Boundary-type sets and product operators in graphs ⋆

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Abstract. In this paper we present the description of some boundary-type sets, such are the extreme vertex set, the boundary, the eccentricity, the periphery an the contour of strong and lexicographic products of graphs, in terms of its factors. Similar descriptions can be found in the literature for the cartesian product of graphs.

Key words: Boundary-type sets, strong product of graphs, lexicographic product of graphs.

1 Introduction.

The problem of rebuilding the vertex set of a graph using convex-type operations has attracted much attention since it was proved by Farber and Jamison (see [6]) that every convex subset in a graph is the convex hull of its extreme vertices if and only if the graph is chordal and contains no induced 3-fan. In other words, *extreme vertices*, that is, vertices whose neighborhood induces a complete graph, can be used to rebuild the vertex set of a graph just in a particular graph class. So the natural question about which vertex set could play a similar role in general graphs arises, and a number of boundary-type sets can be used to answer it. Now we remember definitions of such sets.

Recall that the *eccentricity* of a vertex g of a graph G is

$$ecc(g) = \max\{d(g, g') \colon g' \in V(G)\}.$$

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Also a vertex g' is an *eccentric* vertex of g, if $ecc(g) = d(g, g')$. Now we recall the definitions of some boundary-type sets in a graph. As usual, we denote $N(g)$ the set of all neighbors of vertex g.

The *boundary* (see [4]) $\partial(G)$ of a graph G is

$$\partial(G) = \{g \in V(G)\colon \exists g' \in V(G) \text{ such that} \forall g'' \in N(g)\colon d(g', g'') \leq d(g', g)\}$$

The *periphery* $\mathrm{Per}\,(G)$ of G is

$$\mathrm{Per}\,(G) = \{g \in V(G)\colon ecc(g') \leq ecc(g),\ \forall g' \in V(G)\}$$

The *eccentricity* (see [5]) $\mathrm{Ecc}\,(G)$ of G is

$$\mathrm{Ecc}\,(G) = \{g \in V(G)\colon \exists g' \in V(G) \text{ such that } ecc(g') = d(g', g)\}$$

And finally, *the contour* (see [3]) $\mathrm{Ct}\,(G)$ of G is

$$\mathrm{Ct}\,(G) = \{g \in V(G)\colon ecc(g') \leq ecc(g),\ \forall g' \in N(g)\}$$

Figure 1 shows a general view of the inclusion relations among these sets, where $\mathrm{Ext}(G)$ is the set of extreme vertices in G.

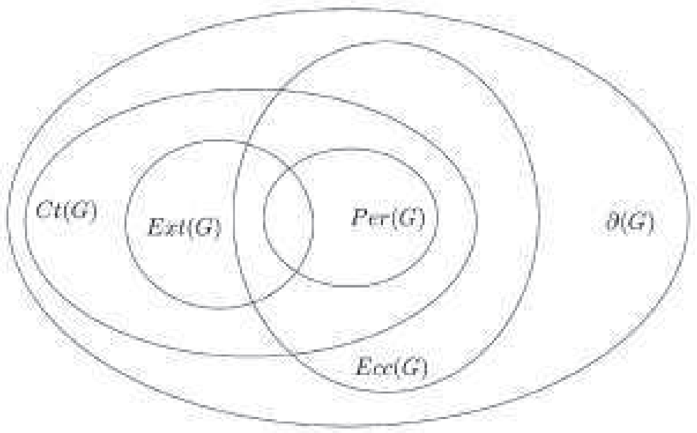

Fig. 1. Inclusions between boundary-type sets.

