
Moving rectangles ^{*}

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Abstract. Consider a set of n pairwise disjoint axis-parallel rectangles in the plane. We call this set the source rectangles \mathcal{S} . The aim is to move \mathcal{S} to a set of (pairwise disjoint) target rectangles \mathcal{T} . A move consists in a horizontal or vertical translation of one rectangle, such that it does not collide with any other rectangle during the move. We study how many moves are needed to transform \mathcal{S} into \mathcal{T} . We obtain bounds on the number of needed moves for labeled and for unlabeled rectangles, and for rectangles of different and of equal dimensions.

Key words: translation, move, rectangle, motion planning

1 Introduction

Consider a set of n pairwise disjoint axis-parallel rectangles in the plane. We call this set the source rectangles \mathcal{S} . The aim is to move the source rectangles to a set of (pairwise disjoint) target rectangles \mathcal{T} . A move consists in a horizontal or vertical translation of one rectangle, such that it does not collide with any other rectangle during the move. We say that two rectangles collide if their intersection is non-empty. We study how many moves are needed to transform \mathcal{S} into \mathcal{T} . This problem can be seen in the context of motion planning.

In a related work, Abellanas et al. [1] studied the number of moves needed to translate coins. There a move consists of translating a coin along a fixed (not necessarily axis-parallel) direction. Another model for reconfiguration of disks, studied by Bereg, Dumitrescu and Pach [4], considers disk slides, where the center of a disk moves along an arbitrary continuous curve. Bereg and Dumitrescu [3] investigated the number of moves needed to reconfigure disks in the lifting model, where a disk is lifted from the plane and placed back in

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the plane at another location. Recent results on translating and sliding coins, and convex bodies in general, are due to Dumitrescu and Jiang [6]. They also show that in the translation model (not restricted to axis-parallel directions) of unlabeled axis-parallel unit squares $2n - 1$ moves are always sufficient and $\lfloor \frac{3n}{2} \rfloor$ moves are sometimes necessary to transform \mathcal{S} into \mathcal{T} . Related works on motion planning are also reconfigurations of modular metamorphic systems in a rectangular model [7] and reconfigurations of modular cube-style robots [2].

We study reconfigurations of rectangles with axis-parallel movements for three settings: In Section 2 we consider labeled rectangles of different dimensions. Here each source rectangle has its uniquely assigned target rectangle. In Section 3 we study labeled rectangles of equal dimensions. And finally in Section 4 we consider unlabeled rectangles. There, all rectangles are translates of a given rectangle and we are free to assign the source rectangles to the target rectangles. The following Table 1 summarizes the obtained bounds. A main tool in our proofs are known results on separability of convex bodies [9,10]. In the figures, source rectangles \mathcal{S} are drawn in white, and target rectangles \mathcal{T} in gray.

	necessary moves	sufficient moves
labeled rectangles of different dimensions	$3n - 1$	$4n - \frac{n}{8 \log_2 n}$
labeled rectangles of equal dimensions	$3n - 1$	$\frac{31n}{8}$
unlabeled rectangles	$2n + 1$	$\frac{17n}{6}$

Table 1. The number of moves that are sometimes necessary (left) and the number of moves that are always sufficient (right) to reconfigure a set of n rectangles.

2 Moving labeled rectangles of different dimensions

In this section each source rectangle has its unique target rectangle. The source rectangles might have different dimensions.

Theorem 1. *If the n rectangles of \mathcal{S} are separated from the n rectangles of \mathcal{T} by an axis-parallel line, then $3n - 1$ moves are always sufficient and sometimes necessary to rearrange \mathcal{S} as \mathcal{T} .*

Proof. Without loss of generality, assume that \mathcal{S} and \mathcal{T} are separated by a vertical line and \mathcal{S} lies to the left of this line. In a first step move each rectangle of \mathcal{S} upwards in such a way that each pair of rectangles is separated by a horizontal line and all rectangles lie above the target. We remark that Guibas and Yao [8] showed that for every set of pairwise disjoint rectangles

