# Miniversal Deformations of Bimodal Picewise Linear Systems

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**Abstract**

Bimodal linear systems are those consisting of two linear systems on each side of a given hyperplane, having continuous dynamics along that hyperplane. In this work, we focus on the derivation of (orthogonal) miniversal deformations, by using reduced forms.

**Keywords:** Bimodal piecewise linear system, miniversal deformations, reduced forms.

## 1 Introduction

Bimodal piecewise linear systems (consisting of two linear dynamics on each side of a given hyperplane) have attracted the interest of researchers because they present a complex dynamical behaviour (see [2], for example).

We consider a natural equivalence in the space of matrices defining those systems, defined by basis changes in the state variables space and preserving the hyperplanes parallel to that where the system has a continuous dynamics. Equivalence classes coincide with the orbits under a suitable Lie group action on the differentiable manifold of matrices defining the systems. Following Arnold's techniques in [1] and [5], miniversal deformations are obtained. In order to be able to explicitly compute them, reduced forms in [3] play a key role.

The structure of the paper is as follows.

In Sec. 2, we define the equivalence relation in the space of matrices defining bimodal piecewise linear systems which corresponds to basis changes of state variables, preserving the hyperplane where the system presents a continuous dynamics and interpret equivalence classes as orbits under a Lie group action on the space of matrices.

In Sec. 3, we obtain miniversal deformations, following Arnold's theory (see [1]) and using reduced forms for the matrices defining bimodal linear systems.

Throughout the paper, $\mathbb{R}$ will denote the set of real numbers, $M_{n\times m}$ the set of matrices having $n$ rows and $m$ columns and entries in $\mathbb{R}$ (in the case where $n=m$, we will simply write $M_n(\mathbb{R})$) and by $Gl_n(\mathbb{R})$ the group of non-singular matrices in $M_n(\mathbb{R})$. We will denote by $e_1,\dots,e_n$ the natural basis of the Euclidean space $\mathbb{R}^n$.

## 2 Equivalence Relation. Geometric Approach

Piecewise bimodal linear systems can be defined by different systems of linear equations. In particular, different state variables can be chosen. This leads to a natural equivalence relation in the space of matrices defining such systems.

We will consider bimodal linear systems

$$\left\{\begin{array}{ll}\dot{\mathbf{x}}(t) & = A_1\mathbf{x}(t)+B_1\\ \mathbf{y}(t) & = C\mathbf{x}(t)\end{array}\right. \text{ if } \mathrm{x}_1(t)\le 0 \quad \left\{\begin{array}{ll}\dot{\mathbf{x}}(t) & = A_2\mathbf{x}(t)+B_2\\ \mathbf{y}(t) & = C\mathbf{x}(t)\end{array}\right. \text{ if } \mathrm{x}_1(t)\ge 0$$

where $A_1, A_2 \in M_n(\mathbb{R})$; $B_1, B_2 \in M_{n\times 1}(\mathbb{R})$; $C = (1\ 0\ \dots\ 0) \in M_{1\times n}(\mathbb{R})$, and assume that the dynamics is continuous along the hyperplane $\mathrm{x}_1 = 0$.

Coincidence in the hyperplane $\mathbf{x}_1 = 0$ is equivalent to:

$$B_2 \;= B_1; \qquad A_2e_i \;= A_1e_i, \quad 2\le i\le n$$

We will simply write $B = B_1 = B_2$. Thus any bimodal piecewise linear system is defined by a triple of matrices $(A_1, A_2, B)$.

Throughout the paper, $\mathcal{X}$ will denote the set of triples of matrices defining bimodal piecewise linear systems,

$$\mathcal{X} = \{(A_1, A_2, B) \in M_n(\mathbb{R}) \times M_n(\mathbb{R}) \times M_{n\times 1}(\mathbb{R}) \,|\, A_1e_i = A_2e_i\ 2\le i\le n\}$$

$\mathcal{X}$ is a $(n^2 + 2n)$-submanifold of $\mathcal{M} = M_n(\mathbb{R}) \times M_n(\mathbb{R}) \times M_{n\times 1}(\mathbb{R})$.

