Resurgence of inner solutions for perturbations of the McMillan map

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Abstract

A sequence of "inner equations" attached to certain perturbations of the McMillan map was considered in [MSS09], their solutions were used in that article to measure an exponentially small separatrix splitting. We prove here all the results relative to these equations which are necessary to complete the proof of the main result of [MSS09]. The present work relies on ideas from resurgence theory: we describe the formal solutions, study the analyticity of their Borel transforms and use Écalle's alien derivations to measure the discrepancy between different Borel-Laplace sums.

0 Introduction

0.1 Motivation

This article is the continuation of [MSS09], which was devoted to the study of a family of area-preserving planar maps $F_{h,\varepsilon}$ obtained as perturbations of the so-called McMillan map $F_{h,0}:\ (x,y)\mapsto \big(y,-x+\frac{2(\cosh h)y}{1+y^2}\big)$. The map $F_{h,0}$ is known to be integrable, with a hyperbolic fixed point at the origin for $h>0$ and a separatrix, i.e. a homoclinic loop. The goal was to investigate the splitting of this separatrix when the real parameter ε is nonzero, a phenomenon which is exponentially small with respect to h.

The main theorem of [MSS09] depended on intermediary results, which were stated in Section 2.7 of that article, and which will be proved in the present article as a consequence of the study of the "full inner equation" associated with $F_{h,\varepsilon}$. This is the equation

$$\phi(z+1)+\phi(z-1)=\mathcal{F}(\phi(z),h,\varepsilon), \tag{FIE}$$

where $z\mapsto\phi(z)$ is the unknown scalar function and

$$\mathcal{F}(y,h,\varepsilon)=\frac{2(\cosh h)y}{1+y^2}+\varepsilon V'(y,h,\varepsilon), \tag{1}$$

with a function V' holomorphic in $B=\{\,(y,h,\varepsilon)\in\mathbb{C}^3 \mid |y|<y_0,\ |h|<h_0,\ |\varepsilon|<\varepsilon_0\,\}$ and satisfying

(A) V' is odd in y and even in h,

(B) there exists $C>0$ such that $|V'(y,0,\varepsilon)|\le C|y|^5$ for $|y|<y_0$, $|\varepsilon|<\varepsilon_0$.

The relationship between equation (**FIE**) and the original problem is as follows: up to a simple rescaling, the perturbed map $F_{h,\varepsilon}$ is $(x,y) \mapsto \big(y, -x + \mathcal{F}(y,h,\varepsilon)\big)$, its stable and unstable separatrices can be parametrized as $t \mapsto P^s(t) = \big(\xi^s(t - h/2), \xi^s(t+h/2)\big)$ and $t \mapsto P^u(t) = \big(\xi^u(t-h/2), \xi^u(t+h/2)\big)$, with functions

$$\xi^s(t) = \xi^s(t,h,\varepsilon) \xrightarrow[t\to+\infty]{} 0, \qquad \xi^u(t) = \xi^u(t,h,\varepsilon) \xrightarrow[t\to-\infty]{} 0$$

(so that the parametrized curves are positively or negatively asymptotic to the hyperbolic fixed point) which satisfy the "outer" difference equation

$$\xi(t+h) + \xi(t-h) = \mathcal{F}(\xi(t),h,\varepsilon)$$

(so that $F_{h,\varepsilon}\big(P^{s,u}(t)\big) = P^{s,u}(t+h)$). The full inner equation was obtained simply by setting

$$\xi(t) = \phi(z), \qquad t = \frac{\mathrm{i}\pi}{2} + hz. \tag{2}$$

The reader is referred to the beginning of [MSS09] for more information on the geometric problem and a motivation of formula (2). We shall now focus on equation (**FIE**).

0.2 The integrable case $\varepsilon = 0$

For $\varepsilon = 0$, we know explicitly the solution of (**FIE**) which is related to the separatrix:

$$\Phi^0(z,h) = -\mathrm{i}\frac{\sinh h}{\sinh(hz)} = -\mathrm{i}z^{-1} + \mathrm{i}\frac{h^2}{6}(z - z^{-1}) - \mathrm{i}\frac{h^4}{360}(7z^3 - 10z + 3z^{-1}) + \dots \tag{3}$$

This is related to the integrability of the McMillan map $F_{h,0}$: the function

$$H(x,y;h) = x^2y^2 + x^2 + y^2 - 2(\cosh h)xy \tag{4}$$

is a first integral of $F_{h,0}$ (see Lemma 2.10 below) and $z \mapsto \big(\Phi^0(z-\frac{1}{2},h), \Phi^0(z+\frac{1}{2},h)\big)$ is a parametrization of part of the complexified homoclinic loop $\{\, H(x,y;h) = 0 \,\}$. (Other solutions of (**FIE**) for $\varepsilon = 0$, corresponding to other levels of H, will be discussed in Section 2.3 below.)

For nonzero ε, we shall construct *formal* solutions of (**FIE**) which are deformations of Φ^0 and from which we shall deduce *analytic* solutions.

0.3 The h^2-expansion

We can expand

$$\mathcal{F}(y,h,\varepsilon) = \sum_{n\geq 0} h^{2n}\mathcal{F}_n(y,\varepsilon). \tag{5}$$

Looking for a solution of (**FIE**) in the form

$$\phi = \sum_{n\geq 0} h^{2n}\phi_n(z,\varepsilon)$$

and expanding in powers of h^2, we get the "inner equation"

$$\phi_0(z+1) + \phi_0(z-1) = \mathcal{F}(\phi_0(z),0,\varepsilon) = \frac{2\phi_0(z)}{1+\phi_0(z)^2} + \varepsilon V'(\phi_0(z),0,\varepsilon) \qquad (\mathbf{IE})_0$$

(we sometimes omit the dependence in ε for notational convenience) and a system of "secondary inner equations"

$$\phi_n(z+1) + \phi_n(z-1) = F_n(z,\varepsilon), \qquad n \geq 1, \qquad (\mathbf{IE})_n$$

where the right-hand sides are determined inductively:

$$F_n = \partial_y \mathcal{F}(\phi_0, 0, \varepsilon)\phi_n + f_n, \tag{6}$$

$$f_n = \mathcal{F}_n(\phi_0, \varepsilon) + \sum \frac{1}{r!} \partial_y^r \mathcal{F}_{n_0}(\phi_0, \varepsilon)\phi_{n_1} \dots \phi_{n_r}, \tag{7}$$

where the sum in (7) is taken over all $n_0 \geq 0, r \geq 1$ such that $n_0 + r \geq 2$ and $n_1, \dots, n_r \geq 1$ such that $n_0 + n_1 + \cdots + n_r = n$. In fact, f_n is the coefficient of h^{2n} in $\mathcal{F}(\phi_0 + h^2\phi_1 + \cdots + h^{2(n-1)}\phi_{n-1}, h, \varepsilon)$ (while F_n is the coefficient of h^{2n} in $\mathcal{F}(\phi_0 + h^2\phi_1 + \cdots + h^{2n}\phi_n, h, \varepsilon)$).

0.4 Aim and structure of the article

We shall determine formal solutions $\tilde{\Phi}_n(z, \varepsilon; b)$ (formal with respect to z) of equations $(\mathbf{IE})_n$, $n \geq 0$, depending on a free parameter $b \in \mathbb{C}^{\mathbb{N}^*}$. These formal series are generically divergent (contrarily to what happens when $\varepsilon = 0$), but their Borel transforms with respect to z are analytic in a certain domain. Borel-Laplace summation then leads to solutions Φ_n^s and Φ_n^u holomorphic in two different domains of the z-plane, the difference between them being related to complex singularities of the Borel transforms. The analysis of the singularities in the Borel plane will be performed with the help of the alien derivations, which are tools introduced by J. Écalle in his resurgence theory, and will give access to the precise asymptotic behavior of $\Phi_n^s - \Phi_n^u$.

In order not to interrupt the flow of the arguments with long and technical explanations, we gather in Section 1 the results on the series $\tilde{\Phi}_n(z, \varepsilon; b)$ and show how they imply the statements which were mentioned in Section 2.7 of [MSS09]. These results are then proved in the subsequent sections of the present article:

- Section 2 is devoted to the formal part of the study (existence and definition of the $\tilde{\Phi}_n$'s);
- Section 3 deals with the analytic study of the formal Borel transforms $\hat{\Phi}_n(\zeta, \varepsilon; b)$;
- Section 4 is devoted to the computation of the singularities of the $\hat{\Phi}_n$'s by means of Écalle's alien derivations;
- the appendix gathers a few technical proofs and reminders on second-order linear difference equations.

1 Main results

1.1 Formal solutions

We are interested in formal solutions of the above equations, more precisely solutions in $z^{-1}\mathbb{C}[[z^{-1}]]$ for $(\mathbf{IE})_0$ (power series involving only negative powers of z), solutions in $\mathbb{C}((z^{-1}))$ for $(\mathbf{IE})_n$, $n \geq 1$ (formal Laurent series, with only finitely many positive powers of z). Here it is understood that the coefficients of these formal series may depend on ε.

Observe that the only nonlinear equation is the first one. Since it involves substitution of the unknown series into $\mathcal{F}(\,.\,,h,\varepsilon)$, it requires that the unknown series belong to the maximal ideal $z^{-1}\mathbb{C}[[z^{-1}]]$ of the ring $\mathbb{C}[[z^{-1}]]$. The field of fractions of this ring is $\mathbb{C}((z^{-1})) = \mathbb{C}[[z^{-1}]][z]$ and the operators $\varphi(z) \mapsto \varphi(z+1)$ or $\varphi(z) \mapsto \varphi(z-1)$ are well-defined[1] in $\mathbb{C}((z^{-1}))$.

Let us use the notation $[\phi]_n$ for the coefficient of h^{2n} in a formal series $\phi \in \mathbb{C}((z^{-1}))[[h^2]]$; thus $[\phi]_n \in \mathbb{C}((z^{-1}))$ and ϕ is invertible in $\mathbb{C}((z^{-1}))[[h^2]]$ iff $[\phi]_0 \neq 0$. We shall determine formal solutions $\phi(z,h)$ of $(\mathbf{FIE})$ in the ring $\mathbb{C}((z^{-1}))[[h^2]]$ with $[\phi]_0 \in z^{-1}\mathbb{C}[[z^{-1}]]$.

Theorem 1.1. *For each value of ε, equation $(\mathbf{IE})_0$ has a unique odd formal solution $\tilde\Phi_0(z,\varepsilon)$ of the form $-\mathrm{i}z^{-1} + O(z^{-3})$. The solutions $\phi \in \mathbb{C}((z^{-1}))[[h^2]]$ of $(\mathbf{FIE})$ which are odd in z and such that $[\phi]_0 = \tilde\Phi_0$ are in one-to-one correspondence with the sequences $b \in \mathbb{C}^{\mathbb{N}^*}$; for each such b, the corresponding solution can be written*

$$\tilde\Phi(z,h,\varepsilon;b) = \tilde\Phi_0(z,\varepsilon) + \sum_{n\geq 1} h^{2n}\tilde\Phi_n(z,\varepsilon;b_1,\dots,b_n), \tag{8}$$

where $\tilde\Phi_n(z,\varepsilon;b_1,\dots,b_n) \in z^{4n-1}\mathbb{C}[[z^{-1}]]$. Moreover,

$$b_1 = 0 \quad \Leftrightarrow \quad \forall n \geq 1,\ \ \tilde\Phi_n(z,\varepsilon;b_1,\dots,b_n) \in z^{2n-1}\mathbb{C}[[z^{-1}]] \tag{9}$$

and, for each $n \geq 0$, the coefficients of the formal series $\tilde\Phi_n$ depend analytically on ε.

The general nonzero solution of $(\mathbf{FIE})$ in $\mathbb{C}((z^{-1}))[[h^2]]$ is $\pm\tilde\Phi(z+a(h),h,\varepsilon;b)$, with arbitrary $a(h) \in \mathbb{C}[[h^2]]$ and $b \in \mathbb{C}^{\mathbb{N}^*}$.

The proof is given in Section 2. Observe that, for any $\phi(z,h), a(z,h) \in \mathbb{C}((z^{-1}))[[h^2]]$ with $[a]_0 \in \mathbb{C}[[z^{-1}]]$ (possibly depending on ε), the substitution $\phi\big(z+a(z,h),h\big)$ makes sense; in case $a = a(h)$ does not depend on z, it is obvious that $\phi\big(z+a(h),h\big)$ is a solution of $(\mathbf{FIE})$ whenever $\phi(z,h)$ is a solution.

When $\varepsilon = 0$, a certain choice $b^*(0)$ of b leads to $\tilde\Phi(z,h,0;b^*(0)) = \Phi^0(z,h)$ as defined by (3). In particular

$$\tilde\Phi_0(z,0) = -\mathrm{i}z^{-1} \tag{10}$$

[1] More generally, if $c(z) \in \mathbb{C}[[z^{-1}]]$, the substitution operator $\varphi(z) \mapsto \varphi\big(z+c(z)\big)$ is a well-defined automorphism of the field $\mathbb{C}((z^{-1}))$; it can be written as a series

$$\varphi\big(z+c(z)\big) = \sum_{p\geq 0} \frac{1}{p!}\partial_z^p\varphi(z)\big(c(z)\big)^p,$$

which is convergent for the Krull topology, i.e. the metrizable topology of $\mathbb{C}((z^{-1}))$ induced by the standard valuation (indeed $\varphi \in z^{-v}\mathbb{C}[[z^{-1}]]$ with $v \in \mathbb{Z}$ implies $(\partial_z^p\varphi)c^p \in z^{-v-p}\mathbb{C}[[z^{-1}]]$; the coefficient of a monomial z^{-m} in the right-hand side is thus given by a finite sum of terms).

and $\tilde{\Phi}_1(z,0;b_1^*(0)) = \frac{\mathrm{i}}{6}(z-z^{-1})$, $\tilde{\Phi}_2(z,0;b_1^*(0),b_2^*(0)) = -\frac{\mathrm{i}}{360}(7z^3-10z+3z^{-1})$, etc. It turns out that all the series $\tilde{\Phi}_n(z,0;b_1,\dots,b_n)$ are convergent; in fact, they are polynomials up to the factor z^{-1}:

Proposition 1.2. *When $\varepsilon=0$, for any $b\in\mathbb{C}^{\mathbb{N}^*}$,*

$$\forall n\geq 1,\ \ \tilde{\Phi}_n(z,0;b_1,\dots,b_n)\in z^{-1}\mathbb{C}[z].$$

The proof is given at the end of Section 2.

We shall see that on the contrary, for generic V' and ε, the series $\tilde{\Phi}_n(z,\varepsilon;b_1,\dots,b_n)$ are divergent, and this divergence will be analyzed through resurgence theory. (As for their dependence on $b_1,\dots,b_n$, it is polynomial, with degree 1 in b_n, as can be seen from formulas (37)–(38) of Section 2.2).

> *As a consequence of Theorem 1.1, we can identify the formal series which were denoted $\tilde{\phi}_n(z,\varepsilon)$ in Section 2.6 of [MSS09] with the formal series $\tilde{\Phi}_n\big(z,\varepsilon;b_1^*(\varepsilon),\dots,b_n^*(\varepsilon)\big)$, for specific values of the b_n^*'s which we need not compute. We only remark that $\varepsilon\mapsto b_n^*(\varepsilon)$ is analytic and, in view of (9), $b_1^*(\varepsilon)=0$ (see [MSS09], formulas (89) and (92) and Proposition 2.12).*

1.2 Borel-Laplace summation

We define the Borel transform $\mathcal{B}:\mathbb{C}((z^{-1}))\to\mathbb{C}[[\zeta]]$ as follows: for $\tilde{\varphi}(z)=\sum\limits_{p\geq -v} a_p z^{-p}$ with $v\in\mathbb{N}$, we set

$$\mathcal{B}\tilde{\varphi}(\zeta)=\hat{\varphi}(\zeta)=\sum_{p\geq 1}a_p\frac{\zeta^{p-1}}{(p-1)!}.$$

This is thus a linear operator which cancels out the polynomial part of $\tilde{\varphi}(z)$.

Observe that $\hat{\varphi}(\zeta)\in\mathbb{C}\{\zeta\}$ simply means that $\tilde{\varphi}(z)$ is Gevrey-1, i.e. there exist $C,K>0$ such that $|a_p|\leq CK^p p!$. On the other hand, if $\tilde{\varphi}(z)$ is convergent for $|z|$ large enough, then $\hat{\varphi}(\zeta)$ must define an entire function of exponential type.

In the case of the formal solutions of $(\mathbf{IE})_n$, $n\geq 0$, we shall see that the Borel transforms converge near the origin, but the holomorphic functions of ζ thus defined are generically not entire: their analytic continuations are singular at $\pm 2\pi\mathrm{i}$ (thus the formal solutions themselves are not convergent). We begin by considering the cut plane $\mathcal{R}^{(0)}=\mathbb{C}\setminus\pm 2\pi\mathrm{i}\,[1,+\infty)$, which will be the common holomorphic star of the $\mathcal{B}\tilde{\Phi}_n$'s. (Later on, we shall see that these functions admit a multivalued analytic continuation in a much larger domain; in fact, only the points of $2\pi\mathrm{i}\,\mathbb{Z}$ can be singular.)

Definition 1.3. For any $\rho\in(0,2\pi)$, we set

$$\mathcal{R}^{(0)}_\rho=\big\{\zeta\in\mathbb{C}\mid \mathrm{dist}\big([0,\zeta],2\pi\mathrm{i}\big)\geq\rho,\ \mathrm{dist}\big([0,\zeta],-2\pi\mathrm{i}\big)\geq\rho\big\}\ \subset\ \mathcal{R}^{(0)}$$

(see Figure 1). We define $\widehat{\mathrm{RES}}^{(0)}$ to be the set of all $\hat{\varphi}\in\mathbb{C}\{\zeta\}$ such that

(i) $\hat{\varphi}(\zeta)$ extends analytically to $\mathcal{R}^{(0)}$,

(ii) for each $\rho\in(0,2\pi)$, there exist $\tau,C>0$ such that $|\hat{\varphi}(\zeta)|\leq C\,\mathrm{e}^{\tau|\zeta|}$ for $\zeta\in\mathcal{R}^{(0)}_\rho$.

