Distance from a Controllable Switched Linear System to an Uncontrollable One

JOSEP CLOTET[1], JOSEP FERRER[1], M. DOLORS MAGRET[1]

[1] *Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, E-08028 Barcelona.*
E-mails: `josep.clotet@upc.edu`, `josep.ferrer@upc.edu`, `m.dolors.magret@upc.edu`.

Abstract

The set of controllable switched linear systems is an open set in the space of all switched linear systems. Then it makes sense to compute the distance from a controllable switched linear system to the set of uncontrollable systems. In this work we obtain an upper bound for such distance.

Keywords: Switched linear system, controllability.

1 Introduction

In different works bounds for the distance between a system with a qualitative property to the set of systems with different qualitative properties are found (see [1], [2], [3], [4], [5]). In [6] a necessary and sufficient condition for controllability of switched linear systems is provided. From this algebraic characterization, one can infer that controllability is a generic property in the space of matrices defining such systems. That is to say, the set of controllable systems is an open and dense subset. The natural question arising then is: how far a controllable system is from the nearest uncontrollable one? The answer to this question is specially important when working with matrices whose coefficients are given with some parameter uncertainty.

Because of the fact that in most applications only real matrices make sense, we will restrict ourselves to consider real perturbations.

The structure of the paper is as follows.

In Section §2, we summarize the definitions and some properties of norm matrices.

In Section §3, we lay the foundations of the problem to be solved.

In Section §4, we obtain an explicit bound for the distance from a controllable system to the set of uncontrollable ones.

Finally, in Section §5, we consider different examples.

2 Preliminaries

Let us consider the vector space of $m \times n$-matrices with coefficients in $k = \mathbb{R}$ or $\mathbb{C}$.

A matrix norm $\| \ \|$ is a mapping associating to each matrix M a nonnegative number $\|M\|$ having the following properties:

1. For all matrix M, $\|M\| \geq 0$ and $\|M\| = 0$ if, and only if, $M = 0$.
2. For all matrix M and $\lambda \in k$, $\|\lambda M\| = |\lambda| \|M\|$.
3. For all matrices M_1 and M_2, $\|M_1 + M_2\| \leq \|M_1\| + \|M_2\|$.

The most frequently used matrix norms are the Frobenius norm and the p-norms.

The Frobenius norm of a matrix $M = (m^i_j)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ is defined as

$$\|M\|_F = \sqrt{\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} |m^i_j|^2}$$

The matrix p-norm is defined for a real number $1 \leq p \leq \infty$ as:

$$\|M\|_p = \sup_{x \neq 0} \frac{\|Mx\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \|Mx\|_p$$

where

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

and $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$.

In particular, $\|M\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |m^i_j|$. The spectral norm $\| \ \|_2$, is the square root of the maximum eigenvalue of $M^t M$ in the real case and $M^H M$ in the complex case, where M^H denotes the conjugate transpose of the matrix M. And $\|M\|_\infty = \max_{1 \leq i \leq m} \sum_{1 \leq j \leq n} |m^i_j|$.

The Frobenius and the p-norms satisfy the following inequality (submultiplicative property):

For all $m \times n$-matrix M_1 and $n \times p$-matrix M_2, $\|M_1 M_2\| \leq \|M_1\| \|M_2\|$

and some further inequalities, relating them and which are commonly used in matrix analysis.

- $\|M\|_2 \leq \|M\|_F \leq \sqrt{n} \|M\|_2$
- $\max_{i,j} |m^i_j| \leq \|M\|_2 \leq \sqrt{mn} \max_{i,j} |m^i_j|$
- $\|M\|_1 = \max_{1 \leq j \leq n} \sum_{1 \leq i \leq m} |m^i_j|$.
- $\|M\|_\infty = \max_{1 \leq i \leq m} \sum_{1 \leq j \leq n} |m^i_j|$.
- $\frac{1}{\sqrt{m}} \|M\|_1 \leq \|M\|_2 \leq \sqrt{n} \|M\|_1$
- $\frac{1}{\sqrt{n}} \|M\|_\infty \leq \|M\|_2 \leq \sqrt{m} \|M\|_\infty$

3 Approaching the problem

Let us consider a switched linear system Σ defined by

$$\left\{ \begin{array}{ll} \dot{x}(t) & = A_\sigma x(t) + B_\sigma u(t) \\ y(t) & = C_\sigma x(t) \end{array} \right.$$

where $A_\sigma \in M_n(\mathbb{R}), B_\sigma \in M_{n\times 1}(\mathbb{R}), C_\sigma \in M_{p\times n}(\mathbb{R})$, with $\sigma \in \{1,\dots,\ell\}$.

