Cartesian approach for constrained mechanical system with three degree of freedom.

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Abstract

In the history of mechanics, there have been two points of view for studying mechanical systems: The Newtonian and the Cartesian.

According the Descartes point of view, the motion of mechanical systems is described by the first-order differential equations in the N dimensional configuration space Q.

In this paper we develop the Cartesian approach for mechanical systems with three degrees of freedom and with constraint which are linear with respect to velocity. The obtained results we apply to discuss the integrability of the geodesic flows on the surface in the three dimensional Euclidian space and to analyze the integrability of a heavy rigid body in the Suslov and the Veselov cases .

1 Introduction.

In "Philosophiae Naturalis Principia Mathematica" (1687), Newton considers that movements of celestial bodies can be described by differential equations of the second order. To determine their trajectory, it is necessary to give the initial position and velocity. To reduce the equations of motion to the investigation of a dynamics system it is necessary to double the dimension of the position space and to introduce the auxiliary phase space.

Descartes in 1644 proposed that the behavior of the celestial bodies be studied from another point of view. These ideas were stated in "Principia Philosophiae" (1644) and in "Discours de la métode" (1637). According to Descarte the understanding of cosmology starts from acceptance of the initial chaos, whose moving elements are ordered according to certain fixed laws and form the Cosmo. He consider that the Universe is filled with a tenuous fluid matter (ether), which is constantly in a vortex motion. This motion moves the largest particle of matter of the vortex axis, and they subsequently form planets. Then, according to what Descartes wrote in his "Treatise on Light", "the material of the Heaven must be rotate the planets not only about the Sun but also about their own centers...and this will hence form several small Heavens rotating in the same direction as the great Heaven."[2]. Thus the equation of motion in the Descartes theory must be of the first order

$$\dot{\mathbf{x}} = \mathbf{v}(x,t). \tag{1.1}$$

Hence, to determine the trajectory from Descartes's point of view it is necessary to give only the initial position.

In the modern scientific literature the study of the Descarte ideas we can find in the monographic of V.V. Kozlov [2] in which the author give the following result.

Theorem 1.1 *The manifold $y = u(x,t)$, where u is a covector on* Q *is an invariant manifold for the canonical Hamiltonian equations with the Hamiltonian $H(x,y,t)$ if and only if field u satisfies the Lamb equation*

$$\partial_t u(x,t) + (rotu(x,t))v(x,t) = -gradh(x,t) \tag{1.2}$$

where $(rotu) = \partial_x u - (\partial_x u)^t$ ia skew-symmetric $n \times n$ matrix,

$$v(x,t) = \partial_y H(x,y)|_{y=u(x,t)}, \quad h(x,t) = H(x,y,t)|_{y=u(x,t)} \tag{1.3}$$

From the physical standpoint, equations (1.1), (1.2) and (1.3)describe the motion of the collisionless medium: particles moving along different trajectories do not interact.

In [2] affirm that "solving dynamics problem is possible inside the configuration space". For this it is necessary to solve Lamb equations which is a system of partial differential equations on Q, and then, using (1.3) to calculate the vector field v from the solution of the Lamb equation to solve (1.1).

In [3] we developed the Cartesian approach for mechanical system with configuration space Q and with constraints linear with respects to velocity. The aim of the present paper is to develop the results obtained for mechanical system with three degrees of freedom in the particular case in which Q $= \mathbb{E}^3$ is the three dimensional Euclidean space and Q $= S0(3)$ is the special orthogonal group of rotations of $\mathbb{E}^3$.

2 Cartesian vector field on three dimensional Euclidean space

Let $\mathbb{E}^3$ be the three dimensional Euclidian space with cartesian coordinates $x = (x_1, x_2, x_3)$.

We consider a particle with Lagrangian function

$$L = \frac{1}{2}||\dot{\mathbf{x}}||^2 - U(x)$$

and constraints

$$(\dot{\mathbf{x}}, \mathbf{a}) = 0 \tag{2.1}$$

where (,) denotes the scalar product in $\mathbb{E}^3$, $\dot{\mathbf{x}} = (\dot{x}_1,\, \dot{x}_2,\, \dot{x}_3)$ and $\mathbf{a}(x) = (a_1(x)\,, a_2(x),\, a_3(x))$ is a smooth vector field in $\mathbb{E}^3$

It is well known that the equations of motions can be deduced from the d'Alembert-Lagrange principle [9]

$$\begin{cases} \ddot{\mathbf{x}} = U_{\mathbf{x}} + \mu\mathbf{a}(x), \\ (\mathbf{a}, \dot{\mathbf{x}}) = 0, , \end{cases} \tag{2.2}$$

where μ is the Lagrangian multiplier, $U_{\mathbf{x}} = (U_{x_1},\, U_{x_2},\, U_{x_3})$, $U_{x_j} = \partial_{x_j} U$.

In [3] we introduce the following definition

Definition 1

We say the smooth vector field $\boldsymbol{v}(x) = (v_1(x),\, v_2(x),\, v_3(x))$ *is the Cartesian vector field for a constrained particle in* $\mathbb{E}^3$ *with the constraints (2.1) if*

$$[\mathbf{v}(x) \times rot\mathbf{v}(x)] = \Lambda(x)\mathbf{a}(x). \tag{2.3}$$

where $[\,\times\,]$ denotes the vector product in $\mathbb{E}^3$,

$$rot\mathbf{v} = (\partial_{x_2}v_3 - \partial_{x_3}v_2,\, \partial_{x_3}v_1 - \partial_{x_1}v_3,\, \partial_{x_1}v_2 - \partial_{x_2}v_1).$$

and Λ is a function:

$$\Lambda = \frac{1}{||\mathbf{a}||^2}([\mathbf{v}(x) \times rot\mathbf{v}(x)], \mathbf{a}(x)).$$

By a simple computation from (2.3) we can see that

$$\begin{cases} (\mathbf{a}(x), \mathbf{v}(x)) = 0, \\ (\mathbf{a}(x), rot\mathbf{v}(x)) = 0. \end{cases} \tag{2.4}$$

Corollary 2.1 *Let* $\boldsymbol{v}$ *the Cartesian vector field. Then the following relations hold*

$$\begin{cases} \ddot{\boldsymbol{x}} = (\frac{1}{2}||\boldsymbol{v}(x)||^2)_x + \Lambda(x)\,\boldsymbol{a}(x) \\ (\dot{\boldsymbol{x}}, \boldsymbol{a}) = 0, \end{cases} \tag{2.5}$$

The proof it is easy to obtain in view of the equality

$$\ddot{\mathbf{x}} = (\frac{1}{2}||\mathbf{v}(x)||^2)_x + [\mathbf{v}(x) \times rot\mathbf{v}(x)]$$

which is deduced after derivation the differential equations generated by the vector field $\mathbf{v}$

$$\dot{\mathbf{x}} = \mathbf{v}(x) \tag{2.6}$$

The system (2.5) can be obtained from the Lagrangian equations with Lagrangian function

$$\mathrm{L} = \frac{1}{2}||\dot{\mathbf{x}} - \mathbf{v}(x)||^2$$

where $\mathbf{v}$ is a Cartesian vector field.

Definition 2

The study of the behavior of the constrained particle in $\mathbb{E}^3$ *by using the equations (2.2) or (2.6),(2.3) or (2.5) say the Classical, Cartesian and Lagrangian approach respectively.*

We illustrate the above ideas in the following example

A non-holonomically constrained particle in $\mathbf{R}^3$.

Consider a particle with the kinetic energy $T = \frac{1}{2}||\dot{\mathbf{x}}||^2$ and non-holonomic constraints

$$\dot{x}_1 + \hat{a}(x_3)\dot{x}_2 = 0$$

This instructive academic example, in the particular case when $\hat{a}(x_3) = x_3$ due to Rosenberg [8]. This example was also used to illustrate the theory in Bates and Sniatycki [1].

The Descartes approach in this case produces the vector field $\mathbf{v}$:

$$\mathbf{v} = \lambda_2(\hat{a}(x_3)\partial_{x_1} - \partial_{x_2}) - \lambda_3\partial_{x_3}$$

and condition (2.4) for this case takes the form

$$(rot\mathbf{v}, \mathbf{a}(x)) = 0 \Longleftrightarrow \frac{1}{2}\partial_{x_3}((1+\hat{a}^2)\lambda_2^2) + (\hat{a}(x_3)\partial_{x_1}\lambda_3 - \partial_{x_2}\lambda_3)\lambda_2 = 0.$$

We shall study the case when this relation holds in view of the equalities

$$\lambda_2 = \frac{A}{\sqrt{1+\hat{a}^2(x_3)}}, \quad \lambda_3 = b_2(x_3),$$

for A an arbitrary constant and b_2 an arbitrary function on x_3.

The equations generated by the vector field $\mathbf{v}$ in this case can be written as

$$\begin{cases} \dot{x_1} = \dfrac{\hat{a}(x_3)A}{\sqrt{1+\hat{a}^2(x_3)}} \\ \dot{x_2} = -\dfrac{A}{\sqrt{1+\hat{a}^2(x_3)}} \\ \dot{x_3} = -b_2(x_3) \end{cases} \tag{2.7}$$

The all trajectories of these equations are easy to obtain.

The Lagrangian approach produces the following differential equations

$$\begin{cases} \ddot{x}_1 = -b(x_3)\partial_{x_3}(\dfrac{A\hat{a}(x_3)}{\sqrt{1+\hat{a}^2(x_3)}}) \\ \ddot{x}_2 = b(x_3)\partial_{x_3}(\dfrac{A}{\sqrt{1+\hat{a}^2(x_3)}}) \\ \ddot{x}_3 = \partial_{x_3}\dfrac{1}{2}b^2(x_3) \end{cases}$$

Corollary 2.2 *All the trajectories of the equation of motion of the constrained Lagrangian system*

$$< \mathrm{E}^3,\, L = \frac{1}{2}||\dot{\boldsymbol{x}}||^2 - U(x_3),\, \{\dot{x}_1 + \hat{a}(x_3)\dot{x}_2 = 0\} >$$

can be obtained from (2.7) with $b(x_3) = \pm\sqrt{h + U(x_3)}$.

In fact, the equations of motion obtained from the D'Alembert-Lagrange Principle are

$$\begin{cases} \ddot{x}_1 = \mu \\ \ddot{x}_2 = \hat{a}(x_3)\mu \\ \ddot{x}_3 = \partial_{x_3}U(x_3) \\ \dot{x}_1 + \hat{a}(x_3)\dot{x}_2 = 0 \end{cases}$$

Therefore,

$$\frac{d}{dt}(\dot{x}_2 - \hat{a}(x_3)\dot{x}_1) = -\frac{d\hat{a}(x_3)}{dx_3}\dot{x}_3\dot{x}_1$$

hence,

$$\dot{x}_2 = \frac{A}{\sqrt{1 + \hat{a}^2(x_3)}}$$

where A is an arbitrary constant.

On the other hand from the equation

$$\ddot{x}_3 = \partial_{x_3}U(x_3)$$

we easily obtain $\dot{x}_3 = \mp\sqrt{2(U(x_3) + h)}$, where h is an arbitrary constant.

Finally by considering the constraints we deduce the system of the first order ordinary differential equations (2.7). In this example the Descartes, the lagrangian and Classical approach coincide .