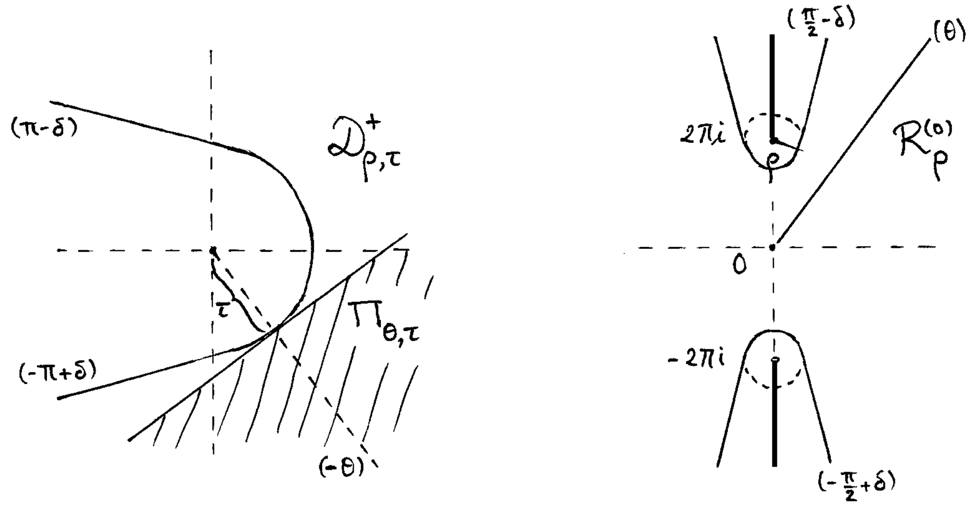

Figure 1: Right: the domain $\mathcal{R}^{(0)}_\rho$ is a part of the cut plane $\mathcal{R}^{(0)}$ in the ζ-plane. Left: the domain $\mathcal{D}^+_{\rho,\tau}$ is the union of the half-planes $\Pi_{\theta,\tau}$ in the z-plane.

We also set $\widetilde{\mathrm{RES}}^{(0)} = \mathcal{B}^{-1}\,\widehat{\mathrm{RES}}^{(0)}$.

Theorem 1.4. *Let $b \in \mathbb{C}^{\mathbb{N}^*}$ and $n \in \mathbb{N}$. Then the Borel transform $\hat\Phi_n(\zeta,\varepsilon;b_1,\dots,b_n)$ of the solution of equation* $(\mathbf{IE})_n$ *described in Theorem 1.1 is convergent for $|\zeta| < 2\pi$ and defines a holomorphic function of two variables in $\{\,(\zeta,\varepsilon)\in\mathbb{C}^2 \mid \zeta\in\mathcal{R}^{(0)},\ |\varepsilon| < \varepsilon_0\,\}$ which depends polynomially on $b_1,\dots,b_n$. Moreover, for any $\varepsilon_0' \in (0,\varepsilon_0)$ and $\rho \in (0,2\pi)$, there exist positive constants C_n, τ_n which depend continuously on $b_1,\dots,b_n$, such that*

$$|\hat\Phi_n(\zeta,\varepsilon;b_1,\dots,b_n)| \le C_n\,\mathrm{e}^{\tau_n|\zeta|}, \qquad \zeta\in\mathcal{R}^{(0)}_\rho,\ \ |\varepsilon|\le\varepsilon_0'.$$

In particular $\tilde\Phi_n(z,\varepsilon;b_1,\dots,b_n) \in \widetilde{\mathrm{RES}}^{(0)}$ for each ε.

The proof is given in Section 3.

We are thus in a position to apply the Borel-Laplace summation process, which can be described as follows. Suppose that $\tilde\varphi(z) = \sum_{p\ge -v} a_p z^{-p}$ belongs to $\widetilde{\mathrm{RES}}^{(0)}$ and let $\rho\in(0,2\pi)$, $\delta = \arcsin\frac{\rho}{2\pi}$ and $\tau = \tau(\rho)$ as in Definition 1.3 (ii). The formula

$$(\mathcal{S}^\theta\tilde\varphi)(z) = \sum_{p=0}^{v} a_{-p} z^p + \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-z\zeta}\hat\varphi(\zeta)\,\mathrm{d}\zeta \tag{11}$$

defines a function $\mathcal{S}^\theta\tilde\varphi$ which is holomorphic in the half-plane $\Pi_{\theta,\tau} = \{\,z\in\mathbb{C} \mid \Re e(z\,\mathrm{e}^{\mathrm{i}\theta}) > \tau\,\}$, provided the angle θ is such that the half-line of integration $\mathrm{e}^{\mathrm{i}\theta}\mathbb{R}^+$ be contained in $\mathcal{R}^{(0)}_\rho$. Such angles correspond to two intervals:

$$\theta\in I^+_\rho = \left[-\tfrac{\pi}{2}+\delta, \tfrac{\pi}{2}-\delta\right] \ \text{ or } \ \theta\in I^-_\rho = \left[\tfrac{\pi}{2}+\delta, \tfrac{3\pi}{2}-\delta\right].$$

The Cauchy theorem shows that the functions $\mathcal{S}^\theta\tilde\varphi$ corresponding to angles θ from the same interval mutually extend, so that we get two holomorphic functions:

$$\mathcal{S}^+\tilde\varphi \text{ holomorphic in } \mathcal{D}^+_{\rho,\tau} = \bigcup_{\theta\in I^+_\rho}\Pi_{\theta,\tau}, \quad \mathcal{S}^-\tilde\varphi \text{ holomorphic in } \mathcal{D}^-_{\rho,\tau} = \bigcup_{\theta\in I^-_\rho}\Pi_{\theta,\tau},$$

defined as

$$(\mathcal{S}^{\pm}\tilde{\varphi})(z) = (\mathcal{S}^{\theta}\tilde{\varphi})(z) \text{ for any } \theta \in I_{\rho}^{\pm} \text{ such that } z \in \Pi_{\theta,\tau}. \qquad (12)$$

Notice that the domains $\mathcal{D}^{+}_{\rho,\tau}$ and $\mathcal{D}^{-}_{\rho,\tau}$ can be considered as sectorial neighborhoods of infinity of opening $2\pi - 2\delta$ centered respectively on $\mathbb{R}^{+}$ and $\mathbb{R}^{-}$ (see Figure 1).

The classical properties of the summation operators $\mathcal{S}^{\theta}$ imply that $\tilde{\varphi}$ is the asymptotic expansion of $\mathcal{S}^{\pm}\tilde{\varphi}$ in the Gevrey-1 sense uniformly in $\mathcal{D}^{\pm}_{\rho,\tau}$, a property which we denote

$$\mathcal{S}^{\pm}\tilde{\varphi}(z) \sim_1 \tilde{\varphi}(z), \qquad z \in \mathcal{D}^{\pm}_{\rho,\tau},$$

and which means that there exist $C, K > 0$ such that, for each $p \in \mathbb{N}^*$,

$$|\mathcal{S}^{\pm}\tilde{\varphi}(z) - \sum_{p'=-v}^{p-1} a_{p'} z^{-p'}| \leq C K^p p! |z|^{-p}, \qquad z \in \mathcal{D}^{\pm}_{\rho,\tau}.$$

The intersection of $\mathcal{D}^{+}_{\rho,\tau}$ and $\mathcal{D}^{-}_{\rho,\tau}$ has two connected components, in which $\mathcal{S}^{+}\tilde{\varphi}$ and $\mathcal{S}^{-}\tilde{\varphi}$ generically do not coincide; in fact, $\mathcal{S}^{+}\tilde{\varphi}$ and $\mathcal{S}^{-}\tilde{\varphi}$ mutually extend if and only if the original series $\tilde{\varphi}$ has positive radius of convergence (then the union $\mathcal{D}^{+}_{\rho,\tau} \cup \mathcal{D}^{-}_{\rho,\tau}$ contains a full neighborhood of infinity, $\{|z| > R\}$, in which $\tilde{\varphi}(z)$ converges to $\mathcal{S}^{\pm}\tilde{\varphi}(z)$).

By letting ρ vary in $(0, 2\pi)$, we see that $\mathcal{S}^{+}\tilde{\varphi}$ and $\mathcal{S}^{-}\tilde{\varphi}$ admit an analytic continuation to $\mathcal{D}^{s} = \bigcup \mathcal{D}^{+}_{\rho,\tau(\rho)}$ and $\mathcal{D}^{u} = \bigcup \mathcal{D}^{-}_{\rho,\tau(\rho)}$.

Moreover, $\widetilde{\mathrm{RES}}^{(0)}$ is a differential subalgebra of $\mathbb{C}((z^{-1}))$ (it is stable by multiplication and differentiation), the operators $\mathcal{S}^{\pm}$ are differential algebra morphisms (they map the product of formal series on the product of analytic functions and they commute with ∂_z) and they commute with the shift operator $\tilde{\varphi}(z) \mapsto \tilde{\varphi}(z+1)$. Consequently, when $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$ can be applied to a formal solution of a (possibly non-linear) difference equation, it yields an analytic solution of this equation. The reader is referred e.g. to [CNP93], [Eca81], [Mal95] or [Sau05] for these properties (for the stability under multiplication and differentiation, see also Section 3.4, Lemma 3.12).

Corollary 1.5. *Let $b \in \mathbb{C}^{\mathbb{N}^*}$. Then there exist two decreasing sequences of domains $\mathcal{D}^s_n$ and $\mathcal{D}^u_n$, each of which contains sectorial neighborhoods of infinity with opening arbitrarily close to 2π centered respectively on $\mathbb{R}^+$ and $\mathbb{R}^-$, such that the functions*

$$\Phi^s_n(z,\varepsilon;b_1,\dots,b_n) := \mathcal{S}^{+}\tilde{\Phi}_n, \quad \Phi^u_n(z,\varepsilon;b_1,\dots,b_n) := \mathcal{S}^{-}\tilde{\Phi}_n \qquad (13)$$

are holomorphic for $z \in \mathcal{D}^s_n$, resp. $z \in \mathcal{D}^u_n$, and $|\varepsilon| < \varepsilon_0$, and solve equations $(\mathbf{IE})_n$, $n \in \mathbb{N}$. Moreover, for each $\rho \in (0, 2\pi)$, there exists $\tau_n > 0$ such that

$$\Phi^{s,u}_n(z,\varepsilon;b_1,\dots,b_n) \sim_1 \tilde{\Phi}_n(z,\varepsilon;b_1,\dots,b_n), \qquad z \in \mathcal{D}^s_{\rho,\tau_n} \text{ or } \mathcal{D}^u_{\rho,\tau_n},$$

and Φ^s_n and Φ^u_n coincide for $\varepsilon = 0$.

Proof. Letting ρ vary in $(0, 2\pi)$, we define $\mathcal{D}^{s,u}_n$ as

$$\mathcal{D}^s_n = \bigcup \mathcal{D}^{+}_{\rho,\tau_n(\rho)}, \qquad \mathcal{D}^u_n = \bigcup \mathcal{D}^{-}_{\rho,\tau_n(\rho)}, \qquad (14)$$

with τ_n given in function of ρ by Theorem 1.4 (there is no loss of generality in assuming that the sequence (τ_n) is increasing). $\square$

There is a characterization of the solutions Φ_n^s and Φ_n^u by the beginning of their asymptotic expansion, without any extra regularity assumption. If, for $\tilde{\varphi}(z) = \sum_{p\geq -v} a_p z^{-p}$, we introduce the notation

$$[\tilde{\varphi}]_{\leq 2} = \sum_{p=-v}^{2} a_p z^{-p}$$

(for instance $\big[\tilde{\Phi}_0(z,\varepsilon)\big]_{\leq 2} = -\mathrm{i}z^{-1}$ by Theorem 1.1), we indeed have

Proposition 1.6. *Let $b_1, \dots, b_{n_0} \in \mathbb{C}$, $\sigma \in (2,3]$, $z_0 \in \mathcal{D}^u_{n_0}$ and $\varepsilon \in \mathbb{C}$ such that $|\varepsilon| < |\varepsilon_0|$. Then the sequence of functions $(\phi_n)_{0\leq n\leq n_0}$ defined by $\phi_n(z) = \Phi_n^u(z,\varepsilon; b_1,\dots,b_n)$ is the only sequence of solutions of* $(\mathbf{IE})_n$*, $0 \leq n \leq n_0$, such that each ϕ_n is defined on the half-line $z_0 + \mathbb{R}^-$ and satisfies*

$$\phi_n(z) = \big[\tilde{\Phi}_n(z,\varepsilon; b_1,\dots,b_n)\big]_{\leq 2} + O\big(|z|^{-\sigma}\big).$$

A similar statement holds for the functions $\Phi_n^s(z,\varepsilon;b_1,\dots,b_n)$, with $z_0+\mathbb{R}^-$ replaced by $z_0+\mathbb{R}^+$.

The proof is given in Appendix A.3.

> *The above Corollary 1.5 and Proposition 1.6 yield Theorem 2.14 of [MSS09]. Indeed, the domain which is denoted $\mathcal{D}^u_{in}(R_n)$ there is clearly contained in $\mathcal{D}^u_n$.*

1.3 The alien derivatives of the formal solution

Definition 1.7. Let $\tilde{\varphi} \in \widetilde{\mathrm{RES}}^{(0)}$. We say that $\hat{\varphi} = \mathcal{B}\tilde{\varphi}$ has a *simply ramified singularity* at $\omega = \pm 2\pi\mathrm{i}$ if there exist $\mathrm{reg}(\zeta) \in \mathbb{C}\{\zeta\}$ and $\tilde{\psi}(z) = \sum_{p\geq -v} b_p z^{-p} \in \mathbb{C}((z^{-1}))$ (with $v \in \mathbb{N}$), such that $\hat{\psi} = \mathcal{B}\tilde{\psi} \in \mathbb{C}\{\zeta\}$ and

$$\hat{\varphi}(\zeta) = \sum_{p=0}^{v} b_{-p}\frac{(-1)^p p!}{2\pi\mathrm{i}(\zeta-\omega)^{p+1}} + \hat{\psi}(\zeta-\omega)\frac{\log(\zeta-\omega)}{2\pi\mathrm{i}} + \mathrm{reg}(\zeta-\omega) \tag{15}$$

for $\zeta \in \mathcal{R}^{(0)}$ with $|\zeta-\omega|$ small enough. In this situation, we use the notation

$$\Delta_\omega\tilde{\varphi} = \tilde{\psi}. \tag{16}$$

Observe that, in the above situation, the Gevrey-1 formal series $\tilde{\psi}$ is indeed determined by $\tilde{\varphi}$ (by $\mathcal{B}\tilde{\varphi}$ in fact): the function $\hat{\varphi}$ extends holomorphically to the universal cover of a punctured disc centered at ω and $\hat{\psi}(\xi)$ is the *variation* (or monodromy) of $\hat{\varphi}$ at $\omega+\xi$ around ω, i.e. the difference between two consecutive branches $\hat{\psi}(\xi) = \hat{\varphi}(\omega+\xi) - \hat{\varphi}(\omega+\xi\,\mathrm{e}^{-2\pi\mathrm{i}})$, while the polynomial part of $\tilde{\psi}(z)$ is determined by the polar part of the Laurent expansion at the origin of $\check{P}(\xi) = \hat{\varphi}(\omega+\xi) - \hat{\psi}(\xi)\frac{\log\xi}{2\pi\mathrm{i}}$ (which is meromorphic in a small disc centered at the origin); but the regular function $\mathrm{reg}(\xi)$ depends on the branch of the logarithm which is chosen in (15).

We thus have two linear operators $\Delta_{2\pi\mathrm{i}}$ and $\Delta_{-2\pi\mathrm{i}}$ defined on the subspace of $\widetilde{\mathrm{RES}}^{(0)}$ consisting of the formal series whose Borel transforms have simply ramified singularities at $\pm 2\pi\mathrm{i}$, with values in the space of Gevrey-1 formal series $\mathbb{C}((z^{-1}))_{\mathrm{Gev}}$. These operators are particular instances of Écalle's *alien derivations*. They are indeed derivations: it can be proved that $\Delta_\omega(\tilde\varphi_1\tilde\varphi_2) = (\Delta_\omega\tilde\varphi_1)\,\tilde\varphi_2 + \tilde\varphi_1\,(\Delta_\omega\tilde\varphi_2)$ (see e.g. [Eca81], [CNP93] or [Sau05]).

It will turn out that the $\hat\Phi_n$'s have simply ramified singularities at $\pm 2\pi\mathrm{i}$. Theorem 1.10 will describe these singularities through the action of the alien derivations $\Delta_{\pm 2\pi\mathrm{i}}$ on $\tilde\Phi_n$ in Formula (20); this formula will involve auxiliary formal series $\tilde\Psi_{1,n}$, $\tilde\Psi_{2,n}$ which we now introduce.

Let $b \in \mathbb{C}^{\mathbb{N}^*}$. Associated with the formal solution $\tilde\Phi(z,h,\varepsilon;b) \in \mathbb{C}((z^{-1}))[[h^2]]$ of (**FIE**), there is a variational equation, which is the linear equation

$$\Psi(z+1) + \Psi(z-1) = \partial_y\mathcal{F}\big(\tilde\Phi(z,h,\varepsilon;b),h,\varepsilon\big)\,\Psi(z), \qquad (\mathbf{FL})_b$$

for an unknown $\Psi = \sum_{n\geq 0} h^{2n}\Psi_n(z) \in \mathbb{C}((z^{-1}))[[h^2]]$. Similarly, one can consider the variational equation associated with the solution $\Phi^u(z,h,\varepsilon;b) = \Phi^u_0(z,\varepsilon) + \sum_{n\geq 1} h^{2n}\Phi^u_n(z,\varepsilon;b_1,\dots,b_n)$ (formal in h, analytic in z):

$$\Psi(z+1) + \Psi(z-1) = \partial_y\mathcal{F}\big(\Phi^u(z,h,\varepsilon;b),h,\varepsilon\big)\,\Psi(z), \qquad (\mathbf{FL})^u_b$$

for an unknown $\Psi = \sum_{n\geq 0} h^{2n}\Psi_n(z)$ with coefficients analytic in z.

