A Generalized Method for the Transient Analysis of Markov Models of Fault-Tolerant Systems with Deferred Repair

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Technical report DMSD_2004_1
last revision: September 15, 2009
appeared in reduced version in
Communications in Statistics—Simulation and Computation

Abstract

Randomization is an attractive alternative for the transient analysis of continuous time Markov models. The main advantages of the method are numerical stability, well-controlled computation error and ability to specify the computation error in advance. However, the fact that the method can be computationally expensive limits its applicability. Recently, a variant of the (standard) randomization method, called *split regenerative randomization* has been proposed for the efficient analysis of reliability-like models of fault-tolerant systems with deferred repair. In this paper, we generalize that method so that it covers more general reward measures: the expected transient reward rate and the expected averaged reward rate. The generalized method has the same good properties as the standard randomization method and, for large models and large values of the time t at which the measure has to be computed, can be significantly less expensive. The method requires the selection of a subset of states and a *regenerative* state satisfying some conditions. For a class of continuous time Markov models, class C_2', including typical failure/repair reliability models with exponential failure and repair time distributions and deferred repair, natural selections for the subset of states and the regenerative state exist and results are available assessing approximately the computational cost of the method in terms of "visible" model characteristics. Using a large model class C_2' example, we illustrate the performance of the method and show that it can be significantly faster than previously proposed randomization-based methods.

Index Terms: Continuous-time Markov chains. Transient analysis. Randomization. Fault-tolerant systems. Deferred repair.

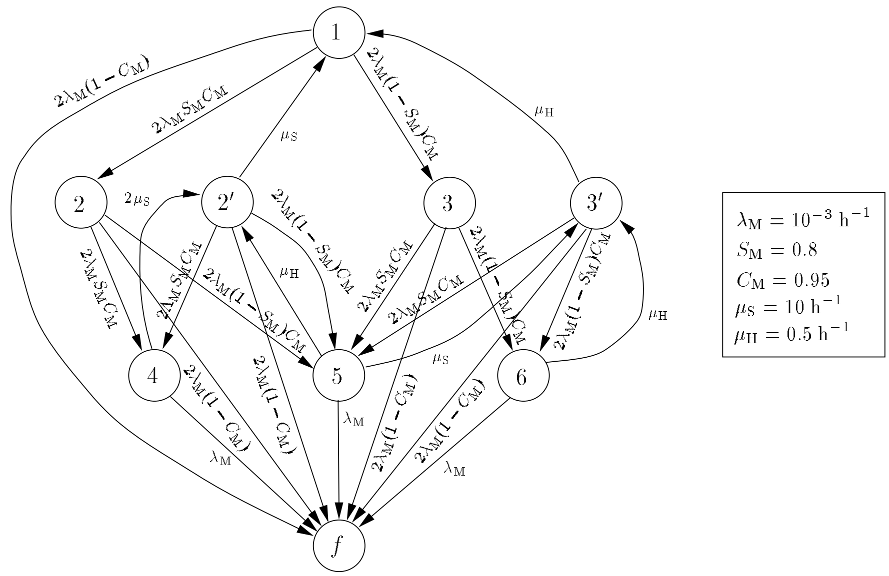

Figure 1: CTMC reliability model of a repairable fault-tolerant system with deferred repair using the pair-and-spare technique.

1 Introduction

Repair deferment is an interesting approach in fault-tolerant systems in which actions of replacement of failed components are expensive, for instance, because the system is located at a remote site. Clearly, there are several tradeoffs that can be analyzed in fault-tolerant systems with deferred repair. One of them could be an appropriate repair-deferment policy: a policy allowing many faults to happen before starting repair could result in too a small system's reliability. These and other tradeoffs can be studied with the aid of models. Homogeneous continuous time Markov chain (CTMC) models are frequently used to analyze the reliability and performability of fault-tolerant systems. To illustrate such models, Figure 1 depicts a small reliability CTMC model of a fault-tolerant system with deferred repair using the pair-and-spare technique [11], in which active modules have failure rate $\lambda_{\rm M}$, the spare module does not fail, the failure of an active module is "soft" with probability $S_{\rm M}$ and "hard" with probability $1 - S_{\rm M}$, and whether soft or hard, the failure of an active module is covered with probability $C_{\rm M}$. Modules in soft failure mode are independently recovered at rate $\mu_{\rm S}$ and modules in hard failure mode are repaired by a single repairman at rate $\mu_{\rm H}$. Repair is deferred till two modules are failed and, when that condition is reached, repair proceeds till reaching the state 1 without failed components, unless the system fails before. The states with deferred repair are states 2 and 3.

Rewarded CTMC models have emerged in the last years as a useful modeling paradigm. Let $X = \{X(t); t \geq 0\}$ be a CTMC with state space Ω modeling the system under study. In this paper, we will consider rewarded CTMC models obtained by defining a reward rate structure $r_i \geq 0, i \in \Omega$. The quantity r_i has the meaning of "rate" at which reward is earned while X is in state i. In that context, two useful measures to consider are the expected transient reward rate $ETRR(t) = E[r_{X(t)}]$ and the expected averaged reward rate $EARR(t) = E[(1/t)\int_0^t r_{X(\tau)}\, d\tau]$. As examples of instances of those generic measures, consider a CTMC modeling a fault-tolerant system with deferred repair that can be up or down, and assume that a reward rate 0 is assigned to the states in which the system is up and a reward rate 1 is assigned to the states in which the system is down. Then, $ETRR(t)$ would be the unavailability of the system at time t and $EARR(t)$ would be the expected interval unavailability at time t (i.e., the expected value of the fraction of time that the system is down in the interval $[0, t]$). The reward rates could also represent the "performance" rate of the system and, then, the $ETRR(t)$ measure would be the expected performance rate of the system at time t and the $EARR(t)$ measure would be the expected averaged performance rate of the system during the time interval $[0, t]$.

Computation of the $ETRR(t)$ and $EARR(t)$ measures involves the transient analysis of X. Randomization (also called uniformization) is a well-known method for performing such analysis. The randomization method is attractive because it is numerically stable and, unlike ODE solvers [14, 15, 21], the computation error is well-controlled and can be specified in advance. It was first proposed by Grassman [9] and has been further developed by Gross and Miller [10]. The randomization method is based on the following result [12, Theorem 4.19]. Let $\lambda_{i,j}$, $i, j \in \Omega$, $j \neq i$, be the transition rate of X from state i to state j and let $\lambda_i = \sum_{j \in \Omega - \{i\}} \lambda_{i,j}$, $i \in \Omega$, be the output rate of X from state i. Consider any $\Lambda \geq \max_{i \in \Omega} \lambda_i$ and define the homogeneous discrete time Markov chain (DTMC) $\widehat{X} = \{\widehat{X}_n; n = 0, 1, 2, \ldots\}$ with same state space and initial probability distribution as X and transition probabilities $P[\widehat{X}_{n+1} = j \mid \widehat{X}_n = i] = P_{i,j} = \lambda_{i,j}/\Lambda$, $i \in \Omega$, $j \neq i$, $P[\widehat{X}_{n+1} = i \mid \widehat{X}_n = i] = P_{i,i} = 1 - \lambda_i/\Lambda$, $i \in \Omega$. Let $Q = \{Q(t); t \geq 0\}$ be a Poisson process with arrival rate Λ independent of $\widehat{X}$ ($P[Q(t) = n] = e^{-\Lambda t}(\Lambda t)^n/n!$). Then, $X = \{X(t); t \geq 0\}$ is probabilistically identical to $\{\widehat{X}_{Q(t)}; t \geq 0\}$. We call this the *randomization result*. We will review next typical implementations of the randomization method for the computation of the $ETRR(t)$ and $EARR(t)$ measures.

Using the randomization result, we can express $ETRR(t)$ as

$$ETRR(t) = \sum_{n=0}^{\infty} d(n)\, e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}\,,$$

with $d(n) = \sum_{i \in \Omega} r_i P[\widehat{X}_n = i]$, and, using $EARR(t) = (1/t) \int_0^t ETRR(\tau)\, d\tau$ and $\int_0^t e^{-\Lambda \tau} (\Lambda \tau)^n/n!\, d\tau = (1/\Lambda) \sum_{l=n+1}^{\infty} e^{-\Lambda t} (\Lambda t)^l/l!$, we can express $EARR(t)$ as

$$EARR(t) = \frac{1}{\Lambda t} \sum_{n=0}^{\infty} d(n) \sum_{l=n+1}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^l}{l!}\,.$$

In a practical implementation of the randomization method, approximate values for $ETRR(t)$, $ETRR_N^a(t)$, and $EARR(t)$, $EARR_N^a(t)$, are obtained by truncating the above summatories:

$$ETRR_N^a(t) = \sum_{n=0}^{N} d(n) e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}\,,$$

$$EARR_N^a(t) = \frac{1}{\Lambda t} \sum_{n=0}^{N} d(n) \sum_{l=n+1}^{N+1} e^{-\Lambda t} \frac{(\Lambda t)^l}{l!} = \frac{1}{\Lambda t} \sum_{n=1}^{N+1} \left(\sum_{l=0}^{n-1} d(l) \right) e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}\,.$$

Taking into account $0 \leq d(n) \leq r_{\max} = \max_{i\in\Omega} r_i$, it can be easily shown that both $ETRR(t) - ETRR_N^a(t)$ and $EARR(t) - EARR_N^a(t)$ are ≥ 0 and are upper bounded by $r_{\max}\sum_{n=N+1}^{\infty} e^{-\Lambda t}(\Lambda t)^n/n!$. Then, being ε an error control parameter, N is chosen as

$$N = \min\left\{ m \geq 0 : r_{\max} \sum_{n=m+1}^{\infty} e^{-\Lambda t}\frac{(\Lambda t)^n}{n!} \leq \varepsilon \right\},$$

guaranteeing an absolute error $\leq \varepsilon$ in both $ETRR(t)$ and $EARR(t)$. Let $\mathbf{q}(n)$ be the row vector $(P[\widehat{X}_n = i])_{i\in\Omega}$ and let $\mathbf{P} = (P_{i,j})_{i,j\in\Omega}$ be the transition probability matrix of $\widehat{X}$. Computation of $ETRR_N^a(t)$ and $EARR_N^a(t)$ requires the knowledge of $\mathbf{q}(n)$, $0 \leq n \leq N$. Vector $\mathbf{q}(0)$ is known, since it is the initial probability row vector of X. Vectors $\mathbf{q}(n)$, $0 < n \leq N$ can be computed from $\mathbf{q}(0)$ using

$$\mathbf{q}(n+1) = \mathbf{q}(n)\mathbf{P}\,. \tag{1}$$

Stable and efficient computation of the Poisson probabilities $e^{-\Lambda t}(\Lambda t)^n/n!$ avoiding overflows and intermediate underflows is a delicate issue and several alternatives have been proposed [3, 8, 13, 19]. Our implementation of all randomization-based methods will use the approach described in [13, pp. 1028–1029] (see also [1]), which has good numerical stability.

For large models, the computational cost of the randomization method is roughly due to the N vector-matrix multiplications (1). The truncation parameter N increases with Λt and, for that reason, Λ is usually taken equal to $\max_{i\in\Omega}\lambda_i$. Using the well-known result [22, Theorem 3.3.5] that $Q(t)$ has for $\Lambda t \to \infty$ an asymptotic normal distribution with mean and variance Λt, it is easy to realize that, for large Λt and $\varepsilon \ll 1$, the required N will be $\approx \Lambda t$. Then, if the model is large and has to be solved for values of t for which Λt is large, the randomization method will be expensive.

Several variants of the (standard) randomization method have been proposed to improve its efficiency. Miller has used selective randomization to solve reliability models with detailed representation of error handling activities [17]. The idea behind selective randomization [16] is to randomize the model only in a subset of the state space. Reibman and Trivedi [21] have proposed an approach based on the multistep concept. The idea is to compute $\mathbf{P}^M$ explicitly, where M is the length of the multistep, and use the recurrence $\mathbf{q}(n+M) = \mathbf{q}(n)\mathbf{P}^M$ to advance $\widehat{X}$ faster for steps which have negligible contributions to the transient solution of X at time t. Since, for large Λt, the number of $\mathbf{q}(n)$'s with significant contributions is of the order of $\sqrt{\Lambda t}$, the multistep concept allows a significant reduction of the required number of vector-matrix multiplications when Λt is large. However, when $\mathbf{P}$ is sparse, significant fill-in can occur when computing $\mathbf{P}^M$. Adaptive uniformization [18] is a method in which the randomization rate is adapted depending on the states in which the randomized DTMC can be at a given step. Numerical experiments have shown that adaptive uniformization can be faster than standard randomization for short to medium mission times. In addition, it can be used to solve models with infinite state spaces and not uniformly bounded output rates. Recently, it has been proposed to combine adaptive uniformization and standard randomization to obtain a method which outperforms both adaptive uniformization and standard randomization for most models [19]. Steady-state detection [14] is another proposal to speed up the standard randomization method. A method based on steady-state detection with error bounds has been developed [23].

Steady-state detection is useful for models which reach their steady-state before the largest time at which the measure has to be computed. Another recently proposed randomization-based method is regenerative randomization [4, 5]. That method covers rewarded CTMC models X with finite state space $\Omega = S \cup \{f_1, f_2, \ldots, f_A\}$, $A \geq 0$, satisfying some conditions. In the method, a truncated transformed model is obtained having the same measure as the original model with some arbitrarily small error and the truncated transformed model is, then, solved by standard randomization. The method requires the selection of a regenerative state $r \in S$ and its performance depends on that selection. The truncated transformed model is constructed by characterizing with enough accuracy the behavior of the original model from $S' = S - \{r\}$ up to state r or a state f_i and from r until next hit of r or a state f_i, and its size depends on how fast the randomized DTMC $\widehat{X}$ of X with a randomization rate slightly larger than $\max_{i \in \Omega} \lambda_i$ hits with high probability r or a state f_i starting at a state in S'. For large enough models and large enough t, regenerative randomization will be significantly more efficient than standard randomization. Furthermore, for a class of models, class C', including typical failure/repair models with exponential failure and repair time distributions and repair in every state with failed components, a natural selection for the regenerative state exists and theoretical results are available assessing approximately the performance of the method for that natural selection in terms of "visible" model characteristics. The bounding regenerative randomization method [6] allows to compute inexpensively tight bounds for a certain class of models, class C", including typical failure/repair reliability-like models with exponential failure and repair time distributions and repair in every state with failed components. Randomization with quasistationarity detection [7] is another recently proposed randomization-based method. The method is applicable to CTMC models with state space $S \cup \{f_1, \ldots, f_A\}$, where the states f_i, $1 \leq i \leq A$, are absorbing and all states in S are transient and reachable from each other, and is based on the existence of a quasistationary distribution in the subset of transient states of DTMCs with a certain structure. For those models and large t the method can be significantly more efficient than the standard randomization method.

Recently, it has been proposed [24] a method called *split regenerative randomization* that is specifically targeted to the transient analysis of CTMC models of fault-tolerant systems with deferred repair. The method covers CTMCs X with finite state space $\Omega = S \cup \{f_1, f_2, \ldots, f_A\}$, $|S| \geq 3$, $A \geq 1$, where f_i are absorbing states and S has to satisfy some conditions, and allows to compute the measure $m(t) = \sum_{i=1}^{A} r_{f_i} P[X(t) = f_i]$, where all r_{f_i} are different and ≥ 0. The method requires the selection of a subset E of states and a regenerative state r. For a class of CTMC models, model class C_2, including typical failure/repair models of fault-tolerant systems with exponential failure and repair time distributions and deferred repair, natural selections for E and r exist and, for those natural selections, theoretical results are available predicting approximately the computational cost of the method. Numerical experiments have shown that, for models in that class, the method can be significantly faster than all other randomization-based methods.

In this paper we generalize the split regenerative randomization method. The generalized method considers the same class of CTMCs as the previously proposed split regenerative randomization method with $A \geq 0$[1] and allows to compute the $ETRR(t)$ and $EARR(t)$ measures with an arbitrary reward rate structure $r_i \geq 0$, $i \in \Omega$. The method has the same good properties as standard randomization (numerical stability, well-controlled computation error, and ability to specify the computation error in advance) and can be much faster than that method. In fact, it can be proved that the computational cost of the method increases smoothly with t. That property is called "benign" behavior. For a class of rewarded CTMC models, class C_2', generalizing model class C_2, the computational cost of the generalized method can be predicted approximately. The rest of the paper is organized as follows. Section 2 develops the generalized method. Section 3 states the benign behavior of the method, discusses qualitatively the efficiency of the method compared with that of standard randomization, defines the model class C_2', and discusses how the computational cost of the method for those models can be predicted approximately. Using a large class C_2' model, Section 4 analyzes the performance of the method and compares it with that of standard randomization, regenerative randomization, randomization with quasistationarity detection and, for $ETRR(t)$, adaptive uniformization, which has been shown [18] to improve the performance of standard randomization for failure/repair models with deferred repair for short to medium mission times. Finally, Section 5 concludes the paper. The Appendix includes a long, technical proof.

2 The generalized method

The method covers rewarded CTMCs X with finite state space Ω and selections of the subset of states E and the regenerative state r such that, letting $E' = E - \{r\}$ and $\overline{E} = S - E$, the following conditions are satisfied:

C1. $\Omega = S \cup \{f_1, \ldots, f_A\}$, $|S| \geq 3$, $A \geq 0$, where the states f_i, $1 \leq i \leq A$, are absorbing and either all states in S are transient or S includes a single recurrent class of states $C \subset S$.

C2. All states are reachable (from some state with nonnull initial probability).

C3. $r_i \geq 0$, $i \in \Omega$, and all r_{f_i} are different.

C4. $E \subset S$.

C5. $r \in E$ and if X includes a single recurrent class of states $C \subset S$, $r \in C$.

C6. $|E| \geq 2$.

C7. $|\overline{E}| \geq 1$.

C8. r can only be entered from $\overline{E}$ ($\lambda_{i,r} = 0$, $i \in E'$).

C9. r is the only entry point in E ($\lambda_{i,j} = 0$, $i \in \overline{E}$, $j \in E'$).

C10. $\lambda_{r,j} > 0$ for some $j \in E'$.

[1] The case $A = 0$ was not previously considered because in that case the $m(t)$ measure is identical to 0. The developments made in [24] for the case $A \geq 1$ carry immediately to the more general case $A \geq 0$ considered here.