Below we determine the Cartesian vector field for a particle on the surface in $\mathbb{E}^3$.

First we introduce the ve ctor fields X, ,Y, Z which are characteristic elements of the 1-form

$$\Omega = a_1(x)dx_1 + a_2(x)dx_2 + a_3(x)dx_3$$

$$\begin{cases} X = a_3\partial_y - a_2\partial_z \\ Y = a_1\partial_z - a_3\partial_x \\ Z = a_2\partial_x - a_1\partial_y \end{cases} \tag{2.8}$$

Clearly, the most general element of the given 1-form Ω is

$$\mathbf{v} = w_1 X + w_2 Y + w_3 Z$$

hence

$$\mathbf{v}(x) = [\mathbf{a}(x) \times \mathbf{w}(x)] \tag{2.9}$$

where $\mathbf{w}(x) = (w_1(x),\, w_2(x),\, w_3(x))$ is an arbitrary smooth vector field which we shall determine in such a way that (2.4) takes place.

By using the identity

$$rot[\mathbf{a}(x) \times \mathbf{b}(x)] = [\mathbf{a},\, \mathbf{b}] + div\mathbf{b}(x)\mathbf{a}(x) - div\mathbf{a}(x)\mathbf{b}(x)$$

where $[\mathbf{a},\, \mathbf{b}]$ is the Lie bracket of the smooth vector field $\mathbf{a}$ and $\mathbf{b}$, one can prove the following assertion

Corollary 2.3 *The condition (2.3),(2.9) can be written as follows*

$$div([\boldsymbol{a}(x) \times [\boldsymbol{a}(x) \times \boldsymbol{w}(x)]]) = ([\boldsymbol{a}(x) \times rot\,\boldsymbol{a}(x)], \boldsymbol{w}(x)) \tag{2.10}$$

Proposition 2.1 *Let us suppose that the vector field $\boldsymbol{a}$ is such that*

$$[\boldsymbol{a}(x) \times rot\,\boldsymbol{a}(x)] = \boldsymbol{0} \tag{2.11}$$

then the Cartesian vector field exist i and only if

$$\boldsymbol{a}(x) = f_{\boldsymbol{x}}(x) \tag{2.12}$$

for a certain smooth function f.

Proof

From (2.10), (2.11) follows that

$$[\mathbf{a}(x) \times [\mathbf{a}(x) \times \mathbf{w}(x)]]$$

is a solenoidal vector field, hence

$$[\mathbf{a}(x) \times [\mathbf{a}(x) \times \mathbf{w}(x)]] = rot\mathbf{W}(x)$$

for arbitrary vector field $\mathbf{W}$, thus the following representation holds

$$\mathbf{w} = \frac{(\mathbf{a}(x), \mathbf{w}(x))}{||\mathbf{a}(x)||^2}\mathbf{a}(x) + \frac{rot\mathbf{W}(x)}{||\mathbf{a}(x)||^2}$$

as a consequence

$$(\mathbf{a}(x), rot\mathbf{W}(x)) = 0. \tag{2.13}$$

Clearly if $(\mathbf{a}(x), rot\mathbf{W}(x)) \neq 0$ then the Cartesian vector field does not exist if (2.11) holds. If we choose

$$\mathbf{a}(x) = f_{\mathbf{X}}(x), \quad \mathbf{W}(x) = \Phi G_{\mathbf{X}}(f, \Phi)$$

when Φ, G are an arbitrary smooth functions, then we obtain that (2.13) holds identically and a consequence the vector $\mathbf{w}$ takes the form

$$\mathbf{w}(x) = \frac{(f_{\mathbf{X}}(x), \mathbf{w}(x))}{||f_{\mathbf{X}}(x)||^2} + \nu(x)[f_{\mathbf{X}}(x) \times \Phi_{\mathbf{X}}], \quad \nu(x) = \frac{\partial_f G(f, \Phi)}{||f_{\mathbf{X}}(x)||^2} \tag{2.14}$$

Corollary 2.4 *The Cartesian vector field for a particle in $\mathbb{E}^3$ which is constrained to move on the surface*

$$f(x) = c, \, c \neq 0 \tag{2.15}$$

generated the following differential system

$$\dot{\boldsymbol{x}} = \nu(x)(||f_{\boldsymbol{x}}(x)||^2 \Phi_{\boldsymbol{x}}(x) - (f_{\boldsymbol{x}}(x), \, \Phi_{\boldsymbol{x}}(x)) f_{\boldsymbol{x}}(x)) \tag{2.16}$$

Corollary 2.5 *The Lagrangian approach for a particle in $\mathbb{E}^3$ which is constrained to move on the surface (2.15) produces the following differential equations*

$$\ddot{\boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}} \Big(\frac{1}{2} \nu^2(x) ||f_{\boldsymbol{x}}(x)||^2 ||[f_{\boldsymbol{x}}(x) \times \Phi_{\boldsymbol{x}}(x)]||^2 \Big) + \Lambda(x) f_{\boldsymbol{x}}(x) \tag{2.17}$$

Corollary 2.6 *If there exist a function G and Φ such that*

$$||[f_{\boldsymbol{x}}(x) \times \Phi_{\boldsymbol{x}}(x)]||^2 = \frac{2h(f) g}{G_f(f, \Phi)} \equiv \Psi(f, \Phi) \, g \tag{2.18}$$

Then the equations (2.17) take the form

$$\ddot{\mathbf{x}} = \lambda_0(x) f_{\mathbf{X}}(x), \quad \lambda_0(x) = h_f(f) + \Lambda(x). \tag{2.19}$$

If one introduce the matrix $A(x)$:

$$A(x) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_1x_2} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_1x_3} & f_{x_2x_3} & f_{x_3x_3}, \end{pmatrix} \tag{2.20}$$

then one checks, that the equations (2.17) may be written as

$$\ddot{x} = \frac{(A(x)\mathbf{v}(x), \mathbf{v}(x))}{||f_{\mathbf{x}}(x)||^2} f_{\mathbf{x}}(x),$$

where $\mathbf{v}$ is the Cartesian vector field generated the differential equations (2.16). The differential equations (2.19) determined the geodesic flows on the surface (2.15) and admits the energy integral

$$||\dot{\mathbf{x}}||^2 = 2h(f).$$

If there is an additional first integral, functionally independent with the energy integral , then the geodesic flow is integrable.

In order to study the integrability of the geodesic flow on the given surface we introduce the following functions which we determine from (2.16)

$$\begin{cases} F_1 = \Big(\dfrac{||[f_{\mathbf{x}} \times \dot{\mathbf{x}}]||}{||f_{\mathbf{x}}||}\Big)^2 \Longleftrightarrow F_1 = ||\dot{x}||^2, & \\ F_2 = \Big(\dfrac{||f_{\mathbf{x}}||||[\Phi_{\mathbf{x}} \times \dot{\mathbf{x}}]||}{(f_{\mathbf{x}},\, \Phi_{\mathbf{x}})}\Big)^2, & \text{if } (f_{\mathbf{x}},\, \Phi_{\mathbf{x}}) \neq 0; \\ F_3 = \Big(\dfrac{||\Phi_{\mathbf{x}}||||[\mathbf{x} \times \dot{\mathbf{x}}]||}{||[\mathbf{x} \times \Phi_{\mathbf{x}}]||}\Big)^2, & \text{if } (f_{\mathbf{x}},\, \Phi_{\mathbf{x}}) = 0, \quad \Phi_{\mathbf{x}} \neq \kappa(x)\mathbf{x}. \end{cases}$$

In view of (2.18), it is easy to show that

$$F_j = 2h(f), \quad j = 1, 2, 3.$$

3 Integrability of the geodesic flow on the homogeneous surface.

We now consider the surface

$$\begin{cases} f(x) = c, \quad c \neq 0, \\ (\mathbf{x}, f_{\mathbf{x}}(x)) = mf(x). \end{cases} \tag{3.1}$$

which we will call the *homogeneous surface of degree* m.

From the Euler formula follow that $c = 0$ is the unique critical value of f. hence for $c \neq 0$ the function

$$g = ||f_{\mathbf{X}}(x)||^2 > 0$$

on the given homogeneous surface.

Taking (3.1) into account we deduce the relations

$$\begin{aligned} A(x)\mathbf{x}^T &= (m-1)f_{\mathbf{X}}^T(x), & (3.2) \\ (\mathbf{x}, g_{\mathbf{X}}(x)) &= 2(m-1)g(x) & (3.3) \end{aligned}$$

Below we use the following notation

$$\{F, G, H\} \equiv \begin{vmatrix} F_{x_1} & F_{x_2} & F_{x_3} \\ G_{x_1} & G_{x_2} & G_{x_3} \\ H_{x_1} & H_{x_2} & H_{x_3} \end{vmatrix}$$

Clearly, if F, G, H are independent functions then $\{F, G, H\} \neq 0$.

The integrability of the geodesic flow on the homogeneous surface we shall study in the following two cases

$$\begin{aligned} \{f, g, r^2\} &= 0, & (3.4) \\ \{f, g, r^2\} &\neq 0 & (3.5) \end{aligned}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$.

We analyze the first case. We study only the particular subcase when the homogeneous surface is such that

$$||f_{\mathbf{X}}(x)||^2 = g(f, r). \qquad (3.6)$$

Hence, in view of (3.3) we give

$$mf\partial_f g + r\partial_r g = 2(m-1)g(f, r). \qquad (3.7)$$

We assume that the arbitrary function Φ is such that

$$\Phi_{\mathbf{X}} = \mathbf{x},$$

thus the differential equation generated by the Cartesian vector field and second order differential equations of the geodesic flows under the indicated condition take the form respectively

$$\dot{\mathbf{x}} = \nu(x)(g\,\mathbf{x} - m\,f f_{\mathbf{X}}), \tag{3.8}$$

$$\ddot{\mathbf{x}} = \frac{m\partial_r g f h(f)}{r^2 g^2} f_{\mathbf{X}} \tag{3.9}$$

where

$$\nu^2 g(f,r)(g(f,r)r^2 - m^2 f^2) \tag{3.10}$$

Proposition 3.1 *The geodesic flow on the homogeneous surface under the assumption (3.6) is integrable*

Proof

First we observe that there is the function ν such that (3.10) holds , i.e.,

$$G_f^2(f,r) = \frac{2h(f)g(f,r)}{g(f,r)r^2 - m^2 f^2},$$

hence exist the additional first integral F_2 which in this case takes the form

$$F_2 = \frac{g(f,r)||[f_{\mathbf{X}}(x) \times \dot{\mathbf{x}}]||^2}{m^2 f^2} \Leftrightarrow g(f,r)||[f_{\mathbf{X}}(x) \times \dot{\mathbf{x}}]||^2 = 2m^2 f^2 h(f).$$

The particular class of the study homogeneous surface are the following. If $m = 1$ then $(\mathbf{x}, g_{\mathbf{X}}) = 0$, in particular this relation holds if

$$g = \Psi(\frac{f}{r})$$

A concrete example we obtain from the celestial mechanics [5]:

$$f(x) = r + (\mathbf{b}, \mathbf{x}) = c, \quad c \neq 0, \tag{3.11}$$

where $\mathbf{b} = (b_1, b_2, b_3)$ is a constant vector field. In this case we have

$$g = \frac{2f}{r} + ||b||^2 - 1$$

The first integral for this particular case are

$$\begin{cases} ||\dot{\mathbf{x}}||^2 = 2h(f), \\ (\frac{2f}{r} + ||\mathbf{b}||^2 - 1)||[\mathbf{x} \times \dot{\mathbf{x}}]||^2 = 2f^2 h(f) \end{cases} \tag{3.12}$$

It is interesting to deduce the equations of motion of a particle constrained to move in the surface (3.11) with the subsidiary condition that there is a nonzero constant vector field $\breve{c} = (c_1, c_2, c_3)$:

$$(\mathbf{x}, \breve{\mathbf{c}}) = 0, \quad (\mathbf{b}, \breve{\mathbf{c}}) = 0 \quad \Rightarrow (f_{\mathbf{X}}(x), \breve{\mathbf{c}}) = 0, \tag{3.13}$$

from the Lagrangian approach.

