Nonlinear q-voter model

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We introduce a nonlinear variant of the voter model, *the q*-voter model, in which q neighbors (with possible repetition) are consulted for a voter to change opinion. If the q neighbors agree, the voter takes their opinion; if they do not have a unanimous opinion, still a voter can flip its state with probability ε. We solve the model on a fully connected network (i.e., in mean field) and compute the exit probability as well as the average time to reach consensus by employing the backward Fokker-Planck formalism and scaling arguments. We analyze the results in the perspective of a recently proposed Langevin equation aimed at describing generic phase transitions in systems with two (Z_2-symmetric) absorbing states. In particular, by deriving explicitly the coefficients of such a Langevin equation as a function of the microscopic flipping probabilities, we find that in mean field the q-voter model exhibits a disordered phase for high ε and an ordered one for low ε with three possible ways to go from one to the other: (i) a unique (generalized-voter-like) transition, (ii) a series of two consecutive transitions, one (Ising-like) in which the Z_2 symmetry is broken and a separate one (in the directed-percolation class) in which the system falls into an absorbing state, and (iii) a series of two transitions, including an intermediate regime in which the final state depends on initial conditions. This third (so far unexplored) scenario, in which a type of ordering dynamics emerges, is rationalized and found to be specific of mean field, i.e., fluctuations are explicitly shown to wash it out in spatially extended systems.

I. INTRODUCTION

In a situation where one has to choose between two alternatives that appear equally agreeable, a rather common way to remove the uncertainty is to copy what somebody else (randomly selected among acquaintances) does. The voter model dynamics [1] describes precisely this situation: Agents placed on the vertices of a graph are characterized by a binary (spin) variable ± 1; at each time step two nearest-neighbor vertices are selected and the first copies the state of the second. This can equivalently be expressed by the flipping probability $f(x)$: A vertex with a fraction x of disagreeing neighbors has a linear probability $f(x)=x$ to flip. The iteration of this simple (parameter free) rule gives rise to nontrivial ordering phenomena, which have drawn the attention of many scholars, both in physics [2] and beyond [3]. From the point of view of statistical physics, voter dynamics stands out as one of the very few nonequilibrium processes amenable of exact analytical treatment in any dimension [4]. In more physical terms, it owes its special character to the absence of surface tension [5]. Contrary to the more common curvature-driven dynamics [6], in voter dynamics curved interfaces do not tend to reduce their curvature and assume a straight shape. This induces a slow domain growth characterized, in two dimensions, by a logarithmic decay of the density of active links (i.e., links connecting sites with opposite opinion states).

A natural and relevant question, which has attracted interest in the past years, has to do with how generic the voter behavior is. Early work [7–9] already pointed out that small changes in the dynamics destroy the voter behavior in two dimensions. The voter model turns out to sit at the transition between a ferromagnetic and a paramagnetic phase but it is only a point in a generalized parameter space and any perturbation leads to a drastically different behavior. These and other results pointing toward the fragility of the voter behavior [10–13] raise the question of whether the voter model is a peculiar exception or the representative of a more generic class of models.

An answer to this question was provided by Dornic *et al.* [5]. The authors of this work showed explicitly the existence of models, different from the pure linear voter, which nevertheless exhibit its typical dynamical features. This led them to conjecture that there is a proper generalized-voter (GV) universality class encompassing systems at "an order-disorder transition driven by the interfacial noise between two absorbing states possessing equivalent dynamical roles, this symmetry being enforced either by Z_2 symmetry of the local rules, or by the global conservation of the magnetization" [5].

Further progress in the understanding of this issue has been made by Al Hammal *et al.* [14], who worked out a generic Langevin equation [15] for critical phenomena with two symmetric absorbing states and identified conditions for having a transition from order to disorder belonging to the GV class. Note that in voterlike models there are two different competing phenomena: One is the breaking of the Z_2 symmetry and the other one is the possibility for the system

to get trapped into an absorbing state. If both occur in unison then the transition point is in the GV class. Instead, if they occur separately, the Z_2 symmetry is broken first (i.e., an Ising-like transition occurs, and the system changes from paramagnetic to ferromagnetic) and afterwards the system falls into the corresponding absorbing state (i.e., a directed-percolation-like transition) [14]. In this sense, the GV class can be rationalized as the superposition of Ising and directed-percolation phase transitions. The fact that the voter transition can be split into two different ones was first reported in [16].

The picture devised in Ref. [14] on the basis of generic symmetry arguments has been recently substantiated by Vázquez and López [17]. Starting from the microscopic spin dynamics of a *nonlinear voter model*, they have derived an explicit Langevin equation for the magnetization, which coincides with the one conjectured in Ref. [14]. In this way, it is possible to precisely determine, depending on the analytical form of the microscopic flipping probability $f(x)$, which of the two scenarios above occurs.