For such linear difference equations, we call *normalized fundamental system of solutions* a pair of solutions (Ψ_1,Ψ_2) such that

$$\Psi_1(z)\Psi_2(z+1) - \Psi_1(z+1)\Psi_2(z) \equiv 1$$

(see Section 2.1 and Appendix A.2 for reminders about the theory of linear difference equations).

Proposition 1.8. *For each $b \in \mathbb{C}^{\mathbb{N}^*}$, there exists a normalized fundamental system of solutions $(\tilde\Psi_1,\tilde\Psi_2)$ for $(\mathbf{FL})_b$, of the form*

$$\tilde\Psi_j(z,h,\varepsilon;b) = \tilde\Psi_{j,0}(z,\varepsilon) + \sum_{n\geq 1} h^{2n}\tilde\Psi_{j,n}(z,\varepsilon;b_1,\dots,b_n), \qquad j=1,2, \tag{17}$$
$$\tilde\Psi_1 = \partial_z\tilde\Phi \textit{ even in } z, \qquad \tilde\Psi_{1,0}(z,\varepsilon) = \mathrm{i}z^{-2} + O(z^{-4}),$$
$$\tilde\Psi_2 \textit{ odd in } z, \qquad \tilde\Psi_{2,0}(z,\varepsilon) = -\frac{\mathrm{i}}{5}z^3 + O(z),$$

with all $\tilde\Psi_{j,n} \in \widetilde{\mathrm{RES}}^{(0)}$. Moreover, $\tilde\Psi_{1,n} \in z^{4n-2}\mathbb{C}[[z^{-1}]]$ and $\tilde\Psi_{2,n} \in z^{4n+3}\mathbb{C}[[z^{-1}]]$ in general, while

$$b_1 = 0 \quad\Rightarrow\quad \forall n\geq 0,\ \ \tilde\Psi_{1,n} \in z^{2n-2}\mathbb{C}[[z^{-1}]],\ \ \tilde\Psi_{2,n} \in z^{2n+3}\mathbb{C}[[z^{-1}]]. \tag{18}$$

The proof is in Section 4.1. We immediately deduce

Corollary 1.9. *The formulas*

$$\Psi^u_j = \sum_{n\geq 0} h^{2n}\Psi^u_{j,n}, \qquad \Psi^u_{j,n} = \mathcal{S}^-\tilde\Psi^u_{j,n},$$

define a normalized fundamental system of solutions (Ψ^u_1,Ψ^u_2) for $(\mathbf{FL})^u_b$.

We thus have at our disposal formal series $\tilde{\Psi}_{1,n}$, $\tilde{\Psi}_{2,n}$, and analytic functions which admit them as Gevrey-1 asymptotic expansions. In fact, the coefficients of these formal series can be determined inductively, as was the case for the formal series $\tilde{\Phi}_n$.

> *Proposition 1.8 and Corollary 1.9 contain Proposition 2.16 and the first part of Theorem 2.17 of [MSS09].*

We are now ready for the main statement of this section:

Theorem 1.10. *Let $b \in \mathbb{C}^{\mathbb{N}^*}$. Then the Borel transforms $\hat{\Phi}_n(\zeta, \varepsilon; b)$ have simply ramified singularities at $\pm 2\pi\mathrm{i}$ and there exist four formal series in h^2, the coefficients of which are complex polynomials in $b_1, b_2 \ldots$ that depend analytically on ε for $|\varepsilon| < \varepsilon_0$ and vanish at $\varepsilon = 0$,*

$$A^{\pm}(h, \varepsilon; b) = \sum_{n \geq 0} A_n^{\pm}(\varepsilon; b_1, \ldots, b_n) h^{2n}, \quad B^{\pm}(h, \varepsilon; b) = \sum_{n \geq 0} B_n^{\pm}(\varepsilon; b_1, \ldots, b_n) h^{2n}, \tag{19}$$

such that

$$\Delta_{\pm 2\pi\mathrm{i}} \tilde{\Phi}_n = \sum_{n_1 + n_2 = n} \big(A_{n_1}^{\pm} \tilde{\Psi}_{1,n_2} + \mathrm{i} B_{n_1}^{\pm} \tilde{\Psi}_{2,n_2} \big), \qquad n \in \mathbb{N}. \tag{20}$$

The analytic functions $A_0^{\pm}(\varepsilon)$ and $B_0^{\pm}(\varepsilon)$ do not depend on b. One has

$$A_0^{\pm}(\varepsilon) = \varepsilon A_{0,1}^{\pm} + O(\varepsilon^2), \qquad A_{0,1}^{\pm} = 2\pi D \hat{V}_0(\pm 2\pi), \tag{21}$$

$$B_0^{\pm}(\varepsilon) = \varepsilon B_{0,1}^{\pm} + O(\varepsilon^2), \qquad B_{0,1}^{\pm} = \pm 4\pi^2 \hat{V}_0(\pm 2\pi), \tag{22}$$

where $\hat{V}_0$ is the entire function obtained as Borel transform with respect to $1/y$ of a primitive of $V'(y, 0, 0)$:

$$V'(y, 0, 0) = \sum_{p \geq 5} v_p y^p, \quad V_0(y) = \sum_{p \geq 5} v_p \frac{y^{p+1}}{p+1}, \quad \hat{V}_0(\xi) = \sum_{p \geq 5} v_p \frac{\xi^p}{(p+1)!}, \tag{23}$$

and $D = \frac{1}{5}\xi\partial_\xi^5 + \partial_\xi^4 + \frac{1}{3}\xi\partial_\xi^3 + \partial_\xi^2 + \frac{2}{15}\xi\partial_\xi + \frac{2}{15}\,\mathrm{Id}$.

The proof is given in Section 4.3. It relies on Écalle's formalism of "singularities" which is briefly described in Section 4.2 (and on auxiliary results contained in Sections 3.3 and 3.5).

Observe that equation (20) can be written in a more compact form if we extend the action of the linear operators Δ_ω to formal series in h^2 by the formula $\Delta_\omega \left(\sum h^{2n} \tilde{\varphi}_n \right) = \sum h^{2n} \Delta_\omega \tilde{\varphi}_n$, namely

$$\Delta_{\pm 2\pi\mathrm{i}} \tilde{\Phi} = A^{\pm} \tilde{\Psi}_1 + \mathrm{i} B^{\pm} \tilde{\Psi}_2. \tag{24}$$

Equation (24) is an example of what is called the *bridge equation* in Écalle's terminology (see Section 1.5).

Remark 1.11. In Theorem 1.1 of [MSS09], the constant $B_{0,1}^{+}$ is given in the form $4\pi^2 \hat{V}(2\pi)$ instead of $4\pi^2 \hat{V}_0(2\pi)$, where $\hat{V}$ is the Borel transform of the original potential $\tilde{V}$, whose y-derivative $\tilde{V}'$ differs slightly from V':

$$\tilde{V}' = V'(\alpha y, h, \varepsilon) + \frac{\cosh h}{\varepsilon} \big(f(\alpha y) - \alpha f(y) \big), \qquad f(y) = \frac{2y}{1 + y^2},$$

with $\alpha = \alpha(h, \varepsilon)$ satisfying $\alpha(0, \varepsilon) = 1 - \frac{v_3}{4}\varepsilon + O(\varepsilon^2)$ for a certain $v_3 \in \mathbb{C}$ (this rescaling of potential is intended to kill the cubic term in the original function $\tilde{V}'(y)$, which was only assumed to be $O(y^3)$). The discrepancy for $\varepsilon = h = 0$ is thus

$$(\tilde{V}' - V')_{|\varepsilon=h=0} = c\big(yf'(y) - f(y)\big),$$

with a constant c, hence a discrepancy $(\tilde{V} - V)_{|\varepsilon=h=0} = cG$ with $G(y) = \int_0^y \big(y_1 f'(y_1) - f(y_1)\big)\,\mathrm{d}y_1$. However, this is coherent with formula (22), since the Borel transform with respect to $1/y$

$$\hat{G}(\xi) = 2\sum_{p\geq 0}(-1)^p \frac{2p}{(2p+2)!}\xi^{2p+1} = 2\sin\xi + \frac{4}{\xi}(\cos\xi - 1)$$

vanishes at 2π.

1.4 Consequences for the splitting of separatrices

Let $n \in \mathbb{N}$. According to (20), $\hat{\Phi}_n$ has a simply ramified singularity at $\omega = 2\pi\mathrm{i}$, the variation of which is $\hat{\psi} = \sum_{n_1+n_2=n}\Big(A^+_{n_1}\hat{\Psi}_{1,n_2} + \mathrm{i}B^+_{n_1}\hat{\Psi}_{2,n_2}\Big) \in \widehat{\mathrm{RES}}^{(0)}$. This implies that $\hat{\Phi}_n$ admits a multivalued analytic continuation through the cut between $2\pi\mathrm{i}$ and $4\pi\mathrm{i}$: if $\zeta = \omega + \xi \in \mathcal{R}^{(0)}$ with $\xi \in \mathcal{R}^{(0)}$, we can consider $\hat{\Phi}_n(\omega + \xi\,\mathrm{e}^{2\pi\mathrm{i}}) = \hat{\Phi}_n(\omega + \xi) + \hat{\psi}(\xi)$ as defining the branch of the analytic continuation of $\hat{\Phi}_n$ which is obtained from the principal one (the branch holomorphic in $\mathcal{R}^{(0)}$) by turning anticlockwise around $2\pi\mathrm{i}$.

Let $\lambda \in (0, 1)$, $\beta \in (0, \pi/2)$. Consider the path $\Gamma_{\lambda,\beta}$ consisting of two half-lines with vertex at $2\pi(1+\lambda)\mathrm{i}$ and angle β with respect to the horizontal, oriented from left to right, as on Figure 2. Let $\varepsilon_0' \in (0, \varepsilon_0)$. We shall see in Sections 3.3 and 3.5 that, for any $n \geq 0$, there exist constants $C_n^*, \tau_n^* > 0$ which depend only on $\lambda, \beta, \varepsilon_0', b_1, \ldots, b_n$ such that

$$|\hat{\Phi}_n(\zeta, \varepsilon; b_1, \ldots, b_n)| \leq C_n^*\,\mathrm{e}^{\tau_n^*|\zeta - 2\pi(1+\lambda)\mathrm{i}|}, \qquad \zeta \in \Gamma_{\lambda,\beta},\ \ |\varepsilon| \leq \varepsilon_0', \tag{25}$$

where the branch of $\hat{\Phi}_n$ considered in (25) is determined by the convention that the right part of $\Gamma_{\lambda,\beta}$ lies in $\mathcal{R}^{(0)}$, while on its left part one should use the branch of $\hat{\Phi}_n$ obtained by crossing the cut from right to left.

We now estimate the differences $\Phi_n^s - \Phi_n^u$ for z belonging to the intersection of half-planes

$$\mathcal{D}_n = \{\, z \in \mathbb{C} \mid \Re e(z\,\mathrm{e}^{\mathrm{i}\beta}) \geq 2\tau_n^* \text{ and } \Re e(z\,\mathrm{e}^{-\mathrm{i}\beta}) \geq 2\tau_n^* \,\}. \tag{26}$$

Taking τ_n^* large enough, we can assume that $\mathcal{D}_n$ is contained in the lower component of the intersection $\mathcal{D}_n^s \cap \mathcal{D}_n^u$ of the domains defined by (14) (see Figure 2).

Theorem 1.12. *Let $n \geq 0$. For any $\varepsilon \in \mathbb{C}$ such that $|\varepsilon| \leq \varepsilon_0'$ and any $z \in \mathcal{D}_n$,*

$$\Phi_n^s - \Phi_n^u = \sum_{n_1+n_2=n}\big(A^+_{n_1}\Psi^u_{1,n_2} + \mathrm{i}B^+_{n_1}\Psi^u_{2,n_2}\big)\,\mathrm{e}^{-2\pi\mathrm{i}z} + R,$$

$$\textit{with } |R| \leq K_n|\varepsilon|\mathrm{e}^{-2\pi(1+\lambda)|\Im m\, z|}, \tag{27}$$

where $K_n = \frac{2C_n^*}{\varepsilon_0'\tau_n^*}$.

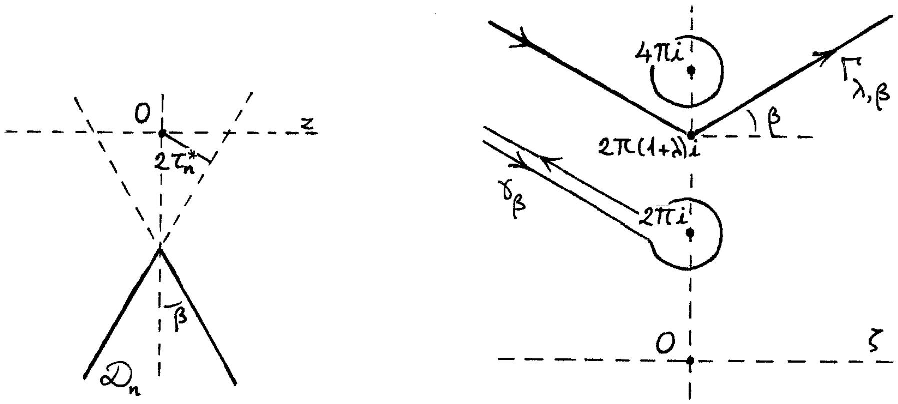

Figure 2: Left: The domain $\mathcal{D}_n$. Right: Computation of $\Phi_n^s - \Phi_n^u$ for $z \in \mathcal{D}_n$ by deformation of the integration contour.

Proof. For such ε and z, in view of (12) and (13), we can write

$$(\Phi_n^s - \Phi_n^u)(z, \varepsilon; b_1, \dots, b_n) = \int_{e^{i(\pi-\beta)}\infty}^{e^{i\beta}\infty} e^{-z\zeta}\, \hat{\Phi}_n(\zeta, \varepsilon; b_1, \dots, b_n)\, d\zeta.$$

By the Cauchy theorem, we can deform the contour: $\Phi_n^s - \Phi_n^u = D + R$ with

$$D = \int_{\gamma_\beta} e^{-z\zeta}\, \hat{\Phi}_n\, d\zeta, \quad R = \int_{\Gamma_{\lambda,\beta}} e^{-z\zeta}\, \hat{\Phi}_n\, d\zeta,$$

where the path $\Gamma_{\lambda,\beta}$ was already defined, while γ_β comes from $e^{i(\pi-\beta)}\infty$ in $\mathcal{R}^{(0)}$, encircles the point $2\pi i$ anticlockwise and goes back to $e^{i(\pi-\beta)}\infty$ (thus on another sheet of the Riemann surface of $\hat{\Phi}_n$—see Figure 2).

Thanks to (20), we can express $\hat{\Phi}_n$ along γ_β by a formula of the form (15) with $\omega = 2\pi i$; the change of variable $\zeta = 2\pi i + \xi$ then yields

$$\int_{\gamma_\beta} e^{-z\zeta} \tfrac{(-1)^p p!}{2\pi i(\zeta-\omega)^{p+1}}\, d\zeta = e^{-2\pi i z}\, z^p,$$

$$\int_{\gamma_\beta} e^{-z\zeta}\, \hat{\psi}(\zeta-\omega) \tfrac{\log(\zeta-\omega)}{2\pi i}\, d\zeta = e^{-2\pi i z} \int_0^{e^{i(\pi-\beta)}\infty} e^{-z\xi} \hat{\psi}(\xi)\, d\xi,$$

thus the contribution of the singularity at $2\pi i$ is given by the operator $\mathcal{S}^-$ of (12) applied to the alien derivative $\Delta_{2\pi i}\tilde{\Phi}_n$ defined by (16):

$$D = e^{-2\pi i z} \mathcal{S}^- \Delta_{2\pi i} \tilde{\Phi}_n = \sum_{n_1+n_2=n} \big(A_{n_1}^+ \Psi_{1,n_2}^u + i B_{n_1}^+ \Psi_{2,n_2}^u\big)\, e^{-2\pi i z}.$$

As for the remainder R, we use the change of variable $\zeta = 2\pi(1+\lambda)i + \xi$ and get

$$R(z,\varepsilon) = e^{-2\pi(1+\lambda)iz} \left(\int_0^{e^{i\beta}\infty} - \int_0^{e^{i(\pi-\beta)}\infty} \right) e^{-z\xi}\, \hat{\Phi}_n(2\pi(1+\lambda)i + \xi)\, d\xi,$$

whence

$$|R(z,\varepsilon)| \le C_n^* \,\mathrm{e}^{-2\pi(1+\lambda)|\,\Im m\, z|} \int_0^\infty \Big(\mathrm{e}^{-t\,\Re e(z\,\mathrm{e}^{\mathrm{i}\beta})} + \mathrm{e}^{-t\,\Re e(z\,\mathrm{e}^{\mathrm{i}(\pi-\beta)})}\Big)\, \mathrm{e}^{\tau_n^* t}\,\mathrm{d}t \\ \le \frac{2C_n^*}{\tau_n^*}\mathrm{e}^{-2\pi(1+\lambda)|\,\Im m\, z|}$$

(using (25) and $z \in \mathcal{D}_n$). We finally get (27) by the Schwarz lemma, since R is analytic for $|\varepsilon| < \varepsilon_0'$ and vanishes for $\varepsilon = 0$. □

Observe that $|\mathrm{e}^{-2\pi\mathrm{i}z}| = \mathrm{e}^{-2\pi|\,\Im m\, z|}$ is exponentially small and the asymptotics of the functions $\Psi_{j,n}^u$'s is known from Proposition 1.8 and Corollary 1.9, while $\mathrm{e}^{-2\pi(1+\lambda)|\,\Im m\, z|}$ is exponentially smaller. The singularity analysis in the Borel plane thus gave us access to the precise measure of the exponentially small splitting phenomenon.

The last part of Theorem 2.17 of [MSS09] follows.

As previously explained, the previous results are sufficient to complete the proof of the main results of [MSS09]. The rest of this article (except Section 1.5, which is a side remark) is devoted to their proof, as announced in Section 0.4.