Remember that the *interval* $I[u, v]$ between two vertices u, v of a graph G is the set of all vertices lying in any shortest path between u and v. Also if $S \subset V(G)$, the interval of S is the set $I[S] = \bigcup_{u,v \in S} I[u, v]$. So S is called a *geodetic set* when $I[S] = V(G)$. On the other hand, a subset S of vertices of G is called *convex* if $I[u, v] \subseteq S$, for any pair of vertices $u, v \in S$, and the *convex hull* $CH(A)$ of any subset of vertices A is the smallest convex set containing it. Finally a subset of vertices A is called a *hull set* if $\mathrm{CH}(A) = V(G)$

The role of the boundary-type sets in rebuilding operations using both intervals and convex hulls, has been studied in [2,3], showing that $\mathrm{Ct}\,(G)$ is a hull set in any graph and it is also a geodetic set in special graph-classes, such are chordal graphs and distance-hereditary graphs. Also $\mathrm{Ct}\,(G) \bigcup \mathrm{Ecc}\,(\mathrm{Ct}\,(G))$ and $\partial(G)$ are geodetic sets in any graph.

Note that the periphery and the eccentricity are natural boundary-type sets defined using the eccentricity of vertices in a graph, however they do not play an remarkable role in term of rebuilding using convex operations in general graphs. The boundary and the contour arise to this end and, although their definitions are less natural, they have proved useful for this problem.

The following result shows the description of these boundary-type sets in cartesian products of graphs.

Theorem 1. *[1] For any graphs G and H:*

- $\partial(G \Box H) = \partial(G) \times \partial(H)$,
- $Ct\,(G \Box H) = Ct\,(G) \times Ct\,(H)$,
- $Ecc\,(G \Box H) = Ecc\,(G) \times Ecc\,(H)$,
- $Per\,(G \Box H) = Per\,(G) \times Per\,(H)$.

In this work we present similar results in strong products and lexicographic products of graphs.

The *strong product* $G \boxtimes H$ (see [8]) of G and H is the graph having $V(G) \times V(H)$ as vertex set and two vertices (g, h) and (g', h') are adjacent if:

- $gg' \in E(G), h = h'$, or
- $g = g', hh' \in E(H)$, or
- $gg' \in E(G), hh' \in E(H)$

An easy consequence of the definition is that there exists a general relation between distance in the strong product and distances in its factor graphs:

$$d\big((g, h), (g', h')\big) = \max\{d(g, g'), d(h, h')\}$$

From this relation, it is derived that the diameter of a strong product is the maximum between both diameters of factor graphs.

The relationship between eccentricity of a vertex of a strong product graph and eccentricities of its factors is the following:

$$ecc(g, h) = \max\{ecc(g), ecc(h)\}$$

The *lexicographic product* $G \circ H$ (see [7]) of G and H is the graph having $V(G) \times V(H)$ as vertex set and two vertices (g, h) and (g', h') are adjacent if:

- $gg' \in E(G)$, or
- $g = g', hh' \in E(H)$.

The relation between distances in a lexicographic product graph and in its factors is the following:

$$d((g,h),(g',h')) = \begin{cases} d(g,g') & \text{if } g \neq g' \\ 1 & \text{if } g = g' \text{ and } h, h' \text{ are neighbors} \\ 2 & \text{if } g = g' \text{ and } h, h' \text{ are not neighbors} \end{cases}$$

It is also easy to obtain the eccentricity of vertices in $V(G \circ H)$:

$$ecc(g,h) = \begin{cases} ecc(g) & \text{if } ecc(g) \geq 2 \\ \min\{ecc(x), 2\} & \text{if } ecc(g) = 1 \end{cases}$$

So the diameter of lexicographic product graph $K_n \circ H$ $(H \neq K_m)$ is equal to 2 and the diameter of $G \circ H$ $(G \neq K_n)$ is equal to the diameter of G.

Additional information about both graph product operators can be found in [7].

2 Boundary-type sets in products of graphs.

In this section we present the description of boundary-type sets in both types of product graphs: strong and lexicographic. Firstly, the following result shows that all these sets can be described in terms of factor graphs in the strong product of two graphs and depend on the relation between diameters and radii of both graphs.