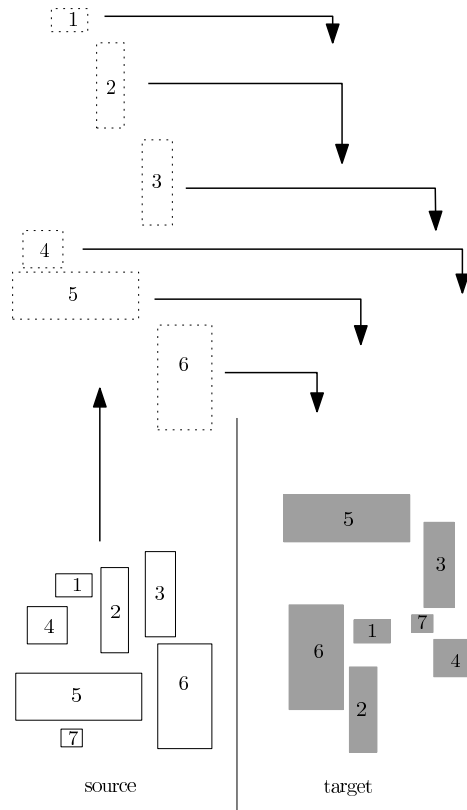


Fig. 1. $3n - 1$ moves are sufficient if \mathcal{S} and \mathcal{T} are separated by a vertical or horizontal line.

and for every fixed direction, there exists an order for the rectangles which allows to translate them, one at a time, in that direction. Then we can choose any rectangle, translate it horizontally and let it ‘drop’ to its target position. We ‘fill up’ the target from the bottom right to the top left. More formally, the insertion order is given by repeated application of the following rule:

Choose the rightmost rectangle among all rectangles p of \mathcal{T} for which the rectangle defined by the lower left corner of p and the point $(\infty, -\infty)$ is empty. Then delete this rectangle from the target list.

In the example of Figure 1 the insertion order is $\{4, 7, 3, 2, 1, 6, 5\}$. The insertion order guarantees that no two rectangles collide. This sequence uses $3n$ moves in total to transform \mathcal{S} into \mathcal{T} . It remains to show that there is one rectangle which only requires two moves. Actually, it is sufficient to only move upwards $n - 1$ rectangles in a first step. One can verify that moving this last rectangle directly to its target position does not produce a conflict with the given insertion order.

A collection of rectangles that requires $3n - 1$ moves is shown in Figure 2. Each pair of source rectangles is separated by a vertical line but not by a horizontal line. The rectangles are labeled $1, \dots, n$ from left to right. The target destinations are obtained from \mathcal{S} by mirroring \mathcal{S} along the separating line and then translating each rectangle slightly in vertical direction. Thus, for each rectangle we need at least two moves to place it in its final position. Assume there are two source rectangles i and j , $i < j$, such that each of them only needs two moves in an optimal sequence of moves. For rectangle i there are two possible ways to reach its target destination, that is, first do the horizontal move or do the vertical move. In either case rectangle j is an obstacle, because there is no horizontal line which separates i and j - neither in the source nor in the target. This implies that the rectangles i and j collide. Hence, for all but one rectangle three moves are needed. \square

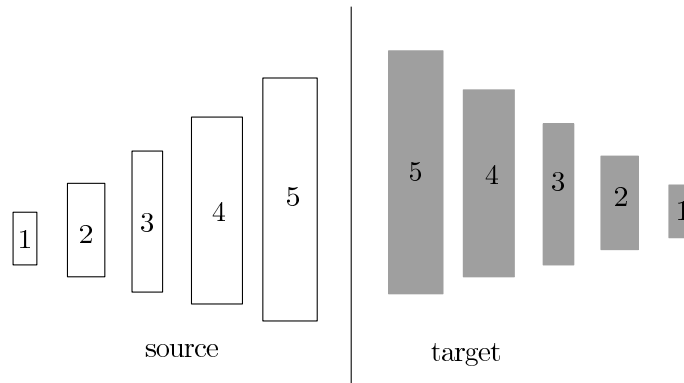


Fig. 2. A collection of rectangles that requires $3n - 1$ moves to reach the target destination.

To bound the number of moves in the general case we use 'separable sets' [5,9,11].