1.5 Rephrasing of the bridge equation and alternative description of the formal solutions

The name "bridge equation" for (24) comes from the fact that it can be interpreted as a bridge between the action of the alien derivations $\Delta_{\pm 2\pi\mathrm{i}}$ and the natural derivations $\frac{\partial}{\partial z}, \frac{\partial}{\partial b_1}, \frac{\partial}{\partial b_2}, \dots$, in view of

Proposition 1.13. *For each $n \ge 1$, there exists a formal series $\beta_n(h,\varepsilon;b) = 1 + O(h^2) \in \mathbb{C}[[h^2]]$ which depends analytically on ε and polynomially on $b_1, b_2, \dots$, such that*

$$\frac{\partial\tilde\Phi}{\partial b_n}(z,h,\varepsilon;b) = h^{2n}\beta_n(h,\varepsilon;b)\tilde\Psi_2(z,h,\varepsilon;b). \tag{28}$$

Consequently, equation (24) can be written

$$\Delta_{\pm 2\pi\mathrm{i}}\tilde\Phi = A^{\pm}\frac{\partial\tilde\Phi}{\partial z} + C_n^{\pm}h^{-2n}\frac{\partial\tilde\Phi}{\partial b_n}, \qquad C_n^{\pm} = \mathrm{i}B^{\pm}/\beta_n$$

with arbitrary $n \in \mathbb{N}^*$.

The proof is given at the end of Section 4.1.

The resurgent analysis could be developed farther, with the help of the alien derivations Δ_ω of index $\omega \in 2\pi\mathrm{i}\,\mathbb{Z}^*$. Indeed, it turns out that the Borel transforms $\hat\Phi_n(\zeta)$ are holomorphic on the whole universal cover of $\mathbb{C}\setminus 2\pi\mathrm{i}\,\mathbb{Z}$ (this property is precisely the definition of a resurgent function with singular support in $2\pi\mathrm{i}\,\mathbb{Z}$; see e.g. [Eca81], [CNP93] or [Sau05]), but we shall not give details about this.

The relations (28) entail a certain functional dependence between the formal series $\tilde\Phi(z,h;b)$, which comes from an alternative description of the formal solutions. The set of all odd solutions ϕ of (**FIE**) such that $[\phi]_0 = \tilde\Phi_0$ can indeed be described using a single sequence of formal series $\tilde G_0, \tilde G_1, \tilde G_2, \dots \in \mathbb{C}((z^{-1}))[[h^2]]$ as follows: for each $b \in \mathbb{C}^{\mathbb{N}^*}$, there exists $c(h) = \sum_{n\geq 1} c_n h^{2n}$ (where each c_n is the sum of b_n and a polynomial in $(b_1, \dots, b_{n-1})$) such that

$$\tilde{\Phi}(z, h; b) = \tilde{G}_0(z, h) + c(h)\tilde{G}_1(z, h) + c(h)^2\tilde{G}_2(z, h) + \cdots$$

(this series is formally convergent in $\mathbb{C}((z^{-1}))[[h^2]]$ because $c(h) \in h^2\mathbb{C}[[h^2]]$). In fact, $\tilde{G}(z, c, h) = \sum_{m\geq 0} c^m \tilde{G}_m(z, h)$ is an odd solution of (**FIE**) in $\mathbb{C}((z^{-1}))[[h^2, c]]$. The general nonzero solution in $\mathbb{C}((z^{-1}))[[h^2]]$ can then be written $\pm\tilde{G}\big(z + a(h), c(h), h\big)$, with arbitrary $a(h) \in \mathbb{C}[[h^2]]$ and $c(h) \in h^2\mathbb{C}[[h^2]]$; the solution $\tilde{\Phi}$ is obtained as $\tilde{\Phi}(z, h; b) = \tilde{G}\big(z, c_b(h), h\big)$ for a certain series $c_b(h)$.

The bridge equation for $\tilde{G}$ takes the form

$$\Delta_\omega \tilde{G} = \mathrm{A}_\omega \frac{\partial \tilde{G}}{\partial z} + \mathrm{i}\mathrm{B}_\omega \frac{\partial \tilde{G}}{\partial c}, \qquad \omega = \pm 2\pi\mathrm{i},$$

with $\mathrm{A}_\omega = \mathrm{A}_\omega(c, h)$ and $\mathrm{B}_\omega = \mathrm{B}_\omega(c, h) \in \mathbb{C}[[h^2, c]]$. The series $A^\pm(h; b)$ and $B^\pm(h; b)$ (for any $b \in \mathbb{C}^{\mathbb{N}^*}$) of the bridge equation for $\tilde{\Phi}$ can be expressed in terms of $\mathrm{A}_{\pm 2\pi\mathrm{i}}\big(c_b(h), h\big)$ and $\mathrm{B}_{\pm 2\pi\mathrm{i}}\big(c_b(h), h\big)$.

2 Formal solutions of the inner equations and related series

This section contains the proof of Theorem 1.1 (Sections 2.1 and 2.2) and Proposition 1.2 (Section 2.3), which are statements on the formal solutions of equations $(\mathbf{IE})_n$, $n \geq 0$.

2.1 The inner equation and its variational equation

Lemma 2.1. *For each value of ε, equation $(\mathbf{IE})_0$ admits a unique formal solution $\tilde{\Phi}_0(z) = \tilde{\Phi}_0(z, \varepsilon)$ of the form $-\mathrm{i}z^{-1} + O(z^{-3})$. This solution is odd and the nonzero formal solutions are exactly the series $\tilde{\Phi}_0(z + c)$ and $-\tilde{\Phi}_0(z + c)$, $c \in \mathbb{C}$.*

Proof. In view of assumption **(B)**, equation $(\mathbf{IE})_0$ can be written

$$\phi_0(z + 1) - 2\phi_0(z) + \phi_0(z - 1) = -2\phi_0(z)^3 + \sum_{n\geq 0} v_n(\varepsilon)\phi_0(z)^{2n+5},$$

where the coefficients $v_n(\varepsilon)$ depend on the Taylor expansion in y of $V'(y, 0, \varepsilon)$. Substituting $\phi_0(z) = a_0 z^{-N} + a_1 z^{-N-1} + \cdots$ with $N \geq 1$ and $a_0 \neq 0$, and taking into account that $(z + 1)^{-N} - 2z^{-N} + (z - 1)^{-N} = N(N + 1)z^{-N-2} + O(z^{-N-3})$, one sees that $N = 1$ and $a_0 = \pm\mathrm{i}$, the coefficient a_1 is free and the next ones are uniquely determined by a_0 and a_1 in terms of the v_n's. In particular, there is a unique solution $\tilde{\Phi}_0(z)$ of the form $-\mathrm{i}z^{-1} + O(z^{-3})$ (corresponding to $a_0 = -\mathrm{i}$ and $a_1 = 0$).

If $\phi(z)$ is a formal solution, so are $\phi(-z)$, $-\phi(z)$, $\overline{\phi(\bar{z})}$ and $\phi(z + c)$ for any c. Uniqueness implies that $-\tilde{\Phi}_0(-z)$ and $-\overline{\tilde{\Phi}_0(\bar{z})}$ coincide with $\tilde{\Phi}_0(z)$, and the general nonzero solution is $\pm\tilde{\Phi}_0(z + c)$. □

Notice that, if V' is real-analytic, then $\tilde{\Phi}_0 \in \mathrm{i}z^{-1}\mathbb{R}[[z^{-1}]]$.

Definition 2.2. The variational equation of $(\mathbf{IE})_0$ along $\tilde{\Phi}_0(z)$ (with the notation of Lemma 2.1) is the homogeneous equation $\mathcal{L}_0\psi = 0$, where $\mathcal{L}_0$ is the linear operator of $\mathbb{C}((z^{-1}))$ defined by

$$\mathcal{L}_0\psi(z) = \psi(z+1) + \psi(z-1) - 2\psi(z) - A_0(z)\psi(z), \tag{29}$$

$$A_0(z) = -2 + \partial_y \mathcal{F}(\tilde{\Phi}_0(z,\varepsilon), 0, \varepsilon). \tag{30}$$

The corresponding inhomogeneous equations are the equations $\mathcal{L}_0\psi = f$ with given $f \in \mathbb{C}((z^{-1}))$.

The secondary inner equations $(\mathbf{IE})_n$ can be written $\mathcal{L}_0\phi_n = f_n$, with $f_n \in \mathbb{C}((z^{-1}))$ inductively determined in terms of $\phi_0 = \tilde{\Phi}_0, \phi_1, \dots, \phi_{n-1}$ according to formula (7). It is thus worth recalling a few classical facts about operators of the form (29), which will be used in Section 2.2 to construct solutions of the secondary inner equations (and also in Section 3.2, to devise a perturbative method in order to study $(\mathbf{IE})_0$ and the Borel transform of $\tilde{\Phi}_0$); the reader is referred to Appendix A.2 for their proofs.

(i) Denoting by T and T^{-1} the mutually inverse shift operators $\psi(z) \mapsto T^{\pm 1}\psi(z) = \psi(z \pm 1)$ and by I the identity operator, we introduce the difference operators

$$\Delta = T - I, \quad P = T - 2I + T^{-1}. \tag{31}$$

Thus $\mathcal{L}_0\psi = P\psi - A_0\psi$. The discrete Wronskian, or Casoratian, is classically defined to be the determinant

$$\mathcal{W}(\psi_1,\psi_2) = \begin{vmatrix} \psi_1 & \psi_2 \\ T\psi_1 & T\psi_2 \end{vmatrix} = \begin{vmatrix} \psi_1 & \psi_2 \\ \Delta\psi_1 & \Delta\psi_2 \end{vmatrix}.$$

The Wronskian $W(z)$ of any two solutions of $\mathcal{L}_0$ satisfies $\Delta W = 0$; when dealing with elements of $\mathbb{C}((z^{-1}))$, this implies that $W(z)$ is constant (this only implies periodicity if we deal with general functions as in Appendix A.3).

(ii) If two solutions ψ_1 and ψ_2 have Wronskian 1, we say that they form a *normalized fundamental system*; ones finds that ψ is solution if and only if $a = \mathcal{W}(\psi,\psi_2)$ and $b = \mathcal{W}(\psi_1,\psi)$ satisfy $\Delta a = \Delta b = 0$, and linear algebra yields $\psi = a\psi_1 + b\psi_2 = (T^{-1}a)\psi_1 + (T^{-1}b)\psi_2$; in the case of formal series, a and b are constant and the set of solutions is thus the linear span of (ψ_1,ψ_2) (in the case of general functions, a and b are arbitrary 1-periodic functions).

(iii) The solutions of an inhomogeneous equation are obtained by adding any solution of the homogeneous equation to a particular solution. If (ψ_1,ψ_2) is a normalized fundamental system, we get a particular solution of $\mathcal{L}_0\psi = f$ in the form $\psi = a^*\psi_1 + b^*\psi_2$ as soon as a^* and b^* satisfy $\Delta a^* = -\psi_2 f$ and $\Delta b^* = \psi_1 f$ (with $\mathcal{W}(\psi,\psi_2) = Ta^*$ and $\mathcal{W}(\psi_1,\psi) = Tb^*$ for this solution[2]).

(iv) If a particular solution ψ_1 is known for the homogeneous equation and if $\psi_1 T\psi_1$ is invertible, a standard method to find a normalized fundamental system consists in "varying the constant": $\psi_2(z) = c(z)\psi(z)$ is solution and $\mathcal{W}(\psi_1,\psi_2) \equiv 1$ as soon as $\Delta c = \frac{1}{\psi_1 T\psi_1}$.

[2] One gets a solution $\psi = a\psi_1 + b\psi_2$ such that $\mathcal{W}(\psi,\psi_2) = a$ and $\mathcal{W}(\psi_1,\psi) = b$ as soon as $(I - T^{-1})a = -\psi_2 f$ and $(I - T^{-1})b = \psi_1 f$.

In our case, since the linear equation $\mathcal{L}_0\psi = 0$ was obtained as variational equational along $\tilde{\Phi}_0$ from $(\mathbf{IE})_0$, it is obvious that a particular solution of $\mathcal{L}_0$ in $\mathbb{C}((z^{-1}))$ is $\tilde{\psi}_1 = \partial_z\tilde{\Phi}_0$. To apply the aforementioned methods, we need to invert Δ in $\mathbb{C}((z^{-1}))$; this could lead in principle to the appearance of logarithms in our formal series, but the symmetries of the problem ($\tilde{\psi}_1(z)$ and $A_0(z)$ are even) will prevent this. We henceforth denote by $[\varphi]_{(m)}$ the coefficient of z^{-m} in a formal series $\varphi \in \mathbb{C}((z^{-1}))$.

Lemma 2.3. *Let $\beta_1, \beta_2, \ldots$ denote the coefficients of the Taylor expansion of the even function $\frac{X}{e^X - 1} + \frac{1}{2}X - 1 = \sum_{\ell\geq 1}\beta_\ell X^{2\ell}$. Let $\mathbb{C}((z^{-1}))_{(0)}$, resp. $\mathbb{C}((z^{-1}))_{(1)}$, denote the subspaces of formal series without constant term, resp. without residuum, i.e.*

$$\mathbb{C}((z^{-1}))_{(m)} = \{\, \varphi \in \mathbb{C}((z^{-1})) \mid [\varphi]_{(m)} = 0 \,\}, \qquad m = 0, 1, \tag{32}$$

and let ∂_z^{-1} be the unique operator $\mathbb{C}((z^{-1}))_{(1)} \to \mathbb{C}((z^{-1}))_{(0)}$ such that $\partial_z \circ \partial_z^{-1} = I$. Then the range of Δ is $\mathbb{C}((z^{-1}))_{(1)}$ and the formulas

$$\Delta^{-1} = \partial_z^{-1} - \frac{1}{2}I + \sum_{\ell\geq 1}\beta_\ell\partial_z^{2\ell-1}, \quad \Delta^{-1}_{(0)} = \partial_z^{-1} \circ \left(I - \frac{1}{2}\partial_z + \sum_{\ell\geq 1}\beta_\ell\partial_z^{2\ell} \right) \tag{33}$$

define two right inverses of Δ on $\mathbb{C}((z^{-1}))_{(1)}$, the range of the second being $\mathbb{C}((z^{-1}))_{(0)}$.

Proof. On can write $\Delta = \partial_z \circ \alpha = \alpha \circ \partial_z$, with $\alpha = \sum_{r\geq 0}\frac{1}{(r+1)!}\partial_z^r$ invertible in $\mathbb{C}((z^{-1}))$: $\alpha^{-1} = I - \frac{1}{2}\partial_z + \sum_{\ell\geq 1}\beta_\ell\partial_z^{2\ell}$ (the coefficients β_ℓ are essentially the Bernoulli numbers). The range of Δ thus coincides with the range of ∂_z, which is invariant by α^{-1}, and $\Delta^{-1} = \alpha^{-1} \circ \partial_z^{-1}$ and $\Delta^{-1}_{(0)} = \partial_z^{-1} \circ \alpha^{-1}$ are right inverses of Δ on $\mathbb{C}((z^{-1}))_{(1)}$. □

Remark 2.4. The two right inverses do not coincide because the operators $\partial_z^{2\ell} \circ \partial_z^{-1}$ and $\partial_z^{-1} \circ \partial_z^{2\ell}$ do not agree on polynomials (nor does $\partial_z^{-1} \circ \partial_z$ coincide with I in $\mathbb{C}[z]$). The restrictions of Δ^{-1} and $\Delta^{-1}_{(0)}$ to $z^{-2}\mathbb{C}[[z^{-1}]]$ agree, whereas they leave $\mathbb{C}[z]$ invariant with $\Delta^{-1}Q - \Delta^{-1}_{(0)}Q = (\Delta^{-1}Q)(0)$ for any polynomial $Q \in \mathbb{C}[z]$. We shall use $\Delta^{-1}_{(0)}$ to find a normalized fundamental system of solutions of $\mathcal{L}_0$, and Δ^{-1} to solve the secondary inner equations.

Remark 2.5. The operators Δ^{-1} and $\Delta^{-1}_{(0)}$ can be extended to the whole space $\mathbb{C}((z^{-1}))$ at the price of admitting multiples of $\log z$ in the target space: indeed, ∂_z^{-1} can be extended to an operator $\mathbb{C}((z^{-1})) \to \mathbb{C}[[z^{-1}]][z, \log z]$ as a right inverse of ∂_z and formulas (33) then yield right inverses $\Delta^{-1}, \Delta^{-1}_{(0)} : \mathbb{C}((z^{-1})) \to \mathbb{C}[[z^{-1}]][z, \log z]$.

To apply the above point (iv) and use the solution $\tilde{\psi}_1 = \partial_z\tilde{\Phi}_0$ to determine an independent solution of $\mathcal{L}_0$, we need to check that $\frac{1}{\tilde{\psi}_1 T\tilde{\psi}_1}$ has no residuum, so as to be able to apply $\Delta^{-1}_{(0)}$ and to set $\tilde{\psi}_2 = \tilde{\psi}_1\Delta^{-1}_{(0)}\left(\frac{1}{\tilde{\psi}_1 T\tilde{\psi}_1}\right)$ (note that $\frac{1}{\tilde{\psi}_1 T\tilde{\psi}_1}$ is well-defined in $\mathbb{C}((z^{-1}))$, this was one of the reasons for introducing the field of fractions of $\mathbb{C}[[z^{-1}]]$).

Lemma 2.6. *If $\varphi \in \mathbb{C}((z^{-1}))$ is even or odd, then $\varphi \cdot T\varphi$ has no residuum and the formal series $\Delta^{-1}_{(0)}(\varphi \cdot T\varphi)$ is odd.*

Proof. We have

$$\varphi\cdot T\varphi=\sum_{p\geq 0}\frac{1}{p!}\varphi\cdot\partial^p\varphi,$$

where ∂ is a shorthand for ∂_z (cf. footnote 1 for the convergence of this series of formal series). If p is even, then $\varphi\cdot\partial^p\varphi$ is even and has no residuum. If p is odd, then the identity

$$\partial\Bigg(\sum_{\substack{p_1+p_2=p-1\\ p_1,p_2\geq 0}}(-1)^{p_1}\partial^{p_1}\varphi_1\cdot\partial^{p_2}\varphi_2\Bigg)=\varphi_1\cdot\partial^p\varphi_2+(-1)^{p+1}\partial^p\varphi_1\cdot\varphi_2 \tag{34}$$

shows that $2\varphi\cdot\partial^p\varphi$ is the derivative of an element of $\mathbb{C}((z^{-1}))$, thus it has no residuum. Hence $\varphi\cdot T\varphi\in\mathbb{C}((z^{-1}))_{(1)}$.