Sun-Ge proved in [6] that system above is controllable if, and only if, the vector space

$$\sum_{\substack{k_1,\dots,k_\ell \in \{0,1\}, k_1+\cdots+k_\ell = 1 \\ j_1,\dots,j_{n-1} \in \{0,1,\dots,n-1\} \\ \mathcal{A}_1,\dots,\mathcal{A}_{n-1} \in \{A_\sigma\}_{\sigma\in\{1,\dots,\ell\}}}} \operatorname{Im}[\mathcal{A}_1^{j_1}\dots\mathcal{A}_{n-1}^{j_{n-1}} B_1^{k_1}\dots B_\ell^{k_\ell}] = \mathbb{R}^n$$

Therefore, the set of matrices $\{(A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,\dots,\ell\}}\}$ defining a controllable system is an open dense subset of $M_n(\mathbb{R})^\ell \times M_{n\times 1}(\mathbb{R})^\ell \times M_{p\times n}(\mathbb{R})^\ell$.In particular, for each set of matrices defining a controllable system there exists an open neighbourhood of this set in $M_n(\mathbb{R})^\ell \times M_{n\times 1}(\mathbb{R})^\ell \times M_{p\times n}(\mathbb{R})^\ell$ with all set of matrices in it defining a controllable system. Given a controllable system, our main goal is to explore the distance from this system to the nearest uncontrollable one, hence deducing a safety neighbourhood.

Note that when studying controllability, only matrices $\{(A_\sigma, B_\sigma)_{\sigma\in\{1,\dots,\ell\}}\}$ are relevant.

From now on, we will consider the metric given by the 2-norm.

Definition 1 The 2-norm of the set of matrices $\{(A_\sigma, B_\sigma)_{\sigma\in\{1,\dots,\ell\}}\}$ is taken as:

$$\|\{(A_\sigma, B_\sigma)_{\sigma\in\{1,\dots,\ell\}}\}\|_2 = \|\,(A_1|B_1|\dots|A_\ell|B_\ell)\,\|_2$$

and thus the distance between two sets of matrices is:

$$d_2(\{(A_\sigma, B_\sigma)_{\sigma\in\{1,\dots,\ell\}}\}, \{(A'_\sigma, B'_\sigma)_{\sigma\in\{1,\dots,\ell\}}\}) = \|\{(A'_\sigma - A_\sigma, B'_\sigma - B_\sigma)_{\sigma\in\{1,\dots,\ell\}}\}\|_2$$

Definition 2 Given a set of matrices $\{(A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,\dots,\ell\}}\}$ defining a controllable system, the distance from this system to the nearest uncontrollable one is

$$\mu((A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,\dots,\ell\}}) \;=\; \inf_{\delta A_\sigma \in M_n(\mathbb{R}), \delta B_\sigma \in M_{n\times 1}(\mathbb{R})} \|\{(\delta A_\sigma, \delta B_\sigma)_{\sigma\in\{1,\dots,\ell\}}\}\|_2$$

with $\{(A_\sigma + \delta A_\sigma, B_\sigma + \delta B_\sigma, C_\sigma + \delta C_\sigma)_{\sigma\in\{1,\dots,\ell\}}\}$ defining an uncontrollable system.

We will restrict ourselves to consider real perturbations, and find a bound for $\mu((A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,\dots,\ell\}})$.

