_{ji}(\mathbf{E})}{\tau_j}\right)\right]. \tag{3}$$

For the FD mode, the power allocation vector is $\mathbf{E} = [E_1, E_2]$, the vector of channel fractions is $\boldsymbol{\tau} = [1, 0]$ and the functions g_i, $i \in \{1,2\}$ are given by

$$g_1(\mathbf{E}) = \hat{\alpha}_{12} E_1, \tag{4a}$$
$$g_2(\mathbf{E}) = \beta_1 E_1 + \beta_2 E_2. \tag{4b}$$

with $\hat{\alpha}_{12} = \alpha_{12}$ for DF and $\hat{\alpha}_{12} = \alpha_{12} + \beta_1$ for the CB.

For the HD mode, the power allocation vector is $\mathbf{E} = [E_{11}, E_{12}, E_{22}]$ and the vector of channel fractions is $\boldsymbol{\tau} = [\tau_1, \tau_2]$. In this case, $g_{ji}(\mathbf{E})$ $i, j \in \{1,2\}$ are given by

$$g_{11}(\mathbf{E}) = \hat{\alpha}_{12} E_{11}, \; g_{21}(\mathbf{E}) = \beta_1 E_{12}, \tag{5a}$$
$$g_{12}(\mathbf{E}) = \beta_1 E_{11}, \; g_{22}(\mathbf{E}) = \beta_1 E_{12} + \beta_2 E_{22}. \tag{5b}$$

4. THE ENERGY EFFICIENCY ANALYSIS

To study the energy efficiency of a communication system we are interested in determining the minimum energy we need to dedicate to each transmitted bit normalized by the noise spectral level N_0 $\left(\frac{E_b}{N_0}\right)$ to obtain a certain spectral efficiency $\mathsf{C}(\frac{E_b}{N_0})$ [bit/s/Hz]. In this work, we consider that the energy belongs to the "network" and is optimally allocated among nodes. Then, the network energy per bit is defined as $E_b \triangleq \frac{E}{R(E)}$, where E is the total network energy $E = \sum_{l \in \{1,2\}} \sum_{j \in \{1,2\}} E_{lj}$ and $R(E)$ is the spectral efficiency in bits[1]. We define the rate per total network energy normalized by the noise spectral level as

$$RPE \triangleq N_0 \log_e 2 \frac{R(E)}{E}. \tag{6}$$

If the energy efficient regime is the low power regime, it was shown in [3] that the minimum energy per bit normalized by the noise spectral level is given by

$$\frac{E_b}{N_0}_{\min} = \frac{\log_e 2}{\dot{R}(0)} \tag{7}$$

where $\dot{R}(0)$ is computed in nats. Then, the maximum RPE can be obtained as

$$\bar{\eta} = \frac{\log_e 2}{\frac{E_b}{N_0}_{\min}} = \dot{R}(0) \tag{8}$$

The slope of the spectral efficiency as a function of the $\frac{E_b}{N_0}$ was shown in [3] to be

$$\bar{S} = \frac{2\left[\dot{R}(0)\right]^2}{-\ddot{R}(0)}. \tag{9}$$

To obtain the low power metrics $\bar{\eta}$ and $\bar{S}$, we need to compute the first and second order derivatives at $E = 0$ of the rate as a function of the total power $R^*(E)$. However, the rate $R^*(E)$ is only available as the solution to the following problem for all E

$$R^*(E) = \max_{\boldsymbol{\tau}, \mathbf{E}} \min_{i \in \{1,2\}} C_i(\boldsymbol{\tau}, \mathbf{E}) \tag{10a}$$
$$\tau_1 + \tau_2 = 1, \quad \tau_1, \tau_2 \geq 0, \tag{10b}$$
$$\sum_{\forall j} [\mathbf{E}]_j = E, \quad [\mathbf{E}]_j \geq 0, \forall j. \tag{10c}$$

where $[\mathbf{x}]_j$ denotes the j-th element in the vector $\mathbf{x}$.

We are unable to find, explicitly, the pair $(\boldsymbol{\tau}, \mathbf{E})$ that maximizes the rate in (10) for all E. Therefore, we can not directly compute the derivatives of $R^*(E)$ at $E = 0$.