Definition 1. *A set of pairwise disjoint rectangles in the plane is separable if the plane can be subdivided by a straight-line and if then each of the two obtained open half-planes can be subdivided recursively by straight-lines (or straight-line segments with endpoints on previously drawn straight-lines), such that each of the obtained regions contains exactly one rectangle in its open interior.*

J. Urrutia conjectured that each set of n pairwise disjoint axis-parallel rectangles has a subset of $\Omega(n)$ separable rectangles (ACCOTA 1996, see [9]). We will use the following slightly weaker result of J. Pach and G. Tardos:

Theorem 2. [9] *Any set of n pairwise disjoint axis-parallel rectangles in the plane has a subset of $\frac{n}{2 \log_2(n)}$ separable rectangles.*

For the separation only axis-parallel lines are used. Furthermore, the separation can be obtained by first using only horizontal lines and then adding vertical segments within stripes defined by two consecutive horizontal lines. We will use this special separation.

Theorem 3. *There always is a sequence of at most $4n - \frac{n}{8\log_2 n}$ moves to rearrange any set \mathcal{S} of n pairwise disjoint, labeled source rectangles as any target set \mathcal{T} .*

Proof. Figures 3 and 4 also illustrate the following proof. Consider a maximal separable subset B of rectangles of \mathcal{S} . Draw all the horizontal lines which separate rectangles of B . This gives a partition of the rectangles of B into horizontal stripes. In Figure 3 these stripes are labeled A to D . Within each stripe the rectangles of B are separated by vertical segments. We show that

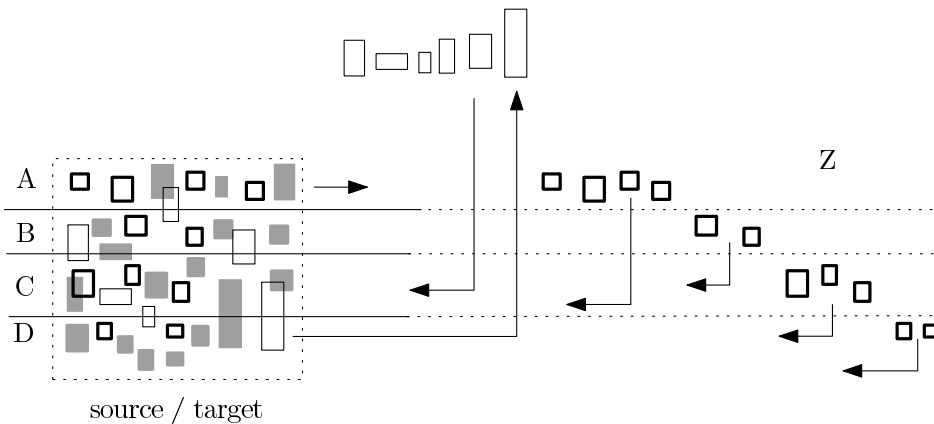


Fig. 3. Transforming \mathcal{S} into \mathcal{T} using at most $4n - \frac{n}{8\log_2 n}$ moves. Case 1.

\mathcal{S} can be transformed into \mathcal{T} by performing 3 moves for $\frac{|B|}{4}$ elements of B and 4 moves for the remaining rectangles. For each rectangle b of B its target either lies in the same stripe - we also assign the target of b to this stripe if it intersects one of the boundary lines of the stripe - or entirely above or entirely below the stripe. Let X denote the rectangles b of B whose target lies in the same stripe as b . Let Y denote the rectangles b of B whose target does not lie in the same stripe as b . One of X or Y contains at least half of the elements of B . We distinguish these two cases.

Case 1) $|Y| \geq \frac{|B|}{2}$.