Condition C10 can be easily circumvented in practice by adding, in case $\lambda_{r,j} = 0$ for all $j \in E'$, a tiny transition rate $\lambda \leq 10^{-10}\varepsilon/(2r_{\max}t_{\max})$ from r to some state in E', where ε is the allowed error, $r_{\max} = \max_{i \in \Omega} r_i$, and $t_{\max}$ is the largest time at which the measure has to be computed, introducing an error $\leq 10^{-10}\varepsilon$ in both $ETRR(t)$ and $EARR(t)$, $t \leq t_{\max}$ (see [5]). Also, if X has a single recurrent class of states $C \subset S$, by conditions C5 and C10, $|C| \geq 2$, since $|C| = 1$ would imply through condition C5 that r would be absorbing, in contradiction with condition C10. Therefore, when the method is applicable, $f_1, f_2, \ldots, f_A$ have to be the only absorbing states. This makes it easy to check whether the method is applicable to a given finite CTMC with given selections for E and r. The part $r_i \geq 0$, $i \in \Omega$, from condition C3 can be circumvented by shifting the reward rates by a positive quantity d so that all new reward rates $r'_i = r_i + d$ are ≥ 0. The $ETRR(t)$ and $EARR(t)$ measures of the original rewarded CTMC are related to the corresponding measures, $ETRR'(t)$ and $EARR'(t)$, of the rewarded CTMC with shifted reward rates by $ETRR(t) = ETRR'(t) - d$ and $EARR(t) = EARR'(t) - d$. The part that all reward rates of states f_i are different from condition C3 can be obviated by merging absorbing states with same reward rate. Finally, condition C2 can be obviated by deleting non-reachable states.

In the following, we will let $\alpha_i = P[X(0) = i]$, $\alpha_C = \sum_{i \in C} \alpha_i$, $C \subset \Omega$, and $\lambda_{i,C} = \sum_{j \in C} \lambda_{i,j}$, $C \subset \Omega - \{i\}$. Also, given a DTMC $Y = \{Y_n; n = 0, 1, 2, \ldots\}$, we will use the notation $Y_{l:m}c$ for the predicate which is true when Y_n satisfies condition c for all n, $l \leq n \leq m$ (by convention, the predicate will be true for $l > m$) and $\#(Y_{l:m}c)$ for the number of indices n, $l \leq n \leq m$, for which Y_n satisfies condition c.

In the generalized method, a truncated transformed rewarded CTMC model is built having with error $\leq \varepsilon/2$ the same $ETRR(t)$ and $EARR(t)$ measures as the original rewarded CTMC model X and the $ETRR(t)$ ($EARR(t)$) measure of the truncated transformed rewarded CTMC model is computed with error $\leq \varepsilon/2$ using the standard randomization method.

Let $\widehat{X}$ be the DTMC obtained by randomizing X with rate Λ_E in E and rate $\Lambda_{\overline{E}}$ in $\overline{E} \cup \{f_1, f_2, \ldots, f_A\}$, where Λ_E is slightly larger than $\max_{i \in E} \lambda_i$ and $\Lambda_{\overline{E}}$ is slightly larger than $\max_{i \in \overline{E}} \lambda_i$, e.g. $\Lambda_E = (1+\theta)\max_{i \in E} \lambda_i$, $\Lambda_{\overline{E}} = (1+\theta)\max_{i \in \overline{E}} \lambda_i$, where θ is a small quantity, say, 10^{-4}. The DTMC $\widehat{X}$ has same state space and initial probability distribution as X and transition probabilities $P_{i,j} = \lambda_{i,j}/\Lambda_E$, $i \in E$, $j \neq i$, $P_{i,i} = 1 - \lambda_i/\Lambda_E$, $i \in E$, $P_{i,j} = \lambda_{i,j}/\Lambda_{\overline{E}}$, $i \in \overline{E} \cup \{f_1, f_2, \ldots, f_A\}$, $j \neq i$, $P_{i,i} = 1 - \lambda_i/\Lambda_{\overline{E}}$, $i \in \overline{E} \cup \{f_1, f_2, \ldots, f_A\}$. Note that $P_{i,i} > 0$, $i \in \Omega$. We will say that $\widehat{X}$ is the randomized DTMC of X with randomization rate Λ_E in E and $\Lambda_{\overline{E}}$ in $\overline{E} \cup \{f_1, f_2, \ldots, f_A\}$ and that X is the derandomized CTMC of $\widehat{X}$ with randomization rate Λ_E in E and $\Lambda_{\overline{E}}$ in $\overline{E} \cup \{f_1, f_2, \ldots, f_A\}$. In the following we will let $P_{i,C} = \sum_{j \in C} P_{i,j}$, $C \subset \Omega$

As in [24], to develop the generalized method we will find it convenient to consider three DTMCs. The first one, $Z = \{Z_n; n = 0, 1, 2, \ldots\}$, follows $\widehat{X}$ from r till re-entry in r. Formally, Z can be defined from a version, $\widehat{X}'$, of $\widehat{X}$ with initial state r as

$$Z_0 = r\,,$$

$$Z_n = \begin{cases} i & \text{if } \widehat{X}'_{1:n} \neq r \wedge \widehat{X}'_n = i, i \in S' \cup \{f_1, f_2, \ldots, f_A\}\,, \\ a & \text{if } \#(\widehat{X}'_{1:n} = r) > 0\,. \end{cases} \tag{2}$$

The DTMC Z has state space $S \cup \{f_1, f_2, \ldots, f_A, a\}$, where f_i, $1 \leq i \leq A$, and a are absorbing states and all states in S are transient (Proposition 5 in [24]), and its (possibly) nonnull transition probabilities are:

$$P[Z_{n+1} = j \mid Z_n = i] = P_{i,j},\ i \in S, j \in S' \cup \{f_1, f_2, \ldots, f_A\}\,,$$
$$P[Z_{n+1} = a \mid Z_n = i] = P_{i,r},\ i \in S\,,$$
$$P[Z_{n+1} = f_i \mid Z_n = f_i] = P[Z_{n+1} = a \mid Z_n = a] = 1,\ 1 \leq i \leq A\,.$$

The second DTMC, $Z' = \{Z'_n; n = 0, 1, 2, \ldots\}$, follows $\widehat{X}$ from E' till its first visit to r. Formally Z' can be defined from $\widehat{X}$ as

$$Z'_n = \begin{cases} i & \text{if } \widehat{X}_0 \in E' \wedge \widehat{X}_{1:n} \neq r \wedge \widehat{X}_n = i, i \in S' \cup \{f_1, f_2, \ldots, f_A\}\,, \\ a & \text{otherwise}\,. \end{cases} \tag{3}$$

The DTMC Z' has state space $S' \cup \{f_1, f_2, \ldots, f_A, a\}$, where f_i, $1 \leq i \leq A$, and a are absorbing states and all states in S' are transient (Proposition 6 in [24]). The initial probability distribution of Z' is $P[Z'_0 = i] = \alpha_i$, $i \in E'$, $P[Z'_0 = i] = 0$, $i \in \overline{E} \cup \{f_1, f_2, \ldots, f_A\}$, $P[Z'_0 = a] = \alpha_{\{r\} \cup \overline{E} \cup \{f_1, f_2, \ldots, f_A\}}$, and its (possibly) nonnull transition probabilities are:

$$P[Z'_{n+1} = j \mid Z'_n = i] = P_{i,j},\ i \in S', j \in S' \cup \{f_1, f_2, \ldots, f_A\}\,,$$
$$P[Z'_{n+1} = a \mid Z'_n = i] = P_{i,r},\ i \in S'\,,$$
$$P[Z'_{n+1} = f_i \mid Z'_n = f_i] = P[Z'_{n+1} = a \mid Z'_n = a] = 1,\ 1 \leq i \leq A\,.$$

The third DTMC, $Z'' = \{Z''_n; n = 0, 1, 2, \ldots\}$, follows $\widehat{X}$ from $\overline{E}$ till its first visit to state r. Z'' can be defined from $\widehat{X}$ as (note that, by condition C9, the only entry point of $\widehat{X}$ in E is state r)

$$Z''_n = \begin{cases} i & \text{if } \widehat{X}_0 \in \overline{E} \wedge \widehat{X}_{1:n} \neq r \wedge \widehat{X}_n = i, i \in \overline{E} \cup \{f_1, f_2, \ldots, f_A\}\,, \\ a & \text{otherwise}\,. \end{cases} \tag{4}$$

The DTMC Z'' has state space $\overline{E} \cup \{f_1, f_2, \ldots, f_A, a\}$, where f_i, $1 \leq i \leq A$, and a are absorbing states and all states in $\overline{E}$ are transient (Proposition 7 in [24]). The initial probability distribution of Z'' is $P[Z''_0 = i] = \alpha_i, i \in \overline{E}$, $P[Z''_0 = f_i] = 0, 1 \leq i \leq A$, $P[Z''_0 = a] = \alpha_{E \cup \{f_1, f_2, \ldots, f_A\}}$, and its (possibly) nonnull transition probabilities are:

$$P[Z''_{n+1} = j \mid Z''_n = i] = P_{i,j},\ i \in \overline{E}, j \in \overline{E} \cup \{f_1, f_2, \ldots, f_A\}\,,$$
$$P[Z''_{n+1} = a \mid Z''_n = i] = P_{i,r},\ i \in \overline{E}\,,$$
$$P[Z''_{n+1} = f_i \mid Z''_n = f_i] = P[Z''_{n+1} = a \mid Z''_n = a] = 1,\ 1 \leq i \leq A\,.$$

Let $\mathbf{P} = (P_{i,j})_{i,j \in \Omega}$ be the transition probability matrix of $\widehat{X}$. Denoting by $\mathbf{P}_{C',C''}$, $C', C'' \subset \Omega$, the subblock of $\mathbf{P}$ collecting the transition probabilities from states in C' to states in C'' and letting $\mathbf{P}'_{E,E}$ the matrix identical to $\mathbf{P}_{E,E}$ except that the elements of the column corresponding to state r are 0, the transition probability matrix of Z restricted to its subset of transient states, S, has, with the ordering of states $E, \overline{E}$, the form:

$$\mathbf{P}_Z = \begin{pmatrix} \mathbf{P}'_{E,E} & \mathbf{P}_{E,\overline{E}} \\ \mathbf{0} & \mathbf{P}_{\overline{E},\overline{E}} \end{pmatrix}, \tag{5}$$

where $\mathbf{0}$ is a matrix of all zeroes of appropriate dimensions. The restriction of the transition probability matrix of Z' to its subset of transient states, S', has with the ordering of states E', $\overline{E}$ the form:

$$\mathbf{P}_{Z'} = \begin{pmatrix} \mathbf{P}_{E',E'} & \mathbf{P}_{E',\overline{E}} \\ \mathbf{0} & \mathbf{P}_{\overline{E},\overline{E}} \end{pmatrix}. \tag{6}$$

The transition probability matrix of Z'' restricted to its subset of transient states, $\overline{E}$, is

$$\mathbf{P}_{Z''} = \mathbf{P}_{\overline{E},\overline{E}}\,.$$

Let $\pi_i(n) = P[Z_n = i]$, $i \in E$, $\pi_i(n,l) = P[Z_n \in E \wedge Z_{n+1:n+l} \in \overline{E} \wedge Z_{n+l} = i]$, $i \in \overline{E}$, $\pi'_i(n) = P[Z'_n = i]$, $i \in E'$, $\pi'_i(n,l) = P[Z'_n \in E' \wedge Z'_{n+1:n+l} \in \overline{E} \wedge Z'_{n+l} = i]$, $i \in \overline{E}$, and $\pi''_i(n) = P[Z''_n = i]$, $i \in \overline{E}$, and consider the row vectors $\boldsymbol{\pi}(n) = (\pi_i(n))_{i\in E}$, $\boldsymbol{\pi}(n,l) = (\pi_i(n,l))_{i\in\overline{E}}$, $\boldsymbol{\pi}'(n) = (\pi'_i(n))_{i\in E'}$, $\boldsymbol{\pi}'(n,l) = (\pi'_i(n,l))_{i\in\overline{E}}$, and $\boldsymbol{\pi}''(n) = (\pi''_i(n))_{i\in\overline{E}}$. Assuming that, within E, state r is numbered first, those vectors, can be computed for $n \geq 0$, $l \geq 1$ using:

$$\boldsymbol{\pi}(0) = (1\;0\;0\;\cdots\;0)\,,$$

$$\boldsymbol{\pi}(n+1) = \boldsymbol{\pi}(n)\mathbf{P}'_{E,E},\; n \geq 0\,,$$

$$\boldsymbol{\pi}(n,1) = \boldsymbol{\pi}(n)\mathbf{P}_{E,\overline{E}},\; n \geq 0\,,$$

$$\boldsymbol{\pi}(n,l+1) = \boldsymbol{\pi}(n,l)\mathbf{P}_{\overline{E},\overline{E}},\; l \geq 1\,,$$

$$\boldsymbol{\pi}'(0) = (\alpha_i)_{i\in E'}\,,$$

$$\boldsymbol{\pi}'(n+1) = \boldsymbol{\pi}'(n)\mathbf{P}_{E',E'},\; n \geq 0\,,$$

$$\boldsymbol{\pi}'(n,1) = \boldsymbol{\pi}'(n)\mathbf{P}_{E',\overline{E}},\; n \geq 0\,,$$

$$\boldsymbol{\pi}'(n,l+1) = \boldsymbol{\pi}'(n,l)\mathbf{P}_{\overline{E},\overline{E}},\; l \geq 1\,,$$

$$\boldsymbol{\pi}''(0) = (\alpha_i)_{i\in\overline{E}}\,,$$

$$\boldsymbol{\pi}''(n+1) = \boldsymbol{\pi}''(n)\mathbf{P}_{\overline{E},\overline{E}},\; n \geq 0\,.$$

To define the truncated transformed model we will consider a discrete-time stochastic process $\widehat{V} = \{\widehat{V}_n; n = 0, 1, 2, \ldots\}$ defined from $\widehat{X}$ as:

$$\widehat{V}_n = \begin{cases} s_k & \text{if } 0 \le k \le n \wedge \widehat{X}_{n-k} = r \wedge \widehat{X}_{n-k+1:n} \in E' \,, \\ s_{k,l} & \text{if } 0 \le k \le n-1 \wedge\ 1 \le l \le n-k \wedge \widehat{X}_{n-k-l} = r \\ & \quad \wedge\ \widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E} \,, \\ s'_n & \text{if } \widehat{X}_{0:n} \in E' \,, \\ s'_{k,n-k} & \text{if } 0 \le k \le n-1 \wedge \widehat{X}_{0:k} \in E' \wedge \widehat{X}_{k+1:n} \in \overline{E} \,, \\ s''_n & \text{if } \widehat{X}_{0:n} \in \overline{E} \,, \\ f_i & \text{if } \widehat{X}_n = f_i \,. \end{cases} \tag{7}$$

In words, $\widehat{V}_n = s_k$ if, by step n, $\widehat{X}$ has not left S, has visited r, the last time it visited r was k steps before, and has not left E since then; $\widehat{V}_n = s_{k,l}$ if $\widehat{X}$ has not left S, has visited r, the last time it visited r was $k+l$ steps before and, since then, has been first $k+1$ steps in E and, after that, l steps in $\overline{E}$; $\widehat{V}_n = s'_n$ if, by step n, $\widehat{X}$ has not left E'; $\widehat{V}_n = s'_{k,n-k}$ if, by step n, $\widehat{X}$ has been in E' the first $k+1$ steps and, after that, has been in $\overline{E}$ $n-k$ steps; $\widehat{V}_n = s''_n$ if, by step n, $\widehat{X}$ has not left $\overline{E}$; and $\widehat{V}_n = f_i$ if, by step n, $\widehat{X}$ has been absorbed into f_i. Note that $\widehat{V}_n = s_0$ if and only if $\widehat{X}_n = r$ and that $\widehat{V}_n = f_i$ if and only if $\widehat{X}_n = f_i$. Let

$$a(k) = \sum_{i \in E} \pi_i(k) \,, \tag{8}$$

$$a(k,l) = \sum_{i \in \overline{E}} \pi_i(k,l) \,, \tag{9}$$

$$a'(k) = \sum_{i \in E'} \pi'_i(k) \,, \tag{10}$$

$$a'(k,l) = \sum_{i \in \overline{E}} \pi'_i(k,l) \,, \tag{11}$$

$$a''(k) = \sum_{i \in \overline{E}} \pi''_i(k) \,, \tag{12}$$

$$w_k = \frac{\sum_{i \in E} \pi_i(k) P_{i,E'}}{a(k)} \,, \tag{13}$$

$$v^i_k = \frac{\sum_{j \in E} \pi_j(k) P_{j,f_i}}{a(k)} \,, \tag{14}$$

$$h_k = \frac{\sum_{i \in E} \pi_i(k) P_{i,\overline{E}}}{a(k)} \,, \tag{15}$$

$$w_{k,l} = \frac{\sum_{i \in \overline{E}} \pi_i(k,l) P_{i,\overline{E}}}{a(k,l)} \,, \tag{16}$$

$$q_{k,l} = \frac{\sum_{i \in \overline{E}} \pi_i(k,l) P_{i,r}}{a(k,l)} \,, \tag{17}$$

$$v^i_{k,l} = \frac{\sum_{j \in \overline{E}} \pi_j(k,l) P_{j,f_i}}{a(k,l)} \,, \tag{18}$$

$$w'_k = \frac{\sum_{i \in E'} \pi'_i(k) P_{i,E'}}{a'(k)} \,, \tag{19}$$

$$v'^i_k = \frac{\sum_{j\in E'} \pi'_j(k) P_{j,f_i}}{a'(k)}\,, \tag{20}$$

$$h'_k = \frac{\sum_{i\in E'} \pi'_i(k) P_{i,\overline{E}}}{a'(k)}\,, \tag{21}$$

$$w'_{k,l} = \frac{\sum_{i\in \overline{E}} \pi'_i(k,l) P_{i,\overline{E}}}{a'(k,l)}\,, \tag{22}$$

$$q'_{k,l} = \frac{\sum_{i\in \overline{E}} \pi'_i(k,l) P_{i,r}}{a'(k,l)}\,, \tag{23}$$

$$v'^i_{k,l} = \frac{\sum_{j\in \overline{E}} \pi'_j(k,l) P_{j,f_i}}{a'(k,l)}\,, \tag{24}$$

$$w''_k = \frac{\sum_{i\in \overline{E}} \pi''_i(k) P_{i,\overline{E}}}{a''(k)}\,, \tag{25}$$

$$q''_k = \frac{\sum_{i\in \overline{E}} \pi''_i(k) P_{i,r}}{a''(k)}\,, \tag{26}$$

$$v''^i_k = \frac{\sum_{j\in \overline{E}} \pi''_j(k) P_{j,f_i}}{a''(k)}\,. \tag{27}$$

Note that, being $P_{r,E'} > 0$ (by condition C10) and $P_{i,i} > 0$, $i \in E'$, there will exist $i \in E$ with $\pi_i(k) > 0$ for all $k \geq 0$, implying $a(k) > 0$ for all $k \geq 0$. Also, for k such that $a(k,1) > 0$, we have $\pi_i(k,1) > 0$ for some $i \in \overline{E}$ and, since $P_{i,i} > 0$, $i \in \overline{E}$, there will exist $i \in \overline{E}$ with $\pi_i(k,l) > 0$ for all $l \geq 1$, implying $a(k,l) > 0$ for all $l \geq 1$. In addition, assuming $\alpha_{E'} > 0$, $\pi'_i(0) > 0$ for some $i \in E'$ and, since $P_{i,i} > 0$, $i \in E'$, there will exist $i \in E'$ with $\pi'_i(k) > 0$ for all $k \geq 0$, implying $a'(k) > 0$ for all $k \geq 0$. Assuming $\alpha_{E'} > 0$, for k such that $a'(k,1) > 0$, $\pi'_i(k,1) > 0$ for some $i \in \overline{E}$ and, since $P_{i,i} > 0$, $i \in \overline{E}$, there will exist $i \in \overline{E}$ with $\pi'_i(k,l) > 0$, implying $a'(k,l) > 0$ for all $l \geq 1$. Finally, assuming $\alpha_{\overline{E}} > 0$, $\pi''_i(0) > 0$ for some $i \in \overline{E}$ and, since $P_{i,i} > 0$, $i \in \overline{E}$, there will exist $i \in \overline{E}$ with $\pi''_i(k) > 0$ for all $k \geq 0$, implying $a''(k) > 0$ for all $k \geq 0$.