To compute these derivatives, we assume that the resources allocation solution for all E are any differentiable function $\boldsymbol{\tau}(E)$ and $\mathbf{E}(E)$ and define $\dot{\mathbf{e}} \triangleq \dot{\mathbf{E}}(0)$, $\ddot{\mathbf{e}} \triangleq \dot{\mathbf{E}}(0)$ and $\mathbf{t} \triangleq \boldsymbol{\tau}(0)$. Then, the first and second order derivatives

[1] We replace the $\mathsf{SNR} = \frac{E}{N_0}$ by the total power E, since we consider that the noise has unit variance $N_0 = 1$.

of the rate constraints in (3) as a function total power E, $C_{E_i}(E) = C_i(\boldsymbol{\tau}(E), \mathbf{E}(E))$ evaluated at $E = 0$ are

$$\dot{C}_{E_i}(E)|_{E=0} = \sum_{j=1}^{2} \mathbb{E}\left[\dot{g}_{E_{ji}}(\dot{\mathbf{e}})\right], \tag{11a}$$

$$\ddot{C}_{E_i}(E)|_{E=0} = \sum_{j=1}^{2} \mathbb{E}\left[\ddot{g}_{E_{ji}}(\dot{\mathbf{e}}, \ddot{\mathbf{e}})\right] - \frac{\mathbb{E}\left[\left(\dot{g}_{E_{ji}}(\dot{\mathbf{e}})\right)^2\right]}{t_j} \tag{11b}$$

with $g_{E_{ji}}(E) = g_{ji}(\mathbf{E}(E))$. Notice that given (11) it is clear that $\dot{C}_{E_i}(E)|_{E=0} = \dot{C}_{E_i}(\dot{\mathbf{e}})$ and $\dot{g}_{E_{ji}}(0) = \dot{g}_{E_{ji}}(\dot{\mathbf{e}})$ only depend on $\dot{\mathbf{e}}$ whereas $\ddot{g}_{E_{ji}}(0) = \ddot{g}_{E_{ji}}(\dot{\mathbf{e}}, \ddot{\mathbf{e}})$ and $\ddot{C}_{E_i}(E)|_{E=0} = \ddot{C}_{E_i}(\dot{\mathbf{e}}, \ddot{\mathbf{e}}, \mathbf{t})$ only depends on $\dot{\mathbf{e}}$, $\ddot{\mathbf{e}}$ and $\mathbf{t}$. To find $\dot{R}^*(0)$ we just need to solve the following problem w.r.t $\dot{\mathbf{e}}$

$$\dot{R}^*(0) = \max_{\dot{\mathbf{e}}} \min_{i\in\{1,2\}} \dot{C}_{E_i}(\dot{\mathbf{e}}) \tag{12a}$$

$$\sum_{\forall j} [\dot{\mathbf{e}}]_j = \mathbf{1}, [\dot{\mathbf{e}}]_j \geq 0, \forall j \tag{12b}$$

where the constraints (12b) on $\dot{\mathbf{e}}$ are obtained computing the first order derivative of the constraints in (10c) at $E = 0$.

After solving (12) we know $\dot{\mathbf{e}}^*$, thus $\ddot{C}_{E_l}(0) = \ddot{C}_{E_l}(\mathbf{t}, \ddot{\mathbf{e}})$ only depends on $\ddot{\mathbf{e}}$ and $\mathbf{t}$. The second derivative $\ddot{R}^*(0)$ can be obtained solving the following problem w.r.t $\ddot{\mathbf{e}}$ and $\mathbf{t}$

$$\ddot{R}^*(0) = \max_{\ddot{\mathbf{e}}, \mathbf{t}} \min_{i\in\{1,2\}} \ddot{C}_{E_i}(\ddot{\mathbf{e}}, \mathbf{t}) \tag{13a}$$

$$t_1 + t_2 = 1,\ t_1, t_2 \geq 0, \tag{13b}$$

$$\sum_{\forall j} [\ddot{\mathbf{e}}]_j = \mathbf{0},\ \ [\ddot{\mathbf{e}}]_j \geq 0 \quad \text{if } [\dot{\mathbf{e}}]_j = 0. \tag{13c}$$

where the constraints (13b) on $\mathbf{t}$ are found computing the first order derivative of the constraints in (10b) at $E = 0$ and the constraints on $\ddot{\mathbf{e}}$ (13c) are found by computing the second order derivative of the constraints in (10c) at $E = 0$.