In this paper we provide an assessment of the picture presented in Refs. [14,17] by proposing a microscopically motivated nonlinear voter model and analyzing it, both at the mean-field level and numerically. The model we consider, the q-voter model, represents a simple generalization of the original voter model: each individual interacts with a set of q of his nearest neighbors; if all q neighbors share the same state, the individual conforms to this state. Otherwise, if the q neighbors do not agree, the individual flips with a probability ε. The q-voter model is directly inspired in models of ordering dynamics in which each update step involves more than two individuals [16,18–22], and represents a practical and simple way to introduce nonlinearity in the voter dynamics at a microscopic level.

We study the model phenomenology analytically via a mean-field approach, and numerically in two dimensions, uncovering a rich phenomenology. Depending on the value of q, the model exhibits all the possible transitions of a system with two symmetric absorbing states, as described above. Interestingly, at mean-field level the voter behavior is restricted only to very specific values of q ($q=2$ and $q=3$), two separated phase transitions occur in $2<q<3$, and, otherwise ($q<2$ and $q>3$) we find a different phenomenology (i.e., dependence on the initial conditions and a double transition of a different type). Direct numerical simulations of the model on a fully connected network are presented to back up the mean-field results. On the other hand, in a $d=2$ lattice, we recover the picture presented in Ref. [14], with a single voterlike critical point.

The paper is structured as follows: After Sec. II, where the model is defined at a microscopic level, in Sec. III we perform a mean-field analysis, by means of both a Fokker-Planck and a Langevin approach; a more detailed study of the case $q=4$ is also presented. Results for finite dimensional systems are presented in Sec. IV. Finally, in Sec. V we discuss our findings.

II. DEFINITION OF THE *q*-VOTER MODEL

We consider a nonlinear voter model defined on a lattice (or network) of N sites. Each site hosts a spin, with value

± 1. The dynamics is given by the following update rule:

(i) At a given time t, choose one spin at random, located at site i.

(ii) Choose at random q neighbors of site i. In order to simplify the numerical analysis, and allow for an arbitrary value of q in regular lattices, we consider here the possibility of repetition, i.e., a given neighbor can be selected more than once.

(iii) If all the q neighbors are in the same state, the original spin takes the value of the q neighbors.

(iv) Otherwise, if the q neighbors are different, the original spin flips with probability ε.

(v) Time is updated $t \rightarrow t + 1/N$.

It is easy to see that this model is nonlinear. Consider the probability that a site flips as a function of the fraction x of disagreeing neighbors, that is,

$$f(x,q) = x^q + \varepsilon[1 - x^q - (1-x)^q]. \tag{1}$$

Notice that, although in the original definition of the model q is an integer, Eq. (1) makes sense for any $q>0$ and can therefore be considered as the definition of the q-voter model for real values of q. For $q=1$, we recover, for any value of ε, the standard voter model, namely, $f(x,1)=x$. Nonlinear behavior arises for $q \neq 1$. Observe also that if the fraction of disagreeing neighbors vanishes, i.e., $x=0$, the configuration is absorbing, $f(0,q)=0$.

The q-voter model bears some resemblance to other opinion dynamics models introduced recently in the literature. For example, the "vacillating voter" model [18] is very similar to the q voter with $q=2$ and $\varepsilon=1$, apart from the possibility to select twice the same neighbor (repetition). The case with $q=2$ and $\varepsilon=0$ is instead similar to the Sznajd model [19], in its formulation where a pair of agreeing agents convince only one of their neighbors [23].

III. MEAN-FIELD THEORY

In order to gain an understanding of the q-voter model behavior, it is useful to consider it first at the mean-field level (that is, on a fully connected network), for which several analytical tools have been recently developed.

A. Backward Fokker-Planck approach

Following Refs. [24–26], we can study the mean-field theory of the q-voter model by applying the backward Fokker-Planck (BFP) technique [15]. Consider a time t, in which there are n spins in state $+1$. The state of the system is fully defined by this quantity, plus the transition rates to go to a state with $n \pm 1$ spins in state $+1$. Denoting this transition probabilities by $p_{n\pm1,n}$, then

$$p_{n+1,n} = (1-x)f(x,q), \tag{2}$$

$$p_{n-1,n} = xf(1-x,q), \tag{3}$$

$$p_{n,n} = 1 - p_{n+1,n} - p_{n-1,n}, \tag{4}$$

all the rest of values of $p_{n',n}$ being equal to zero. Here we consider $x=n/N$ as the probability of selecting a $+1$ spin

when a vertex in randomly chosen, a simplification which provides valid results in the limit of large N. With this definition there are two absorbing states, $n=0$ and $n=N$ (i.e., $x=0$ and $x=1$).