Let $\psi=\Delta^{-1}_{(0)}(\varphi\cdot T\varphi)$: this is the unique element of $\mathbb{C}((z^{-1}))$ such that $[\psi]_{(0)}=0$ and $\Delta\psi=\varphi\cdot T\varphi$. Let $\psi^*(z)=-\psi(-z)$. A straightforward computation shows that $\Delta\psi^*(z)=\varphi(-z)\varphi(-z-1)$. The symmetry assumption implies $\Delta\psi^*=\varphi\cdot T\varphi$, hence $\psi^*=\psi$. ☐

Applying this with $\varphi=1/\partial_z\tilde\Phi_0$, which is even, we finally get

Corollary 2.7. *Let*

$$\tilde\psi_1=\partial_z\tilde\Phi_0=\mathrm{i}z^{-2}+O(z^{-4}), \tag{35}$$

where $\tilde\Phi_0$ is the solution of $(\mathbf{IE})_0$ *determined in Lemma 2.1. Then $\frac{1}{\tilde\psi_1 T\tilde\psi_1}\in\mathbb{C}((z^{-1}))_{(1)}$ and the formula*

$$\tilde\psi_2=\tilde\psi_1\Delta^{-1}_{(0)}\left(\tfrac{1}{\tilde\psi_1 T\tilde\psi_1}\right)=-\frac{\mathrm{i}}{5}z^3+O(z)$$

defines a formal series such that $(\tilde\psi_1,\tilde\psi_2)$ is a normalized fundamental system of solutions in $\mathbb{C}((z^{-1}))$ of the variational equation of $(\mathbf{IE})_0$ *along $\tilde\Phi_0$. Moreover $\tilde\psi_1$ is even and $\tilde\psi_2$ is odd.*

2.2 The formal solutions of the secondary inner equations

Lemma 2.8. *Let $\tilde\Phi_0$ be the odd formal solution of* $(\mathbf{IE})_0$ *determined in Lemma 2.1. Then there exist sequences of odd formal series $\tilde\Phi_1,\tilde\Phi_2,\dots$ in $\mathbb{C}((z^{-1}))$ satisfying equations* $(\mathbf{IE})_1,(\mathbf{IE})_2,\dots$ *All these solutions are obtained inductively and are unique up to the choice of a complex number b_n at each step: for $n\geq 1$, denoting by $\tilde f_n$ the coefficient of h^{2n} in $\mathcal{F}(\tilde\Phi_0+h^2\tilde\Phi_1+\cdots+h^{2(n-1)}\tilde\Phi_{n-1},h,\varepsilon)$ and using the operator Δ^{-1} of Lemma 2.3,*

$$\tilde\Phi_n=-\tilde\psi_1\Delta^{-1}(\tilde\psi_2\tilde f_n)+\tilde\psi_2\Delta^{-1}(\tilde\psi_1\tilde f_n)+b_n\tilde\psi_2, \tag{36}$$

where $b_n\in\mathbb{C}$ is arbitrary; thus $\tilde\Phi_n(z)=\tilde\Phi_n(z;b_1,\dots,b_n)$.

Proof. We argue by induction and assume that, besides the odd solution $\tilde\Phi_0$ of $(\mathbf{IE})_0$, we have odd formal solutions $\tilde\Phi_1,\dots,\tilde\Phi_{n-1}$ of $(\mathbf{IE})_1,\dots,(\mathbf{IE})_{n-1}$, depending on $n-1$ free parameters $b_1,\dots,b_{n-1}$. The nth secondary equation, $(\mathbf{IE})_n$, can be written

$\mathcal{L}_0\Phi_n = \tilde{f}_n$. As a consequence of Corollary 2.7 and of what was explained in Section 2.1 (points (ii), (iii) and Lemma 2.3), its solutions in $\mathbb{C}[[z^{-1}]][z,\log z]$ are exactly the formal series $\Phi^* + a\tilde{\psi}_1 + b\tilde{\psi}_2$, where

$$\Phi^* = \Phi^*(z;b_1,\dots,b_{n-1}) = -\tilde{\psi}_1\Delta^{-1}(\tilde{\psi}_2\tilde{f}_n) + \tilde{\psi}_2\Delta^{-1}(\tilde{\psi}_1\tilde{f}_n) \tag{37}$$

and a and b are arbitrary complex numbers. These solutions will contain logarithmic terms or not according to the values of the residuums of $\tilde{\psi}_2\tilde{f}_n$ and $\tilde{\psi}_1\tilde{f}_n$. It is thus enough to prove that these residuums vanish and to check that Φ^* is an odd formal series: the only odd formal solutions will then correspond to $a=0$, and we shall set

$$\tilde{\Phi}_n(z;b_1,\dots,b_n) = \Phi^*(z;b_1,\dots,b_{n-1}) + b_n\tilde{\psi}_2(z). \tag{38}$$

Let

$$\chi = h^2\tilde{\Phi}_1 + \cdots + h^{2(n-1)}\tilde{\Phi}_{n-1} \in \mathbb{C}((z^{-1}))[h^2],$$

with $\chi = 0$ if $n=1$. Clearly, $\tilde{f}_n = \Big[\mathcal{F}(\tilde{\Phi}_0+\chi,h)\Big]_n$ is an odd formal series. Thus $\tilde{\psi}_2\tilde{f}_n$ is even and has no residuum: $\tilde{\psi}_2\tilde{f}_n \in \mathbb{C}((z^{-1}))_{(1)}$.

We have

$$\tilde{\psi}_1\tilde{f}_n = \Big[\partial_z\tilde{\Phi}_0\,\mathcal{F}(\tilde{\Phi}_0+\chi,h)\Big]_n = A - B,$$

with $A = \Big[(\partial_z\tilde{\Phi}_0+\partial_z\chi)\mathcal{F}(\tilde{\Phi}_0+\chi,h)\Big]_n$ and $B = \Big[\partial_z\chi\,\mathcal{F}(\tilde{\Phi}_0+\chi,h)\Big]_n$. Since $A = \partial_z\Big[\mathcal{F}(\tilde{\Phi}_0+\chi,h)\Big]_n$, this series has no residuum. We now show that B has no residuum.

We observe that, for $1\le k\le n-1$, $\Big[\mathcal{F}(\tilde{\Phi}_0+\chi,h)\Big]_k = F_k$ defined by (6), and that this series coincides with $\tilde{\Phi}_k(z+1)+\tilde{\Phi}_k(z-1)$ (in view of the previous equations). Thus

$$B = \sum_{\substack{j+k=n\\1\le j,k\le n-1}} [\partial_z\chi]_j\Big[\mathcal{F}(\tilde{\Phi}_0+\chi,h)\Big]_k = \sum_{\substack{j+k=n\\1\le j,k\le n-1}} \partial_z\tilde{\Phi}_j\Big(\tilde{\Phi}_k(z+1)+\tilde{\Phi}_k(z-1)\Big),$$

and we can identify B with the coefficient of h^{2n} in

$$\partial_z\chi(z,h)\big[\chi(z+1,h)+\chi(z-1,h)\big] = \sum_{r\ge0}\frac{2}{(2r)!}\partial_z\chi\cdot\partial_z^{2r}\chi \in \mathbb{C}((z^{-1}))[h^2].$$

But the coefficients of this polynomial cannot have a nonzero residuum, because none of the terms $\frac{2}{(2r)!}\partial_z\chi\cdot\partial_z^{2r}\chi$ has: the term with $r=0$ is nothing but $\partial_z(\chi^2)$, and any term with $r\ge1$ can be written $\frac{1}{(2r)!}(\varphi_1\cdot\partial_z^{2r-1}\varphi_2 + \partial_z^{2r-1}\varphi_1\cdot\varphi_2)$, with $\varphi_1=\varphi_2=\partial_z\chi$, which is also the z-derivative of an element of $\mathbb{C}((z^{-1}))[h^2]$ by virtue of (34). Thus B has no residuum.

Since $\tilde{\psi}_2\tilde{f}_n$ and $\tilde{\psi}_1\tilde{f}_n$ belong to $\mathbb{C}((z^{-1}))_{(1)}$, the formula

$$\Phi^* = -\tilde{\psi}_1\Delta^{-1}(\tilde{\psi}_2\tilde{f}_n) + \tilde{\psi}_2\Delta^{-1}(\tilde{\psi}_1\tilde{f}_n)$$

defines a formal series in $\mathbb{C}((z^{-1}))$ which solves $(\mathbf{IE})_n$. Let us check that Φ^* is odd. In view of (33), we can write $\Delta^{-1} = -\frac{1}{2}I + \Gamma$ with an operator $\Gamma : \mathbb{C}((z^{-1}))_{(1)} \to \mathbb{C}((z^{-1}))$ which reverses parity, namely $\Gamma = \partial_z^{-1} + \sum_{\ell\ge1}\beta_\ell\partial_z^{2\ell-1}$. Hence $\Phi^* = -\tilde{\psi}_1\Gamma(\tilde{\psi}_2\tilde{f}_n) + \tilde{\psi}_2\Gamma(\tilde{\psi}_1\tilde{f}_n)$, with $\Gamma(\tilde{\psi}_2\tilde{f}_n)$ odd and $\Gamma(\tilde{\psi}_1\tilde{f}_n)$ even. $\square$

Lemma 2.9. *The formal solutions of Lemma 2.8 with $b_1 = 0$ satisfy*

$$\tilde{\Phi}_n(z; 0, b_2, \dots, b_n) \in z^{2n-1}\mathbb{C}[[z^{-1}]], \qquad n \geq 1$$

for any choice of $b_2, \dots, b_n$, while $\tilde{\Phi}_n(z; b_1, b_2, \dots, b_n) \in z^{4n-1}\mathbb{C}[[z^{-1}]]$ in general.

Proof. Let us choose any sequence $b \in \mathbb{C}^{\mathbb{N}^*}$ with $b_1 = 0$. In view of formula (36) and since $\tilde{\psi}_2 \in z^3\mathbb{C}[[z^{-1}]] \subset z^{2n-1}\mathbb{C}[[z^{-1}]]$ for $n \geq 2$, the conclusion will follow from the property

$$\tilde{f}_n \in z^{2n-3}\mathbb{C}[[z^{-1}]], \qquad n \geq 1, \tag{39}$$

since multiplication by $\tilde{\psi}_1$, resp. multiplication by $\tilde{\psi}_2$, resp. Δ^{-1} adds 2, resp. -3, resp. -1 to the (z^{-1})-valuation.

To prove (39), we shall make use of formula (7). In view of formula (1) and assumption **(A)**, the holomorphic functions $\mathcal{F}_n(y, \varepsilon)$ are odd in y for all $n \geq 0$. Hence $\partial_y^r \mathcal{F}_n(\tilde{\Phi}_0, \varepsilon) \in \mathbb{C}[[z^{-1}]]$ for $r \geq 0$, with $\partial_y^r \mathcal{F}_n(\tilde{\Phi}_0, \varepsilon) \in z^{-1}\mathbb{C}[[z^{-1}]]$ for even r.

Let $n \geq 1$; we argue by induction and suppose that $\tilde{\Phi}_m \in z^{2m-1}\mathbb{C}[[z^{-1}]]$ for $1 \leq m \leq n-1$. Each product $\tilde{\Phi}_{n_1} \cdots \tilde{\Phi}_{n_r}$ involved in (7) thus belongs to the space $z^{2n-(2n_0+r)}\mathbb{C}[[z^{-1}]]$, which is included in $z^{2n-3}\mathbb{C}[[z^{-1}]]$ as soon as $2n_0 + r \geq 3$. Since $n_0 + r \geq 2$, the only terms which have $2n_0 + r < 3$ correspond to $n_0 = 0$ and $r = 2$, but then $\partial_y^r \mathcal{F}_{n_0}(\tilde{\Phi}_0, \varepsilon) \in z^{-1}\mathbb{C}[[z^{-1}]]$, which is sufficient to prove (39).

The case $b_1 \neq 0$ is treated similarly, yielding $\tilde{f}_n \in z^{4n-3}\mathbb{C}[[z^{-1}]]$. □

Theorem 1.1 follows easily from Lemmas 2.1, 2.8 and 2.9, with the help of arguments analogous to those employed in the proof of Lemma 2.1 to get the description of all the formal solutions.

2.3 The formal solutions in the integrable case

We now prove Proposition 1.2. We thus fix $\varepsilon = 0$ and first show how the function H defined by (4) appears in relation with (**FIE**). Let $\mu = \cosh h$.

Lemma 2.10. *Assume $\Phi \in \mathbb{C}((z^{-1}))[[h^2]]$ is not independent of z, i.e. not reduced to an element of $\mathbb{C}[[h^2]]$. Then Φ is solution of* (**FIE**) *for $\varepsilon = 0$ if and only if $H(\Phi, T\Phi; h) = \Phi^2 + T\Phi^2 + \Phi^2 T\Phi^2 - 2\mu\Phi T\Phi$ is independent of z, i.e. of the form $c(h) \in \mathbb{C}[[h^2]]$.*

Proof. Let $c = H(\Phi, T\Phi; h)$: a priori, $c \in \mathbb{C}((z^{-1}))[[h^2]]$. The assumption on Φ implies $T\Phi - T^{-1}\Phi \neq 0$. Thus

$$\begin{aligned}
(\mathbf{FIE}) \Leftrightarrow (1+\Phi^2)(T\Phi + T^{-1}\Phi) - 2\mu\Phi = 0 \Leftrightarrow \\
\big[(1+\Phi^2)(T\Phi + T^{-1}\Phi) - 2\mu\Phi\big](T\Phi - T^{-1}\Phi) = 0 \Leftrightarrow \\
(1+\Phi^2)(T\Phi^2 - T^{-1}\Phi^2) - 2\mu\Phi(T\Phi - T^{-1}\Phi) = 0 \Leftrightarrow \\
\Phi^2 T\Phi^2 + T\Phi^2 - 2\mu\Phi T\Phi + \Phi^2 - \big[\Phi^2 T^{-1}\Phi^2 + T^{-1}\Phi^2 - 2\mu\Phi T^{-1}\Phi + \Phi^2\big] = 0 \\
\Leftrightarrow c - T^{-1}c = 0.
\end{aligned}$$

□

In the case of $\Phi^0 = \tilde{\Phi}(z, h, 0; b)$, we have $[c]_0 = 0$ and, for $n \geq 1$, $[c]_n$ depends on $b_1, \dots, b_n$. In order to take advantage of the symmetries of the problem, we rewrite the equation $H(\Phi, T\Phi; h) = c(h)$ as $H(U^{-1}\Phi, U\Phi; h) = c(h)$, where U is the shift operator $\Phi(z,h) \mapsto \Phi(z+\frac{1}{2}, h)$. Let us divide the equation by $(U^{-1}\Phi^2)(U\Phi^2)$:

$$U\left(\tfrac{1}{\Phi^2}\right) + U^{-1}\left(\tfrac{1}{\Phi^2}\right) + 1 - 2\mu U\left(\tfrac{1}{\Phi}\right)U^{-1}\left(\tfrac{1}{\Phi}\right) = c(h) U\left(\tfrac{1}{\Phi^2}\right)U^{-1}\left(\tfrac{1}{\Phi^2}\right).$$

Since, for $\Phi \in \mathbb{C}((z^{-1}))[[h^2]]$ with $[\Phi]_0 = -\mathrm{i}z^{-1}$,

$$\forall n \geq 1,\ \ [\Phi]_n \in z^{-1}\mathbb{C}[z] \ \ \Leftrightarrow\ \ \forall n \geq 1,\ \ [1/\Phi]_n \in z\mathbb{C}[z],$$

Proposition 1.2 follows from

Lemma 2.11. *Let $c(h) \in h^2\mathbb{C}[[h^2]]$. There exists a unique $\Psi \in \mathbb{C}((z^{-1}))[[h^2]]$ such that $[\Psi]_0 = \mathrm{i}z$, each $[\Psi]_n$ is odd and*

$$1 + U\Psi^2 + U^{-1}\Psi^2 - 2\mu(U\Psi)(U^{-1}\Psi) - c(h)(U\Psi^2)(U^{-1}\Psi^2) = 0. \qquad (40)$$

Moreover, each $[\Psi]_n \in z\mathbb{C}[z]$.