4 Bounding the distance

Let us assume that the switched linear system Σ defined by

$$\begin{cases} \dot{x}(t) & = A_\sigma x(t) + B_\sigma u(t) \\ y(t) & = C_\sigma x(t) \end{cases}$$

where $A_\sigma \in M_n(\mathbb{R}), B_\sigma \in M_{n\times 1}(\mathbb{R}), C_\sigma \in M_{p\times n}(\mathbb{R})$, with $\sigma \in \{1, \dots, \ell\}$ is controllable.

For $1 \leq \sigma \leq \ell$, let us consider the vector subspaces:

$$G_\sigma = \sum_{\substack{k_1,\dots,k_\ell \in \{0,1\} \\ k_1+\dots+k_\ell = 1}} \sum_{\substack{j_1,\dots,j_{n-2} \in \{0,1,\dots,n-2\} \\ j_1+\dots+j_{n-2} \leq n-2 \\ \mathcal{A}_1,\dots,\mathcal{A}_{n-2} \in \{A_\sigma\}_{\sigma\in\{1,\dots,\ell\}} \\ \mathcal{A}_1 \neq A_\sigma}} \left[\sum_{0\leq j\leq n-1} \operatorname{Im} A_\sigma^j [\mathcal{A}_1^{j_1} \dots \mathcal{A}_{n-2}^{j_{n-2}} B_1^{k_1} \dots B_\ell^{k_\ell}] \right]$$

Let us denote by G the matrix having as columns the generators of the vector subspaces G_σ, as above:

$$(< G_1 > \mid \dots \mid < G_\ell >)$$

Remark 1 System Σ is controllable if, and only if, $\operatorname{rk} G = n$.

Let P, Q be orthogonal matrices such that

$$G = P^t(\operatorname{diag}(\lambda_1, \dots, \lambda_n) \mid 0)Q$$

where $\lambda_1 \geq \dots \geq \lambda_n$ are the singular values of G (Singular Value Decomposition of Matrix G).

In order to obtain the desired bound, we will need the following Lemmas.

Lemma 1 $\mu((A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,\dots,\ell\}}) = \mu((PA_\sigma P^t, PB_\sigma, C_\sigma P^t)_{\sigma\in\{1,\dots,\ell\}})$.

Proof: It is straightforward that $(A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,\dots,\ell\}}$ defines a controllable system if, and only if, $(PA_\sigma P^t, PB_\sigma, C_\sigma P^t)_{\sigma\in\{1,\dots,\ell\}}$ does. Then it suffices to prove that

$$\|(A_1 - X_1|B_1 - Y_1|\dots|A_\ell - X_\ell|B_\ell - Y_\ell)\|_2$$

is equal to

$$\|(P(A_1 - X_1)P^t|P(B_1 - Y_1)|\dots|P(A_\ell - X_\ell)P^t|P(B_\ell - Y_\ell))\|_2$$

This follows from the equality:

$$P(A_1 - X_1|B_1 - Y_1|\dots|A_\ell - X_\ell|B_\ell - Y_\ell)\operatorname{diag}\left(\begin{pmatrix} P^t & \\ & I_1 \end{pmatrix}_{1\leq\sigma\leq\ell}\right) =$$

$$(P(A_1 - X_1)P^t|P(B_1 - Y_1)|\dots|P(A_\ell - X_\ell)P^t|P(B_\ell - Y_\ell))$$

and the fact that P and $\operatorname{diag}\left(\begin{pmatrix} P^t & \\ & I_1 \end{pmatrix}_{\sigma\in\{1,\dots,\ell\}}\right)$ are orthogonal matrices.

Lemma 2 *Let us consider* $PB_\sigma = \begin{pmatrix} B_\sigma^1 \\ B_\sigma^2 \end{pmatrix}$*, with* $B_\sigma^1 \in M_{(n-1)\times 1}(\mathbb{R})$*, for* $\sigma \in \{1,\dots,\ell\}$*. Then* $\|B_\sigma^2\|_2 \leq \lambda_n$*.*

Proof: First, note that

$$B_\sigma^2 \quad = e_n^t PB_\sigma = e_n^t PGe_{1+(\sigma-1)n} = e_n^t(\mathrm{diag}(\lambda_1,\dots,\lambda_n)\,|\,0)Qe_{1+(\sigma-1)n}$$

where $e_1,\dots,e_n$ denote the natural basis of the Euclidean space $\mathbb{R}^n$. Then

$$\|B_\sigma^2\|_2 = \lambda_n\|Qe_{1+(\sigma-1)n}\|_2 \leq \lambda_n$$

since Q is an orthogonal matrix.