Assume that at least half of the rectangles of Y have their target in a stripe below (the case that at least half of the rectangles of Y have their target in a stripe above being symmetric). Call this set of rectangles Z . In a first step, move all rectangles of \mathcal{S} far to the right, such that any two rectangles are

separated by a vertical line, and further move each rectangle not in Z upwards, such that all these rectangles lie high above the target. Every rectangle not in Z is moved upwards immediately after the move to the right. The rectangles of Z are not moved upwards. The move to the right for the rectangles of Z is done in such a way that each rectangle of Z can be moved downwards and to the left again without intersecting rectangles of Z . Also, the rectangles of Z lie to the right of all other rectangles after the first step. Figure 3 depicts the situation. The rectangles of Z are drawn in bold. In a second step, fill up the target from the bottom left to the top right. This is done by using an insertion order for the target rectangles similar to the proof of Theorem 1, and moving the source rectangles in this order first downwards and then to the left.

Case 2) $|X| > \frac{|B|}{2}$.

Figure 4 shows this case. Assume that for at least half of the rectangles of

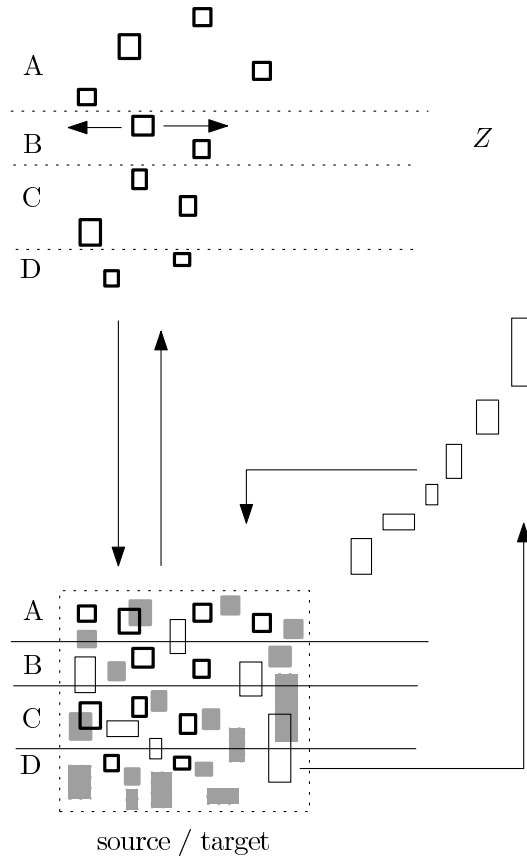


Fig. 4. Transforming \mathcal{S} into \mathcal{T} using at most $4n - \frac{n}{8 \log_2 n}$ moves. Case 2.

X the target does not intersect the upper bound of the stripe containing the source rectangle (the case of intersections with the lower bounds of the stripes

being symmetric). Let Z denote this set of rectangles. In a first step, move all rectangles of Z far upwards and all other rectangles first far to the right and then upwards. This can be done in such a way that each pair of the rectangles not in Z is separated by a vertical line and by a horizontal line after this first step. The first move for each rectangle of Z is far upwards, in such a way that Z lies far above the other rectangles, the rectangles stay within their stripe (in the order A-D in Figure 4), and within each stripe each pair of rectangles is separated by a horizontal line. Further, within each stripe, the rectangles are ordered (vertically) according to the insertion rule for the second step. More precisely, if a rectangle i will be dropped before a rectangle j ($i, j \in Z$) then i lies below j after the upward moves of Z . This guarantees that no two rectangles of a same stripe of Z collide. In a second step, fill up the target rectangles, according to an insertion order similar to Theorem 1. Each source rectangle is first translated horizontally and then moved downwards to its target. Note that there are no collisions when elements of Z are dropped to their target positions because the elements of Z are worked off stripe after stripe.

In both cases $|Z| \geq \frac{|B|}{4} \geq \frac{n}{8 \log_2 n}$. \square

3 Moving labeled rectangles of equal dimensions

Now we consider the case that all rectangles have the same dimensions. This case allows an improvement of Theorem 2. A similar statement to the following lemma can be found in [10].

