Strong product of graphs: geodetic and hull numbers and boundary-type sets☆

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Abstract

In this work, we investigate the behavior of both the geodetic and the hull number with respect to the strong product operation for graphs and we also describe several boundary-type sets, such are periphery, boundary, eccentricity and contour in terms of its factors.

Key words: strong product, geodetic number, hull number, boundary-type sets

1. Introduction

Rebuilding the vertex set of a graph using a vertex subset and a convex operator is a problem that has attracted much attention since it was proved by Farber an Jamison [11] that every convex subset in a graph is the convex hull of its extreme vertices if an only if the graph is chordal and contains no induced 3-fan. Thus in a general graph, this rebuilding problem can be studied from different points of view [15]. On the one hand, geodetic and hull numbers give how many vertices are needed, at least, to rebuild the vertex set of a graph by using the closed interval and the convex hull operations respectively. However, those numbers do not give information about the sets which can be used to this end, so a different point of view consists in finding such sets, even if they are not minimum-sized, and different boundary-type sets can play this role, for example the periphery, the eccentric subgraph, the boundary or the contour.

Both of these sides of the rebuilding problem have been studied in different graph classes obtained by means of graph operations. For example in cartesian products [1, 5, 14], compositions [6] and joins [7] of graphs. In this work we develop these topics in strong products of graphs.

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We consider only finite, simple, connected graphs. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [16]. Given vertices u, v in a graph G we let $d_G(u,v)$ denote the distance between u and v in G. When there is no confusion, subscripts will be omitted. An $x-y$ path of length $d(x,y)$ is called an *$x-y$ geodesic*. The *closed interval* $I[x,y]$ consists of x, y and all vertices lying in some $x-y$ geodesic of G. For $S \subseteq V(G)$, the *geodetic closure* $I[S]$ of S is the union of all closed intervals $I[u,v]$ over all pairs $u, v \in S$, i.e. $I[S] = \bigcup_{u,v\in S} I[u,v]$. A set S of vertices is called *geodetic* if $I[S] = V(G)$ and it is said to be *convex* if $I[S] = S$. A set $A \subseteq V(G)$ is said to be a *hull set* if its *convex hull* $CH(A)$ is the whole vertex set $V(G)$, where $CH(A)$ denotes the smallest convex set containing A [10]. The *geodetic number* $g(G)$ and the *hull number* $h(G)$ are the minimum cardinality of a geodetic set and a hull set, respectively [10, 12]. Certainly, every geodetic set is a hull set, and hence, $h(G) \le g(G)$.

G	P_n	C_{2l}	C_{2l+1}	T_n	K_n	$K_{p,q}$ $(2 \le p \le q)$	$W_{1,p}$
$h(G)$	2	2	3	$\lvert Ext(T_n)\rvert$	n	2	$\lceil \frac{p}{2} \rceil$
$g(G)$	2	2	3	$\lvert Ext(T_n)\rvert$	n	$\min\{4, p\}$	$\lceil \frac{p}{2} \rceil$

Table 1: Hull and geodetic number of paths, cycles, trees, cliques, bicliques and wheels.

The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(a,v) : a \in V(G), b \in V(H)\}$, where (a,v) is adjacent to (b,w) whenever (1) $a = b$ and $vw \in E(H)$, or (2) $v = w$ and $ab \in E(G)$, or (3) $ab \in E(G)$ and $vw \in E(H)$.

Lemma 1. *[13]: Let G and H be two graphs and $(a,v),(b,w) \in V(G \boxtimes H)$. Then, $d_{G\boxtimes H}((a,v),(b,w)) = max\{d_G(a,b), d_H(v,w)\}$.*

From this result, it immediately follows that the diameter of $G \boxtimes H$ is the maximum of the diameters of its factors:

$$diam(G \boxtimes H) = \max\{diam(G), diam(H)\}$$

2. Geodetic and hull numbers for the strong product of graphs.

In this section, we study both behavior of both the geodetic and the hull number with respect to the strong product operation for graphs, in terms of its factors. More precisely, we obtain bounds for both parameters and we give some examples showing that some of the upper bounds are sharp.