Lemma 3 *Let us consider* $PA_\sigma P^t = \begin{pmatrix} A_\sigma^1 & A_\sigma^2 \\ A_\sigma^3 & A_\sigma^4 \end{pmatrix}$*, with* $A_\sigma^1 \in M_{n-1}(\mathbb{R})$*, and assume* $\mathrm{rk}\, G_\sigma = n$ *for* $\sigma \in \{1,\dots,\ell\}$*. Then* $\|A_\sigma^3\|_2 \leq \frac{\lambda_n}{\lambda_{n,\sigma}}\|A_\sigma^C\|_2$*, where* $\lambda_{1,\sigma} \geq \cdots \geq \lambda_{n,\sigma}$ *are the singular values of* G_σ *(*$G_\sigma = P_\sigma^t(\mathrm{diag}(\lambda_{1,\sigma},\dots,\lambda_{n,\sigma})\,|\,0)Q_\sigma$*) and* A_σ^C *is the companion matrix for* A_σ*.*

Proof: We will denote by $M^\dagger$ the Moore-Penrose inverse of any matrix M.

It is straightforward to check that $A_\sigma G_\sigma = G_\sigma(I \otimes A_\sigma^C)$. Then $A_\sigma = G_\sigma(I \otimes A_\sigma^C)G_\sigma^\dagger$ and

$$\begin{aligned} PA_\sigma P^t &= PG_\sigma(I \otimes A_\sigma^C)G_\sigma^\dagger P^t \\ &= PG\begin{pmatrix} I \\ 0 \end{pmatrix}(I \otimes A_\sigma^C)G_\sigma^\dagger P^t \\ &= (\mathrm{diag}(\lambda_1,\dots,\lambda_n)\,|\,0)Q\begin{pmatrix} I \\ 0 \end{pmatrix}(I \otimes A_\sigma^C)G_\sigma^\dagger P^t \end{aligned}$$

Notice that

$$A_\sigma^3 = e_n^t(\mathrm{diag}(\lambda_1,\dots,\lambda_n)\,|\,0)Q\begin{pmatrix} I \\ 0 \end{pmatrix}(I\otimes A_\sigma^C)Q_\sigma^t\begin{pmatrix} \mathrm{diag}\left(\frac{1}{\lambda_{1,\sigma}},\dots,\frac{1}{\lambda_{n,\sigma}}\right) \\ 0 \end{pmatrix}P_\sigma P^t\begin{pmatrix} I_{n-1} \\ 0 \end{pmatrix}$$

Therefore

$$\|A_\sigma^3\|_2 \leq \lambda_n\|(I \otimes A_\sigma^C)\|_2\|G_\sigma^\dagger\|_2 \leq \frac{\lambda_n}{\lambda_{n,\sigma}}\|A_\sigma^C\|_2$$

Finally, we can state the main result.

Theorem 1 *Given a controllable switched linear system defined by a set of matrices* $((A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,\dots,\ell\}})$*, such that* $\mathrm{rk}\, G_\sigma = n$ *for* $\sigma \in \{1,\dots,\ell\}$*,*

$$\mu((A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,\dots,\ell\}}) \leq \sum_{\sigma\in\{1,\dots,\ell\}} \lambda_n\left(1 + \frac{\|A_\sigma^C\|_2}{\lambda_{n,\sigma}}\right)$$

Proof: Let us set the blocks B_σ^2, A_σ^3 to zero, for all σ, it is obvious that the system thus obtained is uncontrollable. The real perturbation we have committed has a norm which is bounded by $\lambda_n + \frac{\|A_\sigma^C\|_2\lambda_n}{\lambda_{n,\sigma}}$ for all $\sigma \in \{1,\dots,\ell\}$, and the statement follows.