Lemma 1. *Any set of n pairwise disjoint axis-parallel rectangles, all of them of equal dimensions, in the plane has a subset of at least $\frac{n}{2}$ separable rectangles.*

Proof. We can assume that each rectangle has base length 1 and that for no rectangle the x-coordinate of the left boundary is i , $i \in \mathbb{Z}$. Consider the set L of vertical lines at distance $2i$ from the origin, for $i \in \mathbb{N}$. Observe that each rectangle is hit by at most one line of L and any two rectangles which are entirely inside the stripe defined by two consecutive lines of L can be separated by a horizontal line. If at least $\frac{n}{2}$ rectangles are hit by lines of L , then these rectangles can be separated by a set of vertical lines at distance $2i - 1$ from the origin, for $i \in \mathbb{N}$, and within each of the resulting stripes the rectangles can be separated by horizontal segments. If at most $\frac{n}{2}$ rectangles are hit by lines of L , then the remaining rectangles can be separated by L and within each of the resulting stripes the rectangles can be separated by horizontal segments. \square

Now the reasoning of the proof of Theorem 3 implies

Theorem 4. *There always is a sequence of at most $\frac{31n}{8}$ moves to rearrange any set \mathcal{S} of n pairwise disjoint, labeled source rectangles of equal dimensions as any target set \mathcal{T} .*

4 Moving unlabeled rectangles

A lower bound of $2n$ moves to move n unlabeled source rectangles \mathcal{S} to their target destinations \mathcal{T} is obvious, if the centers of all rectangles of \mathcal{S} and \mathcal{T} have different coordinates. There exist examples of n rectangles that require at least $2n + 1$ moves, see Figure 5.

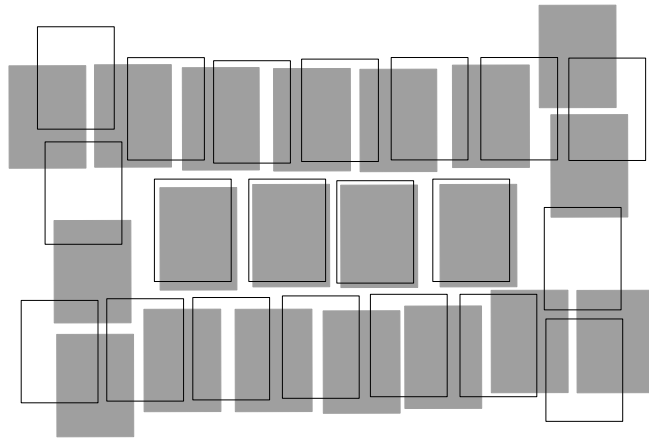


Fig. 5. At least $2n + 1$ moves are needed to reconfigure \mathcal{S} as \mathcal{T} .

Theorem 5. *If the n unlabeled rectangles of \mathcal{S} are separated from the n rectangles of \mathcal{T} by an axis-parallel line, then $2n$ moves are always sufficient and sometimes necessary to rearrange \mathcal{S} as \mathcal{T} .*

Proof. We can assume that \mathcal{S} and \mathcal{T} are separated by a vertical line and that \mathcal{S} lies to the left of that line. We define a removing order for \mathcal{S} and an insertion order for \mathcal{T} . Then we fill up the target destinations by the order of \mathcal{T} and we move rectangles from \mathcal{S} to \mathcal{T} by the order of \mathcal{S} . The order for \mathcal{S} is given by repeated application of the following rule: *Choose the rightmost rectangle among all rectangles p of \mathcal{S} for which the rectangle defined by the lower left corner of p and the point (∞, ∞) does not intersect other rectangles of the source list. Then delete this rectangle from the source list.*

The inverse order for \mathcal{T} is given by repeated application of the following rule: *Choose the leftmost rectangle among all rectangles p of \mathcal{T} for which the rectangle defined by the lower right corner of p and the point $(-\infty, \infty)$ does not intersect other rectangles of the target list. Then delete this rectangle from the target list.* \square

An example for the order of moves is shown in Figure 6. For each move, the two gray regions indicated in Figure 6 are empty, which allows to move the rectangle of \mathcal{S} to \mathcal{T} without collisions in two moves.