We consider the equivalence relation which corresponds to basis changes in the state variables space, preserving the hyperplanes $\mathbf{x}_1 = \delta$, $\delta \in \mathbb{R}$ (*admissible basis changes*), in order the results below can be applied to the cases where a bimodal system is defined by

$$\left\{\begin{array}{ll}\dot{\mathbf{x}}(t) & = A_1\mathbf{x}(t)+B_1\\ \mathbf{y}(t) & = C\mathbf{x}(t)\end{array}\right. \text{ if } \mathrm{x}_1(t)\le \delta \quad \left\{\begin{array}{ll}\dot{\mathbf{x}}(t) & = A_2\mathbf{x}(t)+B_2\\ \mathbf{y}(t) & = C\mathbf{x}(t)\end{array}\right. \text{ if } \mathrm{x}_1(t)\ge \delta$$

with the dynamics continuous along the hyperplane $x_1(t) = \delta$, simply applying the coordinate change $\overline{\mathbf{x}} = \mathbf{x} - \delta e_1$.

This leads to the following natural equivalence relation in $\mathcal{X}$.

**Definition 1** Two triples of matrices $(A_1, A_2, B), ({A'}_1, {A'}_2, B') \in \mathcal{X}$ are said to be *equivalent* if there exists $S \in Gl_n(\mathbb{R})$ representing an admissible basis change such that

$$({A'}_1, {A'}_2, B') = (S^{-1}A_1S, S^{-1}A_2S, S^{-1}B)$$

Note that this is a well-defined equivalence relation. Given a basis change $S \in Gl_n(\mathbb{R})$, the condition of preserving the hyperplane $\mathbf{x}_1(t) = 0$ is equivalent to:

$$S = \begin{pmatrix} 1 & 0\\ U & T\end{pmatrix}, \quad T \in Gl_{n-1}(\mathbb{R})$$

We will denote by $\mathcal{S}$ the set

$$\left\{ S \in\ Gl_n(\mathbb{R}) \,\middle|\, S = \begin{pmatrix} 1 & 0\\ U & T\end{pmatrix}, \quad T \in Gl_{n-1}(\mathbb{R}) \right\}$$

which is a Lie subgroup of $\mathcal{G} = Gl_n(\mathbb{R})$.

Equivalence classes defined above coincide with the orbits under the Lie group action

$$\alpha : \mathcal{S} \times \mathcal{X} \quad \longrightarrow \mathcal{X}$$

defined by $\alpha(S, (A_1, A_2, B)) = (S^{-1}A_1S, S^{-1}A_2S, S^{-1}B)$.

Given a triple of matrices $(A_1, A_2, B) \in \mathcal{X}$, we will denote by $\mathcal{O}(A_1, A_2, B)$ its orbit or equivalence class. As an application of the Closed Orbit Lemma (see [4]), we deduce that equivalence classes are locally closed differentiable submanifolds, and boundaries are a union of equivalence classes or orbits of strictly lower dimension. In particular, equivalence classes or orbits of minimal dimension are closed.

When considering the following scalar product in $\mathcal{M}$ :

$$< (A_1, A_2, B), (A'_1, A'_2, B') >= \operatorname{tr}(A_1^t A'_1) + \operatorname{tr}(A_2^t A'_2) + \operatorname{tr}(B^t B')$$

the normal vector subspace to the orbit of any triple in $\mathcal{X}$ may be described as follows.

**Proposition 1** ([3]) Denoting by $N_{(A_1,A_2,B)}\mathcal{O}(A_1, A_2, B)$ the normal vector subspace to the orbit of the triple $(A_1, A_2, B)$ at $(A_1, A_2, B)$, $N_{(A_1,A_2,B)}\mathcal{O}(A_1, A_2, B) \cap \mathcal{X}$ is the vector subspace consisting of triples $(X_1, X_2, Y)$ such that

$$A_1X_1^t - X_1^tA_1 + A_2X_2^t - X_2^t - BY^t \in \mathcal{A}$$

where $\mathcal{A}$ is the set
$$\left\{ M = (m_i^j) \;\middle|\; \begin{cases} m_i^1 = -m_1^1, \ 2 \leq i \leq n \\ m_i^j = -m_1^1 \ 2 \leq i, j \leq n \end{cases} \right\}$$