5. MAXIMUM RATE PER ENERGY

To obtain $\bar{\eta} = \dot{R}^*(0)$, we need to solve the problem in (12).

For a relay in FD mode, substituting the functions g_{E_i} in (4) into (11a), the first order derivative of the rate constraints at $E = 0$ read

$$\dot{C}_{E_1}(\dot{\mathbf{e}}) = \mathbb{E}[\dot{g}_{E_1}(\dot{\mathbf{e}})] = \mathbb{E}[\hat{\alpha}_{12}]\,\dot{e}_1, \tag{14a}$$

$$\dot{C}_{E_2}(\dot{\mathbf{e}}) = \mathbb{E}[\dot{g}_{E_2}(\dot{\mathbf{e}})] = \mathbb{E}[\beta_1]\,\dot{e}_1 + \mathbb{E}[\beta_2]\,\dot{e}_2. \tag{14b}$$

By substituting (14) into (12), and solving the resultant problem w.r.t $\dot{e}_1, \dot{e}_2$, we obtain

$$\dot{e}_1^* = \frac{\mathbb{E}[\beta_2]}{\mathbb{E}[\beta_2] + \mathbb{E}[\hat{\alpha}_{12}] - \mathbb{E}[\beta_1]}, \tag{15a}$$

$$\dot{e}_2^* = \frac{\mathbb{E}[\hat{\alpha}_{12}] - \mathbb{E}[\beta_1]}{\mathbb{E}[\beta_2] + \mathbb{E}[\hat{\alpha}_{12}] - \mathbb{E}[\beta_1]} \tag{15b}$$

if $\mathbb{E}[\hat{\alpha}_{12}], \mathbb{E}[\beta_2] > \mathbb{E}[\beta_1]$, otherwise the relay is not used $\dot{e}_1^* = 1$ and $\dot{e}_2^* = 0$.

For a relay in HD mode, substituting the functions g_{E_i} in (5) into (11a), the first order derivative of the rate constraints at $E = 0$ read

$$\dot{C}_{E_1}(\dot{\mathbf{e}}) = \mathbb{E}[\hat{\alpha}_{12}]\,\dot{e}_{11} + \mathbb{E}[\beta_1]\,\dot{e}_{12}, \tag{16a}$$

$$\dot{C}_{E_2}(\dot{\mathbf{e}}) = \mathbb{E}[\beta_1]\,\dot{e}_{11} + \mathbb{E}[\beta_1]\,\dot{e}_{12} + \mathbb{E}\left[\hat{\beta}_2\right]\dot{e}_{22}. \tag{16b}$$

Substituting (16) into (12), and solving the resultant problem w.r.t $\dot{e}_{11}$, $\dot{e}_{12}$ and $\dot{e}_{22}$, we obtain

$$\dot{e}_{11}^* = \frac{\mathbb{E}[\beta_2]}{\mathbb{E}[\beta_2] + \mathbb{E}[\hat{\alpha}_{12}] - \mathbb{E}[\beta_1]}, \tag{17a}$$

$$\dot{e}_{12}^* = 0, \tag{17b}$$

$$\dot{e}_{22}^* = \frac{E[\hat{\alpha}_{12}] - E[\beta_1]}{\mathbb{E}[\beta_2] + \mathbb{E}[\hat{\alpha}_{12}] - \mathbb{E}[\beta_1]}, \tag{17c}$$

if $\mathbb{E}[\hat{\alpha}_{12}], \mathbb{E}[\beta_2] > \mathbb{E}[\beta_1]$, otherwise the relay is not used $\dot{e}_{12}^* = 1, \dot{e}_{11}^* = 0$, and $\dot{e}_{22}^* = 0$.