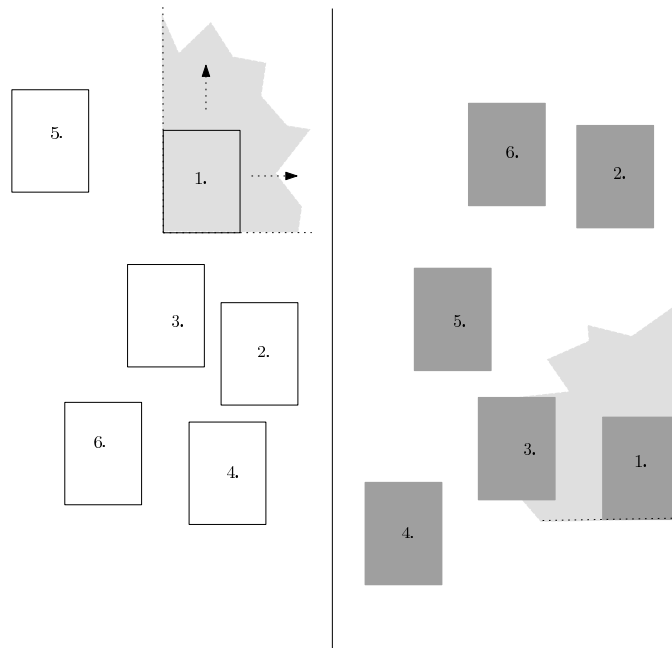


Fig. 6. $2n$ moves are sufficient to rearrange unlabeled rectangles if \mathcal{S} and \mathcal{T} are separated by an axis-parallel line.

Applying Theorem 5 twice, we obtain the following corollary:

Corollary 1. *Given two vertical (or horizontal) lines L_1 and L_2 , such that \mathcal{S} (respectively \mathcal{T}) lies inside the stripe defined by L_1 and L_2 , and \mathcal{T} (respectively \mathcal{S}) lies outside the stripe, then $2n$ moves are sufficient to rearrange the n rectangles of \mathcal{S} as \mathcal{T} .*

Theorem 5 immediately implies that $3n$ moves are always sufficient to rearrange \mathcal{S} as \mathcal{T} , because with a first sequence of n moves we can separate \mathcal{S} from \mathcal{T} by an axis-parallel line. In order to improve this bound we need the following lemmas.

Lemma 2. *If each pair of the n rectangles of \mathcal{T} , or of \mathcal{S} respectively, can be separated by a vertical line, then $2n$ moves are always sufficient and sometimes necessary to rearrange \mathcal{S} as \mathcal{T} .*

Clearly, the statement also holds for horizontal instead of vertical lines. We omit the proof of this lemma due to lack of space.

Lemma 3. *If there exists an axis-parallel line that stabs $c \cdot n$ rectangles of the n rectangles of \mathcal{S} , or of \mathcal{T} respectively, for $0 < c < 1$, then $3n - c \cdot n$ moves are sufficient to transform \mathcal{S} into \mathcal{T} .*

Proof. Assume that a vertical line L stabs $c \cdot n$ source rectangles of \mathcal{S} . Let n_r denote the number of source rectangles to the right of L , and let n_l denote the number of source rectangles to the left of L . In a first step move all n_r rectangles of \mathcal{S} on the right of L far to the right, such that a vertical line separates them from \mathcal{T} , and such that each pair of these n_r rectangles is separated by a vertical line. In the same way, move all n_l rectangles of \mathcal{S} on the left of L far to the left, such that a vertical line separates them from \mathcal{T} and such that each pair of these n_l rectangles is separated by a vertical line. This needs $n - c \cdot n$ moves. Now, assign the n_r rightmost rectangles of \mathcal{T} to the n_r source rectangles on the right. Assign the n_l leftmost rectangles of \mathcal{T} to the n_l source rectangles on the left. Assign the remaining $c \cdot n$ rectangles of \mathcal{T} to the $c \cdot n$ source rectangles that are stabbed by L . Any two of these $c \cdot n$ source rectangles are separated by a horizontal line. Using Lemma 2, these source rectangles can be moved to their assigned target rectangles using $2c \cdot n$ moves. Then, we can move the n_r source rectangles on the right to their assigned targets, using an insertion order for the target as in Theorem 5. Since each pair of these source rectangles is separated vertically, we move them to the target in the order given by x-coordinates. For each rectangle we first perform the vertical move and then the horizontal move. This sequence uses $2n_r$ moves. Analogously, $2n_l$ moves are used to move the source rectangles on the left to their assigned target rectangles. Altogether, $3n - c \cdot n$ moves are used. \square

