DISTURBANCE DECOUPLING FOR SINGULAR SYSTEMS BY PROPORTIONAL AND DERIVATIVE FEEDBACK AND PROPORTIONAL AND DERIVATIVE OUTPUT INJECTION

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We study the disturbance decoupling problem for linear time invariant singular systems. We give necessary and sufficient conditions for the existence of a solution to the disturbance decoupling problem with or without stability via a proportional and derivative feedback and proportional and derivative output injection that also makes the resulting closed-loop system regular and/or of index at most one. All results are based on canonical reduced forms that can be computed using a complete system of invariants that can be implemented in a numerically stable way.

Keywords: Singular Systems; Equivalence relation; Disturbance decoupling.

1. Introduction

We consider linear and time-invariant continuous singular systems of the form

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) + Gg(t), \ x(t_0) = x_0, \ t \geq 0 \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where $E, A \in M_n(C)$, $B \in M_{n\times m}(C)$, $C \in M_{p\times n}(C)$, $G \in M_{n\times q}(C)$ and $\dot{x} = dx/dt$. The term $g(t)$, $t \geq 0$, represents a disturbance, which may represent modeling or measuring errors, noise, or higher order terms in linearization. Singular systems arise naturally in circuits design, mechanical multibody systems and a large variety of the applications (see [5] and [6], for example), and they have been studied under different points of view. The problem of constructing feedbacks and/or output injections that suppress this disturbance in the sense that g(t) does not affect the input-output behavior of the system is analyzed. In the case of standard state space systems the disturbance decoupling problem has been largely studied (see [1],[7],[8] for example), This problem for singular systems has also been studied (see [2], [4] for example). In this paper we study the disturbance decoupling problem for singular systems that can be stated as follows: Find necessary and sufficient conditions under which we can choose state and derivative feedback as well state and derivative output injection such that, the matrix pencil $(E + BF_E^B + F_E^C C, A + BF_A^B + F_A^C C)$ is regular of index at most one and $C(s(E + BF_E^B + F_E^C C) - (A + BF_A^B + F_A^C C))^{-1}G = 0$.

We assume, without loss of generality, that $\operatorname{rank} B = m$, $\operatorname{rank} G = q$, and $\operatorname{rank} C = p$. If this is not the case, then this can be easily achieved, by removing the nullspaces and appropriate renaming of variables.

In the sequel we will use the following notations:

I_n denotes the n-order identity matrix, N denotes a nilpotent matrix in its reduced form $N = \operatorname{diag}(N_1, \dots, N_t)$, $N_i = \begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix} \in M_{n_i}(C)$, J denotes the Jordan matrix $J = \operatorname{diag}(J_1, \dots, J_t)$, $J_i = \operatorname{diag}(J_{i_1}, \dots, J_{i_s})$, $J_{i_j} = \lambda_i I_{i_j} + N$, L denotes the diagonal matrix $L = \operatorname{diag}(L_1, \dots, L_q)$, where $L_j = \begin{pmatrix} I_{n_j} & 0 \end{pmatrix} \in M_{n_j \times (n_j+1)}(C)$, and R denotes the diagonal matrix $R = \operatorname{diag}(R_1, \dots, R_p)$, where $R_j = \begin{pmatrix} 0 & I_{n_j} \end{pmatrix} \in M_{n_j \times (n_j+1)}(C)$.

We represent systems of the form (1) as quadruples of matrices (E, A, B, C) in the case of disturbance do not appear or it is not considered, and a quintuples of matrices (E, A, B, C, G) otherwise.

2. Reduced Form

We recall that, given a singular system (not necessarily square) using standard transformations in state, input and output spaces $x(t) = Px_1(t)$, $u(t) = Ru_1(t)$, $y_1(t) = Sy(t)$, premultiplication by an invertible matrix $QE\dot{x}(t) = QAx(t) + Qu(t)$ making feedbacks $u(t) = u_1(t) - Vx(t)$, $u(t) = u_1(t) - U\dot{x}(t)$ as well as output injections $u(t) = u_1(t) - Wy(t)$, $u(t) = u_1(t) - Z\dot{y}(t)$, it is possible to reduce to $E_r\dot{x}_1(t) = A_r x_1(t) + B_r u_1(t) + G_1$, $y_1 = C_r x(t)$ where

$$E_r = \begin{pmatrix} I_1 &&&&&&& \\ & I_2 &&&&&& \\ && I_3 &&&&& \\ &&& I_4 &&&& \\ &&&& N_1 &&& \\ &&&&& L_1 && \\ &&&&&& L_2^t & \\ &&&&&&& 0 \end{pmatrix}, \quad A_r = \begin{pmatrix} N_2 &&&&&&& \\ & N_3 &&&&&& \\ && N_4 &&&&& \\ &&& J &&&& \\ &&&& I_5 &&& \\ &&&&& R_1 && \\ &&&&&& R_2^t & \\ &&&&&&& 0 \end{pmatrix}$$