By choosing the function Φ as follows

$$\Phi = (c_3 - c_2)x_1 + (c_1 - c_3)x_2 + (c_2 - c_1)x_3$$

and introducing the new time σ as

$$d\sigma = (\frac{x_1 + x_2 + x_3}{r} + b_1 + b_2 + b_3)dt$$

and letting the prime denote differentiation with respect to σ, we have that the equation generated by the Cartesian vector field can be written as

$$\mathbf{x}^{'} = \frac{1}{||\breve{\mathbf{c}}||^2}[f_{\mathbf{X}}(x) \times \breve{\mathbf{c}}] \tag{3.14}$$

By considering that

$$rot[f_{\mathbf{X}} \times \breve{\mathbf{c}}] = \frac{\breve{\mathbf{c}}}{r}$$

we obtain that the Lagrangian approach generated the second order differential equations

$$\mathbf{x}^{''} = -\frac{f(\mathbf{x})x}{||\breve{\mathbf{c}}||^2 r^3}$$

Thus , if

$$f(x) = ||\breve{\mathbf{c}}||^2$$

then we obtain the well known equations

$$\mathbf{x}^{''} = -\frac{\mathbf{x}}{r^3} \tag{3.15}$$

These equations admit the following first integrals

$$\begin{cases} ||\mathbf{x}^{'}||^2 = \dfrac{2}{r} + \dfrac{||\mathbf{b}||^2 - 1}{||\breve{\mathbf{c}}||^2} \\ \\ \left[\mathbf{x}^{'} \times \breve{\mathbf{c}}\right] = -(\dfrac{\mathbf{x}}{r} + \mathbf{b}) \\ \\ \left[\mathbf{x} \times \mathbf{x}^{'}\right] = \breve{\mathbf{c}} \end{cases} \tag{3.16}$$

These relations are easy to obtain from (3.18).

The equations (3.18), after the orthogonal transformation

$$\xi = \frac{(\mathbf{b}, \mathbf{x})}{||\mathbf{b}||}, \quad \eta = \frac{([\breve{\mathbf{c}} \times \mathbf{b}], \mathbf{x})}{||\breve{\mathbf{c}}||\,||\mathbf{b}||}, \quad \zeta = \frac{(\breve{\mathbf{c}}, \mathbf{x})}{||\breve{\mathbf{c}}||},$$

take the form

$$\begin{cases} \xi^{''} = -\dfrac{\xi}{\sqrt{(\xi^2 + \eta^2)^3}}, \\ \\ \eta^{''} = -\dfrac{\eta}{\sqrt{(\xi^2 + \eta^2)^3}} \\ \zeta = 0. \end{cases} \tag{3.17}$$

These equations describe the behavior of the particle with Lagrangian function

$$L = \frac{1}{2}(\xi^{'2} + \eta^{'2}) - \frac{1}{\sqrt{(\xi^2 + \eta^2)}}$$

constrained to move on the one curve of the family of conics

$$\breve{f}(x) = \sqrt{\xi^2 + \eta^2} + ||b||\xi = ||\breve{c}||^2$$

The differential equations generated by the Cartesian vector field in this coordinates can be represented in Hamiltonian form with Hamiltonian function $\breve{f}$ [10]

We now turn to the study the particular case of the homogeneous surface with the condition $g = g(f, r)$.

If

$$g = r^{2(m-1)}\Psi(\frac{f}{r^m}), \quad \Psi(\frac{f}{r^m}) \neq (\frac{f}{r^m})^2$$

then after the change

$$F=\int\frac{d(\xi)}{\sqrt{\Psi(\xi)-\xi^2}},\quad \xi=\frac{f}{r^m},$$

we obtain

$$||F_{\mathbf{x}}(x)||^2=\frac{1}{r^2}.$$

Finally, if

$$g=f^{\frac{2(m-1)}{m}}\Psi(\frac{f}{r^m})$$

then after the change

$$f=F^m$$

we deduce the equation

$$||F_{\mathbf{x}}(x)||^2=\tilde{\Psi}(\frac{F}{r}),$$

which show that this case is equivalent to the first case study above.

We have already studied the case in which $\{f,g,r^2\}=0$. Now we begin to study the case in which the functions f, g, r^2 are independent. Hence

$$\{f,g,r^2\}\neq 0. \tag{3.18}$$

Under this assumption we obtain that

$$x_j=x_j(f,g,r^2),\quad j=1,2,3$$

thus we deduced that

$$\left\{\begin{array}{l}\Phi=\Phi(f,g,r^2)\\ \Phi_{\mathbf{x}}=\partial_f\Phi\, f_{\mathbf{x}}+\partial_g\Phi\, g_{\mathbf{x}}+\dfrac{\partial_r\Phi}{r}\,\mathbf{x}\end{array}\right. \tag{3.19}$$

Proposition 3.2 *If there exists the functions Φ and G such that*

$$\left\{\begin{array}{l}\Phi=\Phi(f,g,r^2),\quad G=G(f,\Phi)\\ g\nu=G_f(f,\Phi)\\ ||f_{\boldsymbol{x}}||^2||\Phi_{\boldsymbol{x}}\times f_{\boldsymbol{x}}||^2\nu^2=2h(f)\end{array}\right. \tag{3.20}$$

then the geodesic flow on the homogeneous surface of degree $m>1$ is integrable.

Proof

We prove this assertion only for the case when

$$(\Phi_{\mathbf{x}},\, f_{\mathbf{x}}) = 0, \tag{3.21}$$

thus the surface $\Phi = c_1$ is orthogonal to the given homogeneous surface. Under this assumption we obtain that the differential equations deduced from the Cartesian and Lagrangian approach are respectively

$$\begin{cases} \dot{\mathbf{x}} = \nu(x)\Phi_{\mathbf{x}}(x) \\ \ddot{\mathbf{x}} = \lambda_0(x) f_{\mathbf{x}}(x) \end{cases} \tag{3.22}$$

where λ_0 can be determined as follows

$$\lambda_0(x) = \nu^2(f_{\mathbf{x}}, \partial_{\mathbf{x}}(\frac{1}{2}||\Phi_x||^2)) = h_f(f) - \frac{\nu||\Phi_{\mathbf{x}}||^2}{g}(f_{\mathbf{x}}, \nu_{\mathbf{x}})$$

After derivation the function

$$F_3 = \Big(\frac{||\Phi_{\mathbf{x}}||||[\mathbf{x}\times\dot{\mathbf{x}}]||}{||[\mathbf{x}\times\Phi_{\mathbf{x}}]||}\Big)^2$$

along the solutions of (3.22) we obtain

$$\frac{dF_3}{dt} = (\Phi_{\mathbf{x}}, \partial_{\mathbf{x}}(\nu^2||\Phi_{\mathbf{x}}||^2)).$$

which is equal to zero in view of (3.20), (3.21) thus the function F_3 is the first integral of the geodesic flow.

Clearly, in order to assess the integrability of the geodesic flow in this case we need first to check whether function ν exists such that (3.20) holds.

4 The geodesic flow on the quadrics and the third-order surface

In order to illustrate the above ideas we consider the algebraic surface of degree three:

$$f(x) = x_1\, x_2\, x_3 = c, \quad c \neq 0. \tag{4.1}$$

This case was examined already by Riemann in his study of motion of a homogeneous liquid ellipsoid. More exactly, Riemann examined the integrability of the geodesic flow on (4.1).

In [7] the author state the following problem.

"Is it true that the geodesic flow on a generic third-order algebraic surface is not integrable?. In particular I do not know a rigorous proof of non-integrability for the surface (4.1)"

By considering that in this case

$$g = (x_1 x_2)^2 + (x_1 x_3)^2 + (x_3 x_2)^2$$

thus the functions f, g and r^2 are independent. The dependence $x_j = x_j(f, g, r^2)$, $j = 1, 2, 3$ we obtain as follows.

We introduce the cubic polynomial in z :

$$P(z) = z^3 - r^2 z^2 + gz - f^2 = (z - x_1^2)(z - x_2^2)(z - x_3^2),$$

and by using Cardano's formula we obtain the require dependence.

In order to construct the Cartesian approach in this case first we observe that the surface

$$\Phi(\xi, \eta, \zeta) = c_1$$

where

$$\xi = \frac{1}{2}(x_1^2 - x_2^2), \quad \eta = \frac{1}{2}(x_3^2 - x_1^2), \quad \zeta = \frac{1}{2}(x_2^2 - x_3^2)$$

is orthogonal to surface (4.1). Thus the differential equations generated by the cartesian vector field are

$$\begin{cases} \dot{x}_1 = \nu(\Phi_\xi - \Phi_\eta)\, x_1 \\ \dot{x}_2 = \nu(\Phi_\zeta - \Phi_\xi)\, x_2 \\ \dot{x}_3 = \nu(\Phi_\eta - \Phi_\zeta)\, x_3 \end{cases} \tag{4.2}$$

To determine the existence the solution of (3.20) or, what is the same,

$$||\Phi_{\mathbf{x}}||^2 = \frac{2h(f)}{G_f^2(f, \Phi)} \equiv \Psi(f, \Phi),$$

is for us an open problem.

Now we study the subcase when the given surface is such that

$$f(x) = \frac{1}{2}(b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2) \tag{4.3}$$

First we state and solve the following problem.

Problem 1

Let X, Y, Z are the vector fields (2.8), (2.12).

We require to determine the function f in such a way that these vector field formed a three dimensional Lie algebra.

The solution of this problem it is easy to obtain in view of the equality

$$\Upsilon_1 = A(x)\,\Upsilon_2, \tag{4.4}$$

where A is the matrix given by the formula (2) and

$$\Upsilon_1 = col([Y,Z],\,[Z,X],\,[X,Y]), \quad \Upsilon_2 = col(X,\,Y,\,Z)$$

and by using the Bianchi representation

$$\Upsilon_3 = B(x)\,\Upsilon_4, \tag{4.5}$$

where $\Upsilon_3 = col([U,V],\,[V,W],\,[W,U])$, $\Upsilon_4 = col(U,\,V,\,W)$, where U, V, W are the vector fields, B is the matrix:

$$B(x) = \begin{pmatrix} 0 & a & b_3 \\ b_1 & 0 & 0 \\ 0 & b_2 & -a \end{pmatrix}$$

and a, b_1, b_2, b_3 are certain constants

From (4.4) and (4.5) after integration we obtain the class of functions which generated the three dimensional Lie algebra.

$$\left\{ \begin{array}{ll} 1. & f = b_1\,x^2 + b_2\,y^2 + b_3\,z^2 \\ 2. & f = b_1\,x^2 + a\,(y^2 - z^2) + 2\,b\,yz \\ 3. & f = 2b\,y\,x + b_3\,z^2 \\ 4. & f = b\,y^2 + 2\,b_1\,z\,x \end{array} \right. \tag{4.6}$$

We construct the Cartesian vector field for the first case.