Assume $\alpha_{E'} > 0$ and $\alpha_{\overline{E}} > 0$. Then, it has been shown in [24] that $\widehat{V}$ is a DTMC with reachable state space $E_V \cup \overline{E}_V \cup \{f_1, f_2, \ldots, f_A\}$, $E_V = \{s_k, k \geq 0\} \cup \{s'_k, k \geq 0\}$, $\overline{E}_V = \{s_{k,l} : k \geq 0 \wedge a(k,1) > 0 \wedge l \geq 1\} \cup \{s'_{k,l} : k \geq 0 \wedge a'(k,1) > 0 \wedge l \geq 1\} \cup \{s''_k, k \geq 0\}$, initial probability distribution $P[\widehat{V}_0 = s_0] = \alpha_r$, $P[\widehat{V}_0 = s'_0] = \alpha_{E'}$, $P[\widehat{V}_0 = s''_0] = \alpha_{\overline{E}}$, $P[\widehat{V}_0 = f_i] = \alpha_{f_i}, 1 \leq i \leq A$, $P[\widehat{V}_0 = i] = 0$, $i \notin \{s_0, s'_0, s''_0, f_1, f_2, \ldots, f_A\}$, and (possibly) non-null transition probabilities $P[\widehat{V}_{n+1} = s_0 \mid \widehat{V}_n = s_0] = P_{r,r}$, $P[\widehat{V}_{n+1} = s_{k+1} \mid \widehat{V}_n = s_k] = w_k$, $P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s_k] = v^i_k$, $P[\widehat{V}_{n+1} = s_{k,1} \mid \widehat{V}_n = s_k] = h_k$, $P[\widehat{V}_{n+1} = s_{k,l+1} \mid \widehat{V}_n = s_{k,l}] = w_{k,l}$, $P[\widehat{V}_{n+1} = s_0 \mid \widehat{V}_n = s_{k,l}] = q_{k,l}$, $P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s_{k,l}] = v^i_{k,l}$, $P[\widehat{V}_{n+1} = s'_{k+1} \mid \widehat{V}_n = s'_k] = w'_k$, $P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s'_k] = v'^i_k$, $P[\widehat{V}_{n+1} = s'_{k,1} \mid \widehat{V}_n = s'_k] = h'_k$, $P[\widehat{V}_{n+1} = s'_{k,l+1} \mid \widehat{V}_n = s'_{k,l}] = w'_{k,l}$, $P[\widehat{V}_{n+1} = s_0 \mid \widehat{V}_n = s'_{k,l}] = q'_{k,l}$, $P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s'_{k,l}] = v'^i_{k,l}$, $P[\widehat{V}_{n+1} = s''_{k+1} \mid \widehat{V}_n = s''_k] = w''_k$, $P[\widehat{V}_{n+1} = s_0 \mid \widehat{V}_n = s''_k] = q''_k$, $P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s''_k] = v''^i_k$, $P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = f_i] = 1$, where $a(k)$, $a(k,l)$, $a'(k)$, $a'(k,l)$, $a''(k)$, w_k, v^i_k, h_k, $w_{k,l}$, $q_{k,l}$, $v^i_{k,l}$, w'_k, v'^i_k, h'_k, $w'_{k,l}$, $q'_{k,l}$, $v'^i_{k,l}$, w''_k, q''_k, and v''^i_k are given by (8)-(27). The state transition diagram of $\widehat{V}$ has, for the case $\alpha_{E'} > 0$ and $\alpha_{\overline{E}} > 0$, two combs and a string of states as illustrated in Figure 2 for the case $A = 1$. The first comb has as a back the states s_k and as teeth the strings of states $s_{k,l}$ with k fixed. The second comb has as a back the states s'_k and as teeth the strings of states $s'_{k,l}$ with k fixed. The string includes the states s''_k. When $\alpha_{E'} = 0$, $\widehat{V}$ loses the second comb. When $\alpha_{\overline{E}} = 0$, $\widehat{V}$ loses the string of states s''_k. Formally, the state space of $\widehat{V}$ can be defined in the general case as $E_V \cup \overline{E}_V \cup \{f_1, f_2, \ldots, f_A\}$, where, when $\alpha_{E'} = 0$, E_V does not include the states s'_k and $\overline{E}_V$ does not include the states $s'_{k,l}$ and, when $\alpha_{\overline{E}} = 0$, $\overline{E}_V$ does not include the states s''_k.

Let $V = \{V(t); t \geq 0\}$ be the CTMC obtained by derandomizing $\widehat{V}$ with rate Λ_E in E_V and rate $\Lambda_{\overline{E}}$ in $\overline{E}_V \cup \{f_1, f_2 \ldots, f_A\}$. The CTMC V has same state space and initial probability distribution as $\widehat{V}$. Figure 3 illustrates the state transition diagram of V for the case $\alpha_{E'} > 0$, $\alpha_{\overline{E}} > 0$ and $A = 1$.

All developments up to now (with the generalization to the case $A \geq 0$) are taken from [24]. Let I_c denote the indicator function returning the value 1 if condition c is satisfied and the value 0 otherwise and let, conventionally, the product of 0 by a non-defined quantity be equal to 0. The key to generalize the method is the following result:

Proposition 1. *For $i \in S$,*

$$\begin{aligned} P[X(t) = i] &= I_{i \in E} \sum_{k=0}^{\infty} \frac{\pi_i(k)}{a(k)} P[V(t) = s_k] + I_{i \in \overline{E}} \sum_{k=0}^{\infty} I_{a(k,1)>0} \sum_{l=1}^{\infty} \frac{\pi_i(k,l)}{a(k,l)} P[V(t) = s_{k,l}] \\ &+ I_{\alpha_{E'}>0} \left(I_{i \in E'} \sum_{k=0}^{\infty} \frac{\pi'_i(k)}{a'(k)} P[V(t) = s'_k] + I_{i \in \overline{E}} \sum_{k=0}^{\infty} I_{a'(k,1)>0} \sum_{l=1}^{\infty} \frac{\pi'_i(k,l)}{a'(k,l)} P[V(t) = s'_{k,l}] \right) \\ &+ I_{\alpha_{\overline{E}}>0} I_{i \in \overline{E}} \sum_{k=0}^{\infty} \frac{\pi''_i(k)}{a''(k)} P[V(t) = s''_k] \,. \end{aligned}$$

Proof. See the Appendix. □

Let $ETRR^V(t)$ and $EARR^V(t)$ be, respectively, the expected transient reward rate and the expected averaged reward rate of V with the reward rate structure:

$$r'_{f_i} = r_{f_i} \,, \tag{28}$$

$$r'_{s_k} = b(k) = \frac{\sum_{i \in E} r_i \pi_i(k)}{a(k)} \,, \tag{29}$$

$$r'_{s_{k,l}} = b(k,l) = \frac{\sum_{i \in \overline{E}} r_i \pi_i(k,l)}{a(k,l)} \,, \tag{30}$$

$$r'_{s'_k} = b'(k) = \frac{\sum_{i \in E'} r_i \pi'_i(k)}{a'(k)} \,, \tag{31}$$

$$r'_{s'_{k,l}} = b'(k,l) = \frac{\sum_{i \in \overline{E}} r_i \pi'_i(k,l)}{a'(k,l)} \,, \tag{32}$$

$$r'_{s''_k} = b''(k) = \frac{\sum_{i \in \overline{E}} r_i \pi''_i(k)}{a''(k)} \,. \tag{33}$$

Then:

Theorem 1. $ETRR^V(t) = ETRR(t)$ *and* $EARR^V(t) = EARR(t)$.

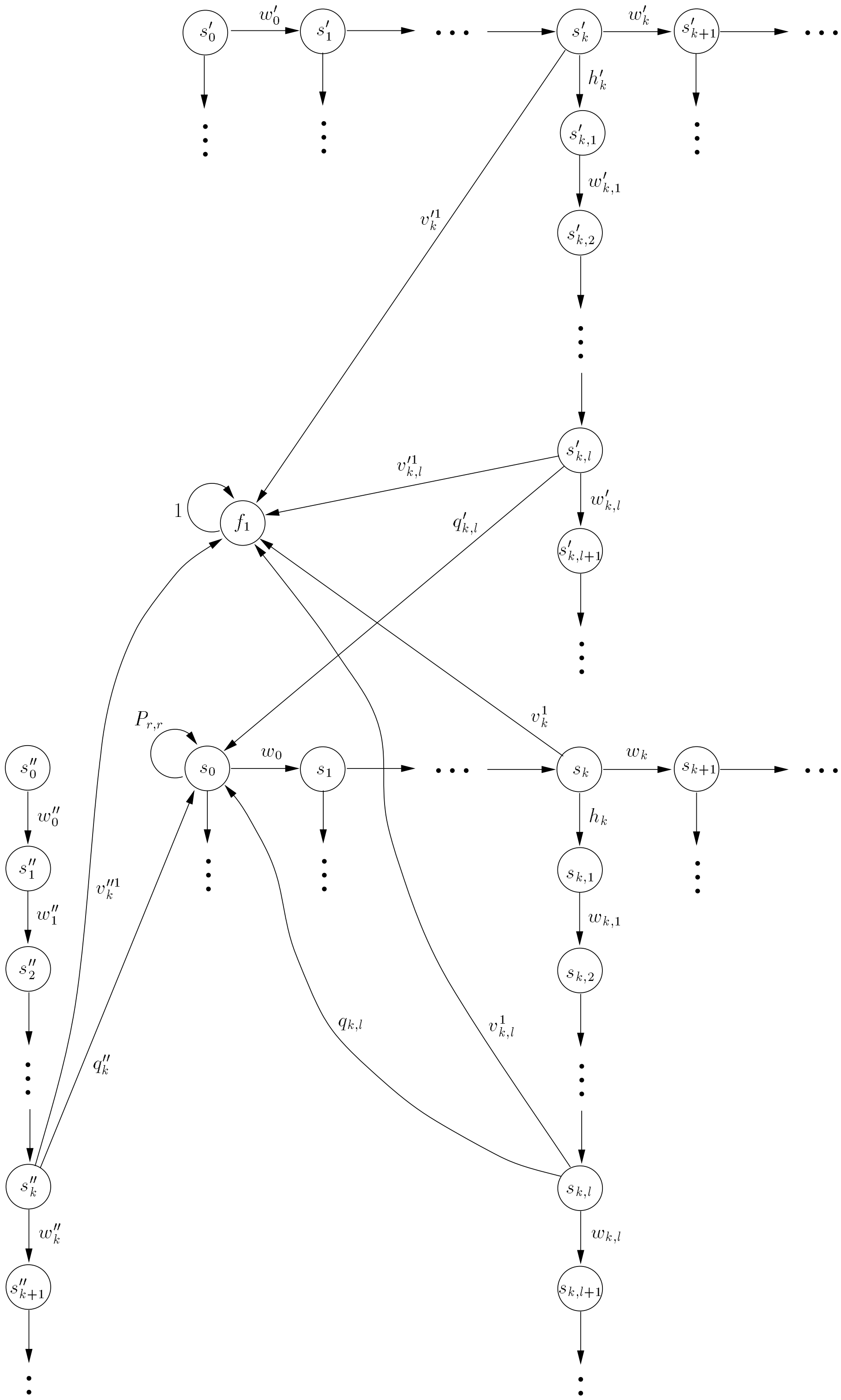

Figure 2: State transition diagram of the DTMC $\widehat{V}$ for the case $\alpha_{E'} > 0$, $\alpha_{\overline{E}} > 0$, and $A = 1$. (There can exist transitions to f_1 from any state and transitions to s_0 from any state $s_{k,l}$, $s'_{k,l}$, and s''_k.)

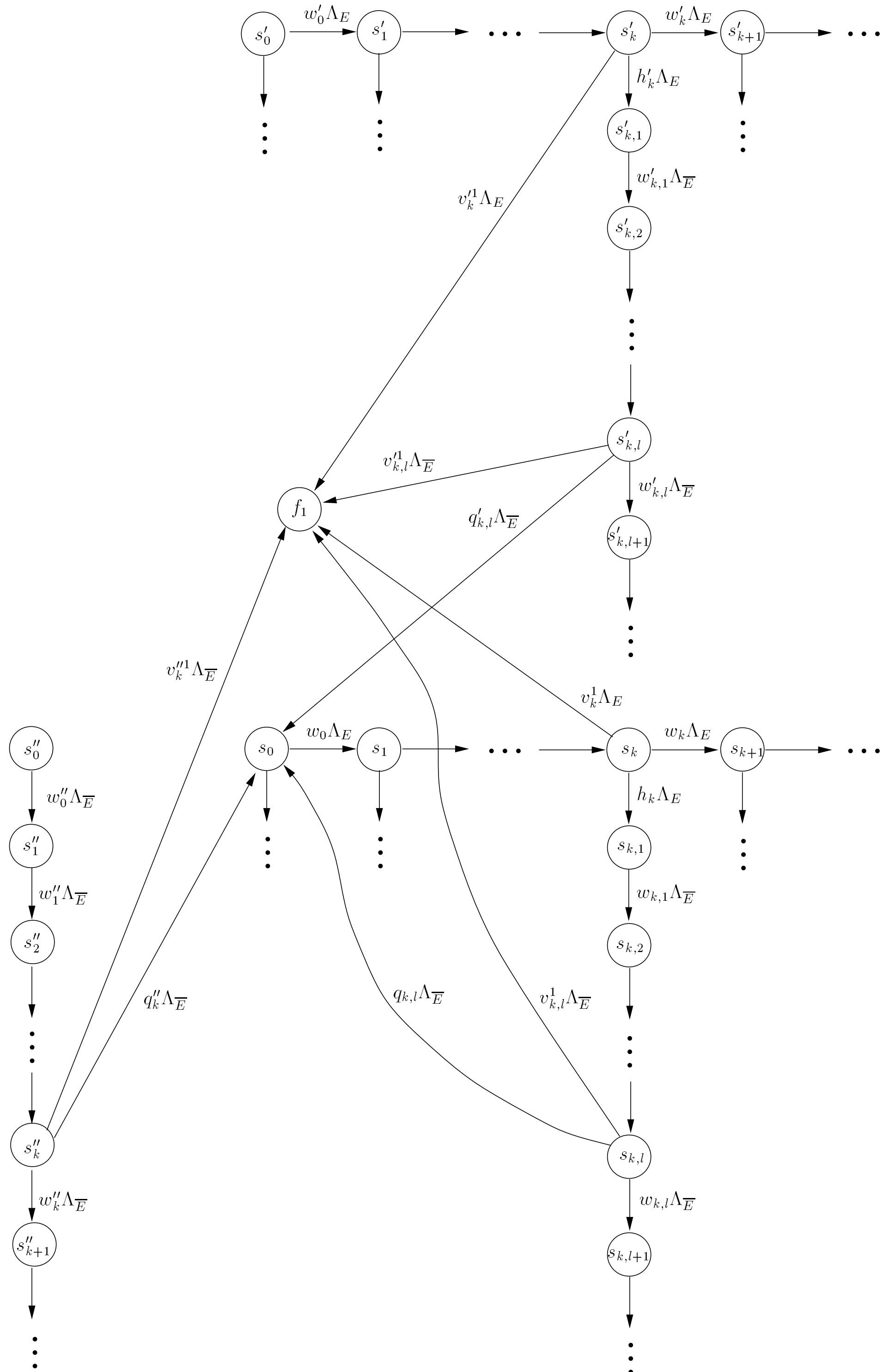

Figure 3: State transition diagram of the CTMC V for the case $\alpha_{E'} > 0$, $\alpha_{\overline{E}} > 0$, and $A = 1$. (There can exist transitions to f_1 from any state and transitions to s_0 from any state $s_{k,l}$, $s'_{k,l}$, and s''_k.)