Finally, provided that if $\mathbb{E}[\hat{\alpha}_{12}], \mathbb{E}[\beta_2] > \mathbb{E}[\beta_1]$, we have $\bar{\eta} = \dot{C}_{E_1}(\dot{\mathbf{e}}^*) = \dot{C}_{E_2}(\dot{\mathbf{e}}^*)$, the maximum RPE with a relay in HD or FD mode is given by

$$\bar{\eta} = \begin{cases} \frac{\mathbb{E}[\hat{\alpha}_{12}]\mathbb{E}[\beta_2]}{\mathbb{E}[\beta_2] + \mathbb{E}[\hat{\alpha}_{12}] - \mathbb{E}[\beta_1]} & \text{if } \mathbb{E}[\hat{\alpha}_{12}], \mathbb{E}[\beta_2] > \mathbb{E}[\beta_1], \\ \mathbb{E}[\beta_1] & \text{otherwise} \end{cases} \tag{18}$$

with $\hat{\alpha}_{12} = \alpha_{12} + \beta_1$ for the CB and $\hat{\alpha}_{12} = \alpha_{12}$ for DF.

6. SLOPE OF THE SPECTRAL EFFICIENCY

The maximum RPE has not revealed the potential gain of the FD capability at the relay nor the impact of the distribution of the channel coefficients. Moreover, obtaining the maximum RPE is possible even without optimizing the fraction of the channel dedicated to each transmission (τ_1, τ_2) and assuming a linear dependence of the power allocation with respect to the total power $\mathbf{E}(E) = \dot{\mathbf{e}}^* E$. Thus, we turn our attention to the analysis of the slope. This metric allows us to compare transmission schemes that have the same maximum RPE and determines the use made of the bandwidth.

The slope of the spectral efficiency can be computed as (9). The first order derivative $\dot{R}^*(0)$ was already computed in the previous section and thus, we just need to obtain $\ddot{R}^*(0)$. To obtain $\ddot{R}^*(0)$, we solve the problem in (13).

For a relay in FD mode, substituting the functions g_{E_i} in (4) into (11b), the second order derivative of the rate constraints at $E = 0$ read

$$\ddot{C}_{E_1}(\ddot{\mathbf{e}}) = \mathbb{E}[\hat{\alpha}_{12}]\,\ddot{e}_1 - \mathbb{E}\left[\left(\dot{g}_{E_1}(\dot{\mathbf{e}}^*)\right)^2\right], \tag{19a}$$

$$\ddot{C}_{E_2}(\ddot{\mathbf{e}}) = \mathbb{E}[\beta_1]\,\ddot{e}_1 + \mathbb{E}[\beta_2]\,\ddot{e}_2 - \mathbb{E}\left[\left(\dot{g}_{E_2}(\dot{\mathbf{e}}^*)\right)^2\right] \tag{19b}$$

with, after substituting the solution to $\dot{\mathbf{e}}^*$ found in (15)

$$\mathbb{E}\left[\left(\dot{g}_{E_1}(\dot{\mathbf{e}}^*)\right)^2\right] = \mathbb{E}\left[\left(\hat{\alpha}_{12}\dot{e}_1^*\right)^2\right] = \kappa_{12}\bar{\eta}^2, \tag{20a}$$

$$\begin{aligned}\mathbb{E}\left[\left(\dot{g}_{E_2}(\dot{\mathbf{e}}^*)\right)^2\right] &= \mathbb{E}\left[\left(\beta_1\dot{e}_1^* + \beta_2\dot{e}_2^*\right)^2\right], \\ &= \left[1 + (\kappa_1 - 1)\Theta^2 + (\kappa_2 - 1)(1 - \Theta)^2\right]\bar{\eta}^2\end{aligned} \tag{20b}$$

with $\Theta = \frac{\mathbb{E}[\beta_1]}{\mathbb{E}[\hat{\alpha}_{12}]}$ and $\kappa_{12} = \frac{\mathbb{E}\left[(\hat{\alpha}_{12})^2\right]}{(\mathbb{E}[\hat{\alpha}_{12}])^2}$, $\kappa_1 = \frac{\mathbb{E}\left[(\beta_1)^2\right]}{(\mathbb{E}[\beta_1])^2}$ and $\kappa_2 = \frac{\mathbb{E}\left[(\beta_2)^2\right]}{(\mathbb{E}[\beta_2])^2}$ are the kurtosis of the channel coefficients.