The quantity n performs in time a biased one-dimensional random walk, between two absorbing states. The random walk is fully defined in terms of a backward master equation, taking the form [15]

$$\frac{\partial P(n,t|n',t')}{\partial t'} = T(n'+1|n')[P(n,t|n'+1,t') - P(n,t|n',t')]$$
$$+ T(n'-1|n')[P(n,t|n'-1,t')$$
$$- P(n,t|n',t')], \tag{5}$$

where $P(n,t|n',t')$ is the probability of having n spins $+1$ at time t, provided there were n' at time $t' \le t$. Equation (5) is given in terms of the transition rates (per unit time)

$$T(n|n') = \frac{p_{n,n'}}{\Delta}, \tag{6}$$

with $\Delta = 1/N$. The master equation can be transformed, via a diffusion approximation, into a BFP equation for the reduced variable $x=n/N$, by expanding Eq. (5) up to second order in Δ. In this expansion, the BFP equation takes the form

$$\frac{\partial P(x,t|x',t')}{\partial t'} = v(x')\frac{\partial P(x,t|x',t')}{\partial x'} + \frac{1}{2}D(x')\frac{\partial^2 P(x,t|x',t')}{\partial x'^2}, \tag{7}$$

with a drift

$$v(x) = \Delta[T(n+1|n) - T(n-1|n)]$$
$$= (1-x)f(x,q) - xf(1-x,q), \tag{8}$$

and a diffusion coefficient

$$D(x) = \Delta^2[T(n+1|n) + T(n-1|n)]$$
$$= \frac{1}{N}[(1-x)f(x,q) + xf(1-x,q)]. \tag{9}$$

For the generic BFP equation Eq. (7), the exit probability $E(x)$, i.e., the probability that, starting from an initial density x of $+1$ vertices, the absorbing state $+1$ is reached, satisfies the differential equation [15]

$$v(x)\partial_x E(x) + \frac{1}{2}D(x)\partial_x^2 E(x) = 0, \tag{10}$$

subject to the boundary conditions $E(0)=0$ and $E(1)=1$, while the average time until consensus, $T(N,x)$ is given by [15]

$$v(x)\partial_x T(N,x) + \frac{1}{2}D(x)\partial_x^2 T(N,x) = -1, \tag{11}$$

with boundary conditions $T(N,0)=T(N,1)=0$.

The standard voter model in Eq. (5) corresponds to $q=1$, for which we find $v(x)=0$ and $D(x)=2x(1-x)/N$. This leads to $E(x)=x$ and $T(N,x)=-N[x \ln x+(1-x)\ln(1-x)]$; thus $T(N,1/2) \sim N$ and $T(N,1/N) \sim \ln N$ [24]. Moreover, it is

easy to see from Eqs. (10) and (11) that the condition $v(x)=0$ is necessary and sufficient to yield, for any diffusion $D(x)$ [as long as $D(x) \propto \frac{1}{N}$], $E(x)=x$ and $T(N,1/2) \sim N$, that are the two main signatures of voter behavior in mean field.

In order to have mean-field voter behavior, we must then consider the cases in which the drift $v(x)$ vanishes. Let us look at the different possibilities.

(i) For $q=2$,

$$v(x) = (-1+2\varepsilon)(1-x)x(1-2x). \tag{12}$$

Therefore, $\varepsilon=1/2$ leads to voter behavior. But for $\varepsilon=1/2$, $f(x,2)=x$, so that in this case the q model *coincides* with the usual voter model.

(ii) For $q=3$,

$$v(x) = (-1+3\varepsilon)(1-x)x(1-2x). \tag{13}$$

Again, $\varepsilon=1/3$ leads to zero drift and hence to voter behavior. However, in this case $f(x,3)=x^3-x^2+x \ne x$, so that the three-voter model is a case belonging nontrivially to the GV class.

(iii) For $q=4$ instead,

$$v(x) = (1-x)x(1-2x) \times [-1+4\varepsilon+x(1-x)(1-2\varepsilon)]. \tag{14}$$

No value of ε can cancel the drift; therefore, voter behavior is, in principle, not possible.

B. Langevin equation approach

Further understanding is provided by applying the formalism developed in Refs. [14,17]. In this approach one focuses on the magnetization $\phi=2x-1$. In this variable, the drift takes the form, at lowest level in powers of ϕ,

$$v(\phi) = (1-\phi^2)(a\phi-b\phi^3). \tag{15}$$

This corresponds to the usual terms in a continuous description for systems with a Z_2 symmetry (i.e., in the Ising class [27]), multiplied by a factor $(1-\phi^2)$ imposing the existence of two absorbing states. The drift can be written as derived from a potential: $v(\phi)=-dV(\phi)/d\phi$, i.e.,

$$V(\phi) = -\frac{a}{2}\phi^2 + \frac{a+b}{4}\phi^4 - \frac{b}{6}\phi^6. \tag{16}$$

This function has five extrema, obtained from the condition $v(\phi)=0$, which are $\phi=0$, $\phi=\pm 1$, and $\phi=\pm\sqrt{\frac{a}{b}}$. Their role is clarified by the concavity of $V(\phi)$, which turns out to be

$$V''(0) = -a, \tag{17}$$

$$V''(\pm 1) = 2(a-b), \tag{18}$$

$$V''\left(\pm\sqrt{\frac{a}{b}}\right) = 2a\left(1-\frac{a}{b}\right). \tag{19}$$

The extrema at 0 (origin) and ± 1 (absorbing barriers) are always relevant. The extrema at $\pm\sqrt{\frac{a}{b}}$ make physical