Theorem 2. *Let G and H be two graphs with diameters D_G and D_H and radii r_G and r_H respectively.*

1. *The extreme vertices of the strong product graph is the following vertex set,*
$$Ext(G \boxtimes H) = Ext(G) \times Ext(H)$$
2. *The boundary has the following description,*
$$\partial(G \boxtimes H) = \big(\partial(G) \times V(H)\big) \bigcup \big(V(G) \times \partial(H)\big)$$
3. *The periphery depends on the relation between both diameters, as follows:*
   a) if $D_G < D_H$,
$$Per(G \boxtimes H) = V(G) \times Per(H)$$
   b) if $D_G = D_H$,
$$Per(G \boxtimes H) = \big(Per(G) \times V(H)\big) \bigcup \big(V(G) \times Per(H)\big)$$
4. *Similarly, the eccentricity depends on the relation between both radii:*
   a) if $r_G \leq r_H$,
$$Ecc(G \boxtimes H) = \Big[\bigcup_{ecc(g) \geq r_H} Ecc(g) \times V(H)\Big] \bigcup \Big[V(G) \times Ecc(H)\Big]$$

b) if $r_G = r_H$, the equality becomes to

$$Ecc\,(G \boxtimes H) = \Big[Ecc\,(G) \times V(H)\Big] \bigcup \Big[V(G) \times Ecc\,(H)\Big]$$

5. *Finally the contour can be described as follows:*

$$\begin{aligned} Ct\,(G \boxtimes H) = {} & \{(x,y) \in G \boxtimes H \colon x \in Ct\,(G), ecc(y) < ecc(x)\} \bigcup \\ & \{(x,y) \in G \boxtimes H \colon y \in Ct\,(H), ecc(x) < ecc(y)\} \bigcup \\ & \big(Ct\,(G) \times Ct\,(H)\big) \end{aligned}$$

Note that this union is not disjoint and that, if $r_G > D_H$, equality becomes $Ct\,(G \boxtimes H) = Ct\,(G) \times V(H)$.

Now we present a similar result for the boundary-type sets in lexicographic products of graphs. In this case, all these sets can be described in terms of factor graphs and there are different cases depending of radii of such graphs. We denote by $Z(G)$ the set of vertices with minimum eccentricity.

Theorem 3. *Let G and H be two graphs with diameters D_G and D_H and radii r_G and r_H respectively.*

1. *The extreme vertices of the lexicographic product graph is the following vertex set,*

$$Ext\,(G \circ H) = Ext\,(G) \times V(H)$$

2. *The boundary has different descriptions depending on radius of H and cardinality of $Z(H)$,*
   a) if $r_H = 1$ and $|Z(H)| \geq 2$ or $r_H \geq 2$,

$$\partial(G \circ H) = V(G) \times V(H)$$

   b) if $r_H = 1$ and $|Z(H)| = 1$,

$$\partial(G \circ H) = \Big[V(G) \times \Big(V(H) \setminus Z(H)\Big)\Big] \bigcup \Big[\partial(G) \times Z(H)\Big]$$

3. *The periphery presents a particular case if G is the complete graph, and also depends on diameter if G and radius of H.*
   a) if $G \neq K_n$ and $D_G \neq 2$,

$$Per\,(G \circ H) = Per\,(G) \times V(H)$$

   b) if $G \neq K_n$, $D_G = 2$ and $r_H \geq 2$,

$$Per(G \circ H) = V(G) \times V(H)$$

c) if $G \neq K_n$, $D_G = 2$ and $r_H = 1$,

$$Per(G \circ H) = \Big(Per(G) \times V(H)\Big) \bigcup \Big(V(G) \times \big(V(H) \setminus Z(H)\big)\Big)$$