In view of the relation

$$g = b_1^2 x_1^2 + b_2^2 x_2^2 + b_3^2 x_3^2$$

we observe that f, g, r^2 are independent functions. The following equalities it is easy to obtain:

$$\begin{cases} x_1^2 = \dfrac{b_2 b_3 r^2 - 2(b_2+b_3)f + g}{(b_1-b_2)(b_1-b_3)} \\ \\ x_2^2 = \dfrac{b_1 b_3 r^2 - 2(b_1+b_3)f + g}{(b_2-b_1)(b_2-b_3)} \\ \\ x_3^2 = \dfrac{b_2 b_1 r^2 - 2(b_2+b_1)f + g}{(b_3-b_2)(b_3-b_1)} \end{cases} \tag{4.7}$$

Notice that

$$\Phi(\xi,\eta,\zeta) = c_1,$$

where

$$\xi = \frac{x_3^{b_2}}{x_2^{b_3}}, \quad \eta = \frac{x_2^{b_1}}{x_1^{b_2}}, \quad \zeta = \frac{x_1^{b_3}}{x_3^{b_1}}$$

is an orthogonal surface to the given surface we obtain that the differential equations generated by the Cartesian vector field in this case can be written as

$$\begin{cases} \dot{x}_1 = \dfrac{\nu}{x_1}(\Phi_\zeta \zeta b_3 - \Phi_\eta \eta b_2) \\ \\ \dot{x}_2 = \dfrac{\nu}{x_2}(\Phi_\eta \eta b_1 - \Phi_\xi \xi b_3) \\ \\ \dot{x}_1 = \dfrac{\nu}{x_3}(\Phi_\xi \xi b_2 - \Phi_\zeta \zeta b_1) \end{cases} \tag{4.8}$$

Now we introduce the elliptic coordinates in $\mathbb{R}^3$:

$$\begin{cases} x_1^2 = \dfrac{(\lambda_1 + b_1^{-1})(\lambda_2 + b_1^{-1})(\lambda_3 + b_1^{-1})}{(b_1^{-1} - b_2^{-1})(b_1^{-1} - b_3^{-1})} \\ x_2^2 = \dfrac{(\lambda_1 + b_2^{-1})(\lambda_2 + b_2^{-1})(\lambda_3 + b_2^{-1})}{(b_2^{-1} - b_1^{-1})(b_2^{-1} - b_3^{-1})} \\ x_3^2 = \dfrac{(\lambda_1 + b_3^{-1})(\lambda_2 + b_3^{-1})(\lambda_3 + b_3^{-1})}{(b_3^{-1} - b_2^{-1})(b_3^{-1} - b_1^{-1})} \end{cases} \tag{4.9}$$

where λ_1, λ_2, λ_3 are the roots of the cubic polynomial in w :

$$\begin{cases} -w^3 + (x_3^2 + x_1^2 + x_2^2 - b_1^{-1} - b_2^{-1} - b_3^{-1})w^2 + (x_1^2(b_2^{-1} + b_3^{-1}) + x_2^2(b_1^{-1} + b_3^{-1}) \\ +x_3^2(b_1^{-1} + b_2^{-1}) - b_1^{-1}b_2^{-1} - b_1^{-1}b_3^{-1} - b_2^{-1}b_3^{-1})w \\ +b_1^{-1}b_2^{-1}b_3^{-1}(b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 - 1) = 0 \end{cases}$$

In this coordinates we obtain

$$||\dot{x}||^2 = g_{11}\dot{\lambda}_1^2 + g_{33}\dot{\lambda}_2^2 + g_{33}\dot{\lambda}_3^2$$

where

$$\begin{cases} g_{11} = \dfrac{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{4(\lambda_1 - b_1^{-1})(\lambda_1 - b_2^{-1})\lambda_1 - b_3^{-1})} \\ g_{22} = \dfrac{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}{4(\lambda_2 - b_1^{-1})(\lambda_2 - b_2^{-1})\lambda_2 - b_3^{-1})} \\ g_{33} = \dfrac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_3)}{4(\lambda_3 - b_1^{-1})(\lambda_3 - b_2^{-1})\lambda_3 - b_3^{-1})} \end{cases} \tag{4.10}$$

The differential equations (4.8) in elliptic coordinates can be transformed to the form

$$\begin{cases} \dot{w} = 0 \\ \dot{u} = \dfrac{2\nu}{b_1 b_2 b_3}((b_2 - b_3)\Phi_\zeta \zeta + (b_1 - b_2)\Phi_\eta \eta + (b_3 - b_1)\Phi_\xi \xi) \equiv \Psi_1(\lambda_1, \lambda_2) \\ \dot{v} = \dfrac{2\nu}{b_1 b_2 b_3}(\dfrac{1}{b_1} + \dfrac{1}{b_2} + \dfrac{1}{b_3})((b_2 - b_3)b_1\Phi_\zeta \zeta + (b_1 - b_2)b_3\Phi_\eta \eta + (b_3 - b_1)b_2\Phi_\xi \xi) \equiv \Psi_2(\lambda_1, \lambda_2) \\ u = \lambda_1 + \lambda_2 + \lambda_3, \quad v = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \quad w = \lambda_1\lambda_2\lambda_3, \end{cases}$$

and by putting

$$\lambda_3 = 0,$$

after some calculations we deduce the planar system

$$\begin{cases} \dot{\lambda}_1 = \dfrac{\Psi_1(\lambda_1, \lambda_2)\lambda_1 - \Psi_2(\lambda_1, \lambda_2))}{\lambda_1 - \lambda_2} \\ \dot{\lambda}_2 = \dfrac{\Psi_1(\lambda_1, \lambda_2)\lambda_2 - \Psi_2(\lambda_1, \lambda_2))}{\lambda_1 - \lambda_2}. \end{cases}$$

In order to deduce the differential equations for the geodesic flow by using the Lagrangian approach first, it is necessary in the first place obtain the solution of the equations (3.20).

The integrability of the geodesic flow on the quadric (m=2) by using the classical approach, was proved by Jacobi and Chasles.

5 The geometrical and physical meaning of the Cartesian vector field

The purpose of this section is to determine the geometrical and physical meaning of the Cartesian vector field constructed above.

Hertz's principle of least curvature is a special case of Gauss' principle, restricted by the two conditions that there be no applied forces and that all masses are identical. (Without loss of generality, the masses may be set equal to one.) Under these conditions, Gauss' minimized quantity can be written

$$\mathrm{Z} = \sum_{j=1}^{N} |\frac{d^2x^j}{dt^2}|^2$$

The kinetic energy

$$T = \frac{1}{2}||\dot{\mathbf{x}}||^2$$

is also conserved under these conditions

Since the line element ds^2 in the $3N$-dimensional space of the coordinates is defined

$$ds^2 = 2Tdt^2 \Longleftrightarrow \frac{ds^2}{dt^2} = 2T$$

by considering the conservation of energy we obtain

$$\frac{ds^2}{dt^2} = 2h$$

Dividing Z by $2T$ yields another minimal quantity

$$\mathrm{K} = \sum_{j=1}^{N} |\frac{d^2x^j}{ds^2}|^2$$

Since $\sqrt{\mathrm{K}}$ is the local curvature of the trajectory in the $3N$-dimensional space of the coordinates, minimization of K is equivalent to finding the trajectory of least curvature (a geodesic) that is consistent with the constraints. Hertz's principle is also a special case of Jacobi's formulation of the least-action principle. Curvature refers to a number of loosely related concepts in different areas of geometry. In mathematics, a geodesic is a generalization of the notion of a straight line to curved spaces. Definition of geodesic depends on the type of curved space. If the space carries a natural metric then geodesics are defined to be (locally) the shortest path between points on the space.

Below we restricted to the case when the configuration space is the three dimensional Euclidean space with Cartesian coordinates $x = (x_1,\, x_2,\, x_3)$. The geodesic flow on the surface $f(x) = c$ is determined by the second- order differential equations

$$\frac{d^2x^j}{dt^2} = \mu f_{x_j}, \quad \mu = \frac{(A(x)\dot{\mathbf{x}},\, \dot{\mathbf{x}})}{||f_{\mathbf{x}}||^2} \quad j = 1, 2, 3$$

which, by considering the energy integral, can be written as follows

$$\frac{d^2x^j}{ds^2} = \tilde{\mu} f_{x_j}, \quad \tilde{\mu} = \frac{(A(x)\tau,\, \tau)}{||f_{\mathbf{x}}||^2} \quad j = 1, 2, 3$$

where

$$\tau = \frac{d\mathbf{x}}{ds}, \quad ||\tau||^2 = 1.$$

Clearly that

$$\sqrt{\mathrm{K}} = \frac{|(A(x)\tau,\, \tau)|}{||f_{\mathbf{x}}||^2}$$

The Hertz's Principle of Least Curvature and problem on the determination the principal directions on the surface lead us to state the following problem.

Problem 2

Determine the

$$extremum(A(x)\tau, \tau)$$

under the conditions

$$\begin{cases} ||\tau||^2 - 1 = 0 \\ (f_{\mathbf{x}}, \tau) = 0 \end{cases}$$

Solution

Note that in this case the Lagrangian function is

$$\mathrm{L} = (A(x)\tau, \tau) + \sigma(f_{\mathbf{x}}, \tau) + z(||\tau||^2 - 1)$$

where σ and z are the Lagrangian multiplier and computer

$$\begin{cases} \dfrac{\partial \mathrm{L}}{\partial \tau_j} = 0, \quad j = 1,2,3 \Longleftrightarrow (A(x) + zI)\tau^T + \sigma f_{\mathbf{X}}^T = 0 \\ \\ \dfrac{\partial \mathrm{L}}{\partial \sigma} = 0 \Leftrightarrow (f_{\mathbf{X}}, \tau) = 0 \\ \\ \dfrac{\partial \mathrm{L}}{\partial z} = 0 \Leftrightarrow ||\tau||^2 - 1 = 0 \end{cases}$$

where $\tau^T = col(\tau_1,\, \tau_2,\, \tau_3)$ and I is the diagonal matrix: $I = diag(1,\, 1,\, 1)$, from the first group of equations, we deduced the following equalities

$$\tau^T(x) = -\sigma(A(x) + zI)^{-1} f_{\mathbf{X}}^T(x)) \tag{5.1}$$

and

$$\mathrm{R}_z \chi = \vec{0} \tag{5.2}$$

if

$$\det\left(A(x) + zI\right) \neq 0,$$

where $\chi = col(\tau_1,\, \tau_2,\, \tau_3,\, \sigma)$ and R_z is the following family of matrixes

$$\mathrm{R}_z = \begin{pmatrix} f_{x_1x_1} + z & f_{x_1x_2} & f_{x_1x_3} & f_{x_1} \\ f_{x_1x_2} & f_{x_2x_2} + z & f_{x_2x_3} & f_{x_2} \\ f_{x_1x_3} & f_{x_2x_3} & f_{x_3x_3} + z & f_{x_3} \\ f_{x_1} & f_{x_2} & f_{x_3} & 0 \end{pmatrix}$$