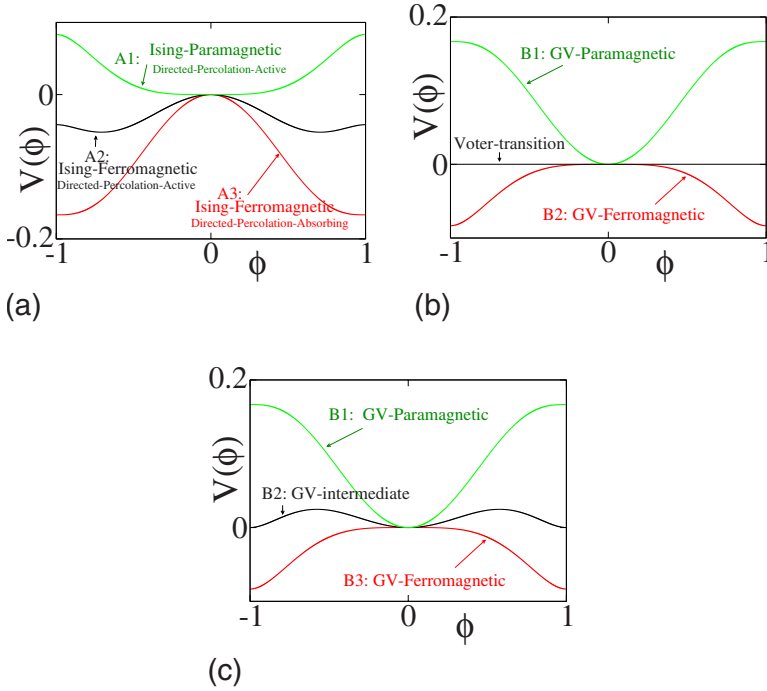FIG. 1. (Color online) Potential $V(\phi)$, as defined by Eq. (16) for $b>0$ (a), $b=0$ (b), and $b<0$ (c).

sense only when $0 < \frac{a}{b} < 1$, otherwise they are imaginary or nonaccessible. According to the interpretation in Ref. [14], there are the following possible scenarios, depending on b (see Fig. 1).

(i) For $b>0$, if $a<0$ the system is paramagnetic; an Ising transition occurs for $a=0$ and (afterwards) an absorbing-state (directed-percolation) transition takes place at $a=b$ [see Fig. 1(a)].

(ii) The case $b=0$ corresponds to the voter case in which the potential identically vanishes at the transition point $a=0$ [see Fig. 1(b)].

(iii) For $b<0$, if a is very negative the system is paramagnetic; then, at a negative value, $a=b$, a pair of symmetric new minima appear at ± 1 (this generates an "intermediate phase" with three competing minima). At $a=0$ the stability of the origin changes [see Fig. 1(c)], and only the minima at ± 1 remain.

In Ref. [14] it was argued that the intermediate phase appearing for $b<0$ is absent in spatially extended systems: fluctuations wash it away, and the central curve in Fig. 1(c) becomes as the lowest one. The reason for this is simple: As soon as minima at the absorbing barriers appear, fluctuations become asymmetric, i.e., they can take the system from the minimum at the origin to the barriers, but not the other way around. Therefore, in a "renormalized" picture the third case is argued to coincide with the $b=0$ case, and thus lead to a unique GV transition. Hence, only two scenarios are expected to exist in the presence of fluctuations.

The coefficients a and b can be explicitly computed for the q voter by combining Eqs. (15), (8), and (1) for generic values of q and ε. This leads to

$$a = 2^{-q+1}(q-1) - 2\varepsilon(1 - 2^{-q+1}), \qquad (20)$$

$$b = 2^{-q}(q-1)(q-2)\left(1 - \frac{q}{3}\right) + 2\varepsilon[1 - 2^{-q}(2 - q + q^2)]. \qquad (21)$$

Before discussing what occurs for generic values of q, we remark that the coefficient b can be made to vanish identically only for $q=1$, $q=2$, and $q=3$, while the coefficient a vanishes for any ε only if $q=1$. Hence, we recover the previous results in Sec. III A. For $q=1$ the potential vanishes; the q-voter model coincides with the usual voter (ε plays no role and the system sits at a critical point). For $q=2$ one has $b=0$ and $a=1/2-\varepsilon$: the potential is exactly zero for $\varepsilon=1/2$ so that one recovers voter behavior at the transition between a paramagnetic and a ferromagnetic phase. For $q=3$, we have $b=0$ and $a=1/2(1-3\varepsilon)$: again the potential vanishes for $\varepsilon=1/3$ (transition point) and voter behavior is found, separating an ordered phase (for small ε) from a disordered one.