Then, we can characterize the triples $(X_1, X_2, Y) \in N_{(A_1,A_2,B)}\mathcal{O}(A_1, A_2, B) \cap \mathcal{X}$ as those such that

$$M(A_1, A_2, B) \begin{pmatrix} X_1e_1 \\ X_1e_2 \\ X_1e_3 \\ X_2e_1 \\ Y \end{pmatrix} = 0$$

for a suitable matrix $M(A_1, A_2, B) \in M_{(1+(n-1)^2)\times(n^2+2n)}(\mathbb{R})$.

## 3 Miniversal Deformations

When using mathematical models of physical systems, the question of how small perturbations of the system may lead to different structures is most interesting. Versal deformations provide all possible structures which can arise from small perturbations and can be applied to the study of singularities and bifurcations.

The main definitions and results about deformations and versality can be found in [1] and [5]. Here we recall them, adapted to our particular case.

**Definition 2** A *deformation* of $(A_1, A_2, B) \in \mathcal{X}$ is a differentiable map $\varphi : U \longrightarrow \mathcal{X}$, with $U$ an open neighborhood of the origin $\mathbb{R}^d$, such that $\varphi(0) = (A_1, A_2, B)$.

A deformation $\varphi : U \longrightarrow \mathcal{X}$ of $(A_1, A_2, B)$ is called *versal* at 0 if for any other deformation of $(A_1, A_2, B)$, $\psi : V \longrightarrow \mathcal{X}$, there exists a neighborhood $V' \subseteq V$ with $0 \in V'$, a differentiable map $\gamma : V' \longrightarrow U$ with $\gamma(0) = 0$ and a deformation of the identity $I \in \mathcal{S}$, $\theta : V' \longrightarrow \mathcal{S}$, such that $\psi(\mu) = \alpha(\theta(\mu), \varphi(\gamma(\mu)))$ for all $\mu \in V'$.

A versal deformation with minimal number of parameters $d$ is called *miniversal deformation.*

Miniversal deformations may be identified with submanifolds which are minitransversal to the orbit of a given triple of matrices (see [1] and [5]). Therefore, the number of parameters of any miniversal deformation is equal to the codimension of the orbit. Hence, this minimal number of parameters does not depend neither on the miniversal deformation we consider nor on the representative of the equivalence class. Thus the dimension of the miniversal deformation of any triple may be computed by calculating the dimension of the miniversal deformation of the canonical form in its equivalence class, or that of any reduced form. Calculations become much simpler when using reduced forms.

A miniversal deformation deduced from a basis of the normal space to the orbit of a given triple is usually called *orthogonal* miniversal deformation.

**Theorem 1** *The mapping*

$$\begin{aligned} \mathbb{R}^d &\longrightarrow \mathcal{X} \\ (u_1, \ldots, u_d) &\longrightarrow (A_1, A_2, B) + u_1 V_1 + \cdots + u_d V_d \end{aligned}$$

*where* $\{V_1, \ldots, V_d\}$ *is any basis of the vectorial space* $N_{(A_1,A_2,B)}\mathcal{O}(A_1, A_2, B)$ *is a miniversal deformation of* $(A_1, A_2, B)$*.*

**Example 1** Let us consider a non-observable system (it is in reduced form, type 2, according to [3]):

$$\left( \begin{pmatrix} a_1 & 0 \\ 0 & \lambda_0 \end{pmatrix}, \begin{pmatrix} \alpha_1 & 0 \\ 1 & \lambda_0 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right)$$

where $b_1 \neq 0$, $\lambda_0 \neq a_1$. Let us denote by $N_1$ the normal space to the orbit of this triple. Then $F_1 = N_1 \cap \mathcal{X}$ consists of triples

$$\left\{ \left( \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}, \begin{pmatrix} x_5 & x_3 \\ x_6 & x_4 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \right\}$$

such that

$$\left.\begin{aligned} x_6 - b_1 y_1 &= 0 \\ (\lambda_0 - a_1)x_2 + (\lambda_0 - \alpha_1)x_6 - b_1 y_2 &= 0 \\ x_6 + b_2 y_2 &= 0 \end{aligned}\right\}$$