In view of that the vector χ is non-zero vector then from (5.2) one can deduce that

$$\det \mathrm{R}_z = 0 \tag{5.3}$$

In order to establish the relation between the vector field with components given by (5.1) and Cartesian vector field (2.16) we introduce the family of vector fields $\mathbf{v}_z$:

$$\mathbf{v}_z = ((A(x) + zI)^{-1} f_x, \partial_x), \quad \partial_x = (\partial_{x_1},\, \partial_{x_2},\, \partial_{x_3}) \tag{5.4}$$

where z is a complex parameter. After some calculations one can prove that v_z admits the representations

$$
\mathbf{v}_z = \begin{vmatrix} f_{x_1x_1}+z & f_{x_1x_2} & f_{x_1x_3} & f_{x_1} \\ f_{x_1x_2} & f_{x_2x_2}+z & f_{x_2x_3} & f_{x_2} \\ f_{x_1x_3} & f_{x_2x_3} & f_{x_3x_3}+z & f_{x_3} \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} & 0 \end{vmatrix} = \frac{z^2X_1 + zX_2 + X_3}{\det(A+zI)} \qquad (5.5)
$$

which are equivalents equivalent to (5.4), where $\triangle f = \partial_{x_1x_1} f + \partial_{x_2x_2} f + \partial_{x_3x_3} f$. and $X_1,\, X_2,\, X_3$ denote the vector fields:

$$
\begin{cases} X_1 = (f_x, \partial_x) \\ X_2 = (g_x, \partial_x) - \triangle f\,(f_x, \partial_x), \quad g = ||f_x||^2 \\ X_3 = \det A(x)\, v_z|_{z=0} \end{cases} \qquad (5.6)
$$

We now introduce the function

$$
F(z) = df(v_z) = \frac{z^2 df(X_1) + z df(X_2) + df(X_3)}{\det(A+zI)} \qquad (5.7)
$$

which in view of (5.5) may be written as

$$
F(z) = \det(A+zI) \det \mathrm{R}_z
$$

Corollary 5.1 *If the function f is a homogeneous function of degree m then*

$$
X_3 = g^2(x)K(x)\,(x,\, \partial_x), \quad g = ||f_x||^2 \qquad (5.8)
$$

where K is the Gaussian curvature of the homogeneous surface which one can calculate as follows

$$
K(x) = \begin{cases} -\dfrac{f\{f_{x_1},\, f_{x_2},\, f_{x_3}\}}{(m-1)g^2} & \text{if } m \neq 1; \\ -\dfrac{f\{f,\, f_{x_2},\, f_{x_3}\}}{x_1\, g^2} & \text{if } m = 1; \end{cases} \qquad (5.9)
$$

Below we shall study only the case when the function f :

$$
(df(X_2))^2 - 4df(X_3)df(X_1) > 0 \qquad (5.10)
$$

Clearly, under this assumption the function F has two different real roots which we shall denote by $z_1,\, z_2$

Corollary 5.2 *Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are the vector fields such that*

$$\boldsymbol{v}_j = \boldsymbol{v}_z|_{z=z_j}, \quad j = 1, 2$$

then the solution of the problem 2 are the vector fields :

$$\begin{cases} \tau^{(j)} = \dfrac{\mathbf{v}_j}{||\mathbf{v}_j(x)||}, \quad j = 1, 2 \\ \\ (\tau^{(1)}, \tau^{(2)}) = 0 \end{cases}$$

The proof it is easy to obtain.

Proposition 5.1 *Let $f(x) = c$, $c \neq 0$ be the homogeneous surface which satisfies (5.10).*

Then the most general vector field tangent to the given surface admits the development

$$\boldsymbol{v}(x) = [f_{\boldsymbol{x}}(x) \times [f_{\boldsymbol{x}}(x) \times (\mu_1(x)\, g_{\boldsymbol{x}}(x) + \mu_2(x)\, \boldsymbol{x})]]. \tag{5.11}$$

where μ_1, μ_2 are arbitrary smooth functions.

Proof

One can check direct from the above that the most general vector field tangent to the given homogeneous surface can be written as

$$\mathbf{v}(x) = a_1(x)\mathbf{v}^{(1)}(x) + a_2(x)v^{(2)}(x)$$

where a_1, a_2 are arbitrary smooth functions.

A brief calculation show that

$$v(x) = \lambda_1(x)\, f_{\mathbf{X}}(x) + \lambda_2(x)\, g_{\mathbf{X}}(x) + \lambda_3(x)\, \mathbf{x} \tag{5.12}$$

where

$$\begin{cases} \lambda_1(x) = \hat{a}_1(x)(z_1^2 - \triangle f\, z_1) + \hat{a}_2(x)(z_2^2 - \triangle f\, z_2) \\ \lambda_2(x) = \hat{a}_1(x)\, z_1 + \hat{a}_2(x)\, z_2 \\ \lambda_3(x) = g^2(x)K(\hat{a}_1(x) + \hat{a}_2(x)) \\ a_j(x) = \hat{a}_j \det(A(x) + z_j I), \quad j = 1, 2 \end{cases}$$

One can see that the equation

$$g\lambda_1 + (f_{\mathbf{X}},\, g_{\mathbf{X}})\lambda_2 + mf\lambda_3 = 0 \tag{5.13}$$

holds identically.

To complete the proof, we show the equivalence of (5.11) and (5.12), (5.13). Indeed, using (5.13) we obtain that

$$\lambda_1 = -(\frac{(f_{\mathbf{X}},\, g_{\mathbf{X}})\lambda_2 + mf\lambda_3}{g}),$$

inserting into (5.12) and introducing the notations

$$\mu_1(x) = \frac{\lambda_3}{g}, \quad \mu_2(x) = \frac{\lambda_2}{g}$$

we get (5.11).

Proposition 5.2 *The vector field (5.11) is Cartesian vector field if the following relation holds*

$$\{f, g, r\}(\partial_r \mu_1 - \partial_g \mu_2 - \frac{\mu_2}{g}) = 0 \tag{5.14}$$

From the definition we obtain that the given vector field is Cartesian (see definition 1) if the following equality takes place

$$(f_{\mathbf{X}}, (rot([f_{\mathbf{X}}(x) \times [f_{\mathbf{X}}(x) \times (\mu_1(x)\, g_{\mathbf{X}}(x) + \mu_2(x)\,\mathbf{x}\,))]]) = 0 \tag{5.15}$$

which is equivalent to (5.14).

From (5.14) after straightforward calculations we can prove the following assertion

Corollary 5.3 *Let us suppose that (5.14) holds, then the vector field (5.12) admits the representation*

$$\mathbf{v}(x) = \kappa\, [f_{\mathbf{X}} \times [f_{\mathbf{X}} \times \Phi_{\mathbf{X}}]] \tag{5.16}$$

where κ and Φ functions such that

$$\kappa = \begin{cases} \kappa(f, \Phi) & \text{if } \{f, g, r\} \neq 0; \\ \mu_1 \dfrac{g_r}{r} + \mu_2 & \text{if } \{f, g, r\} = 0; \end{cases} \tag{5.17}$$

and

$$\Phi_x = \begin{cases} \Phi_{\mathbf{x}}(f, g, r) & \text{if } \{f, g, r\} \neq 0; \\ \\ \mathbf{x} & \text{if } \{f, g, r\} = 0; \end{cases} \tag{5.18}$$

By comparing (5.16), (5.17) with (2.16) we obtain that the solution of the problem 2 under the condition (5.17), (5.18) coincide with the Cartesian vector field constructed in the section 3. In such a way we obtain the physical and geometrical meaning of the constructed above Cartesian vector field.

Remark.

The physical and geometrical meaning of the Cartesian vector field for the case when the given vector field $\mathbf{a}$:

$$(\mathbf{a}, rot\mathbf{a}) \neq 0 \tag{5.19}$$

can be obtained analogously to the case study above by considering that under condition (5.19) the equation deduced from the Lagrangian approach, can be written as follows

$$\ddot{\mathbf{x}} = \frac{(A(x)\dot{\mathbf{x}}, \dot{\mathbf{x}})}{||f_{\mathbf{x}}||^2}\mathbf{a}$$

where

$$A(x) = \begin{pmatrix} \partial_1 a_1 & \frac{1}{2}(\partial_1 a_2 + \partial_2 a_1) & \frac{1}{2}(\partial_1 a_3 + \partial_3 a_1) \\ \frac{1}{2}(\partial_1 a_2 + \partial_2 a_1) & \partial_2 a_2 & \frac{1}{2}(\partial_2 a_3 + \partial_3 a_2) \\ \frac{1}{2}(\partial_1 a_3 + \partial_3 a_1) & \frac{1}{2}(\partial_2 a_3 + \partial_3 a_2) & \partial_3 a_3 \end{pmatrix}$$

The problem 2 in this case we can state as follows

Problem 3

Determine the

$$extremum(A(x)\tau, \tau)$$

under the conditions

$$\begin{cases} ||\tau||^2 - 1 = 0 \\ (\mathbf{a}, \tau) = 0 \end{cases}$$

6 Descartes approach for non-holonomic system with three degree of freedom and one constraints .

Our goal in this section is to extend the Cartesian approach developed above for natural mechanical system with configuration space

$$\mathrm{Q}, \quad \dim \mathrm{Q} = 3$$

in this space the metric (kinetic energy)

$$T = \frac{1}{2}\sum_{j,k=1}^{3} G_{jk}(x)\dot{x}^j\dot{x}^k \equiv \frac{1}{2}||\dot{x}||^2$$

allows calculating the rot of the vector field v on Q. The invariant definition of $rotv$ we can find in [2]. If we assign a covector field $p = (p_1,\, p_2,\, p_3)$ with components

$$p_j = \sum_{k=1}^{3} G_{jk}(x)v^k(x) \tag{6.1}$$

to the vector field $v = (v^1(x),\, v^2(x),\, v^3(x))$, then the components of $rotv$ we can write explicitly

$$rotv(x) = (\frac{1}{\sqrt{G}}(\partial_2 p_3 - \partial_3 p_2),\; \frac{1}{\sqrt{G}}(\partial_3 p_1 - \partial_1 p_3),\, \frac{1}{\sqrt{G}}(\partial_1 p_2 - \partial_2 p_1))$$

where $G = \det(G_{kj}(x))$.

The vector field (2.9) for this mechanical system we shall represented as follows

$$\dot{\mathbf{x}} = [\mathbf{a}(x) \times (\lambda_1\, \mathbf{b}(x) + \lambda_2 \mathbf{c}(x))] \tag{6.2}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are the independent smooth vector in Q, i.e.,

$$\Upsilon = (\mathbf{a}, [\mathbf{b} \times \mathbf{c}]) \neq 0 \tag{6.3}$$

and λ_1, λ_2 are smooth function which we determine as a solution of the equation

$$(\mathbf{a}, rot[\mathbf{a} \times (\lambda_1\, \mathbf{b} + \lambda_2 \mathbf{c})]) = 0 \tag{6.4}$$

The Lagrangian approach produces the following second-order differential equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^k} - \frac{\partial T}{\partial x^k} = \frac{\partial \frac{1}{2}||\mathbf{v}||^2}{\partial x^k} + \Lambda a_k(x), \quad k = 1, 2, 3 \tag{6.5}$$

We shall illustrate this case for the Chapliguin-Caratheodory sleigh and for the heavy rigid body in the Suslov case.