Proof. Let $\Psi = \sum_{n\geq 0} h^{2n}\Psi_n(z) \in \mathbb{C}((z^{-1}))[[h^2]]$. Equation (40) can be written $K(U^{-1}\Psi, U\Psi; h) = 0$ with $K(x,y;h) = 1 + x^2 + y^2 - 2\mu xy - c(h)x^2y^2$. Expanding in powers of h^2, we get $K_0(U^{-1}\Psi_0, U\Psi_0) = 0$, where $K_0(x,y) = 1 + (x-y)^2$, for which $\Psi_0(z) = \mathrm{i}z$ is an obvious solution, and

$$\partial_x K_0(U^{-1}\Psi_0, U\Psi_0)U^{-1}\Psi_n + \partial_y K_0(U^{-1}\Psi_0, U\Psi_0)U\Psi_n = \chi_n, \qquad n \geq 1, \qquad (41)$$

where χ_n is the polynomial in $(\Psi_1, \dots, \Psi_{n-1})$ inductively defined as

$$\chi_n = \left[K(U^{-1}\Psi^{<n}, U\Psi^{<n}; h)\right]_n, \qquad \Psi^{<n} = \Psi_0 + h^2\Psi_1 + \cdots + h^{2(n-1)}\Psi_{n-1}.$$

The choice $\Psi_0(z) = \mathrm{i}z$ yields $-\partial_x K_0(U^{-1}\Psi_0, U\Psi_0) = \partial_y K_0(U^{-1}\Psi_0, U\Psi_0) = 2(U - U^{-1})\Psi_0 = \mathrm{i}$, thus equation (41) can be written

$$(U - U^{-1})\Psi_n = -\mathrm{i}\chi_n,$$

which is equivalent to $\partial_z\Psi_n = -\mathrm{i}\sum_{\ell\geq 0}\gamma_\ell\partial_z^{2\ell}\chi_n$ where $\sum_{\ell\geq 0}\gamma_\ell X^{2\ell} = \frac{X}{\mathrm{e}^{X/2} - \mathrm{e}^{-X/2}}$. By induction on n, one finds a unique odd solution Ψ_n in $\mathbb{C}((z^{-1}))$, because $K(x,y;h) = K(y,x;h) = K(-y,-x;h)$ implies that χ_n is even. Moreover, this unique odd solution is easily seen to be a polynomial in z. ☐

Remark 2.12. In fact, when $\varepsilon = 0$, equation (**FIE**) can be "integrated by quadrature" in the following sense. To compute $\Phi^0 = \tilde{\Phi}(z, h, 0; b)$, we let $c = c(h) \in h^2\mathbb{C}[[h^2]]$ denote the value of $H(\Phi^0, T\Phi^0; h)$ and consider the Hamiltonian vector field generated by $\frac{1}{2}H$,

$$\dot{x} = y + x^2y - \mu x, \quad \dot{y} = -x - xy^2 + \mu y.$$

In the energy level $\{H = c\}$, the first differential equation yields $y = \frac{\dot{x} + \mu x}{1+x^2}$, whence $c = y^2(1+x^2) - 2\mu xy + x^2 = \frac{\dot{x}^2 + x^2(x^2+1) - \mu^2x^2}{x^2+1}$ and

$$\dot{x}^2 = -x^4 + (c + \sinh^2 h)x^2 + c. \qquad (42)$$

We integrate this first-order differential equation in $\mathbb{C}((t^{-1}))[[h^2]]$ by choosing a branch for the square root: let $T(x,h)$ denote the unique odd series in $\mathbb{C}((x^{-1}))[[h^2]]$ such that

$$\partial_x T(x,h) = -\mathrm{i}x^{-2}\big(1-(c+\sinh^2 h)x^{-2}-cx^{-4}\big)^{-1/2},$$

it is of the form

$$t = T(x,h) = -\mathrm{i}x^{-1}(1+h^2P_1+h^4P_2+\cdots) \quad \text{with each } P_n \in x^{-2}\mathbb{C}[x^{-2}]$$

and has a composition inverse (with respect to x) of the form

$$x = X(t,h) = -\mathrm{i}t^{-1}(1+h^2Q_1+h^4Q_2+\cdots), \quad \text{with each } Q_n \in t^2\mathbb{C}[t^2],$$

the solutions of (42) in $\mathbb{C}((t^{-1}))[[h^2]]$ are $t \mapsto \pm X(t-\alpha(h),h)$, with $\alpha(h) \in \mathbb{C}[[h^2]]$ (if $c=0$, then $X(t,h) = -\mathrm{i}\gamma/\sinh(\gamma t)$ with $\gamma = \sinh h$).

Now, for any solution $x(t) = X(t-\alpha(h),h)$, the other component of the solution of the Hamiltonian vector field is $y(t) = \frac{\dot{x}+\mu x}{1+x^2}$; the symmetries of the problem are such that $y(t)$ is solution of the same branch of (42), thus

$$y(t) = X(t-\alpha(h)+\alpha_0(h),h)$$

with a certain $\alpha_0(h) \in \mathbb{C}[[h^2]]$ which can be computed in terms of $c(h)$. One finds $\alpha_0(h) = 1+O(h^2)$ (if $c=0$, then $\alpha_0(h) = h/\gamma$).

Let $\Phi(z) = X\big(\alpha_0(h)z\big)$ and consider $P(z) = \big(\Phi(z-\frac{1}{2}),\Phi(z+\frac{1}{2})\big)$: we have $H\big(P(z);h\big) = H\big(P(z+1);h\big)$ (conservation of energy along the Hamiltonian flow) and $H\big(P(z);h\big) = H\big(F_{h,0}\big(P(z)\big);h\big)$ (conservation of H by the McMillan map); since $P(z+1)$ and $F_{h,0}\big(P(z)\big)$ have the same first component, it is easy, knowing the first terms of the h^2-expansions, to check that they coincide.

Thus $X\big(\alpha_0(h)z\big)$ is an odd solution of (**FIE**), which we can identify with $\Phi^0(z)$ thanks to the uniqueness statement in Lemma 2.11 (this gives a second way of checking that it belongs to $z^{-1}\mathbb{C}[z^2][[h^2]]$).

3 Borel transforms of the formal solutions

This section contains the proof of Theorem 1.4. The general strategy to control the Borel transforms $\hat{\Phi}_n(\zeta)$ of the formal series $\tilde{\Phi}_n(z)$ consists in studying equations in the ζ-plane (i.e. equations in which the unknowns belong to $\mathbb{C}[[\zeta]]$, and hopefully to $\mathbb{C}\{\zeta\}$ too) which are the counterparts of equations $(\mathbf{IE})_n$.

3.1 Preliminary remarks on the Borel transform and the convolution

Our equations involve the operator $T : \tilde{\varphi}(z) \mapsto \tilde{\varphi}(z+1)$. One sees easily that, if $\tilde{\varphi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ and $\hat{\varphi} = \mathcal{B}\tilde{\varphi}$, then

$$\mathcal{B}(\partial_z\tilde{\varphi})(\zeta) = -\zeta\hat{\varphi}(\zeta), \quad \mathcal{B}(T\tilde{\varphi})(\zeta) = \mathrm{e}^{-\zeta}\hat{\varphi}(\zeta). \tag{43}$$

The counterpart in the ζ-plane of linear difference operators is thus manageable.

It is equation $(\mathbf{IE})_0$ that will require more efforts because it is nonlinear: it involves the product of the unknown formal series $\tilde{\Phi}_0$ with itself (through the substitution into $\mathcal{F}(\,.\,,0,\varepsilon)$). We shall thus need to deal repeatedly with the following situation: suppose that $\tilde{\varphi}(z), \tilde{\psi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ with $\hat{\varphi} = \mathcal{B}\tilde{\varphi}, \hat{\psi} = \mathcal{B}\tilde{\psi} \in \mathbb{C}\{\zeta\}$, and let $\tilde{\chi} = \tilde{\varphi}\tilde{\psi}$, $\hat{\chi} = \mathcal{B}\tilde{\chi}$. Then

$$\hat{\chi}(\zeta) = \hat{\varphi} * \hat{\psi}(\zeta) = \int_0^\zeta \hat{\varphi}(\zeta_1)\hat{\psi}(\zeta - \zeta_1)\,\mathrm{d}\zeta_1 \tag{44}$$

for any ζ belonging to the intersection of the discs of convergence of $\hat{\varphi}$ and $\hat{\psi}$. The law $*$ is called convolution; it is bilinear, commutative and associative. For example, in the particular case corresponding to $\tilde{\psi}(z) = z^{-1}$, $\hat{\psi}(\zeta) = 1$, one gets for $\hat{\chi} = 1 * \hat{\varphi}$ the primitive of $\hat{\varphi}$ which vanishes at the origin, and $1 * 1 * \hat{\varphi} = \zeta * \hat{\varphi}$. But the existence of the analytic continuation for a convolution product requires in general stronger assumptions than that for a primitive (except if one of the factor extends to an entire function).

In this section, we are interested in the possibility of following analytic continuation in the domain $\mathcal{R}^{(0)}$, which is star-shaped with respect to the origin, i.e. $[0,\zeta] \subset \mathcal{R}^{(0)}$ for every $\zeta \in \mathcal{R}^{(0)}$. The following elementary result will thus be useful:

Lemma 3.1. *Suppose $\hat{\varphi}$ and $\hat{\psi}$ are holomorphic in $\mathcal{R}^{(0)}$. Then $\hat{\varphi} * \hat{\psi}$ extends holomorphically to $\mathcal{R}^{(0)}$.*

Suppose moreover that $\rho \in (0, 2\pi)$, that $\hat{\Phi}$ and $\hat{\Psi}$ are non-negative continuous functions on $\mathbb{R}^+$ and that τ_1, τ_2 are non-negative constants such that

$$|\hat{\varphi}(\zeta)| \le \hat{\Phi}\big(|\zeta|\big)\,\mathrm{e}^{\tau_1|\zeta|}, \;\; |\hat{\psi}(\zeta)| \le \hat{\Psi}\big(|\zeta|\big)\,\mathrm{e}^{\tau_2|\zeta|}, \qquad \zeta \in \mathcal{R}^{(0)}_\rho.$$

Then

$$|\hat{\varphi} * \hat{\psi}(\zeta)| \le \hat{\Phi} * \hat{\Psi}\big(|\zeta|\big)\,\mathrm{e}^{\tau|\zeta|}, \qquad \zeta \in \mathcal{R}^{(0)}_\rho, \tag{45}$$

where $\tau = \max(\tau_1, \tau_2)$ and $\hat{\Phi} * \hat{\Psi}(\xi) = \int_0^\xi \hat{\Phi}(\xi_1)\hat{\Psi}(\xi - \xi_1)\,\mathrm{d}\xi_1$ for $\xi \in \mathbb{R}^+$.

Proof. Formula (44) makes sense for all $\zeta \in \mathcal{R}^{(0)}$ and defines the analytic continuation of the convolution product. Inequality (45) follows from

$$\hat{\varphi} * \hat{\psi}(\zeta) = \int_0^{|\zeta|} \hat{\varphi}\big(\gamma(s)\big)\hat{\psi}\big(\zeta - \gamma(s)\big)\tfrac{\zeta}{|\zeta|}\,\mathrm{d}s, \qquad \text{where } \gamma(s) = s\tfrac{\zeta}{|\zeta|} \text{ for } s \in \big[0, |\zeta|\big].$$

□

As a consequence of the first statement, $\widehat{\mathrm{RES}}^{(0)}$ is stable by convolution; it is a subring of the ring $\big(\mathbb{C}\{\zeta\}, +, *\big)$. There is no unit for the convolution law in the ring $\mathbb{C}\{\zeta\}$. It is sometimes convenient to adjoin a unit element[3] to it, i.e. to work in $\mathbb{C}\{\zeta\} \oplus \mathbb{C}\delta$, which is in fact a unitary algebra (and $\widehat{\mathrm{RES}}^{(0)} \oplus \mathbb{C}\delta$ is a subalgebra).

The unit δ can be interpreted as the image of 1 by an extended Borel transform $\mathcal{B}_{\mathrm{ext}}$: let $\mathrm{DP} = \{a_0\delta + a_{-1}\delta' + \cdots + a_{-v}\delta^{(v)} \mid v \in \mathbb{N},\ a_0, \ldots, a_{-v} \in \mathbb{C}\}$ denote the free unitary commutative associative algebra generated by the symbol δ' (the symbol $\delta^{(v)}$ represents the "convolution product" of δ' with itself v times—the elements of DP are "Dirac polynomials"), the space of Gevrey-1 series $\mathbb{C}((z^{-1}))_{\mathrm{Gev}}$ is itself a unitary algebra and the extended Borel transform can be defined as

$$\mathcal{B}_{\mathrm{ext}} : \mathbb{C}((z^{-1}))_{\mathrm{Gev}} \to \mathbb{C}\{\zeta\} \oplus \mathrm{DP},$$

$$\mathcal{B}_{\mathrm{ext}}\Big(\sum_{n \ge -v} a_n z^{-n}\Big) = \sum_{n \ge 1} a_n \frac{\zeta^{n-1}}{(n-1)!} + a_0\delta + a_{-1}\delta' + \cdots + a_{-v}\delta^{(v)}.$$

[3] This element δ, unit of the convolution, can be identified with the Dirac mass at the origin.

This is an algebra isomorphism if we define the extended convolution in $\mathbb{C}\{\zeta\} \oplus \mathrm{DP}$ coherently, by

$$\delta * \hat{\varphi} = \hat{\varphi}, \quad \delta^{(n)} * \hat{\varphi} = \hat{\varphi}(0)\delta^{(n-1)} + \hat{\varphi}'(0)\delta^{(n-2)} + \cdots + \hat{\varphi}^{(n-1)}(0)\delta + \hat{\varphi}^{(n)}(\zeta) \quad (46)$$

for $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$ and $n \in \mathbb{N}^*$ (for instance, convolution with δ' is just the counterpart of multiplication by z and can be interpreted as an extended differentiation with respect to ζ; this is due to the fact that $\mathcal{B}(z\tilde{\varphi})$ boils down to $\hat{\varphi}'(\zeta)$ when $\tilde{\varphi}(z) \in z^{-2}\mathbb{C}[[z^{-1}]]$). The counterpart of the derivation ∂_z of $\mathbb{C}((z^{-1}))_{\mathrm{Gev}}$ is the derivation $\partial\big(\hat{\varphi}(\zeta) + a_0\delta + a_{-1}\delta' + \cdots + a_{-v}\delta^{(v)}\big) = -\zeta\hat{\varphi}(\zeta) + a_{-1}\delta + 2a_{-2}\delta' + \cdots + va_{-v}\delta^{(v-1)}$.

The relation with Section 1.2 is that, for any $\tilde{\varphi} \in \mathbb{C}((z^{-1}))_{\mathrm{Gev}}$, $\mathcal{B}\tilde{\varphi}$ is the projection onto $\mathbb{C}\{\zeta\}$ of $\mathcal{B}_{\mathrm{ext}}\tilde{\varphi} \in \mathbb{C}\{\zeta\} \oplus \mathrm{DP}$. We shall see in Section 4.2 how $\mathcal{B}_{\mathrm{ext}}$ and DP fit in Écalle's formalism of singularities.

3.2 The Borel transform of $\tilde{\Phi}_0(z, \varepsilon)$

We first prove the statement relative to $\tilde{\Phi}_0(z, \varepsilon)$ in Theorem 1.4. With the notations of Section 2.1, equation $(\mathbf{IE})_0$ can be written

$$T\phi_0 + T^{-1}\phi_0 = \mathcal{F}(\phi_0, 0, \varepsilon) = \mathcal{F}_{0,0}(\phi_0) + \varepsilon V'(\phi_0, 0, \varepsilon),$$
$$\mathcal{F}_{0,0}(y) = \mathcal{F}(y, 0, 0) = \frac{2y}{1+y^2}.$$

Since $\tilde{\Phi}_0(z, 0) = -\mathrm{i}z^{-1}$ is known to be solution of $(\mathbf{IE})_{0|\varepsilon=0}$ (see (10)), we can set

$$\tilde{\Phi}_0(z, \varepsilon) = -\mathrm{i}z^{-1} + \tilde{\eta}(z, \varepsilon)$$

and look for $\tilde{\eta} = \tilde{\eta}(z, \varepsilon)$ as the unique odd solution in $z^{-3}\mathbb{C}[[z^{-1}]]$ of

$$T\tilde{\eta} + T^{-1}\tilde{\eta} = \mathcal{F}_{0,0}(\Phi_{0,0} + \tilde{\eta}) - \mathcal{F}_{0,0}(\Phi_{0,0}) + \varepsilon V'(\Phi_{0,0} + \tilde{\eta}, 0, \varepsilon),$$

where $\Phi_{0,0}(z) := \tilde{\Phi}_0(z, 0)$. It turns out that it will be convenient to study the more general equation in which $\varepsilon V'(\Phi_{0,0} + \tilde{\eta}, 0, \varepsilon)$ is replaced by $\varepsilon V'(\Phi_{0,0} + \tilde{\eta}, 0, \underline{\varepsilon})$, thus introducing an auxiliary parameter $\underline{\varepsilon}$, to be identified with ε when returning to equation $(\mathbf{IE})_0$:

Proposition 3.2. *One has*

$$\tilde{\Phi}_0(z, \varepsilon) = -\mathrm{i}z^{-1} + \tilde{\eta}(z, \varepsilon, \varepsilon),$$

where $\tilde{\eta}(z, \varepsilon, \underline{\varepsilon})$ is, for each $\varepsilon \in \mathbb{C}$ and $\underline{\varepsilon}$ such that $|\underline{\varepsilon}| < \varepsilon_0$, an odd solution in $z^{-3}\mathbb{C}[[z^{-1}]]$ of

$$T\tilde{\eta} + T^{-1}\tilde{\eta} = \mathcal{F}_{0,0}(\Phi_{0,0} + \tilde{\eta}) - \mathcal{F}_{0,0}(\Phi_{0,0}) + \varepsilon V'(\Phi_{0,0} + \tilde{\eta}, 0, \underline{\varepsilon}). \quad (47)$$

The Borel transform $\hat{\eta}(\zeta, \varepsilon, \underline{\varepsilon})$ is convergent for $|\zeta| < 2\pi$ and defines a holomorphic function of three variables in $\{\, (\zeta, \varepsilon, \underline{\varepsilon}) \in \mathbb{C}^3 \mid \zeta \in \mathcal{R}^{(0)},\ |\underline{\varepsilon}| < \varepsilon_0 \,\}$. Moreover, for any $\varepsilon_0' \in (0, \varepsilon_0)$ and $\rho \in (0, 2\pi)$, there exist positive constants τ_0, τ_1, C such that

$$|\hat{\eta}(\zeta, \varepsilon, \underline{\varepsilon})| \le C|\varepsilon|\frac{|\zeta|^2}{2}\,\mathrm{e}^{(\tau_0+\tau_1|\varepsilon|)|\zeta|}, \qquad \zeta \in \mathcal{R}^{(0)}_\rho,\ \ \varepsilon \in \mathbb{C},\ \ |\underline{\varepsilon}| \le \varepsilon_0'.$$

Proof. We shall expand in powers of ε (but not $\underline{\varepsilon}$). We first write equation (47) as

$$T\tilde{\eta} + T^{-1}\tilde{\eta} = \mathcal{F}'_{0,0}\big(\Phi_{0,0}(z)\big)\tilde{\eta} + \sum_{r\geq 2} f_r(z)\tilde{\eta}^r + \varepsilon \sum_{r\geq 0} w_r(z,\underline{\varepsilon})\tilde{\eta}^r$$

with

$$f_r(z) = \frac{1}{r!}\mathcal{F}^{(r)}_{0,0}\big(\Phi_{0,0}(z)\big), \quad w_r(z,\underline{\varepsilon}) = \frac{1}{r!}\partial_y^r V'\big(\Phi_{0,0}(z), 0, \underline{\varepsilon}\big). \tag{48}$$

Particularizing Definition 2.2 to the case $\varepsilon = 0$, the linear difference operator $\mathcal{L}_{0,0}$ associated with the variational equation of $(\mathbf{IE})_{0|\varepsilon=0}$ along $\Phi_{0,0}$ can be written

$$\mathcal{L}_{0,0}\psi = P\psi - A_{0,0}\psi, \qquad A_{0,0}(z) = -2 + \mathcal{F}'_{0,0}\big(\Phi_{0,0}(z)\big) \tag{49}$$

(see (29)–(31)). We can thus rewrite equation (47) as the system

$$\begin{cases} \mathcal{L}_{0,0}\tilde{\eta} = \tilde{\gamma}, \\ \tilde{\gamma} = \varepsilon \displaystyle\sum_{r\geq 0} w_r(z,\underline{\varepsilon})\tilde{\eta}^r + \sum_{r\geq 2} f_r(z)\tilde{\eta}^r. \end{cases} \tag{50}$$

We only need to study the Borel transform of odd solutions $\tilde{\eta}(z,\varepsilon,\underline{\varepsilon})$, $\tilde{\gamma}(z,\varepsilon,\underline{\varepsilon})$ of this system, with $\tilde{\eta} \in z^{-3}\mathbb{C}[[z^{-1}]]$ and $\tilde{\gamma} \in z^{-5}\mathbb{C}[[z^{-1}]]$.