$$B_r = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_3 \end{pmatrix}, \quad C_r = \begin{pmatrix} C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

or

$$E_r = \begin{pmatrix} I_1 & & & & & & & \\ & I_2 & & & & & & \\ & & I_3 & & & & & \\ & & & I_4 & & & & \\ & & & & N_1 & & & \\ & & & & & L_1 & & \\ & & & & & & L_2^t & 0 \end{pmatrix}, \quad A_r = \begin{pmatrix} N_2 & & & & & & & \\ & N_3 & & & & & & \\ & & N_4 & & & & & \\ & & & J & & & & \\ & & & & I_5 & & & \\ & & & & & R_1 & & \\ & & & & & & R_2^t & 0 \end{pmatrix}$$

$$B_r = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_r = \begin{pmatrix} C_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 \end{pmatrix}$$

and

i) (I_1, N_2, B_1, C_1) is an n_1-completely controllable-observable system.
ii) (I_2, N_3, B_2) is an n_2-completely controllable non observable system.
iii) (I_3, N_4, C_2) is an n_3-completely observable non controllable system.
iv) (I_4, J) is an n_4-system having only finite zeroes.
v) (N_1, I_5) is an n_5-system having only non transferable infinite zeroes.
vi) (L_1, R_1), completely singular systems of n_6 rows.
vii) (L_2^t, R_2^t) completely singular systems of n_7 rows respectively.
viii) $B_3 = \text{diag}(I_{n_{8_1}}, 0_{n_{8_2}})$ or $C_3 = \text{diag}(I_{n_{9_1}}, 0_{n_{9_2}})$.

The regular part of the system is maximal among all possible reductions of the system decomposing it in a regular part and a singular part and not all parts i),..., viii) necessarily appears in the decomposition of a system.

The proof is based in the following proposition:

Proposition 2.1. *Two quadruples of matrices (E_i, A_i, B_i, C_i) are equivalent under equivalence relation considered if and only if the matrix pencils $\lambda \begin{pmatrix} E_i & B_i & 0 \\ C_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} A_i & 0 & B_i \\ 0 & 0 & 0 \\ C_i & 0 & 0 \end{pmatrix}$ are strictly equivalent.*

Based on reduced form, the system (1), is reduced to the following independent subsystems:

$$\begin{cases} \dot{x}_1 = N_2 x_1 + B_1 u_1 + G_1 g_1 \\ y_1 = C_1 x_1 \end{cases} \tag{2}$$

$$\{\dot{x}_2 = N_3 x_2 + B_2 u_2 + G_2 g_2 \tag{3}$$

$$\begin{cases} \dot{x}_3 = N_4 x_3 + G_3 g_3 \\ y_3 = C_2 x_3 \end{cases} \tag{4}$$

$$\{\dot{x}_4 = J x_4 + G_4 g_4 \tag{5}$$

$$\{N_1 \dot{x}_5 = x_5 + G_5 g_5 \tag{6}$$

$$\{L_1 \dot{x}_6 = R_1 x_6 + G_6 g_6 \tag{7}$$

$$\{L_2^t \dot{x}_7 = R_2^t x_7 + G_7 g_7 \tag{8}$$

$$\{B_3 u_3 = 0 \quad \text{or} \quad \{C_3 x_8 = 0. \tag{9}$$

Systems from (2) to (6) are regular and (7), (8) and (9) are completely singular and there are not feedbacks, derivative feedbacks, output injections and derivative output injections regularizing partially or totally the systems (7), (8) and (9).

3. The disturbance decoupling problem

In this section we will use the reduced form for the system in order to analyze the disturbance decoupling problem.

Proposition 3.1. *Consider a system of the form (1). The system can be regularized by means an state and derivative feedback as well as an state and a derivative output injection with index at most one, if and only if the reduced form does not contain parts vi), vii), and viii), and if it contains v) then the nilpotent matrix N_1 is the zero matrix.*

Proof. It suffices to observe that a system is regularisable if and only if the reduced form is regularisable and the index of the system is the index of matrix N_1. □

Let $H(\lambda) = \lambda \begin{pmatrix} E & B & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ C & 0 & 0 \end{pmatrix}$ be a pencil associate to the system (E, A, B, C).

Theorem 3.1. *Consider a system of the form (1). The system can be regularized by means an state and derivative feedback as well as an state and derivative output injection with index at most one if and only if i)* $r_1 - r_0 \geq$

n, ii) $s_k \leq 2(r_B - t)$ *and iii)* $l_k \leq 2(r_C - t)$, *where* $r_0 = rank\begin{pmatrix} E & B \\ C & 0 \end{pmatrix}$, s_k *is the number of column minimal indices of the pencil* $H(\lambda)$, $r_B = rank B$, l_k *is the number of row minimal indices of the pencil* $H(\lambda)$, $r_C = rank C$, $t = r_n - r_{n-1} - n$ *and* $r_\ell = rm\ rank M_\ell, \forall \ell \geq 1$

$$M_\ell = \begin{pmatrix} \begin{smallmatrix} E & B \\ C & 0 \\ A & 0 \end{smallmatrix} & \begin{smallmatrix} E & B \\ C & 0 \\ A & 0 \end{smallmatrix} & & & \\ & & \ddots & & \\ & & & \begin{smallmatrix} E & B \\ C & 0 \\ A & 0 \end{smallmatrix} & \begin{smallmatrix} E & B \\ C & 0 \end{smallmatrix} \end{pmatrix}$$

$\in M_{(\ell+1)(n+p)\times(\ell+1)(n+m)(C)}$.