Lemma 4. *If there does not exist an axis-parallel line that stabs $c \cdot n$ rectangles of \mathcal{S} , or of \mathcal{T} respectively, for $0 < c < \frac{1}{4}$, then $\frac{5n}{2} + 2c \cdot n$ moves are sufficient to transform \mathcal{S} into \mathcal{T} .*

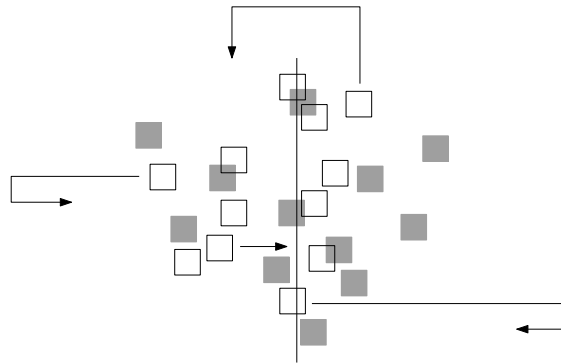


Fig. 7. At most $\frac{5n}{2} + 2c \cdot n$ moves are needed if no axis-parallel line stabs more than $c \cdot n$ rectangles of \mathcal{S} or \mathcal{T} .

Proof. Sweep a vertical line L from right to left until it leaves $\frac{n}{2}$ rectangles of \mathcal{S} or of \mathcal{T} in its open right half-plane. Assume, it leaves $\frac{n}{2}$ rectangles of \mathcal{T} on

its right. The other case follows by first finding a sequence of moves from \mathcal{T} to \mathcal{S} and then considering this sequence in reversed order. The line L crosses at most $c \cdot n$ rectangles of \mathcal{S} and at most $c \cdot n$ rectangles of \mathcal{T} . We denote these two sets with S_L and T_L , respectively. The source and target rectangles on the right (respectively left) of L are denoted with S_r and T_r (respectively S_l and T_l). We will use two moves for at least $\frac{n}{2} - |S_L| - |T_L|$ rectangles of S_l and three moves for the remaining rectangles. In a first step we will move all rectangles of S_r and S_L far upwards or far to the right; which of them are moved upwards will be specified below. Every rectangle of S_r can be moved upwards as well as to the right (at its turn). A rectangle of S_L can only be moved to the right if a rectangle of S_l blocks its upward move. In the worst case we will have to move all S_L rectangles to the right. These moves will be such that each pair of rectangles moved upwards is separated by a horizontal line and each pair of rectangles moved to the right is separated by a vertical line. Then we will move $|T_L|$ rectangles of S_l far to the left, such that again these rectangles are separated by vertical lines. There will be $|S_l| - |T_L|$ yet unmoved rectangles of S_l . We distinguish two cases depending on whether $|S_l| - |T_L| \geq \frac{n}{2} - |S_L|$ or not (here appears $|S_L|$ since this gives the worst case).

Case 1) $|S_l| - |T_L| \geq \frac{n}{2} - |S_L|$.

In the first step, all rectangles of S_r are moved upwards, the rectangles of S_L are moved to the right. Then, all but $\frac{n}{2} - |S_L|$ rectangles of S_l are moved to the left. This uses $|S_r| + |S_L| + |T_L| + (|S_l| - |T_L| - (\frac{n}{2} - |S_L|))$ moves. Now, we fill up the target rectangles of T_r with the yet unmoved rectangles of S_l and with the rectangles of S_L from the right, using Corollary 1. This needs $2|T_r| = n$ moves. It remains to fill up the target rectangles of T_l and T_L . We use rectangles of S_l from the left to fill up the rectangles of T_L and the remaining rectangles to fill up T_l . This can be done, because each target rectangle can be moved to the left, and all but $|T_L|$ rectangles can be moved upwards. (Rectangles of T_L might not be moved upwards because this could give a collision with a rectangle already placed at T_r .) Therefore we can assign the target rectangles accordingly to the source rectangles from S_l on the left and to the source rectangles from S_r from above. We fill up $T_l \cup T_L$ using the insertion order given in Theorem 5. Whenever we need to move a rectangle of S_l from the left to the target, we choose the rightmost available one and first translate it vertically, then horizontally. For moves from S_r above to T_l , we first do the horizontal move and then the vertical move. This guarantees that no collisions occur. It thus takes $2(T_L + T_l) = n$ final moves. Summing up, this gives $|S_r| + |S_L| + |T_L| + (|S_l| - |T_L| - (\frac{n}{2} - |S_L|)) + 2n = \frac{5n}{2} + |S_L| < \frac{5n}{2} + 2c \cdot n$ moves.