Firstly, we relate closed intervals in the strong product of two graphs to closed intervals in factor graphs, which will be a key result to study both geodetic and hull numbers.

Lemma 2. *Let S_1 and S_2 be vertex subsets of G and H, respectively. Then, $I_G[S_1] \times I_H[S_2] \subseteq I_{G\boxtimes H}[S_1 \times S_2]$*

PROOF. Let $(g,h) \in I_G[S_1] \times I_H[S_2]$. Since $g \in I_G[S_1]$, then $g \in I_G[g',g'']$ for some $g',g'' \in V(S_1)$, and thus $d(g',g'') = d(g',g) + d(g,g'')$. Similarly, $d(h',h'') = d(h',h)+d(h,h'')$ for some $h',h'' \in V(S_2)$. From this point onwards, we will assume that $d(g',g) \leq d(g,g'')$ and $d(h',h) \leq d(h,h'')$. Also, we can suppose that $d(g',g) \leq d(h',h)$ without loss of generality.

Then $d((g',h'),(g,h)) = \max\{d(g',g),d(h',h)\} = d(h',h)$ and analogously $d((g,h),(g',h'')) = \max\{d(g,g'),d(h,h'')\} = d(h,h'')$ which means

$$d((g',h'),(g',h'')) = d(h',h'') = d(h',h) + d(h,h'') =$$
$$= d((g',h'),(g,h)) + d((g,h),(g',h'')) \implies$$
$$\implies (g,h) \in I_{G\boxtimes H}[(g',h'),(g',h'')] \subseteq I_{G\boxtimes H}[S_1 \times S_2]$$

□

As a direct consequence of this result, we can relate geodetic sets in the strong product $G \boxtimes H$ to the geodetic sets in its factors.

Proposition 1. *If $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$ are geodetic in G and H, respectively, then $S_1 \times S_2$ is geodetic in $G \boxtimes H$.*

PROOF. Clearly $I_G[S_1] = V(G)$ and $I_H[S_2] = V(H)$, thus $V(G) \times V(H) = I_G[S_1] \times I_H[S_2]$ which, by Lemma 2, is included in $I_{G\boxtimes H}[S_1 \times S_2]$. Hence, $I_{G\boxtimes H}[S_1 \times S_2] = V(G) \times V(H)$, i.e., $S_1 \times S_2$ is geodetic in $G \boxtimes H$. □

This property is far for being true true for minimum geodetic sets, as it is shown in the next example.

Example 1. It is easy to compute that $g(K_3 \boxtimes C_4) = 4$, however $g(K_3) = 3$ y $g(C_4) = 2$. So, the product of minimum geodetic sets is not minimum, in general.

Proposition 2. *If $S \subseteq V(G \boxtimes H)$ is a geodetic set of $G \boxtimes H$, then either the projection $p_G(S)$ of S onto G or the projection $p_H(S)$ of S onto H is geodetic.*

PROOF. Assume that neither $S_1 = p_G(S)$ nor $S_2 = p_H(S)$ is geodetic. We will show that this leads to a contradiction.

In these conditions, let $u_1 \notin V(G)\backslash I_G[S_1]$ and $u_2 \notin V(H)\backslash I_H[S_2]$. Then $(u_1,u_2) \in I(S) = V(G\boxtimes H)$, which implies $(u_1,u_2) \in I_{G\boxtimes H}[(a_1,a_2),(b_1,b_2)]$ for certain $(a_1,a_2),(b_1,b_2) \in S$ whose distance in $G \boxtimes H$ should be:

$$d((a_1,a_2),(b_1,b_2)) = d((a_1,a_2),(u_1,u_2)) + d((u_1,u_2),(b_1,b_2)) =$$
$$= \max\{d(a_1,u_1),d(a_2,u_2)\} + \max\{d(u_1,b_1),d(u_2,b_2)\}$$