Proof. Using (proof of Theorem 1 of [24]) $P[V(t) = f_i] = P[X(t) = f_i]$, $1 \leq i \leq A$, Proposition 1, and (28)–(33):

$$
\begin{aligned}
ETRR(t) &= \sum_{i \in \Omega} r_i P[X(t) = i] = \sum_{i \in S} r_i P[X(t) = i] + \sum_{i=1}^{A} r_{f_i} P[X(t) = f_i] \\
&= \sum_{k=0}^{\infty} \frac{\sum_{i \in E} r_i \pi_i(k)}{a(k)} P[V(t) = s_k] \\
&\quad + \sum_{k=0}^{\infty} I_{a(k,1)>0} \sum_{l=1}^{\infty} \frac{\sum_{i \in \overline{E}} r_i \pi_i(k,l)}{a(k,l)} P[V(t) = s_{k,l}] \\
&\quad + \ I_{\alpha_{E'}>0} \left(\sum_{k=0}^{\infty} \frac{\sum_{i \in E'} r_i \pi'_i(k)}{a'(k)} P[V(t) = s'_k] \right. \\
&\qquad\qquad \left. + \sum_{k=0}^{\infty} I_{a(k,1)>0} \sum_{l=1}^{\infty} \frac{\sum_{i \in \overline{E}} r_i \pi'_i(k,l)}{a'(k,l)} P[V(t) = s'_{k,l}] \right) \\
&\quad + \ I_{\alpha_{\overline{E}}>0} \sum_{k=0}^{\infty} \frac{\sum_{i \in \overline{E}} r_i \pi''_i(k)}{a''(k)} P[V(t) = s''_k] + \sum_{i=1}^{A} r_{f_i} P[V(t) = f_i] \\
&= \sum_{k=0}^{\infty} b(k) P[V(t) = s_k] + \sum_{k=0}^{\infty} I_{a(k,1)>0} \sum_{l=1}^{\infty} b(k,l) P[V(t) = s_{k,l}] \\
&\quad + \ I_{\alpha_{E'}>0} \left(\sum_{k=0}^{\infty} b'(k) P[V(t) = s'_k] + \sum_{k=0}^{\infty} I_{a'(k,1)>0} \sum_{l=1}^{\infty} b'(k,l) P[V(t) = s'_{k,l}] \right) \\
&\quad + \ I_{\alpha_{\overline{E}}>0} \sum_{k=0}^{\infty} b''(k) P[V(t) = s''_k] + \sum_{i=1}^{A} r'_{f_i} P[V(t) = f_i] \\
&= ETRR^V(t)\,.
\end{aligned}
$$

Finally, using $EARR(t) = (1/t) \int_0^t ETRR(\tau)\, d\tau$ and $EARR^V(t) = (1/t) \int_0^t ETRR^V(\tau)\, d\tau$,

$$
EARR(t) = \frac{1}{t} \int_0^t ETRR(\tau)\, d\tau = \frac{1}{t} \int_0^t ETRR^V(\tau)\, d\tau = EARR^V(t)\,. \quad \square
$$

The truncated transformed rewarded CTMC, V_T, is obtained from V by introducing an absorbing state a with null reward rate capturing the truncated behavior and: 1) keeping the states s_k up to s_K, $K \geq 1$, and directing to a the transition rates from s_K; 2) for each k, $0 \leq k \leq K-1$, for which $a(k,1) > 0$, keeping the states $s_{k,l}$ up to $l = K_k \geq 1$ and directing the transition rates from s_{k,K_k} to a; if $\alpha_{E'} > 0$, 3) keeping the states s'_k up to s'_L, $L \geq 1$, and directing to a the transition rates from s'_L and 4) for each k, $0 \leq k \leq L-1$, for which $a'(k,1) > 0$, keeping the states $s'_{k,l}$ up to $l = L_k \geq 1$ and directing the transitions rates from s_{k,L_k} to a; and, if $\alpha_{\overline{E}} > 0$, 5) keeping the states s''_k up to s''_M, $M \geq 1$, and directing to a the transition rates from s''_M. The CTMC V_T can be defined from V as:

$$
V_T(t) = \begin{cases} V(t) & \text{if, by time } t,\ V \text{ has not exited state } s_K, \text{ a state } s_{k,K_k}, \text{ state } s'_L, \\ & \text{a state } s'_{k,L_k}, \text{ or state } s''_M\,; \\ a & \text{otherwise}\,. \end{cases} \tag{34}
$$

The initial probability distribution of V_T is the same as that of V, i.e. $P[V_T(0) = s_0] = \alpha_r$, $P[V_T(0) = s'_0] = \alpha_{E'}$, $P[V_T(0) = s''_0] = \alpha_{\overline{E}}$, $P[V_T(0) = f_i] = \alpha_{f_i}$, $1 \le i \le A$, $P[V_T(0) = i] = 0$, $i \notin \{s_0, s'_0, s''_0, f_1, f_2, \ldots, f_A\}$. Let E_V^T denote the set of states in E_V kept in V_T and let $\overline{E}_V^T$ denote the set of states in $\overline{E}_V$ kept in V_T. Note that the state space of V_T is $E_V^T \cup \overline{E}_V^T \cup \{f_1, f_2, \ldots, f_A, a\}$.

The truncated transformed rewarded CTMC model V_T yields approximate values $ETRR^a(t)$ and $EARR^a(t)$, for, respectively, $ETRR(t)$ and $EARR(t)$. Formally, $ETRR^a(t)$ and $EARR^a(t)$ are, respectively, the expected transient reward rate and expected averaged reward rate of V_T. Let $r_{\max} = \max_{i\in\Omega} r_i$. The following two theorems upper bound the model truncation error for, respectively, the measure $ETRR(t)$ and the measure $EARR(t)$.

Theorem 2. $0 \le ETRR(t) - ETRR^a(t) \le r_{\max} P[V_T(t) = a] = ETRR^e(t)$.

Proof. We can write:

$$\begin{aligned}
ETRR(t) - ETRR^a(t) &= \sum_{i \in E_V \cup \overline{E}_V} r_i P[V(t) = i] + \sum_{i=1}^{A} r_{f_i} P[V(t) = f_i] \\
&\quad - \left(\sum_{i \in E_V^T \cup \overline{E}_V^T} r_i P[V_T(t) = i] + \sum_{i=1}^{A} r_{f_i} P[V_T(t) = f_i] \right) \\
&= \sum_{i \in (E_V - E_V^T) \cup (\overline{E}_V - \overline{E}_V^T)} r_i P[V(t) = i] + \sum_{i \in E_V^T \cup \overline{E}_V^T} r_i \left(P[V(t) = i] - P[V_T(t) = i] \right) \\
&\quad + \sum_{i=1}^{A} r_{f_i} \left(P[V(t) = f_i] - P[V_T(t) = f_i] \right) .
\end{aligned}$$

According to (34), $P[V_T(t) = i] \le P[V(t) = i]$, $i \in E_V^T \cup \overline{E}_V^T$ and $P[V_T(t) = f_i] \le P[V(t) = f_i]$, $1 \le i \le A$, implying $ETRR(t) - ETRR^a(t) \ge 0$. Also, since $\sum_{i \in E_V \cup \overline{E}_V} P[V(t) = i] + \sum_{i=1}^{A} P[V(t) = f_i] = 1$ and $\sum_{i \in E_V^T \cup \overline{E}_V^T} P[V_T(t) = i] + \sum_{i=1}^{A} P[V_T(t) = f_i] + P[V_T(t) = a] = 1$:

$$\begin{aligned}
&ETRR(t) - ETRR^a(t) \\
&\quad \le r_{\max} \Bigg(\sum_{i \in (E_V - E_V^T) \cup (\overline{E}_V - \overline{E}_V^T)} P[V(t) = i] + \sum_{i \in E_V^T \cup \overline{E}_V^T} \left(P[V(t) = i] - P[V_T(t) = i] \right) \\
&\qquad + \sum_{i=1}^{A} (P[V(t) = f_i] - P[V_T(t) = f_i]) \Bigg) \\
&\quad = r_{\max} \Bigg(\sum_{i \in E_V \cup \overline{E}_V} P[V(t) = i] + \sum_{i=1}^{A} P[V(t) = f_i] \\
&\qquad - \sum_{i \in E_V^T \cup \overline{E}_V^T} P[V_T(t) = i] - \sum_{i=1}^{A} P[V_T(t) = f_i] \Bigg) \\
&\quad = r_{\max} \left(1 - \sum_{i \in E_V^T \cup \overline{E}_V^T} P[V_T(t) = i] - \sum_{i=1}^{A} P[V_T(t) = f_i] \right)
\end{aligned}$$

$$= \quad r_{\max} P[V_T(t) = a] = ETRR^e(t)\,. \quad \square$$

Theorem 3. $0 \leq EARR(t) - EARR^a(t) \leq (r_{\max}/t) \int_0^t P[V_T(\tau) = a]\, d\tau = EARR^e(t)$.

Proof. Using $EARR(t) = (1/t) \int_0^t ETRR(\tau)\, d\tau$, $EARR^a(t) = (1/t) \int_0^t ETRR^a(\tau)\, d\tau$, and Theorem 2,

$$EARR(t) - EARR^a(t) = \frac{1}{t} \int_0^t ETRR(\tau)\, d\tau - \frac{1}{t} \int_0^t ETRR^a(\tau)\, d\tau$$

$$= \frac{1}{t} \int_0^t \left(ETRR(\tau) - ETRR^a(\tau) \right)\, d\tau\,,$$

$$0 \leq EARR(t) - EARR^a(t) \leq \frac{r_{\max}}{t} \int_0^t P[V_T(\tau) = a]\, d\tau\,. \quad \square$$

The upper bound for the model truncation error for the $ETRR(t)$ measure given by Theorem 2 is formally identical to the model truncation error upper bound for the less general measure considered in [24]. Then, letting $\gamma_K = \{k \,:\, 0 \leq k \leq K-1 \wedge a(k,1) > 0\}$ and $\gamma'_L = \{k \,:\, 0 \leq k \leq L-1 \wedge a'(k,1) > 0\}$, we can state the following result:

Theorem 4.

$$\begin{aligned} ETRR^e(t) \leq\ & I_{\alpha_{\overline{E}}>0} r_{\max} a''(M) \sum_{k=M+1}^{\infty} e^{-\Lambda_{\overline{E}} t} \frac{(\Lambda_{\overline{E}} t)^k}{k!} \\ & + I_{\alpha_{E'}>0} \left(r_{\max} a'(L) \sum_{k=L+1}^{\infty} e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!} + \sum_{k \in \gamma'_L} r_{\max}\, a'(k, L_k) \sum_{l=k+1}^{\infty} e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!} \right) \\ & + r_{\max} (\alpha_S - a''(M)) a(K) \sum_{k=K+1}^{\infty} (k-K) e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!} \\ & + \sum_{k \in \gamma_K} r_{\max} (\alpha_S - a''(M)) a(k, K_k) \sum_{l=k+1}^{\infty} (l-k) e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!}\,. \end{aligned}$$

The following theorem gives an upper bound for the model truncation error for the $EARR(t)$ measure.

Theorem 5.

$$\begin{aligned} EARR^e(t) \leq\ & I_{\alpha_{\overline{E}}>0} \frac{r_{\max} a''(M)}{\Lambda_{\overline{E}} t} \sum_{k=M+2}^{\infty} (k-M-1) e^{-\Lambda_{\overline{E}} t} \frac{(\Lambda_{\overline{E}} t)^k}{k!} \\ & + I_{\alpha_{E'}>0} \left(\frac{r_{\max} a'(L)}{\Lambda_E t} \sum_{k=L+2}^{\infty} (k-L-1) e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!} \right. \\ & \left. + \sum_{k \in \gamma'_L} \frac{r_{\max} a'(k, L_k)}{\Lambda_E t} \sum_{l=k+2}^{\infty} (l-k-1) e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!} \right) \end{aligned}$$

$$+\frac{r_{\max}(\alpha_S-a''(M))a(K)}{\Lambda_E t}\sum_{k=K+2}^{\infty}\frac{(k-K)(k-K-1)}{2}e^{-\Lambda_E t}\frac{(\Lambda_E t)^k}{k!}$$
$$+\sum_{k\in\gamma_K}\frac{r_{\max}(\alpha_S-a''(M))a(k,K_k)}{\Lambda_E t}\sum_{l=k+2}^{\infty}\frac{(l-k)(l-k-1)}{2}e^{-\Lambda_E t}\frac{(\Lambda_E t)^l}{l!}\,.$$

Proof. From Theorems 2, 3 and 4:

$$EARR^e(t)=\frac{r_{\max}}{t}\int_0^t P[V_T(\tau)=a]\,d\tau=\frac{1}{t}\int_0^t ETRR^e(\tau)\,d\tau$$
$$\leq\ I_{\alpha_{\overline{E}}>0}\frac{r_{\max}a''(M)}{t}\sum_{k=M+1}^{\infty}\int_0^t e^{-\Lambda_{\overline{E}}\tau}\frac{(\Lambda_{\overline{E}}\tau)^k}{k!}\,d\tau$$
$$+I_{\alpha_{E'}>0}\Bigg(\frac{r_{\max}a'(L)}{t}\sum_{k=L+1}^{\infty}\int_0^t e^{-\Lambda_E\tau}\frac{(\Lambda_E\tau)^k}{k!}\,d\tau$$
$$+\sum_{k\in\gamma'_L}\frac{r_{\max}a'(k,L_k)}{t}\sum_{l=k+1}^{\infty}\int_0^t e^{-\Lambda_E\tau}\frac{(\Lambda_E\tau)^l}{l!}\,d\tau\Bigg)$$
$$+\frac{r_{\max}(\alpha_S-a''(M))a(K)}{t}\sum_{k=K+1}^{\infty}(k-K)\int_0^t e^{-\Lambda_E\tau}\frac{(\Lambda_E\tau)^k}{k!}\,d\tau$$
$$+\sum_{k\in\gamma_K}\frac{r_{\max}(\alpha_S-a''(M))a(k,K_k)}{t}\sum_{l=k+1}^{\infty}(l-k)\int_0^t e^{-\Lambda_E\tau}\frac{(\Lambda_E\tau)^l}{l!}\,d\tau\,.$$

Using $\int_0^t e^{-\Lambda\tau}(\Lambda\tau)^k/k!\,d\tau=(1/\Lambda)\sum_{l=k+1}^{\infty}e^{-\Lambda t}(\Lambda t)^l/l!$:

$$\sum_{k=M+1}^{\infty}\int_0^t e^{-\Lambda_{\overline{E}}\tau}\frac{(\Lambda_{\overline{E}}\tau)^k}{k!}\,d\tau=\frac{1}{\Lambda_{\overline{E}}}\sum_{k=M+2}^{\infty}(k-M-1)e^{-\Lambda_{\overline{E}}t}\frac{(\Lambda_{\overline{E}}t)^k}{k!}\,,$$
$$\sum_{k=L+1}^{\infty}\int_0^t e^{-\Lambda_E\tau}\frac{(\Lambda_E\tau)^k}{k!}\,d\tau=\frac{1}{\Lambda_E}\sum_{k=L+2}^{\infty}(k-L-1)e^{-\Lambda_E t}\frac{(\Lambda_E t)^k}{k!}\,,$$
$$\sum_{l=k+1}^{\infty}\int_0^t e^{-\Lambda_E\tau}\frac{(\Lambda_E\tau)^l}{l!}\,d\tau=\frac{1}{\Lambda_E}\sum_{l=k+2}^{\infty}(l-k-1)e^{-\Lambda_E t}\frac{(\Lambda_E t)^l}{l!}\,,$$

$$\sum_{k=K+1}^{\infty}(k-K)\int_0^t e^{-\Lambda_E\tau}\frac{(\Lambda_E\tau)^k}{k!}\,d\tau=\frac{1}{\Lambda_E}\sum_{k=K+2}^{\infty}\left(\sum_{l=K+1}^{k-1}(l-K)\right)e^{-\Lambda_E t}\frac{(\Lambda_E t)^k}{k!}$$
$$=\ \frac{1}{\Lambda_E}\sum_{k=K+2}^{\infty}\frac{(k-K)(k-K-1)}{2}e^{-\Lambda_E t}\frac{(\Lambda_E t)^k}{k!}\,,$$

$$\sum_{l=k+1}^{\infty}(l-k)\int_0^t e^{-\Lambda_E\tau}\frac{(\Lambda_E\tau)^l}{l!}\,d\tau=\frac{1}{\Lambda_E}\sum_{l=k+2}^{\infty}\frac{(l-k)(l-k-1)}{2}e^{-\Lambda_E t}\frac{(\Lambda_E t)^l}{l!}\,,$$

and the result follows. □

The truncation parameters K, L, M, K_k, $k\in\gamma_K$, and L_k, $k\in\gamma'_L$, have to be selected so that the upper bound for the model truncation error given by Theorem 4 for the measure $ETRR(t)$

and by Theorem 5 for the measure $EARR(t)$ is $\leq \varepsilon/2$. For the $ETRR(t)$ measure, the truncation parameters are selected as follows. First, for the case $\alpha_{\overline{E}} > 0$, M is selected using:

$$M = \min\left\{m \geq 1 \,:\, r_{\max} a''(m) \sum_{k=m+1}^{\infty} e^{-\Lambda_{\overline{E}} t} \frac{(\Lambda_{\overline{E}} t)^k}{k!} \leq \varepsilon_1\right\},$$

where $\varepsilon_1 = \varepsilon/6$ if $\alpha_{E'} > 0$ and $\varepsilon_1 = \varepsilon/4$ if $\alpha_{E'} = 0$. The truncation parameter K is, then, chosen using:

$$K = \min\left\{m \geq 1 \,:\, r_{\max}(\alpha_S - a''(M)) a(m) \sum_{k=m+1}^{\infty} (k-m) e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!} \leq \varepsilon_2\right\},$$

where $\varepsilon_2 = \varepsilon/12$ if $\alpha_{E'} > 0$ and $\alpha_{\overline{E}} > 0$, $\varepsilon_2 = \varepsilon/8$ if $\alpha_{E'} > 0$ and $\alpha_{\overline{E}} = 0$ or $\alpha_{E'} = 0$ and $\alpha_{\overline{E}} > 0$, and $\varepsilon_2 = \varepsilon/4$ if $\alpha_{E'} = 0$ and $\alpha_{\overline{E}} = 0$ ($a''(M) = 0$ if $\alpha_{\overline{E}} = 0$). The truncation parameters K_k, $k \in \gamma_K$, are chosen using:

$$K_k = \min\left\{m \geq 1 \,:\, r_{\max}(\alpha_S - a''(M)) a(k,m) \sum_{l=k+1}^{\infty} (l-k) e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!} \leq \frac{\varepsilon_2}{|\gamma_K|}\right\}.$$