The Chapliguin-Caratheodory sleigh

We shall now analyze one of the most classical nonholonomic systems : Chapliguin-Carathodory's sleigh [15]. The idealized sleigh is a body that has three points of contact with the plane. Two of them slide freely but the third, A, behaves like a knife edge subjected to a constraining force $\mathbb{R}$ which does not allow transversal velocity. More precisely, let yoz be an inertial frame and $\xi\, A\eta$ a frame moving with the sleigh. Take as generalized coordinates the Descartes coordinates of the center of mass C of the sleigh and the angle x between the y and the ξ axis. The reaction force $\mathbb{R}$ against the runners is exerted laterally at the point of application A in such a way that the η component of the velocity is zero. Hence, one has the constrained system M with the configuration space $X = S^1 \times \mathbb{R}^2$, with the kinetic energy

$$T = \frac{m}{2}(\dot{y}^2 + \dot{z}^2) + \frac{I_c}{2}\dot{x}^2,$$

and with the constraint

$$\epsilon \dot{x} + \sin x \dot{y} - \cos x \dot{z} = 0,$$

where m is the mass of the system and J_c is the moment of inertia about a vertical axis through C and $\epsilon = |AC|$. Observe that the "javelin" (or arrow or Chapliguin's skate) is a particular case of this mechanical system and can be obtained when $\epsilon = 0$

To apply the Descartes approach for this system, first we determine the vector $\mathbf{b}$ and $\mathbf{a}$ in such a way that the determinant $\Upsilon \neq 0$. In this subcase, we achieve this condition if

$$\mathbf{a} = (\epsilon, \sin x, -\cos x) \quad \mathbf{b} = (0, \cos x, \, sinx), \quad \mathbf{c} = (1, 0, 0).$$

Under these restrictions we obtain that $\Upsilon = 1$ and it is easy to show that the vector field $\mathbf{v}$ takes the form:

$$\mathbf{v} = \lambda_3(\partial_x + \epsilon \sin x \partial_y + \epsilon \cos x \partial_z) - \lambda_2(\cos x \partial_y - \sin x \partial_z).$$

The Descartes approach produce the differential equations [16]

$$\left\{\begin{array}{l} \dot{x} = \lambda_3(x,y,z,\epsilon) \\ \dot{y} = \lambda_2(x,y,z,\epsilon)\cos x - \epsilon\lambda_3\sin x \\ \dot{z} = \lambda_2(x,y,z,\epsilon)\sin x + \epsilon\lambda_3\cos x \end{array}\right. \tag{6.6}$$

where λ_2, λ_3 are solutions of the partial differential equations

$$\sin x(J\partial_z\lambda_3 + \epsilon m\partial_y\lambda_2) + \cos x(J\partial_y\lambda_3 - \epsilon m\partial_z\lambda_2) - m(\partial_x\lambda_2 - \epsilon\lambda_3) = 0 \tag{6.7}$$

where $J = J_C + \epsilon^2 m$.

Clearly,

$$||\mathbf{v}||^2 = (J_C + m\epsilon^2)\lambda_3^2(x,y,z,\epsilon) + m\lambda_2^2(x,y,z,\epsilon)$$

Hence, for the arrow ($\epsilon = 0$) we have

$$\left\{\begin{array}{l} \dot{x} = \lambda_3(x,y,z,0) \\ \dot{y} = \lambda_2(x,y,z,0)\cos x \\ \dot{z} = \lambda_2(x,y,z,0)\sin x \end{array}\right.$$

$$J_C(\sin x\partial_z\lambda_3 + \cos x\partial_y\lambda_3) - m\partial_x\lambda_2 = 0$$

Clearly, the equation (6.7) holds in particular if

$$\left\{\begin{array}{l} \partial_y\lambda_3 = \dfrac{\epsilon m}{J_C + \epsilon^2 m}\partial_z\lambda_2 \\ \partial_z\lambda_3 = -\dfrac{\epsilon m}{J_C + \epsilon^2 m}\partial_y\lambda_2 \\ \partial_x\lambda_2 = \epsilon\lambda_3 \end{array}\right.$$

After some calculations we can prove that the functions

$$\left\{\begin{array}{l} \lambda_2 = \cos\alpha\, V_1(y,z,\epsilon) + \sin\alpha\, V_2(y,z,\epsilon) + a\int K(x,\epsilon)dx \\ \lambda_3 = \dfrac{am}{J_C + a^2 m}\Big(cos\alpha\, V_2(y,z,a) - \sin\alpha\, V_1(y,z,\epsilon)\Big) + K(x,\epsilon), \\ \alpha = \dfrac{\epsilon^2 mx}{J_C + \epsilon^2 m} \end{array}\right.$$

are solutions of (6.7), where K is an arbitrary function and V_1, V_2 are functions which satisfy the Cauchy-Riemann conditions:

$$\left\{\begin{array}{ll} \partial_y V_1(y,z,\epsilon) & = \partial_z V_2(y,z,\epsilon) \\ \partial_z V_1(y,z,\epsilon) & = -\partial_y V_2(y,z,\epsilon). \end{array}\right.$$

Corollary 6.1 *The all trajectories of the Chapliguin skate ($\epsilon = 0$) under the action of the potential field of force with components* $(0, mg, 0)$ *can be obtained from the Descartes approach*

Proof.

In fact, for the case when $\epsilon = 0$ the classical approach for Chapliguin-Carathodory's sleigh gives the following equations of motion

$$\begin{cases} \ddot{x} = 0 \\ \ddot{y} = mg + \sin x\mu \\ \ddot{z} = -\cos x\mu \\ \sin x\dot{y} - \cos x\dot{z} = 0 \end{cases}$$

Hence,by derivation we obtain

$$\frac{d}{dt}(\frac{\dot{z}}{\sin x}) = g\cos x$$

as a consequence,

$$\begin{cases} \dot{x} = C_0, \quad C_0 \neq 0 \\ \dot{y} = (\frac{g\sin x}{C_0} + C_1)\cos x \\ \dot{z} = (\frac{g\sin x}{C_0} + C_1)\sin x \end{cases}$$

or,

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = (gt\cos x_0 + C_1)\cos x_0 \\ \dot{z} = (gt\cos x_0 + C_1)\sin x_0 \end{cases}$$

Clearly, the solutions of these equations coincide with the solutions of (6.6), (6.7) with the subsidiary conditions [16]

$$J_C\lambda_3^2 + m\lambda_2^2 = mgy + h.$$

are particular cases of the equations obtained from the Descartes approach.

Corollary 6.2 *The all trajectories of Chapliguin -Carathodory's sleigh by inertia can be obtained from the Descartes approach*

Proof

Let us suppose that

$$\lambda_j = \lambda_j(x, \epsilon), \quad j = 1, 2$$

then the all trajectories of the equation (6.6) can be obtained from the formula

$$\begin{cases} y = y_0 + \int \dfrac{(\lambda_2(x,\epsilon)\cos x - \epsilon\lambda_3 \sin x)dx}{\lambda_3(x,\epsilon)} \\ z = z_0 - \int \dfrac{(\lambda_2(x,y,z,\epsilon)\sin x - \epsilon\lambda_3 \cos x)dx}{\lambda_3(x,\epsilon)} \\ t = t_0 + \int \dfrac{dx}{\lambda_3(x,\epsilon)} \end{cases}$$

On the other hand, for the Chapliguin- Caratheodory sleigh by inertia from the classical approach we deduce the following equations

$$\begin{cases} J_C\ddot{x} = \epsilon\mu \\ m\ddot{y} = \sin x\mu \\ m\ddot{z} = -\cos x\mu \\ \epsilon\dot{x} + \sin x\dot{y} - \cos x\dot{z} = 0 \end{cases}$$

Hence, after straightforward calculations we obtain the system

$$\begin{cases} \dot{x} = qC_0\cos(q\epsilon x + C), \quad q^2 = \dfrac{m}{J_C + m\epsilon^2} \\ \dot{y} = C_0(\sin(q\epsilon x + C)\cos x - q\epsilon\cos(q\epsilon x + C)\sin x) \\ \dot{z} = C_0(\sin(q\epsilon x + C)\sin x + q\epsilon\cos(q\epsilon + C)\cos x) \end{cases}$$

where $q^2 = \dfrac{m}{J_C + m\epsilon^2}$,

which are particular case of the equations (6.6) with

$$\lambda_2 = C_0\sin(q\epsilon x + C), \quad \lambda_3 = C_0 q\cos(q\epsilon x + C)$$

Evident that in this case

$$2||\mathbf{v}||^2 = (J_C + m\epsilon^2)\lambda_3^2(x,\epsilon) + m\lambda_2^2(x,\epsilon) \equiv mC_0^2$$

7 The rigid body around a fixed point in the Suslov and Veselov cases.

In this section we study one classical problem of non-holonomic dynamics formulated by Suslov [12]. We consider the rotational motion of a rigid body around a fixed point and subject to the non-holonomic constraints $(\tilde{\mathbf{a}}, \omega) = 0$ where ω is a body angular velocity and $\tilde{\mathbf{a}}$ is a constant vector. Suppose the body rotates in an force field with potential $U(\gamma_1, \gamma_2, \gamma_3)$. Applying the method of Lagrange multipliers we write the equations of motion in the form

$$\begin{cases} I\dot{\omega} = [I\omega \times \omega] + [\gamma \times \frac{\partial U}{\partial \gamma}] + \mu\tilde{\mathbf{a}} \\ \dot{\gamma} = [\gamma \times \omega] \\ (\tilde{\mathbf{a}}, \omega) = 0 \end{cases} \tag{7.1}$$

Where

$$I = diag(I_1, I_2, I_3)$$
$$\gamma = (\gamma_1 = \sin z \sin x, \quad \gamma_2 = \sin z \cos x, \quad \gamma_3 = \cos z)$$

I_1, I_2, I_3 are the inertial moment of the body.

If we assume that the vector $\tilde{\mathbf{a}} = (0, 0, 1)$ [12], then

$$\begin{cases} I_1\dot{\omega}_1 = \gamma_3\partial_{\gamma_2}U - \gamma_2\partial_{\gamma_3}U \\ I_2\dot{\omega}_2 = \gamma_1\partial_{\gamma_3}U - \gamma_3\partial_{\gamma_1}U \\ (I_1 - I_2)\omega_1\omega_2 + \gamma_2\partial_{\gamma_1}U - \gamma_1\partial_{\gamma_2}U + \mu = 0 \\ \dot{\gamma}_1 = -\gamma_3\omega_2 \\ \dot{\gamma}_2 = \gamma_3\omega_1 \\ \dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1 \end{cases} \tag{7.2}$$

The above system always has two independent first integrals

$$K_1 = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2) - U(\gamma_1, \gamma_2, \gamma_3)$$
$$K_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$$

For the real motions $K_2 = 1$.

By the Jacobi's theorem about the last multiplier, if there exits a third independent first integral K_3 which is functionally independent together with K_1 and K_2, then the Suslov problem is integrable by quadratures [12]

To determine the integrable cases of the Suslov problem seems interesting the following result which we can prove after straightforward calculations.