Particularizing Corollary 2.7 to the case $\varepsilon = 0$, we get the following normalized fundamental system of solutions of $\mathcal{L}_{0,0}$:

$$\psi_{1,0} = \partial_z \Phi_{0,0}(z) = \mathrm{i}z^{-2}, \quad \psi_{2,0} = \psi_{1,0}\,\Delta^{-1}_{(0)}\frac{1}{\psi_{1,0}T\psi_{1,0}}.$$

Since $\frac{1}{\psi_{1,0}T\psi_{1,0}} = -z^2(z+1)^2$ is a polynomial of degree 4, the computation of $\psi_{2,0}$ is easy and requires only the knowledge of the constants $\beta_1 = 1/12$ and $\beta_2 = -1/720$ involved in Lemma 2.3. One finds

$$\psi_{2,0}(z) = -\frac{\mathrm{i}}{5}z^3 + \frac{\mathrm{i}}{3}z - \frac{2\mathrm{i}}{15}z^{-1}.$$

The method of point (iii) of Section 2.1 allows us to define a right inverse to $\mathcal{L}_{0,0}$ in $z^{-5}\mathbb{C}[[z^{-1}]]$: we set

$$\mathcal{L}^{-1}_{0,0}\tilde{\varphi} = -\psi_{1,0}\Delta^{-1}(\tilde{\varphi}\,\psi_{2,0}) + \psi_{2,0}\Delta^{-1}(\tilde{\varphi}\,\psi_{1,0}), \qquad \tilde{\varphi} \in z^{-5}\mathbb{C}[[z^{-1}]] \tag{51}$$

(recall that Δ^{-1} is defined on $\mathbb{C}((z^{-1}))_{(1)}$; here we use it only in $z^{-2}\mathbb{C}[[z^{-1}]]$). This way, $\mathcal{L}^{-1}_{0,0}\tilde{\varphi}$ is the only preimage of $\tilde{\varphi}$ which lies in $z^{-3}\mathbb{C}[[z^{-1}]]$ (the other preimages are obtained by adding a linear combination of $\psi_{1,0}$ and $\psi_{2,0}$). We can thus replace the first equation in system (50) by

$$\tilde{\eta} = \mathcal{L}^{-1}_{0,0}\tilde{\gamma}.$$

For technical reasons, it will be easier to deal with $\tilde{A} = z^2\tilde{\eta}$ and $\tilde{B} = z^4\tilde{\gamma}$ instead of $\tilde{\eta}$ and $\tilde{\gamma}$, and to use the linear operator $\mathcal{E}$ defined by

$$\tilde{B} \in z^{-1}\mathbb{C}[[z^{-1}]] \ \mapsto\ \mathcal{E}\tilde{B} = z^2\mathcal{L}^{-1}_{0,0}(z^{-4}\tilde{B}) \in z^{-1}\mathbb{C}[[z^{-1}]]. \tag{52}$$

A pair of formal series $(\tilde{\eta}, \tilde{\gamma})$ with $\tilde{\eta} \in z^{-3}\mathbb{C}[[z^{-1}]]$ is thus solution of (50) if and only if $\tilde{A} = z^2\tilde{\eta}$ and $\tilde{B} = z^4\tilde{\gamma}$ satisfy $\tilde{B} \in z^{-1}\mathbb{C}[[z^{-1}]]$ and

$$\begin{cases} \tilde{A} = \mathcal{E}\tilde{B} \\ \tilde{B} = \varepsilon \sum_{r\geq 0} C_r \tilde{A}^r + \sum_{r\geq 2} C_r^* \tilde{A}^r, \end{cases} \tag{53}$$

where

$$C_r = z^{-(2r-4)} w_r(z) = \frac{1}{r!} z^{-(2r-4)} \partial_y^r V'(-\mathrm{i}z^{-1}, 0, \underline{\varepsilon}),$$
$$C_r^* = z^{-(2r-4)} f_r(z) = \frac{1}{r!} z^{-(2r-4)} \mathcal{F}_{0,0}^{(r)}(-\mathrm{i}z^{-1}).$$

Notice that, due to assumptions **(A)** and **(B)**, $C_r \in z^{-r-1}\mathbb{C}\{z^{-1}\}$ for $r = 0, 1, 2$ or 3, that $C_r, C_r^* \in z^{-(2r-4)}\mathbb{C}\{z^{-1}\}$ for $r \geq 2$, and that C_r and C_r^* have the same parity as $r + 1$.

We now observe that, for any $n \in \mathbb{N}^*$,

$$\tilde{B} \in z^{-n}\mathbb{C}[[z^{-1}]] \;\Rightarrow\; \mathcal{E}\tilde{B} \in z^{-n}\mathbb{C}[[z^{-1}]]$$

and that, when acting on odd or even formal series, $\mathcal{E}$ preserves parity. This is due to the properties of the restriction of Δ^{-1} to $z^{-2}\mathbb{C}[[z^{-1}]]$, which can be written $I + \Gamma$ with $\Gamma : z^{-n-1}\mathbb{C}[[z^{-1}]] \to z^{-n}\mathbb{C}[[z^{-1}]]$ defined by $\Gamma = \partial_z^{-1} + \sum_{\ell\geq 1} \beta_\ell \partial_z^{2\ell-1}$, as at the end of the proof of Lemma 2.8, whence $\mathcal{L}_{0,0}^{-1}\tilde{\varphi} = -\psi_{1,0}\Gamma(\tilde{\varphi}\,\psi_{2,0}) + \psi_{2,0}\Gamma(\tilde{\varphi}\,\psi_{1,0})$.

One can thus check by induction that the formulas

$$\tilde{A}_n = \mathcal{E}\tilde{B}_n, \qquad n \geq 1 \tag{54}$$

$$\tilde{B}_1 = C_0 \tag{55}$$

$$\tilde{B}_n = \sum_{\substack{r\geq 1,\, n_1,\dots,n_r\geq 1 \\ n_1+\cdots+n_r=n-1}} C_r \tilde{A}_{n_1} \cdots \tilde{A}_{n_r} + \sum_{\substack{r\geq 2,\, n_1,\dots,n_r\geq 1 \\ n_1+\cdots+n_r=n}} C_r^* \tilde{A}_{n_1} \cdots \tilde{A}_{n_r}, \qquad n \geq 2 \tag{56}$$

define, for each $\underline{\varepsilon}$, odd series $\tilde{B}_n(z, \underline{\varepsilon}), \tilde{A}_n(z, \underline{\varepsilon}) \in z^{-n}\mathbb{C}[[z^{-1}]]$ for $n \geq 1$, such that the formally convergent series

$$\tilde{A}(z, \varepsilon, \underline{\varepsilon}) = \sum_{n\geq 1} \varepsilon^n \tilde{A}_n(z, \underline{\varepsilon}), \;\; \tilde{B}(z, \varepsilon, \underline{\varepsilon}) = \sum_{n\geq 1} \varepsilon^n \tilde{B}_n(z, \underline{\varepsilon}) \;\; \in z^{-1}\mathbb{C}[[z^{-1}]]$$

are odd and solve (53) (in fact, one even has $\tilde{B}_n, \tilde{A}_n \in z^{-2n+1}\mathbb{C}[[z^{-1}]]$).

Correspondingly, the Borel transform $\hat{A}(\zeta, \varepsilon, \underline{\varepsilon})$ can be written as the series

$$\hat{A} = \sum_{n\geq 1} \varepsilon^n \hat{A}_n(\zeta, \underline{\varepsilon}) \; \in \mathbb{C}[[\zeta]] \tag{57}$$

which is formally convergent. The formal series

$$\tilde{\eta}(z, \varepsilon, \underline{\varepsilon}) = z^{-2}\tilde{A}(z, \varepsilon, \underline{\varepsilon})$$

is the desired odd solution of equation (47). We shall show that $\hat{A}$ is holomorphic for $\zeta \in \mathcal{R}^{(0)}$, with holomorphic dependence on $(\varepsilon, \underline{\varepsilon})$ too, and suitably bounded in $\mathcal{R}_\rho^{(0)}$; Proposition 3.2 will then follow by applying Lemma 3.1 to $\hat{\eta}(\zeta, \varepsilon, \underline{\varepsilon}) = \zeta * \hat{A}$.

Proposition 3.3.

(i) Each of the formal series $\hat{A}_n(\zeta, \underline{\varepsilon})$ *defined by* (54)–(56) *has positive radius of convergence and defines a holomorphic function of* $\mathcal{R}^{(0)}$.

(ii) Let $\varepsilon_0' \in (0, \varepsilon_0)$ *and* $\rho \in (0, 2\pi)$. *Then there exist* $c, \tau_1, \tau_0 > 0$ *such that, for every* $n \geq 1$,

$$|\hat{A}_n(\zeta, \underline{\varepsilon})| \leq c\tau_1^n \frac{|\zeta|^{n-1}}{(n-1)!} \, \mathrm{e}^{\tau_0 |\zeta|}, \qquad \zeta \in \mathcal{R}_\rho^{(0)}, \ |\underline{\varepsilon}| \leq \varepsilon_0'. \tag{58}$$

Proof that Proposition 3.3 implies Proposition 3.2: The series of holomorphic functions $\sum \varepsilon^n \hat{A}_n(\zeta, \underline{\varepsilon})$ is uniformly convergent in any compact subset of $\mathcal{R}^{(0)} \times \mathbb{C} \times \{|\underline{\varepsilon}| < \varepsilon_0\}$, its sum is a holomorphic function $\hat{\mathcal{A}}(\zeta, \varepsilon, \underline{\varepsilon})$ which satisfies

$$|\hat{\mathcal{A}}(\zeta, \varepsilon, \underline{\varepsilon})| \leq c\tau_1 |\varepsilon| \, \mathrm{e}^{(\tau_0 + \tau_1 |\varepsilon|)|\zeta|}, \qquad \zeta \in \mathcal{R}_\rho^{(0)}, \ \varepsilon \in \mathbb{C}, \ |\underline{\varepsilon}| \leq \varepsilon_0'.$$

For any $\varepsilon, \underline{\varepsilon}$, the Taylor expansion at the origin of $\zeta \mapsto \hat{\mathcal{A}}(\zeta, \varepsilon, \underline{\varepsilon})$ is nothing but the formal series $\hat{A}(\zeta, \varepsilon, \underline{\varepsilon})$ defined by (57) (by formal convergence, because $\hat{A}_n \in \zeta^{n-1}\mathbb{C}\{\zeta\}$). Thus $\hat{A}(\zeta, \varepsilon, \underline{\varepsilon})$ has positive radius of convergence and the holomorphic germ that it defines extends to the holomorphic function $\hat{\mathcal{A}}(\zeta, \varepsilon, \underline{\varepsilon})$. Consequently, $\hat{\eta} = \zeta * \hat{A}$ is convergent too and Lemma 3.1 yields the conclusion (with $C = c\tau_1$). □

Proof of Proposition 3.3: From now on, we sometimes omit the explicit dependence on $(\varepsilon, \underline{\varepsilon})$.

(i) The formal series $C_r(z), C_r^*(z)$ belong to $z^{-1}\mathbb{C}\{z^{-1}\}$, hence their Borel transforms $\hat{C}_r(\zeta)$, $\hat{C}_r^*(\zeta)$ are entire functions of exponential type and we can write $\hat{B}_1 = \hat{C}_0$ and

$$\hat{B}_n = \sum_{\substack{r \geq 1, \, n_1, \ldots, n_r \geq 1 \\ n_1 + \cdots + n_r = n-1}} \hat{C}_r * \hat{A}_{n_1} * \cdots * \hat{A}_{n_r} + \sum_{\substack{r \geq 2, \, n_1, \ldots, n_r \geq 1 \\ n_1 + \cdots + n_r = n}} \hat{C}_r^* * \hat{A}_{n_1} * \cdots * \hat{A}_{n_r}$$

for $n \geq 2$. Here, convolution is to be understood as the counterpart in $\mathbb{C}[[\zeta]]$ of multiplication in $z^{-1}\mathbb{C}[[z^{-1}]]$, but we shall readily see that the formal series $\hat{A}_n$ belong to $\mathbb{C}\{\zeta\}$, hence the facts indicated in Section 3.1 are in force.

We need to examine the counterpart in $\mathbb{C}[[\zeta]]$ of the operator $\mathcal{E}$ defined by (52). We have $\psi_{1,0} = \mathrm{i}z^{-2}$ and $\psi_{2,0} = -\mathrm{i}z^3 \mathscr{P}(z)$, where $\mathscr{P}(z) = \frac{1}{5} - \frac{1}{3}z^{-2} + \frac{2}{15}z^{-4}$, hence

$$\mathcal{E}\tilde{B} = -\Delta^{-1}(z^{-1}\mathscr{P}\tilde{B}) + z^5 \mathscr{P} \Delta^{-1}(z^{-6}\tilde{B}), \qquad \tilde{B} \in z^{-1}\mathbb{C}[[z^{-1}]].$$

Using the elementary properties of $\mathcal{B}$,

$$\tilde{\varphi} \in z^{-2}\mathbb{C}[[z^{-1}]] \quad \Rightarrow \quad \mathcal{B}(z\tilde{\varphi}) = \partial_\zeta \hat{\varphi}, \ \mathcal{B}(\Delta^{-1}\tilde{\varphi}) = \frac{1}{\mathrm{e}^{-\zeta} - 1} \hat{\varphi}(\zeta) \tag{59}$$

(the second property follows from (43) and the first one is to be used five times), we get

$$\hat{\mathcal{E}}\hat{B} := \mathcal{B}(\mathcal{E}\tilde{B}) = -J(\zeta) \cdot (1 * \overset{\triangledown}{\mathscr{P}} * \hat{B}) + \overset{\triangledown}{\mathscr{P}} * \partial_\zeta^5 \Big[J(\zeta) \cdot \Big(\frac{\zeta^5}{5!} * \hat{B} \Big) \Big], \tag{60}$$

$$J(\zeta) = \frac{1}{\mathrm{e}^{-\zeta} - 1}, \qquad \overset{\triangledown}{\mathscr{P}}(\zeta) = \frac{1}{5}\delta - \frac{1}{3}\zeta + \frac{1}{45}\zeta^3 \tag{61}$$

(with the convention of the end of Section 3.1 to interpret convolution with δ as the identity operator). The counterpart of equation (54) is thus

$$\hat{A}_n = \hat{\mathcal{E}}\hat{B}_n, \qquad n \geq 1.$$

We now observe that $\hat{B} \in \mathbb{C}\{\zeta\}$ implies $\hat{\mathcal{E}}\hat{B} \in \mathbb{C}\{\zeta\}$ (the simple pole of $J(\zeta)$ at 0 is compensated by the vanishing at 0 of the functions with which $J(\zeta)$ is multiplied), hence $\hat{A}_n(\zeta), \hat{B}_n(\zeta) \in \mathbb{C}\{\zeta\}$ by induction. But even if $\hat{B}$ extends to an entire function (as is the case of $\hat{B}_1$ for instance), $\hat{\mathcal{E}}\hat{B}$ is in general singular at $\pm 2\pi\mathrm{i}$.

It is the meromorphic function $J(\zeta)$ which introduces singular points in the ζ-plane, not at the origin, as previously mentioned, but at all non-zero integer multiples of $2\pi\mathrm{i}$. Ultimately, this is the source of the divergence of the formal series $\tilde{A}_n(z)$, $\tilde{\eta}(z)$, $\tilde{\Phi}_0(z)$.

The property of extending holomorphically to $\mathcal{R}^{(0)}$ is preserved by $\hat{\mathcal{E}}$ and by convolution (by virtue of formula (60) and Lemma 3.1). We thus obtain that all the convergent series $\hat{A}_n(\zeta), \hat{B}_n(\zeta)$ define holomorphic functions of $\mathcal{R}^{(0)}$. Moreover, they depend holomorphically on the parameter $\underline{\varepsilon}$ provided $|\underline{\varepsilon}| < \varepsilon_0$.

(ii) We now fix $\varepsilon_0' \in (0, \varepsilon_0)$ and $\rho \in (0, 2\pi)$. We shall use a majorant series method to bound inductively $\hat{A}_n$ in $\mathcal{R}^{(0)}_\rho$ and prove (58).