Finally, for the case $\alpha_{E'} > 0$, the truncation parameter L is chosen using:

$$L = \min\left\{m \geq 1 \,:\, r_{\max} a'(m) \sum_{k=m+1}^{\infty} e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!} \leq \varepsilon_3\right\},$$

where $\varepsilon_3 = \varepsilon/12$ if $\alpha_{\overline{E}} > 0$ and $\varepsilon_3 = \varepsilon/8$ if $\alpha_{\overline{E}} = 0$, and the truncation parameters L_k, $k \in \gamma'_L$, are chosen using:

$$L_k = \min\left\{m \geq 1 \,:\, r_{\max} a'(k,m) \sum_{l=k+1}^{\infty} e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!} \leq \frac{\varepsilon_3}{|\gamma'_L|}\right\}.$$

For the measure $EARR(t)$, for the case $\alpha_{\overline{E}} > 0$, M is selected using:

$$M = \min\left\{m \geq 1 \,:\, \frac{r_{\max} a''(m)}{\Lambda_{\overline{E}} t} \sum_{k=m+2}^{\infty} (k-m-1) e^{-\Lambda_{\overline{E}} t} \frac{(\Lambda_{\overline{E}} t)^k}{k!} \leq \varepsilon_1\right\}.$$

The truncation parameter K is, then, chosen using:

$$K = \min\left\{m \geq 1 \,:\, \frac{r_{\max}(\alpha_S - a''(M)) a(m)}{\Lambda_E t} \sum_{k=m+2}^{\infty} \frac{(k-m)(k-m-1)}{2} e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!} \leq \varepsilon_2\right\}.$$

The truncation parameters K_k, $k \in \gamma_K$, are chosen using:

$$K_k = \min\left\{m \geq 1 \,:\, \frac{r_{\max}(\alpha_S - a''(M)) a(k,m)}{\Lambda_E t} \sum_{l=k+2}^{\infty} \frac{(l-k)(l-k-1)}{2} e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!} \leq \frac{\varepsilon_2}{|\gamma_K|}\right\}.$$

Finally, for the case $\alpha_{E'} > 0$, the truncation parameter L is chosen using:

$$L = \min\left\{m \geq 1 \,:\, \frac{r_{\max} a'(m)}{\Lambda_E t} \sum_{k=m+2}^{\infty} (k-m-1) e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!} \leq \varepsilon_3\right\}$$

and the truncation parameters L_k, $k \in \gamma'_L$, are chosen using:

$$L_k = \min\left\{ m \geq 1 \,:\, \frac{r_{\max} a'(k,m)}{\Lambda_E t} \sum_{l=k+2}^{\infty} (l-k-1) e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!} \leq \frac{\varepsilon_3}{|\gamma'_L|} \right\} .$$

It has been proved in [24] that the upper bound for the model truncation error for the $ETRR(t)$ measure given by Theorem 4 is increasing with t. Since the upper bound for the model truncation error for the $EARR(t)$ measure given by Theorem 5 is the averaged value in the interval $[0,t]$ of the upper bound given by Theorem 4, it follows that the upper bound given by Theorem 5 is also increasing with t. Then, if either $ETRR(t)$ or $EARR(t)$ has to be computed for several values of t, the truncation parameters can be selected using the largest t.

To clarify, Figures 4–5 give a C-like algorithmic description of the method for the $ETRR(t)$ measure. The algorithm has as inputs the CTMC X, the number of absorbing states A, the reward rates r_i, $i \in \Omega$, an initial probability distribution row vector $\boldsymbol{\alpha} = (\alpha_i)_{i\in\Omega}$, the subset $E \subset S$, the regenerative state $r \in E$, the allowed error ε, the number of time points n at which which estimates for the measure have to be computed, and the time points, $t_1, t_2, \ldots, t_n$. The algorithm has as outputs the estimates for the measure at the time points t_i, $\widetilde{ETRR}(t_1), \widetilde{ETRR}(t_2), \ldots, \widetilde{ETRR}(t_n)$. It is assumed that conditions C1–C10 regarding the structure of X and the selection of the subset E and the regenerative state $r \in E$ are satisfied. The truncated transformed CTMC model, called V in the algorithmic description, is built using the functions $add\_state(V,s,p)$ and $add\_transition(V,s,s',\lambda)$. The first function adds to V the state s with initial probability p; the second function adds to V a transition rate λ from state s to state s'. The model truncation error is controlled for $t_{\max} = \max\{t_1, t_2, \ldots, t_n\}$. The algorithm makes two traversals of the backs of the combs: the first one to determine K and $|\gamma_K|$ (called n_k in the algorithm), and, if $\alpha_{E'} > 0$, L and $|\gamma'_L|$ (also called n_k in the algorithm), and the second one to build the teeth. The method for $EARR(t)$ can be described similarly, with the obvious changes.

The method requires the computation of the summatories

$$S(m) = \sum_{k=m+1}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^k}{k!} ,$$

$$S'(m) = \sum_{k=m+1}^{\infty} (k-m) e^{-\Lambda t} \frac{(\Lambda t)^k}{k!} ,$$

$$S''(m) = \sum_{k=m+2}^{\infty} (k-m-1) e^{-\Lambda t} \frac{(\Lambda t)^k}{k!} ,$$

$$S'''(m) = \sum_{k=m+2}^{\infty} \frac{(k-m)(k-m-1)}{2} e^{-\Lambda t} \frac{(\Lambda t)^k}{k!} ,$$

for $\Lambda = \Lambda_E$ or $\Lambda = \Lambda_{\overline{E}}$, $t = t_{\max}$, and increasing values of m. Efficient and numerically stable procedures for computing $S(m)$, $S'(m)$, and $S'''(m)$ are described in [4] and [5]. Since $S''(m) = S'(m+1)$, an efficient and numerically stable procedure for computing $S''(m)$ can be obtained easily by adapting the procedure for computing $S'(m)$.

Inputs: $X, A, r_i, i \in \Omega, \boldsymbol{\alpha}, E, r, \varepsilon, n, t_1, t_2, \ldots, t_n$
Outputs: $\widetilde{ETRR}(t_1), \widetilde{ETRR}(t_2), \ldots, \widetilde{ETRR}(t_n)$
$r_{\max} = \max_{i\in\Omega} r_i$; $t_{\max} = \max\{t_1, t_2, \ldots, t_n\}$;
$\Lambda_E = (1+10^{-4})\max_{i\in E}\lambda_i$; $\Lambda_{\overline{E}} = (1+10^{-4})\max_{i\in\overline{E}}\lambda_i$;
Obtain $\mathbf{P}$;
$\alpha_{E'} = \sum_{i\in E'}\alpha_i$; $\alpha_{\overline{E}} = \sum_{i\in\overline{E}}\alpha_i$; $\alpha_S = \alpha_r + \alpha_{E'} + \alpha_{\overline{E}}$;
for $(i \in E)$ $P_{i,E'} = \sum_{j\in E', P_{i,j}>0} P_{i,j}$; for $(i \in S)$ $P_{i,\overline{E}} = \sum_{j\in\overline{E}, P_{i,j}>0} P_{i,j}$;
Build CTMC V including state s_0 with initial probability α_r, state a with initial probability 0
and states f_i, $1 \le i \le A$, with initial probabilities α_{f_i};
if $(\alpha_{\overline{E}} > 0)\{$
  if $(\alpha_{E'} == 0)$ *tol* $= \varepsilon/4$; else *tol* $= \varepsilon/6$;
  $add\_state(V, s''_0, \alpha_{\overline{E}})$; $\boldsymbol{\pi}'' = (\alpha_i)_{i\in\overline{E}}$; $a'' = \alpha_{\overline{E}}$; $M = 0$;
  do$\{$
    for $(i = 1;\ i \le A;\ i{+}{+})\{$
      $v''^i = \sum_{j\in\overline{E}, P_{j,f_i}>0}\pi''_j P_{j,f_i}/a''$; if $(v''^i > 0)$ $add\_transition(V, s''_M, f_i, v''^i\Lambda_{\overline{E}})$;
    $\}$
    $w'' = \sum_{i\in\overline{E}}\pi''_i P_{i,\overline{E}}/a''$; $q'' = \sum_{i\in\overline{E}}\pi''_i P_{i,r}/a''$; $b''(M) = \sum_{i\in\overline{E}}\pi''_i r_i/a''$
    $add\_state(V, s''_{M+1}, 0)$; $add\_transition(V, s''_M, s''_{M+1}, w''\Lambda_{\overline{E}})$;
    if $(q'' > 0)$ $add\_transition(V, s''_M, s_0, q''\Lambda_{\overline{E}})$;
    $n\boldsymbol{\pi}'' = \boldsymbol{\pi}''\mathbf{P}_{\overline{E},\overline{E}}$; $\boldsymbol{\pi}'' = n\boldsymbol{\pi}''$; $M{+}{+}$; $a'' = \sum_{i\in\overline{E}}\pi''_i$;
  $\}$
  until $(r_{\max}a''\sum_{k=M+1}^{\infty} e^{-\Lambda_{\overline{E}}t_{\max}}(\Lambda_{\overline{E}}t_{\max})^k/k! \le tol)$;
  $b''(M) = \sum_{i\in\overline{E}}\pi''_i r_i/a''$; $add\_transition(V, s''_M, a, \Lambda_{\overline{E}})$;
$\}$
else $a'' = 0$;
if $(\alpha_{E'} > 0$ && $\alpha_{\overline{E}} > 0)$ $tol = \varepsilon/12$; else if $(\alpha_{E'} > 0 \,||\, \alpha_{\overline{E}} > 0)$ $tol = \varepsilon/8$; else $tol = \varepsilon/4$;
$\boldsymbol{\pi} = (I_{i=r})_{i\in E}$; $a = 1$; $K = 0$; $n\_k = 0$;
do$\{$
  for $(i = 1;\ i \le A;\ i{+}{+})\{$
    $v^i = \sum_{j\in E, P_{j,f_i}>0}\pi_j P_{j,f_i}/a$; if $(v^i > 0)$ $add\_transition(V, s_K, f_i, v^i\Lambda_E)$;
  $\}$
  $h_K = \sum_{i\in E}\pi_i P_{i,\overline{E}}/a$; $w = \sum_{i\in E}\pi_i P_{i,E'}/a$; $b(K) = \sum_{i\in E}\pi_i r_i/a$
  $add\_state(V, s_{K+1}, 0)$; $add\_transition(V, s_K, s_{K+1}, w\Lambda_E)$;
  if $(h_K > 0)$ $n\_k{+}{+}$;
  $n\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}'_{E,E}$; $\boldsymbol{\pi} = n\boldsymbol{\pi}$; $K{+}{+}$; $a = \sum_{i\in E}\pi_i$;
$\}$
until $(r_{\max}(\alpha_S - a'')a\sum_{k=K+1}^{\infty}(k-K)e^{-\Lambda_E t_{\max}}(\Lambda_E t_{\max})^k/k! \le tol)$;
$b(K) = \sum_{i\in E}\pi_i r_i/a$; $add\_transition(V, s_K, a, \Lambda_E)$;
$\boldsymbol{\pi} = (I_{i=r})_{i\in E}$;
for $(k = 0;\ k \le K-1;\ k{+}{+})\{$
  if $(h_k > 0)\{$
    $\boldsymbol{\pi}^{\overline{E}} = \boldsymbol{\pi}\mathbf{P}_{E,\overline{E}}$; $K' = 1$; $a = \sum_{i\in\overline{E}}\pi_i^{\overline{E}}$; $add\_state(V, s_{k,1}, 0)$; $add\_transition(V, s_k, s_{k,1}, h_k\Lambda_E)$;
    while $(r_{\max}(\alpha_S - a'')a\sum_{l=k+1}^{\infty}(l-k)e^{-\Lambda_E t_{\max}}(\Lambda_E t_{\max})^l/l! > tol/n\_k)\{$
      for $(i = 1;\ i \le A;\ i{+}{+})\{$
        $v^i = \sum_{j\in\overline{E}, P_{j,f_i}>0}\pi_j^{\overline{E}}P_{j,f_i}/a$; if $(v^i > 0)$ $add\_transition(V, s_{k,K'}, f_i, v^i\Lambda_{\overline{E}})$;
      $\}$
      $w = \sum_{i\in\overline{E}}\pi_i^{\overline{E}}P_{i,\overline{E}}/a$; $q = \sum_{i\in\overline{E}}\pi_i^{\overline{E}}P_{i,r}/a$; $b(k, K') = \sum_{i\in\overline{E}}\pi_i^{\overline{E}}r_i/a$;
      $add\_state(V, s_{k,K'+1}, 0)$; $add\_transition(V, s_{k,K'}, s_{k,K'+1}, w\Lambda_{\overline{E}})$;
      if $(q > 0)$ $add\_transition(V, s_{k,K'}, s_0, q\Lambda_{\overline{E}})$;
      $n\boldsymbol{\pi}^{\overline{E}} = \boldsymbol{\pi}^{\overline{E}}\mathbf{P}_{\overline{E},\overline{E}}$; $\boldsymbol{\pi}^{\overline{E}} = n\boldsymbol{\pi}^{\overline{E}}$; $K'{+}{+}$; $a = \sum_{i\in\overline{E}}\pi_i^{\overline{E}}$;
    $\}$
    $b(k, K') = \sum_{i\in\overline{E}}\pi_i^{\overline{E}}r_i/a$; $add\_transition(V, s_{k,K'}, a, \Lambda_{\overline{E}})$;
  $\}$
  if $(k < K-1)\{$ $n\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}'_{E,E}$; $\boldsymbol{\pi} = n\boldsymbol{\pi}$; $\}$
$\}$

Figure 4: Algorithmic description of split regenerative randomization for the $ETRR(t)$ measure.

if $(\alpha_{E'} > 0)\{$
if $(\alpha_{\overline{E}} > 0)$ $tol = \varepsilon/12$; else $tol = \varepsilon/8$;
$add\_state(V, s'_0, \alpha_{E'})$; $\boldsymbol{\pi}' = (\alpha_i)_{i \in E'}$; $a' = \sum_{i \in E'} \pi'_i$; $L = 0$; $n\_k = 0$;
do$\{$
for $(i = 1;\, i \le A;\, i{+}{+})\{$
$v'^i = \sum_{j \in E', P_{j,f_i} > 0} \pi'_j P_{j,f_i}/a'$; if $(v'^i > 0)$ $add\_transition(V, s'_L, f_i, v'^i \Lambda_E)$;
$\}$
$w' = \sum_{i \in E'} \pi'_i P_{i,E'}/a'$; $h'_L = \sum_{i \in E'} \pi'_i P_{i,\overline{E}}/a'$; $b'(L) = \sum_{i \in E'} \pi'_i r_i/a'$;
$add\_state(V, s'_{L+1}, 0)$; $add\_transition(V, s'_L, s'_{L+1}, w'\Lambda_E)$;
if $(h'_L > 0)$ $n\_k{+}{+}$;
$n\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}_{E',E'}$; $\boldsymbol{\pi}' = n\boldsymbol{\pi}'$; $L{+}{+}$; $a' = \sum_{i \in E'} \pi'_i$;
$\}$
until $(r_{\max} a' \sum_{k=L+1}^{\infty} e^{-\Lambda_E t_{\max}} (\Lambda_E t_{\max})^k / k! \le tol)$;
$b'(L) = \sum_{i \in E'} \pi'_i r_i/a'$; $add\_transition(V, s'_L, a, \Lambda_E)$;
$\boldsymbol{\pi}' = (\alpha_i)_{i \in E'}$;
for $(k = 0;\, k \le L - 1;\, k{+}{+})\{$
if $(h'_k > 0)\{$
$\boldsymbol{\pi}^{\overline{E}} = \boldsymbol{\pi}'\mathbf{P}_{E',\overline{E}}$; $L' = 1$; $a' = \sum_{i \in \overline{E}} \pi_i^{\overline{E}}$; $add\_state(V, s'_{k,1}, 0)$; $add\_transition(V, s'_k, s'_{k,1}, h'_k \Lambda_E)$;
while $(r_{\max} a' \sum_{l=k+1}^{\infty} e^{-\Lambda_E t_{\max}} (\Lambda_E t_{\max})^l / l! > tol/n\_k)\{$
for $(i = 1;\, i \le A;\, i{+}{+})\{$
$v'^i = \sum_{j \in \overline{E}, P_{j,f_i} > 0} \pi_j^{\overline{E}} P_{j,f_i}/a'$; if $(v'^i > 0)$ $add\_transition(V, s'_{k,L'}, f_i, v'^i \Lambda_{\overline{E}})$;
$\}$
$w' = \sum_{i \in \overline{E}} \pi_i^{\overline{E}} P_{i,\overline{E}}/a'$; $q' = \sum_{i \in \overline{E}} \pi_i^{\overline{E}} P_{i,r}/a'$; $b'(k, L') = \sum_{i \in \overline{E}} \pi_i^{\overline{E}} r_i/a'$;
$add\_state(V, s'_{k,L'+1}, 0)$; $add\_transition(V, s'_{k,L'}, s'_{k,L'+1}, w'\Lambda_{\overline{E}})$;
if $(q' > 0)$ $add\_transition(V, s'_{k,L'}, s_0, q'\Lambda_{\overline{E}})$;
$n\boldsymbol{\pi}^{\overline{E}} = \boldsymbol{\pi}^{\overline{E}}\mathbf{P}_{\overline{E},\overline{E}}$; $\boldsymbol{\pi}^{\overline{E}} = n\boldsymbol{\pi}^{\overline{E}}$; $L'{+}{+}$; $a' = \sum_{i \in \overline{E}} \pi_i^{\overline{E}}$;
$\}$
$b'(k, L') = \sum_{i \in \overline{E}} \pi_i^{\overline{E}} r_i/a'$; $add\_transition(V, s'_{k,L'}, a, \Lambda_{\overline{E}})$;
$\}$
if $(k < L - 1)\{$ $n\boldsymbol{\pi}' = \boldsymbol{\pi}'\mathbf{P}_{E',E'}$; $\boldsymbol{\pi}' = n\boldsymbol{\pi}'$;$\}$
$\}$
$\}$
$\Lambda = \max\{\Lambda_E, \Lambda_{\overline{E}}\}$; $N = \min\{m \ge 0 \,:\, r_{\max} \sum_{k=m+1}^{\infty} e^{-\Lambda t_{\max}} (\Lambda t_{\max})^k / k! \le \varepsilon/2\}$;
Let $\widehat{V}$ be the randomized DTMC of V with randomization rate $\Lambda = \max\{\Lambda_E, \Lambda_{\overline{E}}\}$;
Give N steps to $\widehat{V}$ and compute $d(k) = \sum_{l=0}^{K} b(l) P[\widehat{V}_k = s_l] + \sum_{0 \le l \le K-1 : h_l > 0} \sum_{m=1}^{K_l} b(l,m) P[\widehat{V}_k = s_{l,m}]$
$+ I_{\alpha_{E'} > 0} (\sum_{l=0}^{L} b'(L) P[\widehat{V}_k = s'_l] + \sum_{0 \le l \le L-1 : h'_l > 0} \sum_{m=1}^{L_l} b'(l,m) P[\widehat{V}_k = s'_{l,m}])$
$+ I_{\alpha_{\overline{E}} > 0} \sum_{l=0}^{M} b''(l) P[\widehat{V}_k = s''_l] + \sum_{i=1}^{A} r_{f_i} P[\widehat{V}_k = f_i], k = 0, 1, \ldots, N$;
for $(i = 1;\, i \le n;\, i{+}{+})$ for $(k = 0, \widetilde{ETRR}(t_i) = 0;\, k \le N;\, k{+}{+})$ $\widetilde{ETRR}(t_i)$ += $d(k) e^{-\Lambda t_i} (\Lambda t_i)^k / k!$;

Figure 5: Algorithmic description of split regenerative randomization for the $ETRR(t)$ measure (continuation).

