M-Matrix Inverse problem for distance-regular graphs

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1 Introduction

Very often problems in biological, physical and social sciences can be reduced to problems involving matrices which have some special structure. One of the most common situation is where the matrix in question has non–positive off–diagonal and non–negative diagonal entries; that is $L = kI - A$, $k > 0$ and $A \geq 0$, where the diagonal entries of A are less or equal than k. These matrices appear in relation to systems of equations or eigenvalue problems in a broad variety of areas including finite difference methods for solving partial differential equations, input–output production and growth models in economics or Markov processes in probability and statistics. Of course, the combinatorial community can recognize within this type of matrices, the combinatorial Laplacian of a k–regular graph where A is its adjacency matrix.

If k is at least the spectral radius of A, then L is called an *M–matrix*. We remark that M–matrices arise naturally in some discretizations of differential operators, particularly those with a minimum/maximum principle, such as the Laplacian, and as such are well–studied in scientific computing. In fact M–matrices satisfy monotonicity properties that are the discrete counterpart of the minimum principle, and it makes them suitable for the resolution of large sparse systems of linear equations by iterative methods.

As well as a symmetric, irreducible and non–singular M–matrix appears as the discrete counterpart of a Dirichlet problem for a self–adjoint elliptic operator, its inverse corresponds with the Green operator associated with the boundary value problem. On the other hand, when the M–matrix is singular, it can be seen as a discrete analogue of the Poisson equation for a self–adjoint elliptic operator on a manifold without boundary and then, its Moore–Penrose inverse corresponds with the Green operator too. A well–known property of an irreducible non–singular M–matrix is that its inverse is non–negative, [4]. However, the scenario changes dramatically when the matrix is an irreducible and singular M–matrix. In this case, it is known that the matrix has a generalized inverse which is non–negative, but this is not always true for any generalized inverse. For instance, it may happens that the Moore–Penrose inverse has some negative entries. We focus here in studying when the Moore–Penrose inverse of a symmetric, singular and irreducible M–matrix is itself an M–matrix. In particular, we study the case of distance–regular graphs and more specifically strongly regular graphs.

2 Preliminaries

The triple $\Gamma = (V, E, c)$ denotes a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V, whose cardinality equals n, and edge set E, in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x, y) > 0$. So, the conductance can be considered as a symmetric function $c\colon V \times V \longrightarrow [0, +\infty)$ such that $c(x, x) = 0$ for any $x \in V$ and moreover, $x \sim y$, that is vertex x is adjacent to vertex y, iff $c(x, y) > 0$.

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y) \Big(u(x) - u(y) \Big), \quad x \in V.$$

It is well–known that $\mathcal{L}$ is a positive semi–definite self–adjoint operator and has 0 as its lowest eigenvalue whose associated eigenfunctions are constant. So, $\mathcal{L}$ can be interpreted as an irreducible, symmetric, diagonally dominant and singular M–matrix, L. Therefore, the Poisson equation $\mathcal{L}(u) = f$ on V has solution iff $\sum\limits_{x \in V} f(x) = 0$ and, when this happens, there exists a unique solution $u \in \mathcal{C}(V)$ such that $\sum\limits_{x \in V} u(x) = 0$, see [1].

The *Green operator* is the linear operator $\mathcal{G} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to any $f \in \mathcal{C}(V)$ the unique solution of the Poisson equation with data $f - \frac{1}{n}\sum_{x\in V} f(x)$ such that $\sum_{x\in V} u(x) = 0$. It is easy to prove that $\mathcal{G}$ is a positive semi–definite self–adjoint operator and has 0 as its lowest eigenvalue whose associated eigenfunctions are constant. Moreover, if $\mathcal{P}$ denotes the projection on the subspace of constant functions then,

$$\mathcal{L} \circ \mathcal{G} = \mathcal{G} \circ \mathcal{L} = \mathcal{I} - \mathcal{P}.$$

In addition, we define the *Green function* as $G : V \times V \longrightarrow \mathbb{R}$ given by $G(x, y) = \mathcal{G}(\varepsilon_y)(x)$, where ε_y stands for the Dirac function at y. Therefore, interpreting $\mathcal{G}$ or G as a matrix, G, it is nothing else but the Moore–Penrose inverse of L, the matrix associated with $\mathcal{L}$. In consequence, G is an M–matrix iff $G(x, y) \leq 0$ for any $x, y \in V$ with $x \neq y$.

In [1] it was proved that for any $x \in V$, there exists $\nu^x \in \mathcal{C}(V)$ such that $\nu^x(x) = 0$, $\nu^x(y) > 0$ for any $y \neq x$ and verifying

$$\mathcal{L}(\nu^x) = \mathsf{1} - n\varepsilon_x \quad \text{on } V. \tag{1}$$

We call ν^x the *equilibrium measure of* $V \setminus \{x\}$ and then we define *capacity* as the function $\mathsf{cap} \in \mathcal{C}(V)$ given by $\mathsf{cap}(x) = \sum_{y\in V} \nu^x(y)$.

3 The Moore-Penrose inverse of distance–regular graphs

We aim here at characterizing when the Moore–Penrose inverse of the combinatorial Laplacian matrix of a distance–regular graph is a M–matrix.