Proposition 7.1 *Let us suppose that the potential function U in (7.2) is determine as follows*

$$U = \frac{1}{2I_1I_2}(I_1\mu_1^2 + I_2\mu_2^2) - h \tag{7.3}$$

where μ_1, μ_2 are solutions of the partial differential equations

$$\gamma_3(\frac{\partial\mu_1}{\partial\gamma_2} - \frac{\partial\mu_2}{\partial\gamma_1}) - \gamma_2\frac{\partial\mu_1}{\partial\gamma_3} + \gamma_1\frac{\partial\mu_2}{\partial\gamma_3} = 0, \tag{7.4}$$

then the equations (7.2), (7.3) admits the first integrals

$$I_1\omega_1 = \mu_2, \quad I_2\omega_2 = -\mu_1 \tag{7.5}$$

The aim of this apartat is to propose the Descartes approach for heavy rigid body in the Suslov case.

Let us suppose that Q $= SO(3)$, with the Riemann metric

$$G = \begin{pmatrix} I_3 & I_3\cos z & 0 \\ I_3\cos z & (I_1\sin^2 x + I_2\cos^2 x)\sin^2 z + I_3\cos^2 z & (I_1 - I_2)\sin x\cos x\sin z \\ 0 & (I_1 - I_2)\sin x\cos x\sin z & I_1\cos^2 x + I_2\sin^2 x \end{pmatrix}$$

$$\det G = I_1I_2I_3\sin^2 z,$$

In this case we have that the constraints are

$$\omega_3 = 0 \Leftrightarrow \dot{x} + \cos z\dot{y} = 0$$

Hence $a = (1,\, \cos z,\, 0)$. By choosing the vector b and c as follow

$$b = (0, 1, 0), \quad c = (0, 0, 1)$$

we obtain that $\Upsilon = 1$. Consequently

$$v = \lambda_2(\cos\, z\,\partial_x - \partial_y) - \lambda_3\partial_z$$

The differential equations generated by v and condition (6.4) in this cases take the form respectively

$$\begin{cases} \dot{x} = \cos z\,\lambda_2, \\ \dot{y} = -\lambda_2, \\ \dot{z} = -\lambda_3 \end{cases} \tag{7.6}$$

$$(a, rot\mathbf{v}) = 0 \Leftrightarrow \partial_z p_2 - \partial_y p_3 + \cos z \partial_x p_3 = 0 \quad (7.7)$$

After the change $\gamma_1 = \sin z \sin x, \quad \gamma_2 = \sin z \cos x, \quad \gamma_3 = \cos z$ the system (7.6) and condition (7.7) can be written as follow

$$\begin{cases} \dot{\gamma_1} = \frac{I_1}{I_1 I_2}\mu_1\gamma_3 \\ \\ \dot{\gamma_2} = \frac{I_2}{I_1 I_2}\mu_2\gamma_3 \\ \\ \dot{\gamma_3} = \frac{-1}{I_1 I_2}(I_1\mu_1\gamma_1 + I_2\mu_2\gamma_2) \end{cases} \quad (7.8)$$

$$\sin z(\gamma_3(\frac{\partial\mu_1}{\partial\gamma_2} - \frac{\partial\mu_2}{\partial\gamma_1}) - \gamma_2\frac{\partial\mu_1}{\partial\gamma_3} + \gamma_1\frac{\partial\mu_2}{\partial\gamma_3}) - \cos x\partial_y\mu_2 - \sin x\partial_y\mu_1 = 0 \quad (7.9)$$

where

$$\begin{cases} \mu_2 = -I_1(\cos x\lambda_3 + \sin x\lambda_2), \\ \mu_1 = I_2(-\sin x\lambda_3 + \cos x\lambda_2) \end{cases}$$

We shall study only the case when

$$\mu_j = \mu_j(x, z),\ j = 1, 2$$

Hence, we obtain the equation (7.4).

Corollary 7.1 *The function μ_1, μ_2 :*

$$\begin{cases} \mu_1 = \dfrac{\partial S(\gamma_1,\gamma_2)}{\partial\gamma_1} + \Psi_1(\gamma_2^2 + \gamma_3^2, \gamma_1) \\ \mu_2 = \dfrac{\partial S(\gamma_1,\gamma_2)}{\partial\gamma_2} + \Psi_2(\gamma_1^2 + \gamma_3^2, \gamma_2) \end{cases} \quad (7.10)$$

satisfies the equation (7.4).

Corollary 7.2 *Let μ_1, μ_2 are such that*

$$\mu_j = \frac{\partial S(\gamma_1,\gamma_2)}{\partial\gamma_j}, \quad j = 1, 2$$

then the potential function (7.3) and first integrals (7.5) are respectively

$$\begin{cases} U = \frac{1}{2I_1I_2}(I_1(\frac{\partial S}{\partial \gamma_1})^2 + I_2(\frac{\partial S}{\partial \gamma_2})^2) - h \\ I_1\omega_1 = \frac{\partial S}{\partial \gamma_2}, \\ I_2\omega_2 = -\frac{\partial S}{\partial \gamma_1}, \end{cases} \tag{7.11}$$

The following particular cases produces the well known integrable cases [12]: The Suslov, Kharlamova-Zabelina and Kozlov subcase.

The Suslov Subcase

If

$$S = C_1\gamma_1 + C_2\gamma_2, \quad C_j = const,\ j = 1,2$$

then

$$\begin{cases} \mu_1 = C_1, \quad \mu_2 = C_2 \\ U = const. \end{cases}$$

which correspond to the Suslov subcase.

The integration of the equations (7.8) in this case produces the following solutions

$$\begin{cases} \omega_1 = \frac{C_2}{I_1}, \quad \omega_2 = -\frac{C_1}{I_2} \\ \gamma_1 = \frac{C_1I_1}{\sqrt{I_1^2C_1^2 + I_2^2C_2^2}}\sin\beta\sin(\frac{\sqrt{I_1^2C_1^2 + I_2^2C_2^2}}{I_1I_2}t + \alpha) + \frac{I_2C_2\cos\beta}{\sqrt{I_1^2C_1^2 + I_2^2C_2^2}} \\ \gamma_2 = \frac{C_2I_2}{\sqrt{I_1^2C_1^2 + I_2^2C_2^2}}\sin\beta\sin(\frac{\sqrt{I_1^2C_1^2 + I_2^2C_2^2}}{I_1I_2}t + \alpha) - \frac{I_1C_1\cos\beta}{\sqrt{I_1^2C_1^2 + I_2^2C_2^2}} \\ \gamma_3 = \sin\beta\cos((\sqrt{\frac{I_1^2C_1^2 + I_2^2C_2^2}{I_1I_2}}t + \alpha) \end{cases}$$

where C_1, C_2, α, β, are the arbitrary real constants.

The Kharlamova-Zabelina Subcase

If

$$S = \frac{2}{3\sqrt{I_1C_1^2 + I_2C_2^2}}(\sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2})^3 + \frac{CC_2I_2}{C_1^2I_1 + C_2^2I_2}\gamma_1 - \frac{CC_1I_1}{C_1^2I_1 + C_2^2I_2}\gamma_2$$

where $\tilde{h}$, C_1, C_2, C are arbitrary constants, then

$$\begin{cases} \mu_1 = \dfrac{C_1}{\sqrt{I_1C_1^2 + I_2C_2^2}}\sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2} + \dfrac{CC_2I_2}{C_1^2I_1 + C_2^2I_2} \\ \mu_2 = \frac{C_2}{\sqrt{I_1C_1^2+I_2C_2^2}}\sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2} - \frac{CC_1I_1}{C_1^2I_1+C_2^2I_2} \\ U = \tilde{h} + C_1\gamma_1 + C_2\gamma_2 \end{cases}$$

As a consequence we deduce the Kharlamova-Zabelina subcase [6].

The solutions of the equation (7.2), (7.8) are

$$\begin{cases} I_1\omega_1 = \dfrac{C_1}{\sqrt{I_1C_1^2 + I_2C_2^2}}\sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2} + \dfrac{CC_2I_2}{C_1^2I_1 + C_2^2I_2}, \\ I_2\omega_2 = -(\dfrac{C_2}{\sqrt{I_1C_1^2 + I_2C_2^2}}\sqrt{\tilde{h} + C_1\gamma_1 + C_2\gamma_2} - \dfrac{CC_1I_1}{C_1^2I_1 + C_2^2I_2}), \\ U = \tilde{h} + C_1\gamma_1 + C_2\gamma_2 \\ \gamma_j = a_j(\tau - C_3)^2 + b_j(\tau - C_4) + d_j = \gamma_j(\tau, C_1, C_2, C_3, C_4), \quad j = 1, 2 \\ \gamma_3 = \sqrt{1 - \gamma_1^2(\tau, C_1, C_2, C_3, C_4) - \gamma_2^2(\tau, C_1, C_2, C_3, C_4)} \equiv \sqrt{P_4(\tau, C_1, C_2, C_3, C_4)} \\ t = t_0 + \dfrac{I_1I_2}{2}\displaystyle\int \dfrac{d\tau}{\sqrt{P_4(\tau, C_1, C_2, C_3, C_4)}} \end{cases}$$

where

$$a_j = \frac{I_jC_j}{4}, \quad b_j = \frac{CI_1I_2C_1C_2}{C_j(I_1C_1^2 + I_2C_2^2)}, \quad d_j = -\frac{\tilde{h}I_jC_j}{I_1C_1^2 + I_2C_2^2},$$

P_4 is a polynomial of four degree in τ.