We note that, once $\mathbf{P}$ has been computed, the transition rates of the truncated transformed model are obtained without subtractions. Thus, the method has the same excellent numerical stability as the standard randomization method. In addition, the computation error is well-controlled and can be specified in advance.

3 Theoretical properties

The model truncation error bound for the $ETRR(t)$ measure is formally identical to the model truncation error bound for the less general measure considered in [24]. Then, letting $K_{\overline{E}} = \sum_{k\in\gamma_K} K_k$ and $L_{\overline{E}} = \sum_{k\in\gamma'_L} L_k$, we have the following result:

Theorem 6. *The number of steps, K, L, M, $K_{\overline{E}}$, and $L_{\overline{E}}$, required in the split regenerative randomization method for the $ETRR(t)$ measure are, respectively, $O(\log(\Lambda_E t/\varepsilon))$, $O(\log(1/\varepsilon))$, $O(\log(1/\varepsilon))$, $O((\log(\Lambda_E t/\varepsilon))^2)$, and $O((\log(1/\varepsilon))^2)$.*

A similar result is available regarding the $EARR(t)$ measure:

Theorem 7. *The number of steps, K, L, M, $K_{\overline{E}}$, and $L_{\overline{E}}$, required in the split regenerative randomization method for the $EARR(t)$ measure are, respectively, $O(\log(\Lambda_E t/\varepsilon))$, $O(\log(1/\varepsilon))$, $O(\log(1/\varepsilon))$, $O((\log(\Lambda_E t/\varepsilon))^2)$, and $O((\log(1/\varepsilon))^2)$.*

Proof. The terms of the model truncation error bound used in the split regenerative randomization method for the $EARR(t)$ measure are the averaged values in the interval $[0, t]$ of the corresponding terms of the model truncation error bound for the $ETRR(t)$ measure. Furthermore, the terms of the model truncation error bound for the $ETRR(t)$ measure increase with t. Then, the terms of the model truncation error bound for $EARR(t)$ are not greater than the corresponding terms of the model truncation error bound for $ETRR(t)$ and the result follows from Theorem 6. □

Theorems 6 and 7 tell that K, L, M, $K_{\overline{E}}$, and $L_{\overline{E}}$ are all smooth functions of t and ε for both $ETRR(t)$ and $EARR(t)$. That property is called benign behavior and implies that, for large enough X and large enough t, the proposed method will be significantly less costly than standard randomization. This is because 1) the cost of the first phase of the method (generation of the truncated transformed model) is made up of components approximately proportional to, respectively, K, L, M, $K_{\overline{E}}$ and $L_{\overline{E}}$, while the cost of standard randomization is, for large t, approximately proportional to $\max_{i\in\Omega}\lambda_i t$, and 2) being the maximum output rate of the truncated transformed model at most $(1 + \theta)$ times the maximum output rate of the original model, the cost of the second phase of the method (solution of the truncated transformed model by standard randomization) will scale with the cost of standard randomization at most as the size of the truncated transformed model scales with the size of the original model, X.

The performance of the method depends, of course, on the selections for the subset E and the regenerative state r, since those selections influence the behavior of $a(k)$, $a'(k)$, $a''(k)$, $a(k, l)$, and

$a'(k,l)$, and, then, the required values for the truncation parameters K, L, M, K_k, $k \in \gamma_K$, and L_k, $k \in \gamma'_L$. Ideally, E and r should be chosen so that $a(k)$, $a'(k)$, $a''(k)$, $a(k,l)$, and $a'(k,l)$ decrease as fast as possible. For general models, automatic selection of E and r does not seem to be easy in general. A model class, class C'_2, can, however, be defined for which natural selections for E and r exist, and for models in that class and those natural selections, theoretical results are available assessing approximately the performance of the method in terms of "visible" model characteristics.

The model class C'_2 includes all CTMCs X with finite state space Ω satisfying the following conditions:

C11. $\Omega = S \cup \{f_1, f_2, \ldots, f_A\}$, $|S| \geq 3$, $A \geq 0$, where the states f_i, $1 \leq i \leq A$, are absorbing and either all states in S are transient or X has a single recurrent class of states $C \subset S$.

C12. All states are reachable (from some state with nonnull initial probability).

C13. $r_i \geq 0$, $i \in \Omega$ and all r_{f_i} are different.

C14. There exists a partition $S_0 \cup S_1 \cup \ldots \cup S_{N_C} \cup \overline{S}_1 \cup \overline{S}_2 \cup \ldots \cup \overline{S}_{\overline{N}_C}$ for S satisfying the following properties:

- P1. $|S_0| = \{o\}$ (i.e. $|S_0| = 1$).
- P2. If X has a single recurrent class of states $C \subset S$, then $o \in C$.
- P3. $|S_0 \cup S_1 \cup \ldots \cup S_{N_C}| \geq 2$, and $|\overline{S}_1 \cup \overline{S}_2 \cup \ldots \cup \overline{S}_{\overline{N}_c}| \geq 1$.
- P4. $\lambda_{o,S_1 \cup \ldots \cup S_{N_C}} > 0$
- P5. For each $i \in S_k$, $0 < k \leq N_C$, $\lambda_{i,S_0 \cup \cdots \cup S_k} = 0$.
- P6. For each $i \in \overline{S}_k$, $1 \leq k \leq \overline{N}_C$, $\lambda_{i,S_1 \cup \cdots \cup S_{N_c}} = 0$.
- P7. $\max_{1 \leq k \leq \overline{N}_C} \max_{i \in \overline{S}_k} \lambda_{i,\overline{S}_k - \{i\} \cup \overline{S}_{k+1} \cup \cdots \cup \overline{S}_{\overline{N}_C}}$ is significantly smaller than $\min_{1 \leq k \leq \overline{N}_C} \min_{i \in \overline{S}_k} \lambda_{i,S_0 \cup \overline{S}_1 \cup \cdots \cup \overline{S}_{k-1} \cup \{f_1, f_2, \ldots, f_A\}} > 0$.

The class includes failure/repair models with exponential failure and repair time distributions in which repair is deferred until some condition on the subset of failed components is fulfilled and, then, proceeds till the state in which no component is failed is reached, when failure rates are significantly smaller than repair rates. For those models, a partition for S for which properties P1—P7 would be satisfied is the partition in which S_k includes the states without repair and the same number of failed components, with the subsets S_k ordered following increasing number of failed components, and $\overline{S}_k$ includes the states with repair and the same number of failed components, with the subsets $\overline{S}_k$ similarly ordered following increasing number of failed components. Similar failure/repair models with exponential failure time distributions and repair times with acyclic phase-type distributions [20] (which can be used to fit distributions of non-exponential positive random variables [2]), are also covered by model class C'_2, provided that failure rates are significantly smaller than the transition rates of the transient CTMCs defining the phase-type distributions.

With the selection $E = S_0 \cup S_1 \cup \cdots \cup S_{N_C}$ and $r = o$, models in class C_2' satisfy the conditions making the method applicable. Furthermore, with those selections, the models move "fast" from states in $\overline{E}$ to either state o or a state f_i, making those selections natural ones. Let

$$R_E = \frac{\max_{0\leq k\leq N_C} \max_{i\in S_k} \lambda_i}{\min_{0<k\leq N_C} \min_{i\in S_k} \lambda_i}\,,$$
$$R_{\overline{E}} = \frac{\max_{1\leq k\leq \overline{N}_C} \max_{i\in \overline{S}_k} \lambda_i}{\min_{1\leq k\leq \overline{N}_C} \min_{i\in \overline{S}_k} \lambda_i}\,.$$

Note that once E and r have been identified, both R_E and $R_{\overline{E}}$ are model characteristics that can be easily estimated. Let

$$\delta = \frac{\max_{1\leq k\leq \overline{N}_C} \max_{i\in \overline{S}_k} \lambda_{i,\overline{S}_k-\{i\}\cup\overline{S}_{k+1}\cup\cdots\cup\overline{S}_{\overline{N}_C}}}{\min_{1\leq k\leq \overline{N}_C} \min_{i\in \overline{S}_k} \lambda_{i,S_0\cup\overline{S}_1,\cup\cdots\cup\overline{S}_{k-1}\cup\{f_1,\ldots,f_A\}}}\,.$$

The δ can be regarded as a "rarity" parameter measuring how strongly property P7 is satisfied. Then, it has been shown in [24] that with the natural selections for E and r, 1) both $a(k)$ and $a'(k)$ are, for $k \to \infty$, upper bounded by functions of the form $C\binom{k}{p-1}q_E^k$, $C > 0$, p integer ≥ 1, where $q_E \approx 1 - 1/R_E$, and 2) $a(k,l)$, $a'(k,l)$, and $a''(l)$ are, for $l \to \infty$, upper bounded by functions of the form $C(\delta)\binom{l}{p(\delta)-1}\rho(\delta)$, $C(\delta) > 0$, $p(\delta)$ integer ≥ 1, with $\lim_{\delta\to 0}\rho(\delta) = q_{\overline{E}} \approx 1-1/R_{\overline{E}}$. Then, for R_E close to 1, the required K and L should be small and, as R_E gets apart from 1, the required K and L should increase. A similar behavior exhibit M, K_k and L_k with respect to $R_{\overline{E}}$. Moreover, for small ϵ, the required M, K_k, and L_k will be mainly determined by the decay rate of, respectively, $a(k,l)$, $a'(k,l)$, and $a''(l)$ and, following the discussion done in [24], for $R_{\overline{E}} \gg 1$, the required M, K_k, and L_k can be roughly upper bounded by $30R_{\overline{E}}$. Regarding the truncation parameters K and L, for small ε, they can be upper bounded roughly using $a(k) = a'(k) = q_E^k \approx (1-1/R_E)^k$. Then, for class C_2' models with the natural selections for E and r, the computational cost of split regenerative randomization can be estimated roughly.

4 A Large Example

In this section we analyze the performance of the method and will compare it with that of standard randomization, regenerative randomization, randomization with quasistationarity detection, and, for the $ETRR(t)$ measure, adaptive uniformization using a class C_2' performability model of a fault-tolerant multiprocessor including 16 processors interconnected by a 8-node hypercube, as shown in Figure 6. Processors fail with rate λ_P; nodes of the hypercube fail with rate λ_N; links of the hypercube fail with rate λ_L. A fault of a processor is covered with probability C_P; a fault of a node of the hypercube is covered with probability C_N. Coverage to link faults is assumed perfect. There is an unlimited number of repairmen. Repair starts when the number of failed components gets ≥ 2. The repair rate is μ_P for processors, μ_N for nodes, and μ_L for links. A completely down system because there was an uncovered fault is brought to a fully operational state without failed components at rate μ_G. It is assumed the availability of diagnosis and reconfiguration procedures to both determine a subset of interconnected unfailed processors of maximal size and to reconfigure the multiprocessor

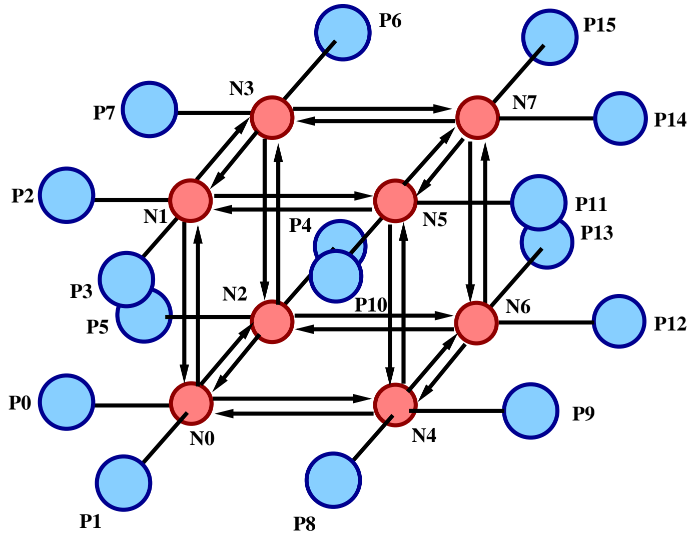

Figure 6: Architecture of the fault-tolerant multiprocessor system.

so that it works using that maximal healthy subset. As reward rates, we take the speedup function of the number of processors in the maximal subset shown in Table 1. Then, $ETRR(t)$ will be the expected speedup of the system at time t and $EARR(t)$ will be the expected speedup of the system averaged over the time interval $[0, t]$. As model parameters we use $\lambda_\mathrm{P} = 2 \times 10^{-5}$ h^{-1}, $\lambda_\mathrm{N} = 10^{-5}$ h^{-1}, $\lambda_\mathrm{L} = 5 \times 10^{-6}$ h^{-1}, $C_\mathrm{P} = 0.99$, $C_\mathrm{N} = 0.995$, $\mu_\mathrm{P} = 0.1$ h^{-1}, $\mu_\mathrm{N} = 0.05$ h^{-1}, $\mu_\mathrm{L} = 0.05$ h^{-1}, and $\mu_\mathrm{G} = 0.2$ h^{-1}. Regarding the initial probability distribution, we will consider two cases: 1) the initial state of the system is the state without failed components, and 2) with probability 0.5 the initial state is the state without failed components, with probability 0.25 the initial state is the state with deferred repair in which processor P0 is the only failed component, and with probability 0.25 the initial state is the state in which processor P0 is the only failed component and repair is underway.

An exact model of the multiprocessor system has an unmanageable size and we will consider instead bounding models with state space $S \cup \{f_1\}$, where S includes the states with up to N_F covered faults and the state in which the system is down due to an uncovered fault and entry into the absorbing state f_1 occurs when the exact model enters a state with more than N_F covered faults. A lower (upper) bound for $ETRR(t)$ and $EARR(t)$ is obtained by assigning to the absorbing state f_1 a reward rate equal to 0 (12). The bounding models belong to model class C_2'. Taking $N_F = 4$ is enough to get very tight bounds. Thus, for case 1 and $t = 100{,}000$ h, the lower and upper bounds thus obtained for $ETRR(t)$ are 11.760559 h^{-1} and 11.760562 h^{-1} and the lower and upper bounds for $EARR(t)$ are 11.762899 h^{-1} and 11.762901 h^{-1}. With that value of N_F, the bounding models have 213,104 states. The reported results are identical for the lower and the upper bounding models. For split regenerative randomization we take for r and E the natural selections, i.e. r is the single state o without failed components and E includes the states in S without repair. With that natural selection, we have $\alpha_{E'} = 0$ and $\alpha_{\overline{E}} = 0$ for case 1 and $\alpha_{E'} > 0$ and $\alpha_{\overline{E}} > 0$ for case 2. For regenerative randomization we use the selection $r = o$. All CPU times are measured on a Sun-Blade 1000, 4 GB workstation running each method with a unique target time t. For all methods we use

Table 1: Speedups of the multiprocessor system as a function of the maximum number of connected operational processors.

processors	speedup
1	1
2	1.96667
3	2.9
4	3.8
5	4.66667
6	5.5
7	6.3
8	7.06667
9	7.8
10	8.5
11	9.16667
12	9.8
13	10.4
14	10.96667
15	11.5
16	12

$\varepsilon = 10^{-10}$.

We start by discussing the dependence on t of the truncation parameters of split regenerative randomization. Table 2 gives the values of the truncation parameters K, L and M, $K_{\overline{E}} = \sum_{k\in\gamma_K} K_k$, and $L_{\overline{E}} = \sum_{k\in\gamma'_L} L_k$ for the method for the $ETRR(t)$ measure; Table 3 gives the corresponding values for the method for the $EARR(t)$ measure. We can note that for both measures and in all cases the truncation parameters increase smoothly with t. Also, the truncation parameters K and L have very small values. This is because having the system many components with quite similar failure rates, the output rates from states in E are very similar and, therefore, R_E is only slightly larger than 1 and q_E is very small. The truncation parameters M, K_k, and L_k have also reasonably small values. In all cases, the truncation parameters for the method for the $EARR(t)$ measure are non-greater than the truncation parameters in the method for the $ETRR(t)$ measure. This can be explained by recalling that the model truncation error bounds for the method for the $EARR(t)$ measure are non-greater than the respective model truncation error bounds for the method for the $ETRR(t)$ measure.