By substituting (19) into (13), and solving the resultant problem w.r.t $\ddot{e}_1, \ddot{e}_2$, we obtain

$$\ddot{e}_1^* = \frac{\mathbb{E}\left[\left(\dot{g}_{E_1}(\dot{\mathbf{e}}^*)\right)^2\right] - \mathbb{E}\left[\left(\dot{g}_{E_2}(\dot{\mathbf{e}}^*)\right)^2\right]}{\mathbb{E}[\hat{\alpha}_{12}] - \mathbb{E}[\beta_1] + \mathbb{E}[\beta_2]}, \tag{21}$$

$$\ddot{e}_2^* = -\ddot{e}_1^* \tag{22}$$

if $\mathbb{E}[\hat{\alpha}_{12}], \mathbb{E}[\beta_2] > \mathbb{E}[\beta_1]$ (the relay is used) and $\ddot{e}_1^* = 1, \ddot{e}_2^* = 0$ otherwise. Substituting, this solution into (19) we have $\ddot{R}^*(0) = \ddot{C}_{E_1}(0) = \ddot{C}_{E_2}(0)$ and the slope (9) reads

$$\frac{2}{\bar{S}_{FD}} = 1 + (\kappa_{12} - 1)(1-\Gamma) + (\kappa_1 - 1)\Gamma\Theta^2 + (\kappa_2 - 1)\Gamma(1-\Theta)^2. \tag{23}$$

with $\Gamma = \frac{\bar{\eta}}{\mathbb{E}[\beta_2]}$.

Instead, if the power allocation is restricted to be linear in E, $E_1 = \dot{e}_1^* E$, $E_2 = \dot{e}_2^* E$, then substituting $\ddot{e}_2^* = \ddot{e}_1^* = 0$ into (19), we have

$$\ddot{R}^*(0) = -\max_{i\in\{1,2\}} \mathbb{E}\left[(\dot{g}_{E_i}(\dot{\mathbf{e}}^*))^2\right] \tag{24}$$

with $\mathbb{E}\left[(\dot{g}_{E_i}(\dot{\mathbf{e}}^*))^2\right]$ in (20). Substituting (24) into the slope definition (9), we obtain

$$\frac{2}{\bar{S}_{FD_e}} = \max\left(\kappa_{12}, \left[1 + (\kappa_1 - 1)\Theta^2 + (\kappa_2 - 1)(1-\Theta)^2\right]\right). \tag{25}$$

This last result was first found in [10, eq. 33].

For a relay in HD mode, substituting the functions g_{E_i} in (5) into (11b), the second order derivative of the rate constraints at $E = 0$ read

$$\ddot{C}_{E_1}(\ddot{\mathbf{e}}, \mathbf{t}) = \mathbb{E}[\hat{\alpha}_{12}]\ddot{e}_{11} + \mathbb{E}[\beta_1]\ddot{e}_{12} - \frac{\mathbb{E}\left[\left(\dot{g}^*_{E_{11}}(\dot{\mathbf{e}}^*)\right)^2\right]}{t_1} - \frac{\mathbb{E}\left[\left(\dot{g}^*_{E_{21}}(\dot{\mathbf{e}}^*)\right)^2\right]}{t_2}, \tag{26a}$$

$$\ddot{C}_{E_2}(\ddot{\mathbf{e}}, \mathbf{t}) = \mathbb{E}[\beta_1]\ddot{e}_{11} + \mathbb{E}[\beta_1]\ddot{e}_{12} + \mathbb{E}[\beta_2]\ddot{e}_{22} - \frac{\mathbb{E}\left[\left(\dot{g}^*_{E_{12}}(\dot{\mathbf{e}}^*)\right)^2\right]}{t_1} - \frac{\mathbb{E}\left[\left(\dot{g}^*_{E_{22}}(\dot{\mathbf{e}}^*)\right)^2\right]}{t_2} \tag{26b}$$

with, after substituting the solution to $\dot{\mathbf{e}}^*$ found in (17)