On the other hand, $u_1 \notin I[a_1,b_1]$ and therefore $d(a_1,b_1) < d(a_1,u_1) + d(u_1,b_1)$. Similarly, $d(a_2,b_2) < d(a_2,u_2) + d(u_2,b_2)$. Then

$$\max\{d(a_1,b_1),d(a_2,b_2)\} < \max\{d(a_1,u_1)+d(u_1,b_1),d(a_2,u_2)+d(u_2,b_2)\} \leq$$
$$\leq \max\{d(a_1,u_1),d(a_2,u_2)\} + \max\{d(u_1,b_1),d(u_2,b_2)\}$$

which contradicts the previous expression for the distance between (a_1,a_2) and (b_1,b_2). □

As a direct consequence of these propositions, we obtain bounds for the geodetic number of the strong product of two graphs , in terms of the geodetic numbers of its factor graphs.

Theorem 1. *For any two graphs G and H,*

$$\min\{g(G), g(H)\} \leq g(G \boxtimes H) \leq g(G)g(H).$$

Furthermore, the upper bound is sharp.

Proof. First, we prove the upper bound. Let S_1 and S_2 be geodetic sets of G and H with minimum cardinality, that is, such that $|S_1| = g(G)$ and $|S_2| = g(H)$. By Proposition 1, $S_1 \times S_2$ is a geodetic set of $G \boxtimes H$ with cardinality $|S_1 \times S_2| = |S_1||S_2| = g(S_1)g(S_2)$, hence $g(G \boxtimes H) \leq g(G)g(H)$.

To prove the lower bound, take a minimum geodetic set S of $G \boxtimes H$. According to Proposition 1, we may suppose, without loss of generality, that $p_G(S)$ is a geodetic set of G. Hence: $\min\{g(G), g(H)\} \leq g(G) \leq |p_G(S)| \leq |S| = g(G \boxtimes H)$.

Finally, to show the sharpness of the upper bound, take $G = K_m$ and $H = K_n$. Then, $g(K_m \boxtimes K_n) = g(K_{mn}) = mn = g(K_m)g(K_n)$ □

The following example shows that the geodetic number of the strong product may be strictly between both bounds.

Proposition 3. *For $m, n \geq 4$,*

$$g(K_m \boxtimes C_n) = \begin{cases} 4 & \textit{if } n \textit{ is even} \\ 5 & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. Let $V(K_m) = \{u_1, \ldots, u_m\}$ and $V(C_n) = \{v_0, \ldots, v_{n-1}\}$ and assume first that n is even. Our claim is that the following set is geodetic:

$$S = \{(u_1, v_0), (u_1, v_1), (u_1, v_{n/2}), (u_1, v_{1+n/2})\}$$

To prove it, take a vertex $(u_i, v_j) \in V(K_m \boxtimes C_n)$. If $j \neq 0$ and $j \neq \frac{n}{2}$ then it is contained in a $(u_1, v_0) - (u_1, v_{n/2})$ geodesic, and if $j = 0$ or $j = \frac{n}{2}$, the vertex lies in a $(u_1, v_1) - (u_1, v_{1+n/2})$ geodesic.

This proves that $g(K_m \boxtimes C_n) \leq 4$. However, it there exists a geodetic set with three or fewer vertices it will imply that is its projection onto the cycle is geodetic which is impossible. Hence, the above set S is geodetic and minimum.