We compare next the performance of split regenerative randomization (SRR) with those of standard randomization (SR), regenerative randomization (RR), randomization with quasistationarity detection (RQD), and, for the $ETRR(t)$ measure, adaptive uniformization (AU). For AU we choose the AU layered uniformization variant for AU processes with converged rate described in [18], since this ensures for AU the same numerical stability as all other three methods have. Figure 7 gives the CPU times for the $ETRR(t)$ measure; Figures 8 gives the CPU times for the $EARR(t)$ measure. We start discussing the results for case 1. Although not clearly seen in Figure 7, for $ETRR(t)$, AU

Table 2: Truncation parameters as a function of t for $ETRR(t)$.

	case 1		case 2				
t (h)	K	$K_{\overline{E}}$	K	$K_{\overline{E}}$	L	$L_{\overline{E}}$	M
1	2	111	2	121	2	199	9
5	3	174	3	194	3	253	16
10	3	202	3	220	3	280	21
50	4	292	4	319	3	351	46
100	4	339	4	367	4	410	69
500	5	500	5	533	5	553	154
1,000	6	603	6	645	5	624	154
5,000	8	904	8	961	7	865	154
10,000	9	1,050	9	1,116	8	958	154
50,000	10	1,276	11	1,380	9	1,021	154
100,000	11	1,359	11	1,443	9	1,021	154

Table 3: Truncation parameters as a function of t for $EARR(t)$.

	case 1		case 2				
t (h)	K	$K_{\overline{E}}$	K	$K_{\overline{E}}$	L	$L_{\overline{E}}$	M
1	2	102	2	112	2	186	8
5	3	155	3	173	2	222	14
10	3	182	3	201	3	258	19
50	4	263	4	287	3	328	42
100	4	308	4	335	4	376	64
500	5	456	5	489	5	508	144
1,000	6	547	6	586	5	576	150
5,000	7	805	8	884	7	802	153
10,000	8	949	9	1,032	8	902	154
50,000	10	1,214	10	1,290	9	1,007	154
100,000	10	1,276	11	1,380	9	1,014	154

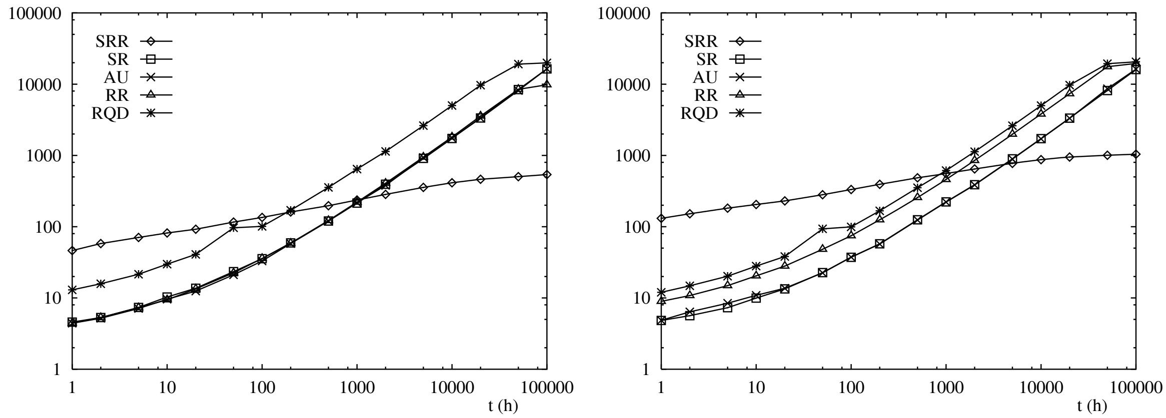

Figure 7: CPU times in seconds for the $ETRR(t)$ measure: case 1 (left), case 2 (right).

is, with few exceptions, the fastest method for t non larger than about 1,000 h. Compared with SR, there is a crossing point at about 5,000 h below which AU is faster and above which AU is slower. This fact is in accordance with the known behavior of AU with respect to SR [18]. RR performs not much worse than SR for both $ETRR(t)$ and $EARR(t)$. In addition, since the size of the truncated transformed model built in RR is logarithmic in t and the number of steps required in SR grows linearly with t, for t large enough RR will eventually become faster than SR. In the example, RR becomes faster than SR for t larger than about 50,000 h for both $ETRR(t)$ and $EARR(t)$. For the considered values of t, RQD is the more expensive method, but it would outperform also SR for larger t's. Finally, SRR is the fastest method for t beyond approximately 1,000 h. For $t = 100{,}000$ h, SRR is, for the $ETRR(t)$ measure, about 18.2 times faster than the fastest of the other methods (RR) and, for the $EARR(t)$ measure, about 19.3 times faster than the fastest of the other methods (RR). In case 2, there is almost no difference in performance between AU and SR for the $ETRR(t)$ measure. This is because, in that case, the adapted randomization rate used in AU is large from the initial steps. In that case RR compares worse with SR than it did in case 1. The reason is that when the initial probability distribution is not concentrated in the regenerative state (the state without failed components), the truncated transformed model built in RR is larger than when that initial probability distribution is concentrated in the regenerative state [5]. The performance of RQD is, however, very similar to the performance of that method in case 1. As in case 1, for t large enough, SRR is the fastest method. However, the time beyond which SRR is the fastest method is now about 5,000 h for both measures, larger than in case 1. The reason is that the truncated transformed model is larger than in case 1 because of the presence of the comb having as back the states $s'_0, s'_1, \ldots, s'_L$ and the string of states $s''_0, s''_1, \ldots, s''_M$. The gain in performance of SRR over the other methods is significant albeit smaller than in case 1. Thus, for $t = 100{,}000$ h, SRR is, for the $ETRR(t)$ measure, about 15.4 times faster than the fastest of the other methods (SR) and, for the $EARR(t)$ measure, also about 15.4 times faster than the fastest of the other methods (SR). For the example, $R_{\overline{E}} \approx 8$. Were the repair rates more different, $R_{\overline{E}}$ would be greater, M, $K_{\overline{E}}$ and $L_{\overline{E}}$ would be greater and split regenerative randomization would be relatively more costly.

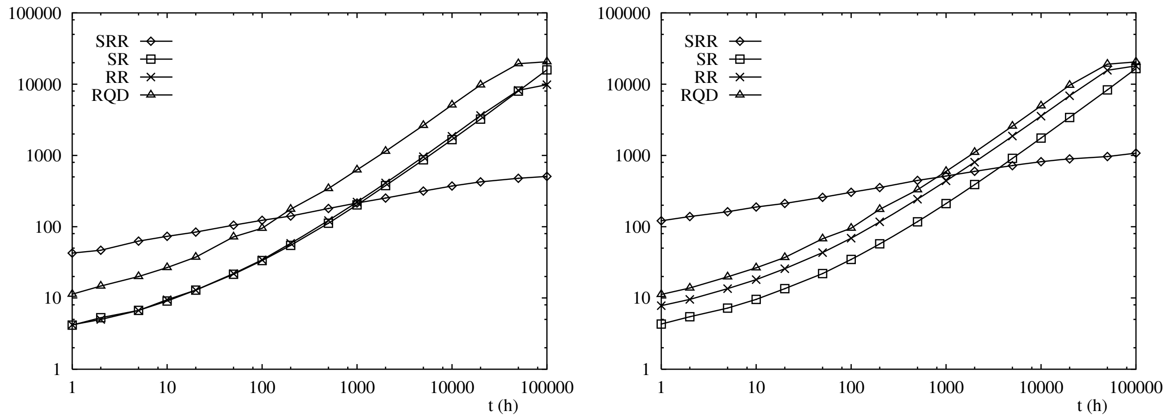

Figure 8: CPU times in seconds for the $EARR(t)$ measure: case 1 (left), case 2 (right).

5 Conclusions

We have generalized a method called split regenerative randomization which is specifically targeted at the transient analysis of rewarded CTMC models of fault-tolerant systems with deferred repair. The generalized method covers a slightly wider type of CTMC models and allows to compute two transient measures: the expected transient reward rate and the expected averaged reward rate. The method has the same good properties as the randomization method (numerical stability, well-controlled computation error, and ability to specify the computation error in advance) and can be significantly less costly than that method. The method requires the selection of a subset of states and a regenerative state and its performance depends on those selections. For a class of rewarded CTMC models, class C_2', including typical failure/repair models with exponential failure and repair time distributions and deferred repair, natural selections for the subset of states and the regenerative state exist and, for those natural selections, theoretical results are available assessing approximately the computational cost of the method in terms of "visible" model characteristics. Using a large class C_2' model, we have shown that, for models in that class, the method can be significantly faster than other randomization-based methods.

Appendix

Proof of Proposition 1. It suffices to prove

$$P[X(t)=i]=\sum_{k=0}^{\infty}\frac{\pi_i(k)}{a(k)}P[V(t)=s_k]+I_{\alpha_{E'}>0}I_{i\in E'}\sum_{k=0}^{\infty}\frac{\pi_i'(k)}{a'(k)}P[V(t)=s_k']\,,\quad i\in E \tag{35}$$

and

$$\begin{aligned}P[X(t)=i]=&\sum_{k=0}^{\infty}I_{a(k,1)>0}\sum_{l=1}^{\infty}\frac{\pi_i(k,l)}{a(k,l)}P[V(t)=s_{k,l}]\\&+I_{\alpha_{E'}>0}\sum_{k=0}^{\infty}I_{a'(k,1)>0}\sum_{l=1}^{\infty}\frac{\pi_i'(k,l)}{a'(k,l)}P[V(t)=s_{k,l}']\end{aligned}$$

$$+ I_{\alpha_{\overline{E}}>0}\sum_{k=0}^{\infty}\frac{\pi''_i(k)}{a''(k)}P[V(t)=s''_k]\,, \quad i\in\overline{E}\,. \tag{36}$$

We will start by proving (35). Using the interpretation of X as the result of composing the state visiting process $\widehat{X}$ with independent visit durations with parameter Λ_E in the states in E and parameter $\Lambda_{\overline{E}}$ in the states in $\overline{E}\cup\{f_1,\ldots,f_A\}$ and letting X^E_j, $j=1,2,\ldots$ and $X^{\overline{E}}_j$, $j=1,2,\ldots$ independent exponential random variables with, respectively, parameters Λ_E and $\Lambda_{\overline{E}}$:

$$P[X(t)=i]=\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}P[\#(\widehat{X}_{0:n}\in E)=k\wedge\widehat{X}_n=i]$$
$$P\left[\sum_{j=1}^{k-1}X^E_j+\sum_{j=1}^{n-k+1}X^{\overline{E}}_j\le t\wedge\sum_{j=1}^{k}X^E_j+\sum_{j=1}^{n-k+1}X^{\overline{E}}_j>t\right]\,,\quad i\in E\,. \tag{37}$$

Noting that, according to the definition of $\widehat{V}$ (7), $\widehat{X}_n\in E$ implies $\widehat{V}_n\in\{s_m, 0\le m\le n\}\cup\{s'_n\}$, $\widehat{V}_n=s_m$, $0\le m\le n$, if and only if $\widehat{X}_{n-m}=r$ and $\widehat{X}_{n-m+1:n}\in E'$, and $\widehat{V}_n=s'_n$ if and only if $\widehat{X}_{0:n}\in E'$, and that $\widehat{X}'$ is probabilistically identical to $\{\widehat{X}_{n-m+l}; l=0,1,\ldots\}$ conditioned on $\widehat{X}_{n-m}=r$, we have:

$$\begin{aligned}
&P[\#(\widehat{X}_{0:n}\in E)=k\wedge\widehat{X}_n=i]\\
&\quad=\sum_{m=0}^{n}P[\widehat{V}_n=s_m\wedge\#(\widehat{X}_{0:n}\in E)=k\wedge\widehat{X}_n=i]\\
&\qquad\quad+P[\widehat{V}_n=s'_n\wedge\#(\widehat{X}_{0:n}\in E)=k\wedge\widehat{X}_n=i]\\
&\quad=\sum_{m=0}^{n}P[\#(\widehat{X}_{0:n-m-1}\in E)=k-m-1\wedge\widehat{X}_{n-m}=r\wedge\widehat{X}_{n-m+1:n}\in E'\wedge\widehat{X}_n=i]\\
&\qquad\quad+I_{i\in E'}I_{\alpha_{E'}>0}I_{k=n+1}P[\widehat{X}_{0:n}\in E'\wedge\widehat{X}_n=i]\\
&\quad=\sum_{m=0}^{n}P[\#(\widehat{X}_{0:n-m-1}\in E)=k-m-1\wedge\widehat{X}_{n-m}=r]\\
&\qquad\quad P[\widehat{X}_{n-m+1:n}\in E'\wedge\widehat{X}_n=i\mid\widehat{X}_{n-m}=r]\\
&\qquad\quad+I_{i\in E'}I_{\alpha_{E'}>0}I_{k=n+1}P[\widehat{X}_{0:n}\in E'\wedge\widehat{X}_n=i]\\
&\quad=\sum_{m=0}^{n}P[\#(\widehat{X}_{0:n-m-1}\in E)=k-m-1\wedge\widehat{X}_{n-m}=r]P[\widehat{X}'_{1:m}\in E'\wedge\widehat{X}'_m=i]\\
&\qquad\quad+I_{i\in E'}I_{\alpha_{E'}>0}I_{k=n+1}P[\widehat{X}_{0:n}\in E'\wedge\widehat{X}_n=i]\,,\quad i\in E\,.
\end{aligned}$$

From the definition of Z (2), taking into account (5), which implies $Z_{1:m}\in E'$ if and only if $Z_m\in E$, $m\ge 1$, and the definition of Z' (3), taking into account (6), which implies $Z'_{0:n}\in E'$ if and only if $Z'_n\in E'$:

$$\begin{aligned}
&P[\#(\widehat{X}_{0:n}\in E)=k\wedge\widehat{X}_n=i]\\
&\quad=\sum_{m=0}^{n}P[\#(\widehat{X}_{0:n-m-1}\in E)=k-m-1\wedge\widehat{X}_{n-m}=r]P[Z_{1:m}\in E'\wedge Z_m=i]\\
&\qquad+I_{i\in E'}I_{\alpha_{E'}>0}I_{k=n+1}P[Z'_{0:n}\in E'\wedge Z'_n=i]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{n} P[\widehat{X}_{n-m} = r \wedge \#(\widehat{X}_{0:n-m-1} \in E) = k-m-1]\pi_i(m) \\
&\quad + I_{i\in E'} I_{\alpha_{E'}>0} I_{k=n+1} \pi'_i(n)\,, \quad i \in E\,. \qquad (38)
\end{aligned}
$$

Using the facts that, according to the definition of $\widehat{V}$ (7), $\widehat{V}_n \in E_V$ if and only if $\widehat{X}_n \in E$ and $\widehat{V}_n = s_m$ if and only if $\widehat{X}_{n-m} = r$ and $\widehat{X}_{n-m+1:n} \in E'$, that $\widehat{X}'$ is probabilistically identical to $\{\widehat{X}_{n-m+l}; l = 0, 1, \ldots\}$ conditioned on $\widehat{X}_{n-m} = r$, and, finally, the definition of Z (2), taking into account that $Z_{1:m} \in E'$ if and only if $Z_m \in E$, $m \geq 1$:

$$
\begin{aligned}
&P[\widehat{V}_n = s_m \wedge \#(\widehat{V}_{0:n} \in E_V) = k] \\
&= \sum_{i\in E} P[\widehat{V}_n = s_m \wedge \#(\widehat{X}_{0:n} \in E) = k \wedge \widehat{X}_n = i] \\
&= \sum_{i\in E} P[\widehat{X}_{n-m} = r \wedge \widehat{X}_{n-m+1:n} \in E' \wedge \#(\widehat{X}_{0:n-m-1} \in E) = k-m-1 \wedge \widehat{X}_n = i] \\
&= P[\widehat{X}_{n-m} = r \wedge \wedge \#(\widehat{X}_{0:n-m-1} \in E) = k-m-1] \\
&\quad \sum_{i\in E} P[\widehat{X}_{n-m+1:n} \in E' \wedge \widehat{X}_n = i \mid \widehat{X}_{n-m} = r] \\
&= P[\widehat{X}_{n-m} = r \wedge \#(\widehat{X}_{0:n-m-1} \in E) = k-m-1] \sum_{i\in E} P[\widehat{X}'_{1:m} \in E' \wedge \widehat{X}'_m = i] \\
&= P[\widehat{X}_{n-m} = r \wedge \#(\widehat{X}_{0:n-m-1} \in E) = k-m-1] \sum_{i\in E} P[Z_{1:m} \in E' \wedge Z_m = i] \\
&= P[\widehat{X}_{n-m} = r \wedge \#(\widehat{X}_{0:n-m-1} \in E) = k-m-1] \sum_{i\in E} \pi_i(m)\,. \qquad (39)
\end{aligned}
$$

Using the facts that, according to the definition of $\widehat{V}$ (7), $\widehat{V}_n \in E_V$ if and only if $\widehat{X}_n \in E$ and $\widehat{V}_n = s'_n$ if and only if $\widehat{X}_{0:n} \in E'$, and the definition of Z' (3), taking into account that $Z'_{0:n} \in E'$ if and only if $Z'_n \in E'$:

$$
\begin{aligned}
&P[\widehat{V}_n = s'_n \wedge \#(\widehat{V}_{0:n} \in E_V) = k] \\
&= \sum_{i\in E'} P[\widehat{V}_n = s'_n \wedge \#(\widehat{X}_{0:n} \in E) = k \wedge \widehat{X}_n = i] \\
&= I_{k=n+1} \sum_{i\in E'} P[\widehat{X}_{0:n} \in E' \wedge \widehat{X}_n = i] \\
&= I_{k=n+1} \sum_{i\in E'} P[Z'_{0:n} \in E' \wedge Z'_n = i] = I_{k=n+1} \sum_{i\in E'} \pi'_i(n). \qquad (40)
\end{aligned}
$$

Combining (38), (39) and (40):

$$
\begin{aligned}
P[\#(\widehat{X}_{0:n} \in E) = k \wedge \widehat{X}_n = i] &= \sum_{m=0}^{n} P[\widehat{V}_n = s_m \wedge \#(\widehat{V}_{0:n} \in E_V) = k] \frac{\pi_i(m)}{\sum_{i\in E} \pi_i(m)} \\
&\quad + I_{i\in E'} I_{\alpha_{E'}>0}\, P[\widehat{V}_n = s'_n \wedge \#(\widehat{V}_{0:n} \in E_V) = k] \frac{\pi'_i(n)}{\sum_{i\in E'} \pi'_i(n)}\,, \quad i \in E\,. \qquad (41)
\end{aligned}
$$

Plugging (41) into (37) and using the fact that V can be interpreted as the composition of the state visiting process $\widehat{V}$ with independent exponential visit durations with parameter Λ_E in the states in

E_V and parameter $\Lambda_{\overline{E}}$ in the states in $\overline{E}_V \cup \{f_1, \dots, f_A\}$:

$$P[X(t)=i] = \sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\sum_{m=0}^{n} P[\widehat{V}_n = s_m \wedge \#(\widehat{V}_{0:n} \in E_V) = k] \frac{\pi_i(m)}{\sum_{i\in E}\pi_i(m)}$$
$$P\left[\sum_{j=1}^{k-1} X_j^E + \sum_{j=1}^{n-k+1} X_j^{\overline{E}} \le t \wedge \sum_{j=1}^{k} X_j^E + \sum_{j=1}^{n-k+1} X_j^{\overline{E}} > t\right]$$
$$+ I_{i\in E'} I_{\alpha_{E'}>0} \sum_{n=0}^{\infty}\sum_{k=0}^{n+1} P[\widehat{V}_n = s'_n \wedge \#(\widehat{V}_{0:n} \in E_V) = k] \frac{\pi'_i(n)}{\sum_{i\in E}\pi'_i(n)}$$
$$P\left[\sum_{j=1}^{k-1} X_j^E + \sum_{j=1}^{n-k+1} X_j^{\overline{E}} \le t \wedge \sum_{j=1}^{k} X_j^E + \sum_{j=1}^{n-k+1} X_j^{\overline{E}} > t\right]$$
$$= \sum_{k=0}^{\infty} \frac{\pi_i(k)}{a(k)} P[V(t)=s_k] + I_{i\in E'} I_{\alpha_{E'}>0} \sum_{k=0}^{\infty} \frac{\pi'_i(k)}{a'(k)} P[V(t)=s'_k]\,, \quad i \in E\,,$$

completing the proof of (35).