$$\mathbb{E}\left[\left(\dot{g}^*_{E_{11}}(\dot{\mathbf{e}}^*)\right)^2\right] = \mathbb{E}\left[(\hat{\alpha}_{12})^2\right](\dot{e}^*_{11})^2 = \kappa_{12}\bar{\eta}^2, \tag{27a}$$

$$\mathbb{E}\left[\left(\dot{g}^*_{E_{21}}(\dot{\mathbf{e}}^*)\right)^2\right] = \mathbb{E}\left[(\beta_1)^2\right](\dot{e}^*_{12})^2 = 0, \tag{27b}$$

$$\mathbb{E}\left[\left(\dot{g}^*_{E_{12}}(\dot{\mathbf{e}}^*)\right)^2\right] = \mathbb{E}\left[(\beta_1)^2\right](\dot{e}^*_{11})^2 = \kappa_1\Theta^2\bar{\eta}^2, \tag{27c}$$

$$\mathbb{E}\left[\left(\dot{g}^*_{E_{22}}(\dot{\mathbf{e}}^*)\right)^2\right] = \mathbb{E}\left[(\beta_2)^2\right](\dot{e}^*_{22})^2 = \kappa_2(1-\Theta)^2\bar{\eta}^2. \tag{27d}$$

Substituting (26) into (13) and solving the resultant problem, first w.r.t $\ddot{e}_{11}, \ddot{e}_{12}, \ddot{e}_{22}$ assuming constant $\mathbf{t}$, we obtain

$$\ddot{R}^*(\mathbf{t}) = \ddot{C}_{E_1}(\ddot{\mathbf{e}}^*, \mathbf{t}) = \ddot{C}_{E_1}(\ddot{\mathbf{e}}^*, \mathbf{t}) = -\frac{(\bar{\Psi}_1)^2}{t_1} - \frac{(\bar{\Psi}_2)^2}{t_2} \tag{28}$$

with

$$(\bar{\Psi}_1)^2 = (1-\Gamma)\kappa_{12}\bar{\eta}^2 + \Gamma\kappa_1\Theta^2\bar{\eta}^2, \tag{29a}$$

$$(\bar{\Psi}_2)^2 = \Gamma\kappa_2(1-\Theta)^2\bar{\eta}^2. \tag{29b}$$

Maximizing (28) w.r.t $\mathbf{t}$ we get $\frac{1}{t_1^*} = 1 + \frac{\Psi_2}{\Psi_1}$ which substituted into (28) yields $\ddot{R}^*(0) = -(\Psi_1 + \Psi_2)^2$ and the slope is

$$\frac{2}{\bar{S}_{HD}} = \left(\sqrt{\kappa_{12}(1-\Gamma) + \kappa_1\Gamma\Theta^2} + \sqrt{\kappa_2\Gamma(1-\Theta)^2}\right)^2. \tag{30}$$

Instead, if the power allocation is restricted to be linear in E, $E_{11} = \dot{e}^*_{11}E$, $E_{12} = 0$, and $E_{22} = \dot{e}^*_{22}E$ then, substituting $\ddot{e}^*_{11} = \ddot{e}^*_{12} = \ddot{e}^*_{22} = 0$ into (26), we have

$$\ddot{C}_{E_1}(\mathbf{0}, \mathbf{t}) = -\frac{\kappa_{12}\bar{\eta}^2}{t_1}, \tag{31a}$$

$$\ddot{C}_{E_2}(\mathbf{0}, \mathbf{t}) = -\frac{\kappa_1\Theta^2\bar{\eta}^2}{t_1} - \frac{\kappa_2(1-\Theta)^2\bar{\eta}^2}{t_2}. \tag{31b}$$