For analyzing what happens for generic values of q, it is useful to calculate the boundaries of the interval of values of ε for which the extrema in $\pm\sqrt{\frac{a}{b}}$ have physical values. We define as ε_1 the value at which $a=0$ so that the extrema are in 0. From Eq. (20) we obtain

$$\varepsilon_1 = \frac{q-1}{2^q - 2}. \qquad (22)$$

Instead, ε_2 is the value for which the stability at the origin changes, i.e., for which $a=b$. From Eq. (21) we find

$$\varepsilon_2 = \frac{\frac{q^3}{3} - 2q^2 + \frac{17}{3}q - 4}{2^{q+2} - 2(4 - q + q^2)}. \qquad (23)$$

The behavior of ε_1 and ε_2 as a function of q is plotted in Fig. 2. It is important also to notice that, for $q>1$, a is smaller
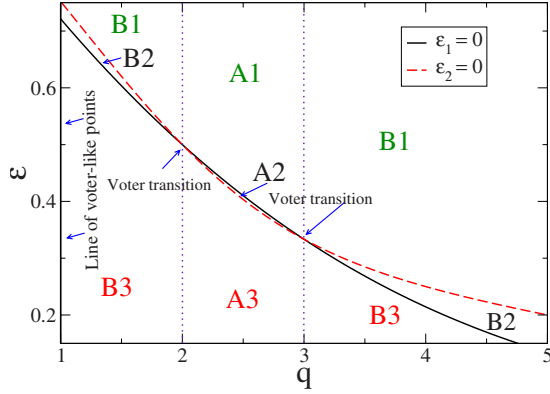
FIG. 2. (Color online) Mean field phase diagram of the q-voter model. At $q=1$, $q=2$, and $q=3$ (marked with vertical lines) GV transitions occur. For any other value of q there are two different transitions: at $\varepsilon_1=0$ (solid line) the up-down (Z_2) symmetry is broken, while at $\varepsilon_2=0$ (dashed line) the barriers at ± 1 become absorbing states. For $2<q<3$ there are an Ising transition followed by a directed percolation one. For $q<2$ and $q>3$ an intermediate phase exists.

than zero for large ε so that there is a paramagnetic phase above the black solid line and a ferromagnetic one for small ε. This corresponds to intuition: ε plays the role of a sort of noise in the dynamics. Instead, for $q<1$ (not represented in Fig. 2) the situation is reversed: the paramagnetic phase is now for small ε and the ferromagnetic one is for large values of ε.

The nature of the transition between the two phases varies depending on the value of q. We have already established that $q=1$ represents a voter line, while for $q=2$, and $q=3$ (marked with vertical lines in Fig. 2) there is a voter transition point for the appropriate values of ε.

For $2<q<3$, ε_1 is larger than ε_2, and the nature of the extrema is therefore as follows:

(i) For $\varepsilon>\varepsilon_1$, 0 is a minimum and ± 1 are maxima. The model is then in the paramagnetic phase (case A1 in Fig. 1).

(ii) For $\varepsilon_2<\varepsilon<\varepsilon_1$, 0 and ± 1 are maxima, and $\pm\sqrt{\frac{a}{b}}$ are minima. The mode is in the ferromagnetic phase (case A2 in Fig. 1).

(iii) For $\varepsilon<\varepsilon_2$, 0 is a maximum and ± 1 are minima. The model is in the ferromagnetic absorbing phase (case A3 in Fig. 1).

Obviously, in this case the double transition scenario described in Ref. [14] applies: Starting from large values of ε, first (at $\varepsilon=\varepsilon_1$) a transition in the Ising class, from a paramagnet to a ferromagnet, occurs; then (at ε_1) a transition of directed percolation type appears and the system becomes fully ordered. The same scenario occurs for $q<1$, where, as mentioned above, the paramagnetic phase is for $\varepsilon<\varepsilon_1$ and the absorbing one is for $\varepsilon>\varepsilon_2$.

On the other hand, for $1<q<2$ and $q>3$, the relative positions of ε_1 and ε_2 are swapped, $\varepsilon_1<\varepsilon_2$. There is an intermediate interval $\varepsilon_1<\varepsilon<\varepsilon_2$ such that 0 and ± 1 are minima separated by maxima in $\pm\sqrt{a/b}$. The nature of the extrema is then as follows:

(i) For $\varepsilon>\varepsilon_2$, 0 is a minimum and ± 1 are maxima. The model is in the paramagnetic phase (case B1 in Fig. 1).
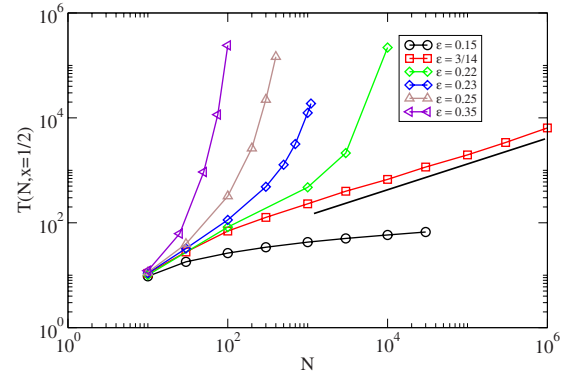

FIG. 3. (Color online) Average consensus time $T(N,x=1/2)$ as a function of N, for a fully connected system with $q=4$ and different values of ε. The straight line has slope 1/2. From bottom to top: $\epsilon=0.15$, 3/14, 0.22, 0.23, 0.25, and 0.35.

(ii) For $\varepsilon_1<\varepsilon<\varepsilon_2$, 0 and ± 1 are minima, and $\pm\sqrt{\frac{a}{b}}$ are maxima (case B2 in Fig. 1).