Suppose now that n is odd, then we will prove that S is geodetic where

$$S = \{(u_1, v_0), (u_1, v_1), (u_1, v_{(n-1)/2}), (u_1, v_{(n+1)/2}), (u_1, v_{(n+3)/2})\}$$

For any $(u_i, v_j) \in V(G \boxtimes H)$, it will contained in a $(u_1, v_0) - (u_1, v_{(n-1)/2})$ geodesic if $0 < j < \frac{n-1}{2}$, or in a $(u_1, v_{(n+1)/2}) - (u_1, v_0)$ geodesic if $\frac{n+1}{2} < j < n$. Finally, all the vertices with $j = 0$ lie in the $(u_1, v_{(n+3)/2}) - (u_1, v_1)$ geodesics, and the vertices with $j = \frac{n-1}{2}$ are in the $(u_1, v_1) - (u_1, v_{(n+1)/2})$ geodesics.

By a similar reasoning as above, it is straightforward to check that no set with four or fewer vertices can be geodetic. □

Now, we apply similar techniques to obtain an upper bound for the hull number of a strong product. Firstly, we need the following Lemma, that describes the relationship between closed intervals in the strong product and closed intervals in its factors.

Lemma 3. *For any vertex subsets $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$ of two graphs G and H, and for all $r \in \mathbb{N}$, we have $I_G^r[S_1] \times I_H^r[S_2] \subseteq I_{G\boxtimes H}^r[S_1 \times S_2]$.*

PROOF. The proof is done by induction in r. For the case $r = 1$, we refer the reader to the Lemma 2. Assume it holds true for any $k < r$, then we prove it for r. By Lemma 2, $I_G[I_G^{r-1}[S_1]] \times I_H[I_H^{r-1}[S_2]] \subseteq I_{G\boxtimes H}[I_G^{r-1}[S_1] \times I_H^{r-1}[S_2]]$, and then by hypothesis of induction, $I_{G\boxtimes H}[I_G^{r-1}[S_1] \times I_H^{r-1}[S_2]] \subseteq I_{G\boxtimes H}[I_{G\boxtimes H}^{r-1}[S_1 \times S_2]] = I_{G\boxtimes H}^r[S_1 \times S_2]$.

Hence $I_G^r[S_1] \times I_H^r[S_2] \subseteq I_{G\boxtimes H}^r[S_1 \times S_2]$. □

As in the case of the geodetic number, this result gives a similar upper bound for the hull number.

Proposition 4. *For any two graphs G and H, $h(G \boxtimes H) \leq h(G)h(H)$. Furthemore, this bound is sharp.*

PROOF. Let $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$ be minimum hull sets, i.e., such that there exist $r, k \in \mathbb{N}$ satisfying $I_G^r[S_1] = V(G)$ and $I_H^k[S_2] = V(H)$. Assume $r \leq k$, so $I_G^k[S_1] = V(G)$.

By Lemma 3, we have that $V(G) \times V(H) = I_G^k[S_1] \times I_H^k[S_2] \subseteq I_{G\boxtimes H}^k[S_1 \times S_2]$, and therefore $I_{G\boxtimes H}^k[S_1 \times S_2] = V(G) \times V(H)$, i.e., $S_1 \times S_2$ is a hull set. Finally, $h(G \boxtimes H) \leq |S_1 \times S_2| = h(G)h(H)$.

To prove the sharpness of the upper bound, take $G = K_m$ and $H = K_n$. Then, $h(K_m \boxtimes K_n) = h(K_{mn}) = mn = h(K_m)h(K_n)$ □