Kozlov Subcase

If we suppose that $I_1 = I_2$ and

$$\begin{cases} S = -2C\arctan\frac{\gamma_1}{\gamma_2} + \int D(\gamma_1^2 + \gamma_2^2)d(\gamma_1^2 + \gamma_2^2) \\ (D(u))^2 = \dfrac{hu^2 + \sqrt{1-u}u - C^2}{u^2} \end{cases}$$

where h and C are arbitrary real constant, hence,

$$\begin{cases} \mu_1 = -\dfrac{\gamma_2C}{\gamma_1^2 + \gamma_2^2} + \gamma_1D(\gamma_1^2 + \gamma_2^2) \\ \mu_2 = \dfrac{\gamma_1C}{\gamma_1^2 + \gamma_2^2} + \gamma_2D(\gamma_1^2 + \gamma_2^2) \\ U = -h + \sqrt{1 - \gamma_1^2 - \gamma_2^2} = -h + \gamma_3 \end{cases}$$

which correspond to the Kozlov subcase.
The equations (7.8) in this case take the form:

$$\begin{cases} \dot{x} = \dfrac{C \cos z}{\sin^2 z} \\ \dot{y} = \dfrac{-C}{\sin^2 z} \\ \dot{z} = \dfrac{(\gamma_1^2 + \gamma_2^2) D(\gamma_1^2 + \gamma_2^2)}{\sin z} \end{cases} \tag{7.12}$$

which are easy to integrate.
The solutions of the equation of motions are:

$$\begin{cases} \omega_1 = \dfrac{\gamma_1 C}{\gamma_1^2 + \gamma_2^2} + \gamma_2 D(\gamma_1^2 + \gamma_2^2) \\ \omega_2 = \dfrac{\gamma_2 C}{\gamma_1^2 + \gamma_2^2} - \gamma_1 D(\gamma_1^2 + \gamma_2^2) \\ x = x_0 + C \int \dfrac{\gamma_3 d\gamma_3}{(1-\gamma_3^2)^2 D(1-\gamma_3^2)} = x_0 + C \displaystyle\int \dfrac{\gamma_3 d\gamma_3}{\sqrt{(1-\gamma_3^2) P_4(\gamma_3, h, C)}} \\ y = y_0 - C \int \dfrac{d\gamma_3}{(1-\gamma_3^2)^2 D(1-\gamma_3^2)} = y_0 - C \displaystyle\int \dfrac{d\gamma_3}{\sqrt{(1-\gamma_3^2) P_4(\gamma_3, h, C)}} \\ t = t_0 + I_1 I_2 \int \dfrac{d\gamma_3}{\sqrt{P_4(\gamma_3, h, C)}} \\ P_4(\gamma_3, h, C) \equiv h\gamma_3^4 - 2\gamma_3^3 - 2h\gamma_3^2 + 2\gamma_3 + h - C^2 \end{cases}$$

Corollary 7.3 *Let μ_1, μ_2 are the functions:*

$$\begin{cases} \mu_1 = \Psi_1(\gamma_2^2 + \gamma_3^2, \gamma_1) \\ \mu_2 = \Psi_2(\gamma_1^2 + \gamma_3^2, \gamma_2) \end{cases}$$

then the solutions of (7.8) are the following functions:

$$\begin{cases} \int \dfrac{d\gamma_j}{F_j(\gamma_j)} = \dfrac{I_j}{I_1 I_2}(\tau - \tau_0), \quad j = 1, 2 \\ \gamma_3 = \sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)} \\ t = t_0 + \int \dfrac{d\tau}{\sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)}} \end{cases}$$

where

$$\begin{cases} F_1(\gamma_1) = \Psi_1(\gamma_2^2 + \gamma_3^2, \gamma_1)|_{\gamma_2^2 + \gamma_3^2 = 1 - \gamma_1^2} \\ F_2(\gamma_2) = \Psi_2(\gamma_1^2 + \gamma_3^2, \gamma_2)|_{\gamma_1^2 + \gamma_3^2 = 1 - \gamma_2^2} \end{cases}$$

As a particular case we obtain the Tisserand Subcase.

Tisserand Subcase

The interesting solution of the equation (7.4) are

$$\begin{cases} \mu_1 = \sqrt{h_1 + a_1(\gamma_3^2 + \gamma_2^2) + b_1\gamma_1^2 + f_1(\gamma_1)} \equiv \Psi_1(\gamma_2^2 + \gamma_3^2, \gamma_1) \\ \mu_2 = \sqrt{h_2 + a_2(\gamma_3^2 + \gamma_1^2) + b_2\gamma_2^2 + f_2(\gamma_2)} \equiv \Psi_2(\gamma_1^2 + \gamma_3^2, \gamma_2) \end{cases}$$

which produce the following potential function U :

$$U = I_1h_1 + I_2h_2 + (I_1b_1 + I_2a_2)\gamma_1^2 + (I_1a_1 + I_2b_2)\gamma_2^2 + (I_1a_1 + I_2a_2)\gamma_3^2 + I_1f_1(\gamma_1) + I_2f_2(\gamma_2)$$

where a_j, b_j, h_j, $j = 1, 2$ are arbitrary real constants and f_j, $j = 1, 2$ are arbitrary functions.

The case when $f_j(\gamma_j) = \alpha_j\gamma_j$, $j = 1, 2$ was studied in [11], where α_j, $j = 1, 2$ are real constants.

The case when $f_j = 0$, $j = 1, 2$ is well known as Tisserands case [12].

After integration the equation (7.8) in the Tisserand case we obtain the following solutions

$$\begin{cases} I_1\omega_1 = \sqrt{h_2 + a_2(\gamma_3^2 + \gamma_1^2) + b_2\gamma_2^2} \\ I_2\omega_2 = -\sqrt{h_1 + a_1(\gamma_3^2 + \gamma_2^2) + b_1\gamma_1^2} \\ \gamma_1 = \sqrt{\frac{h_1 + a_1}{a_1 - b_1}} \sin(\sqrt{a_1 - b_1} I_1\tau + C_1) = \gamma_1(\tau) \\ \gamma_2 = \sqrt{\frac{h_2 + a_2}{a_2 - b_2}} \sin(\sqrt{a_2 - b_2} I_2\tau + C_2) = \gamma_2(\tau) \\ \gamma_3 = \sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)} \\ t = t_0 + I_1I_2 \int \frac{d\tau}{\sqrt{1 - \gamma_1^2(\tau) - \gamma_2^2(\tau)}} \end{cases}$$

Heavy rigid body in the Veselov case

In this example we study the problem of non-holonomic dynamics formulated by Veselov in [13] which in certain sense is opposite to the Suslov problem. In this problem we consider the rotational motion of a rigid body around a fixed point and subject to the non-holonomic constraints

$$(\gamma, \omega) \equiv \dot{y} + \cos z\dot{x} = 0$$

Suppose the body rotates in an force field with potential $U(\gamma_1, \gamma_2, \gamma_3)$. Applying the method of Lagrange multipliers we write the equations of motion in the form

$$\begin{cases} I\dot{\omega} = [I\omega \times \omega] + [\gamma \times \frac{\partial U}{\partial \gamma}] + \lambda\gamma \\ \dot{\gamma} = [\gamma \times \omega] \end{cases}$$

where I is a matrix such that $I = diag(I_1, I_2, I_3)$.

The Descartes approach for this system produces the following equations:

$$\begin{cases} \dot{x} = \lambda_2 \\ \dot{y} = -\cos z \lambda_2 \\ \dot{z} = \lambda_3 \end{cases}$$

and

$$\frac{\partial p_3}{\partial x} - \frac{\partial p_1}{\partial z} + \cos z (\frac{\partial p_2}{\partial z} - \frac{\partial p_3}{\partial y}) = 0$$

where

$$\begin{cases} p_1 = I_3 \sin^2 z \lambda_2 \\ p_2 = (I_3 - I_1 + (I_1 - I_2) \cos^2 x) \cos z \sin^2 z \lambda_2 + (I_1 - I_2) \cos x \sin x \sin z \lambda_3 \\ p_3 = (I_2 \sin^2 x + I_1 \cos^2 x) \lambda_3 + (I_2 - I_1) \sin x \cos x \sin z \cos z \lambda_2 \end{cases}$$

Finally it is interesting to observe that the construction the Descartes approach for the Federov case [14], i.e.,

$$(\omega, \gamma) = a$$

it is necessary in the above example make the change $y \to y + at$, $a = const.$. Hence we obtain that that the equations generated by the Descartes vector field are

$$\begin{cases} \dot{x} = \lambda_2 \\ \dot{y} = -\cos z \lambda_2 + a \\ \dot{z} = \lambda_3 \end{cases}$$

and

$$\frac{\partial p_3}{\partial x} - \frac{\partial p_1}{\partial z} + \cos z (\frac{\partial p_2}{\partial z} - \frac{\partial p_3}{\partial y})) = 0 \tag{7.13}$$

where

$$\begin{cases} p_1 = I_3 \sin^2 z \lambda_2 \\ p_2 = (I_3 - I_1 + (I_1 - I_2) \cos^2 x) \cos z \sin^2 z \lambda_2 + (I_1 - I_2) \cos x \sin x \sin z \lambda_3 + \\ a((I_1 \sin^2 x + I_2 \cos^2 x) \sin^2 z + I_3 \cos^2 z) \\ p_3 = (I_2 \sin^2 x + I_1 \cos^2 x) \lambda_3 + (I_2 - I_1) \sin x \cos x \sin z \cos z \lambda_2 \end{cases} \tag{7.14}$$

The equation (7.13) can be represented as follow

$$\begin{cases} sin^2 z(I_3 \sin^2 z + \cos^2 z(I_1 \sin^2 x + I_2 \cos^2 x)\partial_z\lambda_2(x,z)+ \\ (I_2 \sin^2 x + I_1 \cos^2 x)\partial_x\lambda_3(x,z)+ \\ \cos x \sin\, x \sin z \cos\, z(I_1 - I_2)(\partial_z\lambda_3(x,z) - \partial_x\lambda_2(x,z))+ \\ \sin z \cos\, z(3(I_1 - I_2)\sin^2 z \cos^2 x + 3I_1 - I_3)\sin^2 z + I_1 + I_2)\lambda_2(x,z) \\ \cos x \sin\, x(-2 + \cos^2 z)(I_1 - I_2)\lambda_3(x,z) + 2a \sin z \cos^2 z(I_1 \sin^2 x + I_2 \cos^2 x) = 0. \end{cases} \tag{7.15}$$

Proposition 7.2 *Let* λ_2, λ_3 *are the solutions of the linear partial differential equations (7.15) then the functions*

$$\begin{cases} \omega_1 = \gamma_2 \frac{\lambda_3}{\sin z} - \gamma_1\gamma_3\lambda_2 - a\gamma_1 \\ \omega_2 = -\gamma_1 \frac{\lambda_3}{\sin z} - \gamma_2\gamma_3\lambda_2 - a\gamma_2 \\ \omega_3 = \sin^2 z\lambda_2 - a\gamma_3 \end{cases} \tag{7.16}$$

are the first integral of the equations of the rigid body in the Veselov-Fedorov case.

In particular if

$$I_1 = I_2,$$

then the solutions of the above equation are

$$\begin{cases} \lambda_3 = \sin z\, C(z) \\ \sqrt{I_3 \sin^2 z + \cos^2 z I_2} \sin^2 z\lambda = a\Omega(z) + K(x) \end{cases}$$

where $C(z)$ and $K(x)$ are arbitrary functions and

$$\Omega(z) \equiv \int \frac{I_2\sqrt{(I_3 \sin^2 z + \cos^2 z I_2)}(2 \sin\, z - \sin\, 5z + \sin\, 3z)dz}{(-I_3 \cos\, 4z + 4I_3 \cos\, 2z - 3I_3 - I_2 + I_2 \cos\, 4z)}$$

Hence, from (7.16) we obtain

$$\begin{cases} \omega_1 = \gamma_2 C(z) - \gamma_1\gamma_3 \dfrac{a\Omega(z) + K(x)}{\sqrt{I_3 \sin^2 z + \cos^2 z I_2} \sin^2 z} - a\gamma_1 \\ \omega_2 = -\gamma_1 C(z) - \gamma_2\gamma_3 \dfrac{a\Omega(z) + K(x)}{\sqrt{I_3 \sin^2 z + \cos^2 z I_2} \sin^2 z} - a\gamma_2 \\ \omega_3 = \dfrac{a\Omega(z) + K(x)}{\sqrt{I_3 \sin^2 z + \cos^2 z I_2}} - a\gamma_3 \end{cases} \tag{7.17}$$

Thus, we easily deduce the relation

$$(I_3 \sin^2 z + \cos^2 z I_2)(\omega_3 + a\gamma_3)^2 = (a\Omega(z) + K)^2.$$

In particular if $a = 0$ and $K(x) = C_1 = const$ then we obtain the well known first integral in the Veselov case [4]

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