(iii) For $\varepsilon<\varepsilon_1$, 0 is a maximum and ± 1 are minima. The model is then in the ferromagnetic absorbing phase (case B3 in Fig. 1).

In the intermediate interval the (mean-field) system exhibits ferromagnetic or paramagnetic behavior depending on the initial condition, with basins of attraction determined by the separatrices $\pm\sqrt{a/b}$. The transition is complicated and there is no voter behavior at a mean-field level, as will be illustrated in the forthcoming subsection.

C. Analysis of the $q=4$ case

The case $q=4$ is the smallest integer value of q for which the coefficients a and b cannot be made identically equal to zero simultaneously,

$$a=\frac{3-14\varepsilon}{8} \quad \text{and} \quad b=\frac{2\varepsilon-1}{8}. \tag{24}$$

Correspondingly we have $\varepsilon_1=3/14$ and $\varepsilon_2=1/4$. In Fig. 3 we plot the average consensus time as a function of N for $q=4$ and several values of ε, obtained by numerical simulations of the q-voter model on fully connected networks of different size. For $\varepsilon>1/4$ the growth is exponential, as expected in the paramagnetic phase. For $\varepsilon<3/14$ the growth is logarithmic, as expected in the ferromagnetic phase. For $3/14<\varepsilon<1/4$ there is a crossover, but asymptotically the growth is exponential. For $\varepsilon=3/14$ the growth is proportional to $N^{1/2}$, different from the voter linear behavior.

A simple scaling argument allows us to understand the growth law $N^{1/2}$ for the average consensus time at the transition point $\varepsilon=3/14$. For $q=4$ and $\varepsilon=3/14$ the potential has parameters $a=0$, $b=-1$, hence $V(\phi)=-\phi^4/4$ at leading order. For initial conditions $x=1/2$, $\phi=0$, so that at the beginning only diffusion matters. When ϕ becomes sufficiently large drift comes in, the motion becomes ballistic, and in an interval depending logarithmically on N consensus is reached. How much time is spent in the diffusion stage? This interval lasts a time t^* that is estimated by equating the drift $v\sim\phi^3$ with the effective velocity of the diffusion motion, $\sqrt{Dt^*}/t^*$
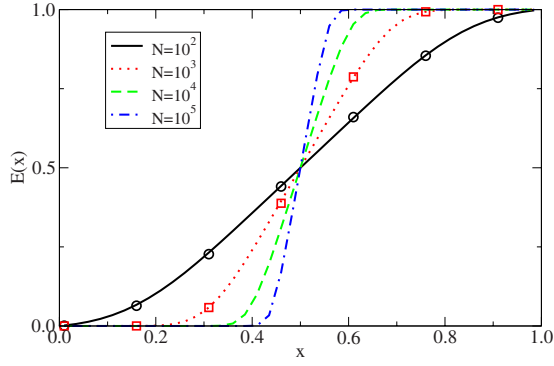
FIG. 4. (Color online) Exit probability $E(x)$ as a function of x, and different system sizes N, for a fully connected system with q =4 and $\varepsilon=3/14$: the larger the size the steeper the slope at x =1/2. Full lines represent the numerical integration of Eq. (28); symbols stand for direct numerical simulations.

$=\sqrt{D/t^*}$. Hence $t^*=D/v^2 \sim D/\phi^6$. During this time interval the diffusive motion gives a displacement $\phi^*=\sqrt{Dt^*}$, yielding $t^*=(\phi^*)^2/D$. Equating the two expressions gives $D \sim (\phi^*)^4$. On the other hand the diffusion coefficient D is D $=(\Delta\phi)^2/(\Delta t)=(1/N)^2/(1/N)=1/N$, from which $\phi^* \sim N^{-1/4}$. Hence the total time to consensus is

$$T(N,1/2) \approx (\phi^*)^2/D + \log_{10}(N) \sim N^{1/2} + \log_{10}(N) \sim N^{1/2}.$$
(25)

With respect to the exit probability $E(x)$, we can apply the formalism in Sec. III A, solving Eq. (10) for the drift and diffusion functions

$$v(x) = -\frac{1}{7}(1-x)x(1-2x)^3,$$
(26)

$$D(x) = \frac{1}{7N}(1-x)x(24x^2-24x+13),$$
(27)

to obtain, applying standard stochastic techniques [28],

$$E(x) = \frac{\int_0^x \exp\left[-\frac{N}{12}(2y-1)^2\right](24y^2-24y+13)^{7N/72}dy}{\int_0^1 \exp\left[-\frac{N}{12}(2y-1)^2\right](24y^2-24y+13)^{7N/72}dy}.$$
(28)

In Fig. 4 we show the results of the numerical integration of Eq. (28) for different values of N (solid lines), together with results form direct numerical simulations of the q-voter model in fully connected networks. It is apparent that $E(x)$ tends to a step function at $x=1/2$ in the limit $N \to \infty$. No voter behavior is therefore observed in this mean-field case.