An interesting example takes place when both graphs, G and H, are extreme geodesic graphs (see [9]). A vertex v of a graph G is an *extreme vertex* if the subgraph induced by its neighborhood $N(v)$ is a clique. It is easily seen that every hull set (and hence every geodesic set) must contain the set $Ext(G)$ of extreme vertices of G. A graph G is called *extreme geodesic* if the set of its extreme vertices is geodesic. Note that, in this case, (1) the set $Ext(G)$ is the unique minimum geodetic set (and also the unique minimum hull set) and (2) $h(G) = g(G) = |Ext(G)|$. Trees and complete graphs are basic examples of extreme geodesic graphs. On the other hand, observe that a vertex (u, v) is an extreme vertex of $G \boxtimes H$ if and only if both u and v are extreme vertices of G and H, respectively, i.e., $Ext(G) \times Ext(H) = Ext(G \boxtimes H)$. As a direct consequence of this equality and Proposition 1, we have that two graphs G and H are extreme geodesic if and only if $G \boxtimes H$ is extreme geodesic, which means that $h(G \boxtimes H) = g(G \boxtimes H) = |Ext(G \boxtimes H)| = |Ext(G) \times Ext(H)| = |Ext(G)| \cdot |Ext(H)| = g(G)g(H) = h(G)h(H)$. As a direct consequence of these facts, the results showed in Table 2 are obtained.

G/H	P_n	T_n^k	K_n
P_m	4	$2k$	$2n$
T_m^h	$2h$	hk	hn
K_m	$2m$	mk	mn

Table 2: Hull (and geodetic number) of some strong products. P_m denotes the path of order m, T_m^h an arbitrary tree with n vertices and h leaves and K_m the clique of order m.

3. Boundary-type sets of the strong product of graphs.

We devote this section to describe a number of boundary-type sets for the strong product of two graphs, in terms of its factors. These sets have been used in rebuilding operations (see [2, 3, 4]).

Let $G = (V, E)$ be a connected graph and $u, v \in V$. The vertex v is said to be a *boundary vertex* of u if no neighbor of v is further away from u than v [8]. By $\partial(u)$ we denote the set of all boundary vertices of u. A vertex v is called a *boundary vertex of* G if $v \in \partial(u)$ for some vertex $u \in V(G)$. The *boundary* $\partial(G)$ of G is the subgraph induced by all boundary vertices of G.

Given $u, v \in V$, the vertex v is called an *eccentric vertex of* u if no vertex in V is further away from u than v, that is, if $d(u, v) = ecc(u) = \max\{d(u, v) \mid v \in V\}$. A vertex v is called an *eccentric vertex* of G if it is the eccentric vertex of some vertex $u \in V$. The *eccentric subgraph* $\mathrm{Ecc}\,(G)$ of G is the subgraph induced by all eccentric vertices of G [8].

A vertex $v \in V$ is called a *contour vertex* of G if no neighbor of v has eccentricity greater than $ecc(v)$ [4]. The contour $\mathrm{Ct}\,(G)$ of G is the subgraph induced by all contour vertices of G.

A vertex $v \in V$ is called a *peripheral vertex* of G if no vertex in V has eccentricity greater than $ecc(v)$, that is, if the eccentricity of v is exactly equal to the diameter of G. The *periphery* $\mathrm{Per}\,(G)$ of G is the subgraph induced by all peripheral vertices of G.

For simplicity, we also use $\partial(G)$, $\mathrm{Ecc}\,(G)$, $\mathrm{Ct}\,(G)$ and $Per(G)$ to denote the respective collections of vertices. Notice that every extreme vertex is a contour vertex, i.e., $Ext(G) \subseteq \mathrm{Ct}\,(G)$. It is also clear that: $\mathrm{Per}\,(G) \subseteq \mathrm{Ct}\,(G) \cap \mathrm{Ecc}\,(G)$ and $\mathrm{Ecc}\,(G) \cup \mathrm{Ct}\,(G) \subseteq \partial(G)$.