Now, we substitute (31) into (13), and solve the resultant problem only w.r.t $\mathbf{t}$. By substituting $\mathbf{t} = [t_1, 1-t_1]$ into (31b) it is easy to show that $-\ddot{C}_{E_2}(\mathbf{0}, \mathbf{t})$ has a unique minimum over $t_1 \in (0,1)$ at $t_1 = t_1^{c2}$

$$\frac{1}{t_1^{c2}} = 1 + \frac{1-\Theta}{\Theta}\sqrt{\frac{\kappa_2}{\kappa_1}} \tag{32}$$

at this point, we have

$$\ddot{C}_{E_2}(\mathbf{0}, \mathbf{t}^{c_2}) = -\left(\sqrt{\kappa_1\Theta^2} + \sqrt{(1-\Theta)^2\kappa_2}\right)^2\bar{\eta}^2 \tag{33}$$

if $\ddot{C}_{E_1}(\mathbf{0}, \mathbf{t}^{c_2}) > \ddot{C}_{E_2}(\mathbf{0}, \mathbf{t}^{c_2})$ or equivalently

$$\kappa_{12} - \kappa_1\Theta^2 < (1-\Theta)\Theta\sqrt{\kappa_1\kappa_2} \tag{34}$$

then $\ddot{R}(0) = \ddot{C}_{E_2}(\mathbf{0}, \mathbf{t}^{c_2})$, otherwise $\ddot{R}(0) = \ddot{C}_{E_1}(\mathbf{0}, \mathbf{t}^{c_1})$ where $\mathbf{t}^{c_1}$ is found by setting $\ddot{C}_{E_1}(\mathbf{0}, \mathbf{t}) = \ddot{C}_{E_2}(\mathbf{0}, \mathbf{t})$ at

$$\frac{1}{t_1^{c_1}} = 1 + \frac{\kappa_2(1-\Theta)^2}{\kappa_{12} - \kappa_1\Theta^2}. \tag{35}$$

Finally, substituting $\ddot{R}(0)$ into the slope definition, we obtain

$$\frac{2}{\bar{S}_{HD_e}} = \begin{cases} \left(\sqrt{\kappa_1\Theta^2} + \sqrt{(1-\Theta)^2\kappa_2}\right)^2 & \text{if } \frac{\kappa_{12} - \kappa_1\Theta^2}{(1-\Theta)\Theta\sqrt{\kappa_1\kappa_2}} < 1, \\ \kappa_{12}\left(1 + \frac{\kappa_2(1-\Theta)^2}{\kappa_{12} - \kappa_1\Theta^2}\right) & \text{otherwise} \end{cases}. \tag{36}$$

This last result was first found in [10, Proposition 4].

Consider, now, that the channel fractions are fixed (independent of the channel gains) $\mathbf{t}_0 = [\frac{1}{2}, \frac{1}{2}]$ but the energies $\ddot{\mathbf{e}}$ are optimally allocated. It is direct from (28) that

$$\ddot{R}^*(\mathbf{t}_0) = -2(\Psi_1)^2 - 2(\Psi_2)^2 \tag{37}$$

and the slope is

$$\frac{1}{\bar{S}_{HD_\tau}} = \kappa_{12}(1-\Gamma) + \kappa_1\Gamma\Theta^2 + \kappa_2\Gamma(1-\Theta)^2. \tag{38}$$

Finally, consider that, the power allocation is linear and the channel fractions are fixed $\mathbf{t} = \mathbf{t}_0$. In this case, we have

$$\ddot{C}_{E_1}(0, \mathbf{t}_0) = -2\kappa_{12}\bar{\eta}^2, \tag{39a}$$

$$\ddot{C}_{E_2}(0, \mathbf{t}_0) = -2\kappa_1\Theta^2\bar{\eta}^2 - 2\kappa_2(1-\Theta)^2\bar{\eta}^2 \tag{39b}$$

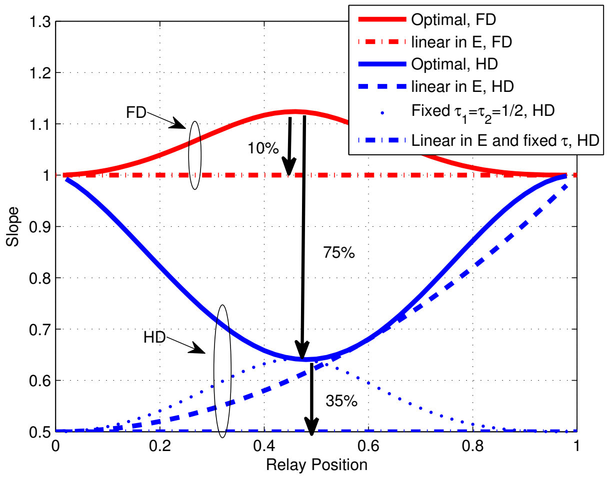

Figure 2: Maximum slope versus source to relay distance over Rayleigh fading $\kappa = 2$.