Since $F_1$ has dimension equal to 6, this is the minimal number of parameters of any miniversal deformation of the triple. If we consider a basis of this vector subspace: $(V_1, \ldots, V_6)$ then an orthogonal miniversal deformation of the triple is given by

$$\begin{aligned} \varphi : U \subseteq \mathbb{R}^6 &\longrightarrow \mathcal{X} \\ u = (u_1, \ldots, u_6) &\longrightarrow \sum_{i=1}^{6} u_i V_i \end{aligned}$$

For example, we can take

$$V_1 = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right), V_2 = \left( \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$V_3 = \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right), V_4 = \left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$$

$$V_5 = \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right), V_6 = \left( \begin{pmatrix} 0 & 0 \\ \frac{(\lambda_0-\alpha_1)b_2+b_1}{\lambda_0-a_1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -b_2 & 0 \end{pmatrix}, \begin{pmatrix} \frac{b_2}{b_1} \\ 1 \end{pmatrix} \right)$$

**Example 2** Let us consider the following triple (as shown in [3], it is in reduced form of type 3, case $r = 2$):

$$A_1 = \begin{pmatrix} a_1 & 1 & 0 \\ a_2 & 0 & 0 \\ 1 & 0 & \lambda_0 \end{pmatrix}, A_2 = \begin{pmatrix} \alpha_1 & 1 & 0 \\ \alpha_2 & 0 & 0 \\ 1 & 0 & \lambda_0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

where $\lambda_0 \neq 0$. We denote by $N_2$ the normal subspace to orbit. Then $F_2 = N_2 \cap \mathcal{X}$ is the vector subspace defined by the matrix equation:

$$M(A_1, A_2, B) \begin{pmatrix} X_1 e_1 \\ X_1 e_2 \\ X_1 e_3 \\ X_2 e_1 \\ Y \end{pmatrix} = 0$$

where $M(A_1, A_2, B)$ is:

$$\begin{pmatrix} 0 & a_2 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2 & 1 & -1 & 0 & 0 \\ 1 & -a_1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & -\alpha_1 & 0 & 0 & -1 & 0 \\ 0 & -a_2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -2\lambda_0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_0 - a_1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & \lambda_0 - \alpha_1 & 0 & 0 & -1 \\ 0 & 0 & -a_2 & 0 & 0 & 2\lambda_0 & 0 & 0 & 0 & 0 & 0 & -\alpha_2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Then a basis of $F$ is:

$$V_1 = \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right),$$

$$V_2 = \left( \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right),$$

$$V_3 = \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right),$$

$$V_4 = \left( \begin{pmatrix} 2a_1 & a_2 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right),$$

$$V_5 = \left( \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right),$$

$$V_6 = \left( \begin{pmatrix} -2\lambda_0 a_1 & -\lambda_0 a_2 & 0 \\ -2\lambda_0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\lambda_0 a_2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$V_7 = \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\lambda_0}{\delta} & \frac{a_2-\alpha_2}{2\delta} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\lambda_0}{\delta} & \frac{a_2-\alpha_2}{2\delta} & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right),$$

$$V_8 = \left( \begin{pmatrix} \alpha_1 - a_1 & \frac{\alpha_2-a_2}{2} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{\alpha_2-a_2}{2} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

where $\delta = \alpha_2 - a_2 + \lambda_0(\alpha_1 - a_1)$. Thus an orthogonal miniversal deformation is given by

$$\begin{aligned} \varphi : U \subseteq \mathbb{R}^8 &\longrightarrow \mathcal{X} \\ u = (u_1, \ldots, u_8) &\longrightarrow \sum_{i=1}^{8} u_i V_i \end{aligned}$$

## Acknowledgements

Partially supported by MTM2007-67812-C02-02.

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