Proof. It suffices to observe that the controllable and observable subsystem joint with subsystem (N_1, I_5), correspond to the infinite zeros of the pencil associate. Controllable non observable subsystem corresponds to the column singular part of the pencil and observable non controllable subsystem corresponds to the row singular part of the pencil. □

Using quadruples in its reduced form, extending the equivalence to the quintuples of matrices (i.e. $QG = \overline{G}$) and taking into account [2], lemma 2.4, we have the following proposition.

Proposition 3.2. *Assume* $\overline{G} = \big(G_1 \ldots G_5\big)^t$ *according to the subsystems (2), ..., (6). Let* $s \in C$ *such that* $\det(sI_{n_1} - N_2) \neq 0$, $\det(sI_{n_2} - N_3) \neq 0$, $\det(sI_{n_3} - N_4) \neq 0$, $\det(sI_n - J) \neq 0$ *and* $\det(sN_1 I_{n_5}) \neq 0$, *(it exists because of regularity of the subsystems (2),..., (6)). Then*

i) $C_1(sI_{n_1} - N_2)^{-1}G_1 = 0$ *if and only if rank* $\begin{pmatrix} sI_{n_1} - N_2 & G_1 \\ C_1 & 0 \end{pmatrix} = n_1$,
ii) $(sI_{n_2} - N_3)^{-1}G_2 = 0$ *if and only if* $G_2 = 0$
iii) $C_1(sI_{n_3} - N_4)^{-1}G_3 = 0$ *if and only if rank* $\begin{pmatrix} sI_{n_3} - N_4 & G_3 \\ C_2 & 0 \end{pmatrix} = n_3$
iv) $(sI_{n_4} - J)^{-1}G_4 = 0$ *if and only if* $G_4 = 0$
v) $(sN_1 - I_{n_5})^{-1}G_5 = 0$ *if and only if* $G_5 = 0$

As a consequence we have.

Corollary 3.1. *Let* (E, A, B, C, G) *be a quintuple of matrices in its reduced form, and we assume* $\overline{G} = \big(G_1 \ldots G_5\big)^t$ *according to the decomposition of the system. If* $G_2 = 0$, $G_4 = 0$, $G_5 = 0$, *rank* $\begin{pmatrix} sI_{n_1} - N_2 & G_1 \\ C_1 & 0 \end{pmatrix} = n_1$ *and rank* $\begin{pmatrix} sI_{n_3} - N_2 & G_3 \\ C_1 & 0 \end{pmatrix} = n_3$, *then the given system is trivially disturbance decoupled.*

The disturbance decoupling problem is called with stability if one imposes the additional constraint that the close-loop $(E+BF_E^B+F_E^C C)\dot{x}(t) = (A+BF_A^B+F_A^C C)x(t)+Bu(t)+Gg(t)$, $y(t)=Cx(t)$ system is stable. Remember that a singular system is stable if and only if the spectrum of the system lies in C^{-1}.

Proposition 3.3. *Given a singular system (E, A, B, C). There exist a proportional and derivative feedback as well as a proportional and derivative output injection such that the close-loop system $(E+BF_E^B+F_E^C C, A+BF_A^B+F_A^C C, B, C)$ is stable (and we call stable under proportional and derivative feedback, and proportional and derivative output injection) if and only if rank $\begin{pmatrix} sE-A & B \\ C & 0 \end{pmatrix} = n$, $\forall s \in C^+$.*

Proof. The spectrum of a system coincides with the spectrum of the associate pencil, and the spectrum is invariant under equivalence relation. □

As a consequence we have.

Corollary 3.2. *Let (E, A, B, C, G) be a quintuple of matrices in its reduced form, and we assume $\overline{G} = \begin{pmatrix} G_1 & \dots & G_5 \end{pmatrix}^t$ according to the decomposition of the system. If $G_2 = 0$, $G_4 = 0$, $G_5 = 0$, rank $\begin{pmatrix} sI_{n_1}-N_2 & G_1 \\ C_1 & 0 \end{pmatrix} = n_1$, rank $\begin{pmatrix} sI_{n_3}-N_4 & G_3 \\ C_1 & 0 \end{pmatrix} = n_3$ and $\sigma(J) \subset C^{-1}$. Then the given system is trivially disturbance decoupled with stability.*

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