Recall that a connected graph Γ is called *distance–regular* if there are integers b_i, c_i, $i = 0, \dots, d$ such that for any two vertices $x, y \in \Gamma$ at distance $i = d(x, y)$, there are exactly c_i neighbours of y in $\Gamma_{i-1}(x)$ and b_i neighbours of y in $\Gamma_{i+1}(x)$, where for any vertex $x \in \Gamma$ the set of vertices at distance i from it is denoted by $\Gamma_i(x)$. Moreover, $|\Gamma_i(x)|$ will be denoted by k_i. In particular, Γ is regular of degree $k = b_0$. The sequence

$$\iota(\gamma) = \{b_0, b_1, \dots, b_{d-1}; c_1, \dots, c_d\},$$

is called the *intersection array* of Γ. In addition, $a_i = k - c_i - b_i$ is the number of neighbours of y in $\Gamma_i(x)$, for $d(x, y) = i$. Clearly, $b_d = c_0 = 0$, $c_1 = 1$ and the diameter of Γ is d.

Lemma 1 ([1, Prop. 4.1]) *Let Γ be a distance–regular graph. Then, for all $y \in V$*

$$\nu^x(y) = \sum_{j=0}^{d(x,y)-1} \frac{n-|B_j|}{|\partial B_j|} \quad \textit{and} \quad \mathsf{cap}(x) = \sum_{j=0}^{d-1} \frac{(n-|B_j|)^2}{|\partial B_j|}$$

where $|B_j|$ is the number of vertices at distance at most j from a given vertex and $|\partial B_j| = k_j b_j$.

The following result has been proved in [3] in a more general context. However, we prove it here for the sake of completeness.

Theorem 2 *The Moore–Penrose inverse of L is an M–matrix iff for any $x \in V$*

$$\mathsf{cap}(x) \leq n\nu^x(y) \qquad \textit{for any } y \sim x.$$

Proof

The Green function is given by

$$G(x,y) = \frac{1}{n^2}\big(\mathsf{cap}(x) - n\,\nu^x(y)\big),$$

see [1]. Therefore, G is an M–matrix iff

$$\mathsf{cap}(x) \leq n \min_{y \in V\setminus\{x\}} \big\{\nu^x(y)\big\}.$$

The result follows by keeping in mind that $\min\limits_{y \in V\setminus\{x\}} \big\{\nu^x(y)\big\} = \min\limits_{y \sim x} \big\{\nu^x(y)\big\}$, since if the minimum is attained at $z \not\sim x$, then

$$1 = \mathcal{L}(\nu^x)(z) = \sum_{y \in V} c(x,y)\big(\nu^x(z) - \nu^x(y)\big) \leq 0,$$

which is a contradiction. □

Let L be the matrix associated with the combinatorial Laplacian of a distance–regular graph. Then, from Theorem 2 we get the following result.

Proposition 3 *The Moore–Penrose inverse of L is an M–matrix iff*

$$\sum_{i=1}^{d-1} \frac{(n-|B_i|)^2}{|\partial B_i|} \leq \frac{n-1}{k}.$$

In particular, for a strongly regular graph with parameters (n,k,a_1,c_2)*, the Moore–Penrose inverse of* L *is an* M*–matrix iff*

$$a_1 \le 3k - \frac{k^2}{n-1} - n.$$

Proof

From Theorem 2 the Moore–Penrose inverse of L is an M–matrix iff

$$\sum_{j=0}^{d-1} \frac{(n-|B_j|)^2}{|\partial B_j|} \le \frac{n(n-1)}{k}$$

that is, iff

$$\frac{(n-1)^2}{k} + \sum_{j=1}^{d-1} \frac{(n-|B_j|)^2}{|\partial B_j|} \le \frac{n(n-1)}{k}.$$

If Γ is a strongly regular graph, then $d=2$ and hence the Moore–Penrose inverse of L is an M–matrix iff $(n-k-1)^2 \le b_1(n-1)$ and the result follows keeping in mind that $b_1 = k-1-a_1$. $\square$

The above conclusion for strongly regular graphs also appeared in [6, Theorem 2.4], expressed in terms of the eigenvalues of the combinatorial Laplacian.

If Γ is the n–cycle with vertices labeled $\{x_1,\dots,x_n\}$, then it is easy to verify that

$$\nu^{x_i}(x_j) = \frac{1}{2}\,|i-j|\big(n-|i-j|\big) \quad \text{and} \quad \mathsf{cap}(x_i) = \frac{n(n^2-1)}{12}, \quad i,j=1,\dots,n.$$

Therefore, by applying the above proposition, we obtain that the Moore–Penrose inverse of the combinatorial Laplacian of a n–cycle is a M–matrix iff

$$\frac{n(n^2-1)}{12} \le \frac{n(n-1)}{2};$$

that is, iff $n \le 5$. This result was already obtained in [2, 5]. In addition, the Moore–Penrose inverse of M is $\mathsf{M}^\dagger = (g_{ij})$ where

$$g_{ij} = \frac{1}{12n}\Big(n^2 - 1 - 6|i-j|(n-|i-j|)\Big), \quad i,j=1,\dots,n.$$

We point out that *Petersen Graph* does not fulfill the above condition, since its parameters is $(10,3,0,1)$. Notice that the Green function of the Petersen Graph is $G(x,x) = 0.33$; $G(x,y) = 0.03$ if $d(x,y) = 1$ and $G(x,y) = -0.07$ if $d(x,y) = 2$, since for any $x, y \in V$, $\nu^x(y) = 3$ if $d(x,y) = 1$, $\nu^x(y) = 4$ if $d(x,y) = 2$ and $\mathsf{cap}(x) = 33$.

As an example of a strongly regular graph that fulfills the above condition we consider the family of *Conference Graphs* whose parameters are

$$\left(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4}\right).$$

Corollary 4 *The Moore–Penrose inverse of the Laplacian matrix of a conference graph is an M–matrix.*

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