IV. BEHAVIOR IN FINITE DIMENSIONS

A. $d=1$

Even if in $d=1$ the number of different nearest neighbors (in a square lattice) is 2, the parameter q can be kept arbi-
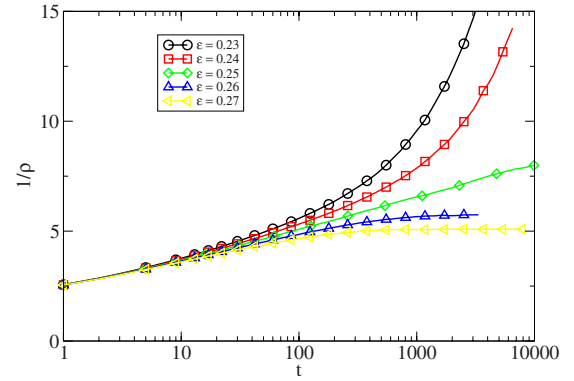

FIG. 5. (Color online) Plot of the inverse of the density of active links $\rho(t)$ for different values of ε. Parameters: $q=4$, $\varepsilon=1/4$, L =5000.

trary. It is easy to see that the one-dimensional q-voter model can be mapped onto the model of nonconservative voters recently introduced in Ref. [29]. In such a model, the relevant parameter γ is given by the ratio p_2/p_1, where p_i is the flipping probability for a site surrounded by i disagreeing neighbors. In the q-voter model p_2 trivially equals 1 for any q, while $p_1=2^{-q}+\varepsilon(1-2^{1-q})$. Hence

$$\gamma = \frac{2^q}{1+\varepsilon(2^q-2)}.$$
(29)

This equation leads to the conclusion that the value $\varepsilon=1/2$ yields, for any q, a voter behavior ($\gamma=2$). Analogously one can see that no ε can give the value $\gamma=1$, implying the "vacillating voter" behavior [18]. The mapping from q and ε to γ allows us to deduce, from the results of Ref. [29], the nontrivial shape of the exit probability $E(x)$ in $d=1$.

B. $d=2$

As in $d=1$, here q is kept arbitrary. We have performed numerical simulations of the ordering dynamics of the q-voter model on a square lattice of size $L \times L$ with L =5000, for several values of $q>1$. In all cases, we find a transition separating a paramagnetic phase for large ε from a ferromagnetic one at low ε. In order to investigate the nature of the transition we concentrate on the case $q=4$, which in a fully connected graph yields nonvoter behavior. In Fig. 5, we plot the temporal behavior of the inverse of ρ, the fraction of active links in the system. At the critical point $\varepsilon=\varepsilon_1=1/4$, $1/\rho$ grows logarithmically, as expected for the voter universality class [4]. Analog results are found for other values of q (data not shown). Additional evidence proving that for ε =1/4 the q-voter model behaves exactly as the usual voter model is provided by measuring the correlation function $C(r,t)$. From the exact solution of the voter model in $d=2$ [30] it turns out that two different length scales are present in the system, leading to the nonstandard scaling form

$$C(r,t) = \frac{1}{\ln(16t)}\widetilde{f}[r^2/(2t)],$$
(30)

where the $\widetilde{f}(x)=E_1(x)$ is the exponential integral function [31]. Figure 6 demonstrates that Eq. (30) is nicely obeyed by
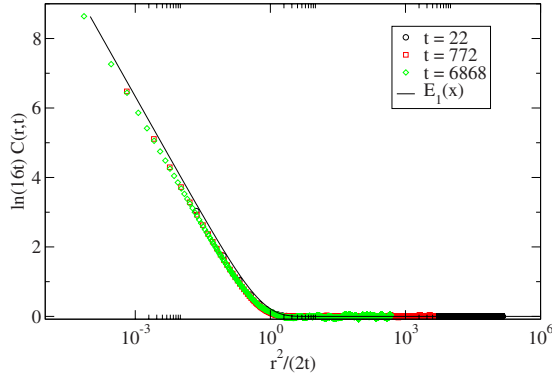
FIG. 6. (Color online) Collapse plot of the correlation function $C(r,t)$ for different times, according to Eq. (30). Parameters: $q=4$, $\varepsilon=1/4$, $L=5000$.

numerical simulations of the q voter at the transition point for $q=4$. This evidence, further confirmed by the analysis of the exit probability, showing a linear behavior, leads to strong numerical confirmation that the scenario predicted by Al Hammal *et al.* [14] is correct in $d=2$: In finite dimensions, fluctuations renormalize the deterministic potential so that the transition is in the GV class for any value of q. This renormalization effect can be directly observed by measuring numerically the drift term. From the Fokker-Planck equation, we can obtain the equation for the time evolution of the magnetization, $\langle\phi\rangle$, namely, [15]