and $\ddot{R}^*(0) = \min_{i\in\{1,2\}} \ddot{C}_{E_i}(\mathbf{0}, \mathbf{t}_0)$. Then, the slope decreases to

$$\frac{1}{\bar{S}_{HD_{\tau,e}}} = \max(\kappa_{12}, \kappa_1 (\Theta)^2 + \kappa_2 (1-\Theta)^2). \qquad (40)$$

From this result, we can state the following conclusions:

- The optimal power allocation is not linear in the total power E.
- For a relay in HD mode, regardless of the channel distribution, the source never transmits in the second in interval $E_{12}(E) \underset{E\to 0}{=} \dot{e}_{12}^* E + \ddot{e}_{12}^* \frac{E^2}{2} + o(E^3) = o(E^3)$. Therefore, orthogonal transmission are optimal in the low power regime in terms of maximum RPE and maximum slope.

7. NUMERICAL RESULTS

In this section, we present numerical results to show the performances losses incurred by the HD transmission mode and the suboptimal resource allocation solutions.

The slope of the spectral efficiency depends on the mean of the channel fading and the kurtosis of the fading coefficients. We consider Rayleigh fading channels; i.e. $\alpha_{12}, \beta_1, \beta_2$ are exponential distributed random variables. Thus, the kurtosis of the channel coefficients is $\kappa = 2$. The relay is placed on the line connecting the source and the transmitters, and the path-loss exponent is $\nu = 2$. The mean of β_1 is normalized to $\mathbb{E}[\beta_1] = 1$, then $\mathbb{E}[\alpha_{12}] = d^{-\nu}$ and $\mathbb{E}[\beta_2] = (1-d)^{-\nu}$ where d is the distance from the source to the relay. In Fig. 2, we depict the slope of the spectral efficiency as a function of the distance from the source to the relay. The figure includes all the scenarios under analysis: the relay in FD or HD modes and the optimal or the approximated resource allocation solutions. For a relay in FD mode, the slope obtained with the linear power allocation approximation does not depend on the distance d, note that substituting $\kappa = \kappa_{12} = \kappa_1 = \kappa_2$ into (25), we obtain $\frac{2}{\bar{S}_{FD_e}} = k_{12}$ if $\mathbb{E}[\beta_1] < \mathbb{E}[\hat{\alpha}_{12}]$, which is always the case if the relay cooperates. Although, an efficient use of the bandwidth is done if the relay is near to the source or the destination, if the relay is located in the middle the linear approximation requires 10% more the minimum bandwidth. Compared to the FD mode, a relay in HD mode, requires up to 75% more the minimum bandwidth, when the relay is in the middle point between the source and the destination. For a relay in HD mode, the bandwidth losses incurred by suboptimal resource allocation solutions depend strongly on the distance (d). If the relay is near to the destination, the best suboptimal solution is to only allocate the channel fraction. However, if the relay is in the middle point, the best option is to consider fixed channel fractions but optimal power allocation. If both approximation are taken the system requires 35% more the minimum possible bandwidth with a HD relay.

8. CONCLUSIONS

We studied the spectral efficiency of relayed transmissions over fading channels in the power efficient regime. The study includes DF relaying and the CB upper bound on the relay channel capacity with relays in full duplex or in half duplex mode. We found the optimal resource allocation that maximizes the spectral efficiency at $E = 0$ and computed the maximum RPE and the slope of the spectral efficiency with the $\frac{E_b}{N_0}$. We showed that a linear power allocation, as assumed in previous works [10] is not sufficient to characterize the slope of the spectral efficiency. It is found that while the maximum RPE does not depend on the duplexing capability. The slope reveals that the HD mode requires up to 75% more the minimum bandwidth under Rayleigh fading channels. Besides, we showed that for a relay in HD mode, orthogonal transmissions are optimal in the low power regime.

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