Case 2) $|S_l| - |T_L| < \frac{n}{2} - |S_L|$.

Here, in a first step, $\frac{n}{2} - |S_L| - (|S_l| - |T_L|)$ rectangles of S_r and all rectangles of S_L are moved to the right, the remaining rectangles of S_r are moved upwards.

Then, $|T_L|$ rectangles of S_l are moved to the left. This uses altogether $|S_r| + |S_L| + |T_L|$ moves. Now, we fill up T_r with the rectangles on the right and with the unmoved rectangles from S_l , using Corollary 1. This uses $2|T_r| = n$ moves. The remaining n moves to fill up T_l and T_L are as in Case 1. In total, $|S_r| + |S_L| + |T_L| + n + n \leq \frac{5n}{2} + |S_L| + |T_L| \leq \frac{5n}{2} + 2c \cdot n$ moves are used. \square

Theorem 6. *There always is a sequence of at most $\frac{17n}{6}$ moves to rearrange any set \mathcal{S} of n pairwise disjoint, unlabeled source rectangles as any target set \mathcal{T} .*

Proof. If there exists an axis-parallel line that stabs at least $\frac{n}{6}$ rectangles of \mathcal{S} , or of \mathcal{T} , then Lemma 3 gives the claimed bound. Otherwise, Lemma 4 can be applied with $c = \frac{1}{6}$. \square

References

- [1] M. Abellanas, S. Bereg, A. García, F. Hurtado, D. Rappaport, and J. Tejel. Moving coins. *Computational Geometry - Theory and Applications*, vol. 34: 35–48, 2006.
- [2] G. Aloupis, S. Collette, M. Damian, E. D. Demaine, R. Flatland, S. Langerman, J. O’Rourke, S. Ramaswami, V. Sacristán, and S. Wuhler. Linear reconfiguration of cube-style modular robots. *Computational Geometry - Theory and Applications*, vol. 42: 652–663, 2009.
- [3] S. Bereg and A. Dumitrescu. The lifting model for reconfiguration. *Discrete and Computational Geometry*, vol. 35: 653–669, 2006.
- [4] S. Bereg, A. Dumitrescu, and J. Pach. Sliding disks in the plane. *International Journal of Computational Geometry and Applications*, vol. 18: 373–387, 2008.
- [5] J. Czyzowicz, E. Rivera-Campo, J. Urrutia, and J. Zaks. Separating convex sets in the plane. *Discrete and Computational Geometry*, vol. 7: 189–195, 1992.
- [6] A. Dumitrescu and M. Jiang. On reconfiguration of disks in the plane and related problems. *11th Workshop on Algorithms and Data Structures (WADS)*, Lecture Notes in Computer Science, vol. 5664: 254–265, 2009.
- [7] A. Dumitrescu and J. Pach. Pushing squares around. *Graphs and Combinatorics*, vol. 22: 37–50, 2006.
- [8] L. J. Guibas and F. F. Yao. On translating a set of rectangles. *12th ACM Symposium on Theory of Computing (STOC)*, 154–160, 1980.
- [9] J. Pach and G. Tardos. Cutting glass. *Discrete and Computational Geometry*, vol. 24: 481–496, 2000.
- [10] J. Pach and G. Tardos. Separating convex sets by straight lines. *Discrete Mathematics*, vol. 241: 427–433, 2001.
- [11] H. Tverberg. A separation property of plane convex sets. *Mathematica Scandinavica*, vol. 45: 255–260, 1979.