5 Examples

1. Let us consider a switched linear system Σ defined by

$$\left\{\begin{array}{ll} \dot{x}(t) & = A_\sigma x(t) + B_\sigma u(t) \\ y(t) & = C_\sigma x(t) \end{array}\right.$$

with $\sigma \in \{1,2,3\}$, where

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \; B_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \; B_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \; A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$B_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $C_1 = C_2 = C_3 = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

Note that neither system defined by (A_1, B_1, C_1) nor system defined by (A_2, B_2, C_2) or (A_3, B_3, C_3) are controllable. Nevertheless, Σ is controllable, since matrix

$$G = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 2 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 2 & -1 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}$$

has full rank.

Then, according to Theorem above, the distance from this system to the nearest uncontrollable one, $\mu((A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,2,3\}})$, is bounded by $\lambda_2\left(1+\frac{\|A_1^C\|_2}{\lambda_{2,1}}\right)+\lambda_2\left(1+\frac{\|A_2^C\|_2}{\lambda_{2,2}}\right)+\lambda_2\left(1+\frac{\|A_3^C\|_2}{\lambda_{2,3}}\right)$ where A_1^C, A_2^C, A_3^C denote the companion matrices of A_1, A_2, A_3 and $\lambda_2 = 4$, $\lambda_{2,1} = 1.66250775$, $\lambda_{2,2} = 1.66250775$, $\lambda_{2,3} = 2$. That is to say, taking into account that $\|A_1^C\|_2 = \sqrt{2}$, $\|A_2^C\|_2 = \sqrt{2}$ and $\|A_3^C\|_2 = \sqrt{5}$, $\mu((A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,2,3\}}) \leq 23.27734242$.

2. Let us consider now a switched linear system Σ defined by

$$\left\{\begin{array}{ll} \dot{x}(t) & = A_\sigma x(t) + B_\sigma u(t) \\ y(t) & = C_\sigma x(t) \end{array}\right.$$

with $\sigma \in \{1,2\}$, where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$C_1 = C_2 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$.

Note that neither system defined by (A_1, B_1, C_1) nor system defined by (A_2, B_2, C_2) are controllable. Nevertheless, Σ is controllable, since matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

has full rank.

Then, according to Theorem above, the distance from this system to the nearest uncontrollable one, $\mu((A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,2\}})$, is bounded by $\lambda_3\left(1+\frac{\|A_1^C\|_2}{\lambda_{3,1}}\right)+\lambda_3\left(1+\frac{\|A_2^C\|_2}{\lambda_{3,2}}\right)$ where A_1^C, A_2^C denote the companion matrices of A_1, A_2 and $\lambda_3 = 1.68455404$, $\lambda_{3,1} = 1$, $\lambda_{3,2} = 1.20773289$. That is to say, taking into account that $\|A_1^C\|_2 = 1$ and $\|A_2^C\|_2 = \sqrt{2}$, $\mu((A_\sigma, B_\sigma, C_\sigma)_{\sigma\in\{1,2\}}) \leq 7.02621680$.

References

[1] D. Boley, *Estimating the Sensitivity of the Algebraic Structure of Pencils with Simple Eigenvalue Estimates*, SIAM J. Matrix Anal. Appl. 11, n. 4 (1990), 632–643.

[2] D. Boley, W.-S. Lu, *Measuring how far a controllable system is from an uncontrollable one*, IEEE Trans. on Automatic Control AC-31 (1986), 249–251.

[3] J. Clotet, Mª I. García-Planas, M. D. Magret, *Estimating distances from quadruples satisfying stability properties to quadruples not satisfying them*, Linear Algebra and its Applications 332-334 (2001), 541-567.

[4] R. Eising, *Between controllable and uncontrollable*, Syst. Control Letters 4, no. 5 (1984), 263-264.

[5] R. Eising, *The distance between a system and the set of uncontrollable systems*, Memo COSOR 82-19, Eindhoven Univ. Technol., Eindhoven, The Netherlands, 1982.

[6] Z. Sun, S. S. Ge, *Switched Linear Systems*, Springer, London, England, 2005.