We will prove next (36). Using the interpretation of X as the result of composing the state visiting process $\widehat{X}$ with independent visit durations with parameter Λ_E in the states in E and parameter $\Lambda_{\overline{E}}$ in the states in $\overline{E} \cup \{f_1, \dots, f_A\}$ and letting X_j^E, $j = 1, 2, \dots$ and $X_j^{\overline{E}}$, $j = 1, 2, \dots$ independent exponential random variables with, respectively, parameters Λ_E and $\Lambda_{\overline{E}}$:

$$P[X(t)=i] = \sum_{n=0}^{\infty}\sum_{k=0}^{n+1} P[\#(\widehat{X}_{0:n} \in E) = k \wedge \widehat{X}_n = i]$$
$$P\left[\sum_{j=1}^{k} X_j^E + \sum_{j=1}^{n-k} X_j^{\overline{E}} \le t \wedge \sum_{j=1}^{k} X_j^E + \sum_{j=1}^{n-k+1} X_j^{\overline{E}} > t\right]\,, \quad i \in \overline{E}\,. \tag{42}$$

Noting that, according to the definition of $\widehat{V}$ (7), $\widehat{X}_n \in \overline{E}$ implies $\widehat{V} \in \{s_{m,l}, 0 \le m \le n-1, 1 \le l \le n-m\} \cup \{s'_{m,n-m}, 0 \le m \le n-1\} \cup \{s''_n\}$, $\widehat{V}_n = s_{m,l}$ if and only if $\widehat{X}_{n-m-l} = r$, $\widehat{X}_{n-m-l+1:n-l} \in E'$ and $\widehat{X}_{n-l+1:n} \in \overline{E}$, $\widehat{V}_n = s'_{m,n-m}$ if and only if $\widehat{X}_{0:m} \in E'$ and $\widehat{X}_{m+1:n} \in \overline{E}$, and $\widehat{V}_n = s''_n$ if and only if $\widehat{X}_{0:n} \in \overline{E}$, and that $\widehat{X}'$ is probabilistically identical to $\{\widehat{X}_{n-m-l+p}; p = 0, 1, \dots\}$ conditioned on $\widehat{X}_{n-m-l} = r$, we have,

$$P[\#(\widehat{X}_{0:n} \in E) = k \wedge \widehat{X}_n = i]$$
$$= \sum_{m=0}^{n-1}\sum_{l=1}^{n-m} P[\widehat{V}_n = s_{m,l} \wedge \#(\widehat{X}_{0:n} \in E) = k \wedge \widehat{X}_n = i]$$
$$+ \sum_{m=0}^{n-1} P[\widehat{V}_n = s'_{m,n-m} \wedge \#(\widehat{X}_{0:n} \in E) = k \wedge \widehat{X}_n = i]$$
$$+ P[\widehat{V}_n = s''_n \wedge \#(\widehat{X}_{0:n} \in E) = k \wedge \widehat{X}_n = i]$$
$$= \sum_{m=0}^{n-1}\sum_{l=1}^{n-m} P[\widehat{X}_{n-m-l} = r \wedge \widehat{X}_{n-m-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E}$$
$$\wedge \#(\widehat{X}_{0:n-m-l-1} \in E) = k-m-1 \wedge \widehat{X}_n = i]$$

$$+ \; I_{\alpha_{E'}>0} \sum_{m=0}^{n-1} I_{k=m+1} P[\widehat{X}_{0:m} \in E' \wedge \widehat{X}_{m+1:n} \in \overline{E} \wedge \widehat{X}_n = i]$$
$$+ I_{\alpha_{\overline{E}}>0} I_{k=0} P[\widehat{X}_{0:n} \in \overline{E} \wedge \widehat{X}_n = i]$$
$$= \sum_{m=0}^{n-1} \sum_{l=1}^{n-m} P[\#(\widehat{X}_{0:n-m-l-1} \in E) = k-m-1 \wedge \widehat{X}_{n-m-l} = r]$$
$$P[\widehat{X}_{n-m-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E} \wedge \widehat{X}_n = i \mid \widehat{X}_{n-m-l} = r]$$
$$+ \; I_{\alpha_{E'}>0} \sum_{m=0}^{n-1} I_{k=m+1} P[\widehat{X}_{0:m} \in E' \wedge \widehat{X}_{m+1:n} \in \overline{E} \wedge \widehat{X}_n = i]$$
$$+ \; I_{\alpha_{\overline{E}}>0} I_{k=0} P[\widehat{X}_{0:n} \in \overline{E} \wedge \widehat{X}_n = i]$$
$$= \sum_{m=0}^{n-1} \sum_{l=1}^{n-m} P[\#(\widehat{X}_{0:n-m-l-1} \in E) = k-m-1 \wedge \widehat{X}_{n-m-l} = r]$$
$$P[\widehat{X}'_{1:m} \in E' \wedge \widehat{X}'_{m+1:m+l} \in \overline{E} \wedge \widehat{X}'_{m+l} = i]$$
$$+ \; I_{\alpha_{E'}>0} \sum_{m=0}^{n-1} I_{k=m+1} P[\widehat{X}_{0:m} \in E' \wedge \widehat{X}_{m+1:n} \in \overline{E} \wedge \widehat{X}_n = i]$$
$$+ \; I_{\alpha_{\overline{E}}>0} I_{k=0} P[\widehat{X}_{0:n} \in \overline{E} \wedge \widehat{X}_n = i]\,, \quad i \in \overline{E}\,. \tag{43}$$

Using the definition of Z (2), taking into account that $Z_{1:m} \in E'$ if and only if $Z_m \in E$, $m \geq 1$, the definition of Z' (3), taking into account that $Z'_{0:n} \in E'$ if and only if $Z'_n \in E'$, and the definition of Z'' (4), taking into account that $Z''_{0:n} \in \overline{E}$ if and only if $Z''_n \in \overline{E}$, and using $\pi_i(m,l) = 0$, $l \geq 1$, for $a(m,1) = 0$ and $\pi'_i(m,l) = 0$, $l \geq 1$, for $a'(m,1) = 0$:

$$P[\#(\widehat{X}_{0:n} \in E) = k \wedge \widehat{X}_n = i]$$
$$= \sum_{m=0}^{n-1} \sum_{l=1}^{n-m} P[\#(\widehat{X}_{0:n-m-l-1} \in E) = k-m-1 \wedge \widehat{X}_{n-m-l} = r]$$
$$P[Z_{1:m} \in E' \wedge Z_{m+1:m+l} \in \overline{E} \wedge Z_{m+l} = i]$$
$$+ I_{\alpha_{E'}>0} \sum_{m=0}^{n-1} I_{k=m+1} P[Z'_{0:m} \in E' \wedge Z'_{m+1:n} \in \overline{E} \wedge Z'_n = i]$$
$$+ I_{\alpha_{\overline{E}}>0} I_{k=0} P[Z''_{0:n} \in \overline{E} \wedge Z''_n = i]$$
$$= \sum_{m=0}^{n-1} \sum_{l=1}^{n-m} P[\#(\widehat{X}_{0:n-m-l-1} \in E) = k-m-1 \wedge \widehat{X}_{n-m-l} = r] I_{a(m,1)>0} \pi_i(m,l)$$
$$+ I_{\alpha_{E'}>0} \sum_{m=0}^{n-1} I_{k=m+1} I_{a'(m,1)>0} \pi'_i(m, n-m) + I_{\alpha_{\overline{E}}>0} I_{k=0} \pi''_i(n)\,, \quad i \in \overline{E}\,. \tag{44}$$

Using the facts that, according to the definition of $\widehat{V}$ (7), $\widehat{V}_n \in E_V$ if and only if $\widehat{X}_n \in E$ and $\widehat{V}_n = s_{m,l}$ if and only if $\widehat{X}_{n-m-l} = r$, $\widehat{X}_{n-m-l+1:n-l} \in E'$, and $\widehat{X}_{n-l+1:n} \in \overline{E}$, using the fact that $\widehat{X}'$ is probabilistically identical to $\{\widehat{X}_{n-m-l+p}; p = 0, 1, \ldots\}$ conditioned on $\widehat{X}_{n-m-l} = r$, and using the definition of Z (2), taking into account that $Z_{1:m} \in E'$ if and only if $Z_m \in E$, $m \geq 1$:

$$P[\#(\widehat{V}_{0:n} \in E_V) = k \wedge \widehat{V}_n = s_{m,l}]$$

$$
\begin{aligned}
&= \sum_{i\in\overline{E}} P[\#(\widehat{X}_{0:n-m-l-1}\in E) = k-m-1 \wedge \widehat{X}_{n-m-l} = r \wedge \widehat{X}_{n-m-l+1:n-l}\in E' \\
&\qquad \wedge \widehat{X}_{n-l+1:n}\in \overline{E} \wedge \widehat{X}_n = i] \\
&= \sum_{i\in\overline{E}} P[\#(\widehat{X}_{0:n-m-l-1}\in E) = k-m-1 \wedge \widehat{X}_{n-m-l} = r] \\
&\qquad P[\widehat{X}_{n-m-l+1:n-l}\in E' \wedge \widehat{X}_{n-l+1:n}\in \overline{E} \wedge \widehat{X}_n = i \mid \widehat{X}_{n-m-l} = r] \\
&= \sum_{i\in\overline{E}} P[\#(\widehat{X}_{0:n-m-l-1}\in E) = k-m-1 \wedge \widehat{X}_{n-m-l} = r] \\
&\qquad P[\widehat{X}'_{1:m}\in E' \wedge \widehat{X}_{m+1:m+l}\in \overline{E} \wedge \widehat{X}_{m+l} = i] \\
&= \sum_{i\in\overline{E}} P[\#(\widehat{X}_{0:n-m-l-1}\in E) = k-m-1 \wedge \widehat{X}_{n-m-l} = r] \\
&\qquad P[Z_{1:m}\in E' \wedge Z_{m+1:m+l}\in \overline{E} \wedge Z_{m+l} = i] \\
&= P[\#(\widehat{X}_{0:n-m-l-1}\in E) = k-m-1 \wedge \widehat{X}_{n-m-l} = r] \sum_{i\in\overline{E}} \pi_i(m,l)\,. \qquad (45)
\end{aligned}
$$

Using the facts that, according to the definition of $\widehat{V}$ (7), $\widehat{V}_n \in E_V$ if and only if $\widehat{X}_n \in E$ and $\widehat{V}_n = s'_{m,n-m}$ if and only if $\widehat{X}_{0:m}\in E'$ and $\widehat{X}_{m+1:n}\in\overline{E}$, and using the definition of Z' (3), taking into account that $Z'_{0:m}\in E'$ if and only if $Z'_m\in E'$:

$$
\begin{aligned}
&P[\#(\widehat{V}_{0:n}\in E_V) = k \wedge \widehat{V}_n = s'_{m,n-m}] \\
&\qquad = I_{\alpha_{E'}>0} I_{k=m+1} \sum_{i\in\overline{E}} P[\widehat{X}_{0:m}\in E' \wedge \widehat{X}_{m+1:n}\in\overline{E} \wedge \widehat{X}_n = i] \\
&\qquad = I_{\alpha_{E'}>0} I_{k=m+1} \sum_{i\in\overline{E}} P[Z'_{0:m}\in E' \wedge Z'_{m+1:n}\in\overline{E} \wedge Z'_n = i] \\
&\qquad = I_{\alpha_{E'}>0} I_{k=m+1} \sum_{i\in\overline{E}} \pi'_i(m,n-m)\,. \qquad (46)
\end{aligned}
$$

Using the facts that, according to the definition of $\widehat{V}$ (7), $\widehat{V}_n \in E_V$ if and only if $\widehat{X}_n \in E$ and $\widehat{V}_n = s''_n$ if and only if $\widehat{X}_{0:n}\in\overline{E}$, and using the definition of Z'' (4), taking into account that $Z''_{0:n}\in\overline{E}$ if and only if $Z''_n\in\overline{E}$:

$$
\begin{aligned}
&P[\#(\widehat{V}_{0:n}\in E_V) = k \wedge \widehat{V}_n = s''_n] \\
&\qquad = I_{\alpha_{\overline{E}}>0} I_{k=0} \sum_{i\in\overline{E}} P[\widehat{X}_{0:n}\in\overline{E} \wedge \widehat{X}_n = i] \\
&\qquad = I_{\alpha_{\overline{E}}>0} I_{k=0} \sum_{i\in\overline{E}} P[Z''_{0:n}\in\overline{E} \wedge Z''_n = i] = I_{\alpha_{\overline{E}}>0} I_{k=0} \sum_{i\in\overline{E}} \pi''_i(n)\,. \qquad (47)
\end{aligned}
$$

Combining (44), (45), (46), and (47):

$$
\begin{aligned}
&P[\#(\widehat{X}_{0:n}\in E) = k \wedge \widehat{X}_n = i] \\
&\quad = \sum_{m=0}^{n-1}\sum_{l=1}^{n-m} I_{a(m,1)>0} P[\#(\widehat{V}_{0:n}\in E_V) = k \wedge \widehat{V}_n = s_{m,l}] \frac{\pi_i(m,l)}{\sum_{i\in\overline{E}} \pi_i(m,l)} \\
&\qquad + I_{\alpha_{E'}>0} \sum_{m=0}^{n-1} I_{k=m+1} I_{a'(m,1)>0} P[\#(\widehat{V}_{0:n}\in E_V) = k \wedge \widehat{V}_n = s'_{m,n-m}] \frac{\pi'_i(m,n-m)}{\sum_{i\in\overline{E}} \pi'_i(m,n-m)}
\end{aligned}
$$

$$+ I_{\alpha_{\overline{E}}>0} I_{k=0} P[\#(\widehat{V}_{0:n} \in E_V) = k \wedge \widehat{V}_n = s''_n] \frac{\pi''_i(n)}{\sum_{i\in\overline{E}} \pi''_i(n)}\,, \quad i \in \overline{E}\,. \tag{48}$$

Plugging (48) into (42) and using the fact that V can be interpreted as the composition of the state visiting process $\widehat{V}$ with independent exponential visit durations with parameter Λ_E in the states in E_V and parameter $\Lambda_{\overline{E}}$ in the states in $\overline{E}_V \cup \{f_1, \ldots, f_A\}$:

$$\begin{aligned}
&P[X(t) = i] \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\sum_{m=0}^{n-1}\sum_{l=1}^{n-m} I_{a(m,1)>0} P[\#(\widehat{V}_{0:n} \in E_V) = k \wedge \widehat{V}_n = s_{m,l}] \frac{\pi_i(m,l)}{\sum_{i\in\overline{E}} \pi_i(m,l)} \\
&\qquad P\left[\sum_{j=1}^{k} X_j^E + \sum_{j=1}^{n-k} X_j^{\overline{E}} \le t \wedge \sum_{j=1}^{k} X_j^E + \sum_{j=1}^{n-k+1} X_j^{\overline{E}} > t\right] \\
&\quad + I_{\alpha_{E'}>0} \sum_{n=0}^{\infty}\sum_{k=0}^{n+1}\sum_{m=0}^{n-1} I_{k=m+1} I_{a'(m,1)>0} P[\#(\widehat{V}_{0:n} \in E_V) = k \wedge \widehat{V}_n = s'_{m,n-m}] \\
&\qquad \frac{\pi'_i(m,n-m)}{\sum_{i\in\overline{E}} \pi'_i(m,n-m)} \\
&\qquad P\left[\sum_{j=1}^{k} X_j^E + \sum_{j=1}^{n-k} X_j^{\overline{E}} \le t \wedge \sum_{j=1}^{k} X_j^E + \sum_{j=1}^{n-k+1} X_j^{\overline{E}} > t\right] \\
&\quad + I_{\alpha_{\overline{E}}>0} \sum_{n=0}^{\infty}\sum_{k=0}^{n+1} I_{k=0} P[\#(\widehat{V}_{0:n} \in E_V) = k \wedge \widehat{V}_n = s''_n] \frac{\pi''_i(n)}{\sum_{i\in\overline{E}} \pi''_i(n)} \\
&\qquad P\left[\sum_{j=1}^{k} X_j^E + \sum_{j=1}^{n-k} X_j^{\overline{E}} \le t \wedge \sum_{j=1}^{k} X_j^E + \sum_{j=1}^{n-k+1} X_j^{\overline{E}} > t\right] \\
&= \sum_{k=0}^{\infty} I_{a(k,1)>0} \sum_{l=1}^{\infty} \frac{\pi_i(k,l)}{a(k.l)} P[V(t) = s_{k,l}] + I_{\alpha_{E'}>0} \sum_{k=0}^{\infty} I_{a'(k,1)>0} \sum_{l=1}^{\infty} \frac{\pi'_i(k,l)}{a'(k,l)} P[V(t) = s'_{k,l}] \\
&\quad + I_{\alpha_{\overline{E}}>0} \sum_{k=0}^{\infty} \frac{\pi''_i(k)}{a''(k)} P[V(t) = s''_k]\,, \quad i \in \overline{E}\,,
\end{aligned}$$

completing the proof of (36). □

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