$$\frac{d\langle\phi\rangle}{dt}=\langle v(\phi)\rangle. \tag{31}$$

Therefore, a numerical evaluation of $d\langle\phi\rangle/dt$ yields an estimate of $\langle v(\phi)\rangle$. In Fig. 7 we plot the average drift as a function of the magnetization for the q-voter model in $d=2$ and for the mean-field fully connected case. In the latter, $\langle v(\phi)\rangle$ shows a functional dependence compatible with the theoretical expectation $v(\phi)\sim(1-\phi^2)\phi^3$ for sufficiently large network size N. In the $d=2$ case, on the other hand, fluctuations are able to quickly cancel the drift term, induc-
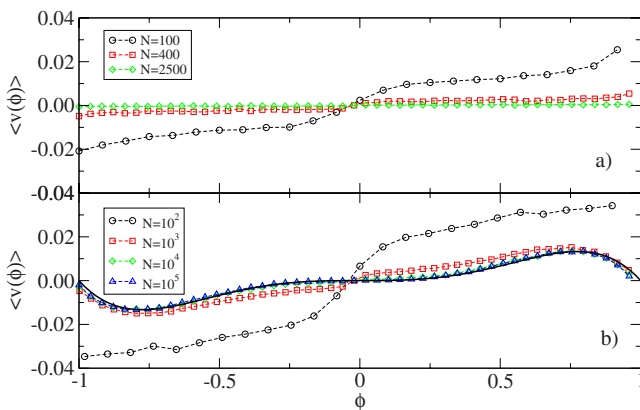

FIG. 7. (Color online) Numerical estimate of the average drift in the q-voter model, as a function of the magnetization. (a) Square lattice. (b) Fully connected network. In this last plot, the full line represents the expected theoretical value $v(\phi)\sim(1-\phi^2)\phi^3$.

ing thus an effective voter behavior in the limit of large N: $b<0$ renormalizes on large scales to $b=0$, as predicted in [14].

V. CONCLUSIONS AND DISCUSSION

We have introduced a nonlinear variant of the voter model in which the opinion of q neighbors (with possible repetition) is taken into account for a voter to change its own opinion. In particular, if all his q neighbors share the same state, an individual conforms to this state; otherwise, if the q neighbors do not agree, he flips with a probability ε. Note that the model includes a noise effect controlled by ε; still voters are not allowed to break the absorbing-state condition and a consensus state remains indefinitely so. While the original definition of the model is meaningful only for integer values of q, analytical generalization to arbitrary values of q, with $q \in [0,\infty]$ is possible. In particular, after taking the continuous limit for the transition rates, q becomes a, not necessarily integer, parameter.

We have studied the model analytically by applying a mean-field analysis based on the backward Fokker-Plank formalism and the Langevin approach developed in Refs. [14,17]. These two approaches permit us to uncover the rich and variate phenomenology of the q-voter model.

(i) For $q=1$ the model reduces to the standard voter model with exit probability proportional to x and average consensus time, starting from $x=1/2$, growing linear with system size.

(ii) For $q=2$, the model coincides with the voter one if $\varepsilon=1/2$. Instead, the system remains disordered for $\varepsilon>1/2$, or it orders exponentially fast for $\varepsilon<1/2$. Therefore the two-voter model exhibits a "generalized-voter transition." Note that our results also clarify the behavior of the Sznajd model. For $q=2$ and $\varepsilon=0$ the q-voter model practically coincides with Sznajd model, at least in the formulation of Ref. [23], according to which one has to select a pair of neighbors and, if they are in the same state, another neighbor of the pair is set in the same state. From this point of view, Sznajd model is just a ferromagnetic model in its ordered phase.

(iii) For $q=3$ there is a voterlike transition at $\varepsilon=1/3$, separating two phases as those described for $q=2$ but, contrarily to the cases before, the flipping probabilities are nonlinear: the three-voter model is an example belonging nontrivially to the generalized-voter class.

(iv) For $2<q<3$ there is no voter transition. Instead the system experiences a sequence of two transitions. Starting with large values of ε and reducing it progressively, first the Z_2 symmetry is broken (Ising transition) and afterward the system orders into an absorbing state at a directed-percolation-like transition.

(v) For $1<q<2$ and $q>3$ the mean-field approach predicts a nonvoter transition, characterized by an exit probability with a Heaviside Θ-function shape and a consensus time which increases with system size as $N^{1/2}$ rather than linearly.

All these results have been verified in numerical simulations of the model on a fully connected lattice. On the other hand, in spatially extended systems this last (third) scenario does not appear, as predicted by Al Hammal *et al.* [14]. The reason for this is, as we have numerically verified, that fluc-

tuations wash out the intermediate regime in which three stable states exist. Indeed, as fluctuations can take the system from the origin to any of the absorbing states but not the other way around, effectively, on sufficiently large scales, the stable state at the origin plays no role, and the system exhibits a single ordering transition in the generalized-voter class.

Previous results were obtained under the rule that, among the q neighbors involved in the dynamical step, each given neighbor may be selected more than once. If the possibility of repetition is explicitly forbidden, mean-field results clearly do not change, but in finite dimensions variations are possible. We have performed numerical simulations in $d=2$, showing that for $q<4$ no qualitative change occurs: there is a GV transition between a disordered phase for large ε and an ordered one for small ε. Things change only in the special case $q=4$, where all neighbors of a site are considered. The ordered phase disappears and for any value of ε a disordered state is reached.

In summary, the q-voter model is a simple nonlinear extension of the voter model exhibiting a rich and interesting phenomenology and illustrating how apparently innocuous changes in the microscopic dynamics can lead to different types of collective phenomena, and in particular to different paths to reach consensus.

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