d) if $G = K_n$ and $r_H = 1$,

$$Per(G \circ H) = V(G) \times \Big(V(H) \setminus Z(H)\Big)$$

e) if $G = K_n$ and $r_H \geq 2$,

$$Per(G \circ H) = V(G) \times V(H)$$

4. The eccentricity depends only on the relation between both radii,
a) if $r_G \geq 3$,

$$Ecc(G \circ H) = Ecc(G) \times V(H)$$

b) if $r_G = 2$ and $r_H \geq 2$,

$$Ecc(G \circ H) = \Big(Ecc(G) \cup Z(G)\Big) \times V(H)$$

c) if $r_G = 2$ and $r_H = 1$, or $r_G = 1$ and $r_H \geq 2$,

$$Ecc(G \circ H) = \Big(Ecc(G) \times V(H)\Big) \bigcup \Big(Z(G) \times \big(V(H) \setminus Z(H)\big)\Big)$$

d) if $r_G = r_H = 1$,

$$Ecc(G \circ H) = \Big(Ecc(G) \times V(H)\Big) \bigcup \Big(Z(G) \times Z(H)\Big)$$

5. Finally, the contour also depends on radii of G and H,
a) if $r_G \geq 2$,

$$Ct(G \circ H) = Ct(G) \times V(H)$$

b) if $r_G = 1$ and $r_H \geq 2$,

$$Ct(G \circ H) = \Big(\big(Ct(G) \setminus Z(G)\big) \times V(H)\Big) \bigcup \Big(Z(G) \times V(H)\Big)$$

c) if $r_G = r_H = 1$,

$$\begin{aligned} Ct(G \circ H) = &\Big(\big(Ct(G) \setminus Z(G)\big) \times V(H)\Big) \cup \\ &\Big(\big(Ct(G) \cap Z(G)\big) \times \big(Ct(H) \cap Z(H)\big)\Big) \cup \\ &\Big(Z(G) \times \big(V(H) \setminus Z(H)\big)\Big) \end{aligned}$$

The sum of graphs is also an usual operation in graphs, although it is not product-type, and it could be a natural question if boundary-type sets can be described in the way than above. Remember that the sum $G + H$ of two graphs has $V(G) \cup V(H)$ as vertex set and two vertices x, y are neighbors in the sum graph if both are neighbors in G or if they are neighbors in H or if $x \in V(G)$ and $y \in V(H)$. It is easy to prove the following equalities about boundary-type sets in the sum of two graphs.

Theorem 4. *Let G and H be two graphs with radii r_G and r_H respectively.*

1. If $r_G \geq 2$ and $r_H \geq 2$ then:

$$\partial(G+H) = Ecc(G+H) = Per(G+H) = Ct(G+H) = V(G+H)$$

2. If $r_G \geq 2$ and $r_H = 1$, there are two cases:

a) if $|Z(H)| = 1$, then

$$\partial(G+H) = Ecc(G+H) = Per(G+H) = Ct(G+H) = V(G+H) \setminus Z(H)$$

b) if $|Z(H)| \geq 2$, then

$$\partial(G+H) = Ecc(G+H) = V(G+H)$$

$$Per(G+H) = Ct(G+H) = V(G+H) \setminus Z(H)$$

3. If $r_G = 1$, $G \neq K_n$ and $r_H = 1$ then:

$$\partial(G+H) = Ecc(G+H) = V(G+H)$$

$$Per(G+H) = Ct(G+H) = V(G+H) \setminus \big(Z(G) \cup Z(H)\big)$$

4. If both G and H are complete graphs, then $G + H$ is also complete and

$$\partial(G+H) = Ecc(G+H) = Per(G+H) = Ct(G+H) = V(G+H)$$

Note that $Ext(G + H) = \emptyset$ is both G and H are different from complete graphs. If both are complete graphs, then $Ext(G+H) = V(G+H)$ and if just G is complete, then $Ext(G + H) = Ext(H)$

This result shows that boundary-type sets of the sum of two graphs are in general too big to play a non trivial role in the rebuilding problem.

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