Theorem 2. *Let G and H be two graphs with diameters D_G and D_H and radii r_G and r_H, respectively, such that $D_G \leq D_H$ and $r_G \leq r_H$. Then,*

1. $\partial(G \boxtimes H) = \big[\partial(G) \times V(H)\big] \bigcup \big[V(G) \times \partial(H)\big]$
2. (a) *If $D_G < D_H$, $Per\,(G \boxtimes H) = V(G) \times Per\,(H)$*
   (b) *If $D_G = D_H$, $Per\,(G \boxtimes H) = \big[Per\,(G) \times V(H)\big] \bigcup \big(\big[V(G) \times Per\,(H)\big)\big]$*
3. $Ecc\,(G \boxtimes H) = \Big[Ecc_{r_H}(G) \times V(H)\Big] \bigcup \Big[V(G) \times Ecc\,(H)\Big]$, *where* $Ecc_{r_H}(G) = \{g \in V(G) \colon \exists g' \in V(G) \text{ such that } r_H \leq ecc(g') = d(g', g)\}$

4. $Ct(G \boxtimes H) = \{(g,h) \in V(G \boxtimes H) \colon g \in Ct(G), ecc(h) < ecc(g)\} \bigcup$
$\{(g,h) \in V(G \boxtimes H) \colon h \in Ct(H), ecc(g) < ecc(h)\} \bigcup$
$\big(Ct(G) \times Ct(H)\big)$

PROOF. 1. Let $(g,h) \in V(G \boxtimes H)$ such that $g \notin \partial(G)$ and $h \notin \partial(H)$. Then, for every $g' \in V(G)$, there exists $g'' \in N(g)$ such that $d(g',g'') > d(g',g)$ and similarly, for every $h' \in V(H)$, there exists $h'' \in N(h)$ such that $d(h',h'') > d(h',h)$. Hence, for every $(g',h') \in V(G \boxtimes H)$ there exists $(g'',h'') \in N((g,h))$ s. t.: $d((g',h'),(g'',h'')) = \max\{d(g',g''), d(h',h'')\} > \max\{d(g',g), d(h',h)\} = d((g',h'),(g,h))$, i.e., $(g,h) \notin \partial(G \boxtimes H)$.

For the reverse inclusion, take $g \in \partial(G)$ and $h \in V(H)$. Hence, there exists $g' \in V(G)$ such that, for every $g'' \in N(g)$, $d(g',g) \geq d(g',g'')$. Let us see that (g,h) is a boundary vertex of (g',h): if $(g'',h'') \in N((g,h))$, then $d((g',h),(g'',h'')) = \max\{d(g',g''), d(h,h'')\} = \max\{d(g',g''), 1\}$, so $d((g',h)),(g,h)) = d(g',g) \geq \max\{d(g',g''),1\} = d((g',h),(g'',h''))$, as desired.

2. (a) Suppose that $D_G < D_H$. Then:
$(g,h) \in \mathrm{Per}\,(G \boxtimes H) \Leftrightarrow ecc(g,h) = D_{G\boxtimes H} = \max\{D_G, D_H\} = D_H \Leftrightarrow ecc(g,h) = \max\{ecc(g), ecc(h)\} = ecc(h) = D_H \Leftrightarrow h \in \mathrm{Per}\,(H)$.

   (b) Assume now that $D_G = D_H$. Then:
$(g,h) \in \mathrm{Per}\,(G \boxtimes H) \Leftrightarrow ecc(g,h) = D_{G\boxtimes H} = \max\{D_G, D_H\} = D_G = D_H \Leftrightarrow ecc(g,h) = \max\{ecc(g), ecc(h)\} = D_G = D_H \Leftrightarrow g \in \mathrm{Per}\,(G)$ or $h \in \mathrm{Per}\,(H)$.

3. Take $(g,h) \in \mathrm{Ecc}\,(G \boxtimes H)$. Then, there exists $(g',h') \in V(G \boxtimes H)$ such that $ecc(g',h') = d((g',h'),(g,h))$. Suppose that $h \notin \mathrm{Ecc}\,(H)$, i.e., $d(h',h) < ecc(h')$, for every $h' \in V(H)$. Let us see that $r_H \leq ecc(g') = d(g',g)$. If $ecc(g') < ecc(h')$, then $ecc(h') = \max\{ecc(g'), ecc(h')\} = ecc(g',h') = d((g',h'),(g,h)) = \max\{d(g',g), d(h',h)\} \geq d(h',h)$, which is not possible. Thus, $r_H \leq ecc(h') \leq ecc(g') = \max\{ecc(g'), ecc(h')\} = ecc(g',h') = d((g',h'),(g,h)) = \max\{d(g',g), d(h',h)\} = d(g',g)$ as desired.

For the reverse inclusion, take first $h \in V(H)$ and $g \in Ecc_{r_H}(G)$. Hence, there exists $g' \in V(G)$ such that $r_H \leq ecc(g') = d(g',g)$. Take $h' \in V(H)$ such that $ecc(h') = r_H$. Then, $d((g',h'),(g,h)) = \max\{d(g',g), d(h',h)\} = d(g',g) = ecc(g') = \max\{ecc(g'), ecc(h')\} = ecc(g',h')$, i.e., $(g,h) \in \mathrm{Ecc}\,(G \boxtimes H)$. Take now $g \in V(G)$ and $h \in \mathrm{Ecc}\,(H)$. Hence, there exists $h' \in V(H)$ such that $ecc(h') = d(h',h)$. Take $g' \in V(G)$ such that $ecc(g') = r_G$. Then, $d((g',h'),(g,h)) = \max\{d(g',g), d(h',h)\} = d(h',h) = ecc(h') = \max\{ecc(g'), ecc(h')\} = ecc(g',h')$, as desired.

4. Take $(g,h) \in \mathrm{Ct}\,(G \boxtimes H)$. Suppose that $g \notin \mathrm{Ct}\,(G)$, i.e., there exists $g' \in N(g)$ such that $ecc(g) < ecc(g')$. Let us see that $h \in \mathrm{Ct}\,(H)$ and $ecc(g) < ecc(h)$. If $h' \in N(h)$, then $(g',h') \in N(g,h)$ and thus $\max\{ecc(g'), ecc(h')\} = ecc(g',h') \leq ecc(g,h) = \max\{ecc(g), ecc(h)\}$, which implies that $ecc(h) = \max\{ecc(g), ecc(h)\} \geq \max\{ecc(g'), ecc(h')\}$, as desired.

For the reverse inclusion, take first $g \in \mathrm{Ct}\,(G)$, $h \in V(H)$ such that $ecc(h) < ecc(g)$ and $(g', h') \in N(g, h)$. If $ecc(g') \geq ecc(h')$, then $ecc(g', h') = \max\{ecc(g'), ecc(h')\} = ecc(g') \leq ecc(g)$ and otherwise, $ecc(g', h') = \max\{ecc(g'), ecc(h')\} = ecc(h') \leq ecc(h)+1 \leq ecc(g)$. Hence, $ecc(g', h') \leq ecc(g) = \max\{ecc(g), ecc(h)\} = ecc(g, h)$, as desired. The case $h \in \mathrm{Ct}\,(H)$ and $g \in V(G)$ with $ecc(g) < ecc(h)$ is similarly proved. Finally, take $g \in \mathrm{Ct}\,(G)$, $h \in \mathrm{Ct}\,(H)$ and $(g', h') \in N(g, h)$. Then, $ecc(g', h') = \max\{ecc(g'), ecc(h')\} \leq \max\{ecc(g), ecc(h)\} = ecc(g, h)$, as desired. □

It is known that the contour is a hull set in any graph [4], and it is also a geodetic set in certain graph families, such are chordal and distance-hereditary graphs [3, 4]. However it is open the characterization of graphs whose contour is a geodetic set. So it is interesting to obtain conditions for $Ct(G \boxtimes H)$ being geodetic.

Note that by Lemma 2, if both $Ct(G)$ and $Ct(H)$ are geodetic sets, then $Ct(G \boxtimes H)$ is also geodetic. As a particular case, if $r_G > D_H$, a geodetic $Ct(G)$ implies that $Ct(G \boxtimes H)$ is geodetic.

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