Definition 3.4. Let $\tau \geq 0$. We say that a function $\hat{A}(\zeta)$ is τ-majorized by $\hat{\mathscr{A}}(\zeta)$, and we write $\hat{A} \preccurlyeq_\tau \hat{\mathscr{A}}$, if

- $\hat{A}$ is a holomorphic function of $\mathcal{R}^{(0)}$,
- $\hat{\mathscr{A}}$ is an entire function with real non-negative Taylor coefficients at the origin,
- $|\hat{A}(\zeta)| \leq \hat{\mathscr{A}}\big(|\zeta|\big)\,\mathrm{e}^{\tau|\zeta|}$ for all $\zeta \in \mathcal{R}^{(0)}_\rho$.

For $\tilde{A}, \tilde{\mathscr{A}} \in \mathbb{C}[[z^{-1}]]$ with constant terms $A_0, \mathscr{A}_0$ and Borel transforms $\hat{A} = \mathcal{B}\tilde{A}, \hat{\mathscr{A}} = \mathcal{B}\tilde{\mathscr{A}} \in \mathbb{C}\{\zeta\}$, we write $\tilde{A} \preccurlyeq_\tau \tilde{\mathscr{A}}$ if

$$|A_0| \leq \mathscr{A}_0, \quad \hat{A} \preccurlyeq_\tau \hat{\mathscr{A}}.$$

In this last situation, we also write $A_0\delta + \hat{A}(\zeta) \preccurlyeq_\tau \mathscr{A}_0\delta + \hat{\mathscr{A}}(\zeta)$.

Lemma 3.5. *Suppose $\tilde{A}, \tilde{B} \in \mathbb{C}[[z^{-1}]]$ satisfy $\tilde{A} \preccurlyeq_\tau \tilde{\mathscr{A}}$ and $\tilde{B} \preccurlyeq_\tau \tilde{\mathscr{B}}$. Then*

$$\tilde{A} \cdot \tilde{B} \preccurlyeq_\tau \tilde{\mathscr{A}} \cdot \tilde{\mathscr{B}}. \tag{62}$$

Suppose now $\hat{A} \preccurlyeq_\tau \hat{\mathscr{A}}$ (the corresponding formal series $\tilde{A}, \tilde{\mathscr{A}}$ have no constant term) and $p \in \mathbb{N}$. Then

$$\frac{\zeta^p}{p!} * \hat{A} \preccurlyeq_\tau \hat{\mathscr{A}}_p, \qquad \hat{\mathscr{A}}_p(\xi) = \frac{\xi^{p+1}}{(p+1)!}\hat{\mathscr{A}}(\xi). \tag{63}$$

Proof of Lemma 3.5: The first statement follows from Lemma 3.1, since $(A_0\delta + \hat{A}) * (B_0\delta + \hat{B}) = A_0B_0\delta + A_0\hat{B} + B_0\hat{A} + \hat{A} * \hat{B}$. The second statement stems from the inequalities

$$\left|\frac{\zeta^p}{p!} * \hat{A}\right| \leq \int_0^{|\zeta|} \frac{s^p}{p!}\hat{\mathscr{A}}\big(|\zeta| - s\big)\mathrm{e}^{\tau(|\zeta|-s)}\,\mathrm{d}s \leq \hat{\mathscr{A}}\big(|\zeta|\big)\mathrm{e}^{\tau|\zeta|}\int_0^{|\zeta|} \frac{s^p}{p!}\,\mathrm{d}s$$

(where we used the fact that $\hat{\mathscr{A}}$ is monotonic non-decreasing on $\mathbb{R}^+$). $\square$

Lemma 3.6. *Let $\tau \geq 1+3\max\left\{1, \frac{1}{y_0}\right\}$ with y_0 as in assumption* **(B)** *of Section 0.1. Then there exist positive constants c, κ such that, for $|\underline{\varepsilon}| \leq \varepsilon_0'$,*

$$C_r(z,\underline{\varepsilon}) \preccurlyeq_\tau \mathscr{C}_r(z) = c\kappa^r z^{-1}, \qquad r \geq 0 \tag{64}$$

$$C_r^*(z) \preccurlyeq_\tau \mathscr{C}_r^*(z) = c\kappa^{r-2} z^{-1}, \qquad r \geq 2. \tag{65}$$

Notice that $\hat{\mathscr{C}}_r$ and $\hat{\mathscr{C}}_r^*$ are the constant functions $c\kappa^r$ and $c\kappa^{r-2}$.

Proof of Lemma 3.6: Let

$$c_0 = \max\left\{|V'(y,0,\underline{\varepsilon})|;\ |y| \leq \tfrac{2y_0}{3},\ |\underline{\varepsilon}| \leq \varepsilon_0'\right\}.$$

The Cauchy inequalities yield $\left|\frac{1}{r!}\partial_y^r V'(y,0,\underline{\varepsilon})\right| \leq c_0\left(\frac{y_0}{3}\right)^{-r}$ for each $r \in \mathbb{N}$ and $(y,\underline{\varepsilon})$ such that $|y| \leq \frac{y_0}{3}$ and $|\underline{\varepsilon}| \leq \varepsilon_0'$. One can apply again the Cauchy inequalities to bound the coefficients of the Taylor expansion of the function $y \mapsto \frac{1}{r!}\partial_y^r V'(y,0,\underline{\varepsilon})$ at the origin. Since $w_r(z,\underline{\varepsilon})$ is obtained by replacing y by $-\mathrm{i}z^{-1}$ in this function (see (48)), we get

$$w_r(z,\underline{\varepsilon}) = \sum_{p\geq 0} w_{r,p}(\underline{\varepsilon}) z^{-p} \quad \text{for } |z^{-1}| < y_0, \quad \text{with } |w_{r,p}(\underline{\varepsilon})| \leq c_0\left(\tfrac{y_0}{3}\right)^{-r-p}. \tag{66}$$

As a consequence,

$$k \geq 1,\ \tau \geq 1 + \tfrac{3}{y_0} \text{ and } |\underline{\varepsilon}| \leq \varepsilon_0' \quad \Rightarrow \quad z^{-k} w_r(z,\underline{\varepsilon}) \preccurlyeq_\tau c_0\left(\tfrac{y_0}{3}\right)^{-r} z^{-1}. \tag{67}$$

Indeed, $\mathcal{B}(z^{-k}w_r)$ is the entire function $\sum_{p\geq 0} w_{r,p}(\underline{\varepsilon})\frac{\zeta^{p+k-1}}{(p+k-1)!}$, the modulus of which is less than $\frac{|\zeta|^{k-1}}{(k-1)!}\sum |w_{r,p}(\underline{\varepsilon})|\frac{|\zeta|^p}{p!}$ (because binomial coefficients are ≥ 1), and (67) follows from (66) and $\frac{|\zeta|^{k-1}}{(k-1)!} \leq \mathrm{e}^{|\zeta|}$.

Now, for each $r \geq 3$, we can apply this to $C_r = z^{-k}w_r(z,\underline{\varepsilon})$ with $k = 2r-4 \geq 2$. For the remaining cases we must make use of assumption **(B)**: $V'(y,0,\underline{\varepsilon}) = O(y^5)$ implies

$$w_0 = \sum_{p\geq 0} w_{0,p+5} z^{-p-5}, \quad w_1 = \sum_{p\geq 1} w_{1,p+3} z^{-p-3}, \quad w_2 = \sum_{p\geq 2} w_{2,p+1} z^{-p-1}.$$

The Borel transforms of $C_0 = z^4 w_0$, $C_1 = z^2 w_1$ and $C_2 = w_2$ thus satisfy, by virtue of (66),

$$|\hat{C}_0(\zeta,\underline{\varepsilon})| \leq c_0\left(\tfrac{y_0}{3}\right)^{-5}\mathrm{e}^{\tau|\zeta|}, \quad |\hat{C}_1(\zeta,\underline{\varepsilon})| \leq c_0\left(\tfrac{y_0}{3}\right)^{-4}\mathrm{e}^{\tau|\zeta|}, \quad |\hat{C}_2(\zeta,\underline{\varepsilon})| \leq c_0\left(\tfrac{y_0}{3}\right)^{-3}\mathrm{e}^{\tau|\zeta|},$$

provided $\tau \geq \frac{3}{y_0}$. We have thus checked that (64) holds if $\tau \geq 1 + \frac{3}{y_0}$ and c and κ are large enough.

We treat C_r^* by following the same steps. The function $\mathcal{F}_{0,0}(y) = \frac{2y}{1+y^2}$ is holomorphic in the unit disc and bounded by 3 for $|y| \leq \frac{2}{3}$, thus $\left|\frac{1}{r!}\partial_y^r \mathcal{F}_{0,0}(y)\right| \leq 3^{r+1}$ for $|y| \leq \frac{1}{3}$, and

$$f_r(z) = \sum_{p\geq 0} f_{r,p} z^{-p} \quad \text{for } |z^{-1}| < 1, \quad \text{with } |f_{r,p}| \leq 3^{r+p+1}.$$

As previously, this implies that $C_r^* \preccurlyeq_\tau 3^{r+1}z^{-1}$ for $r \geq 3$, provided $\tau \geq 4$. As for $C_2^* = f_2$, since this convergent series is odd, it has no constant term and we can write $\hat{C}_2^*(\zeta) = \sum_{p\geq 0} f_{2,p+1}\frac{\zeta^p}{p!}$, hence $|\hat{C}_2^*(\zeta)| \leq 27\,\mathrm{e}^{3|\zeta|}$. □

Lemma 3.7. *There exists $\lambda > 0$ such that, for any $\tau \geq 0$ and $\tilde{B} \in z^{-1}\mathbb{C}[[z^{-1}]]$,*

$$\tilde{B} \preccurlyeq_\tau \tilde{\mathscr{B}} \quad \Rightarrow \quad \mathcal{E}\tilde{B} \preccurlyeq_\tau \mathscr{E} \cdot \tilde{\mathscr{B}},$$

with $\mathscr{E} = \lambda(1 + z^{-1})^5$.

Proof of Lemma 3.7: Let us assume $\tau \geq 0$ and $\hat{B} \preccurlyeq_\tau \hat{\mathscr{B}}$. We should prove $\hat{\mathcal{E}}\hat{B} \preccurlyeq_\tau \overset{\triangledown}{\mathscr{E}} * \hat{\mathscr{B}}$, where $\overset{\triangledown}{\mathscr{E}} = \lambda(\delta + 1)^{*5} = \delta + 5 + 10\zeta + 5\zeta^2 + \frac{5}{6}\zeta^3 + \frac{1}{24}\zeta^4$ and, in view of (60),

$$\hat{\mathcal{E}}\hat{B} = -\hat{D}_0 + \overset{\triangledown}{\mathscr{P}} * \hat{D}, \qquad \hat{D}_0 = J \cdot (1 * \overset{\triangledown}{\mathscr{P}} * \hat{B}), \qquad \hat{D} = \sum_{p=0}^{5} \binom{5}{p} J^{(p)} \cdot \Big(\frac{\zeta^p}{p!} * \hat{B}\Big).$$

Formula (61) shows that $\overset{\triangledown}{\mathscr{P}} \preccurlyeq_\tau (\delta + 1)^{*4}$, and J is a meromorphic function which has a simple pole at the origin and which is holomorphic in $\mathcal{R}^{(0)} \setminus \{0\}$. Writing $J(\zeta) = \sum_{n\geq 1} \mathrm{e}^{n\zeta}$ for $\Re e\, \zeta < 0$ and $J(\zeta) = -\sum_{n\geq 0} \mathrm{e}^{-n\zeta}$ for $\Re e\, \zeta > 0$, we see that the function J is bounded and its derivatives $J^{(p)}$ are exponentially small as $|\Re e\, \zeta| \to \infty$, thus we can find $K > 0$ such that

$$|J(\zeta)| \leq K\big(1 + |\zeta|^{-1}\big), \quad |J^{(p)}(\zeta)| \leq K|\zeta|^{-p-1}, \quad \text{for } 1 \leq p \leq 5,\ \zeta \in \mathcal{R}^{(0)}_\rho \tag{68}$$

(by treating separately the unbounded domain $\mathcal{R}^{(0)}_\rho \cap \big\{|\Re e\, \zeta| > 1\big\}$, the disc $\big\{|\zeta| < 1\big\}$ and the compact set $\mathcal{R}^{(0)}_\rho \cap \big\{|\Re e\, \zeta| \leq 1,\ |\zeta| \geq 1\big\}$).

We now observe that, as a consequence of Lemma 3.5,

$$\hat{A} \preccurlyeq_\tau \hat{\mathscr{A}} \quad \Rightarrow \quad \begin{cases} J \cdot \big(1 * \hat{A}\big) \preccurlyeq_\tau K(\delta + 1) * \hat{\mathscr{A}} = K\Big(\hat{\mathscr{A}} + 1 * \hat{\mathscr{A}}\Big) & \\ J^{(p)} \cdot \Big(\dfrac{\zeta^p}{p!} * \hat{A}\Big) \preccurlyeq_\tau \dfrac{K}{(p+1)!}\hat{\mathscr{A}}, & 1 \leq p \leq 5 \end{cases}$$

(for the first inequality we used both (62) and (63), $1 * \hat{A} \preccurlyeq_\tau 1 * \hat{\mathscr{A}}$ and $1 * \hat{A} \preccurlyeq_\tau \xi\hat{\mathscr{A}}$, before multiplying by $|J(\zeta)| \leq K + K/|\zeta|$).

Applying this with $\hat{A} = \overset{\triangledown}{\mathscr{P}} * \hat{B} \preccurlyeq_\tau (\delta + 1)^{*4} * \hat{\mathscr{B}}$, we get $\hat{D}_0 \preccurlyeq_\tau K(\delta + 1)^{*5} * \hat{\mathscr{B}}$. Applying it with $\hat{A} = \hat{B}$, we get $\hat{D} \preccurlyeq_\tau K\big(1 * \hat{\mathscr{B}}\big) + \sum_{p=0}^{5} \binom{5}{p} K\hat{\mathscr{B}}$, and a fortiori

$$\hat{D} \preccurlyeq_\tau K'(\delta + 1) * \hat{\mathscr{B}}, \qquad K' = 2^5 K,$$

whence the conclusion follows, with $\lambda = K + K'$. □

End of the proof of Proposition 3.3: Let us choose τ, c, κ as in Lemma 3.6. We define inductively formal series $\tilde{\mathscr{A}}_n$, $\tilde{\mathscr{B}}_n$ by the formulas

$$\tilde{\mathscr{A}}_n = \mathscr{E}\tilde{\mathscr{B}}_n, \qquad n \geq 1$$

$$\tilde{\mathscr{B}}_1 = \tilde{\mathscr{C}}_0$$

$$\tilde{\mathscr{B}}_n = \sum_{r\geq 1,\, n_1+\cdots+n_r=n-1} \tilde{\mathscr{C}}_r \tilde{\mathscr{A}}_{n_1} \cdots \tilde{\mathscr{A}}_{n_r} + \sum_{r\geq 2,\, n_1+\cdots+n_r=n} \tilde{\mathscr{C}}^*_r \tilde{\mathscr{A}}_{n_1} \cdots \tilde{\mathscr{A}}_{n_r}, \qquad n \geq 2.$$

The previous lemmas show that

$$\tilde{A}_n \preccurlyeq_\tau \tilde{\mathscr{A}}_n, \quad \tilde{B}_n \preccurlyeq_\tau \tilde{\mathscr{B}}_n, \qquad n \geq 1. \tag{69}$$

Let us consider the generating series $\tilde{\mathscr{A}}(z,\varepsilon) = \sum_{n\geq 1} \varepsilon^n \tilde{\mathscr{A}}_n(z)$: it is the unique solution in $\varepsilon\mathbb{C}[[z^{-1},\varepsilon]]$ of the equation

$$\tilde{\mathscr{A}} = \mathscr{E} \cdot \Big(\varepsilon \sum_{r\geq 0} \mathscr{C}_r \tilde{\mathscr{A}}^r + \sum_{r\geq 2} \mathscr{C}_r^* \tilde{\mathscr{A}}^r \Big),$$

in which the right-hand side can be written $cz^{-1}\mathscr{E}\big(\varepsilon + \tilde{\mathscr{A}}^2\big)\big(1-\kappa\tilde{\mathscr{A}}\big)^{-1}$ by virtue of (64)–(65). We thus get the quadratic equation

$$\tilde{\mathscr{A}} = c\varepsilon z^{-1}\mathscr{E} + (\kappa + cz^{-1}\mathscr{E})\tilde{\mathscr{A}}^2$$

and the solution can be written explicitly: using $R(x) = \frac{1-(1-4x)^{1/2}}{2x} = \sum_{n\geq 0} R_n x^n$, we have

$$\tilde{\mathscr{A}}(z,\varepsilon) = \mathscr{U}(z,\varepsilon z^{-1}), \qquad \mathscr{U}(z,t) = ct\mathscr{E}(z) R\Big(ct\mathscr{E}(z)\big(\kappa + cz^{-1}\mathscr{E}(z)\big)\Big).$$

We have thus found

$$\tilde{\mathscr{A}}_n(z) = z^{-n}\mathscr{U}_n(z) = \sum_{p\geq 0} \mathscr{U}_{n,p} z^{-n-p}, \qquad n \geq 1,$$

with the notations $\mathscr{U}_n(z) = R_{n-1}\big(c\mathscr{E}(z)\big)^n\big(\kappa + cz^{-1}\mathscr{E}(z)\big)^{n-1} = \sum_{p\geq 0} \mathscr{U}_{n,p} z^{-p}$. Now, we can consider $\mathscr{U}$ as a holomorphic function of two variables in the polydisc $\big\{|z^{-1}| \leq \frac{1}{2}\big\} \times \big\{|t| \leq \frac{1}{\tau_1}\big\}$, continuous on the closure of this polydisc, for an appropriate $\tau_1 > 0$ (determined by c, κ, λ). Hence $|\mathscr{U}_{n,p}| \leq \text{const}\, 2^p \tau_1^n$ and

$$0 \leq \hat{\mathscr{A}}_n(\xi) = \sum_{p\geq 0} \mathscr{U}_{n,p} \frac{\xi^{n+p-1}}{(n+p-1)!} \leq \text{const}\, \tau_1^n \frac{\xi^{n-1}}{(n-1)!}\, \mathrm{e}^{2\xi}, \qquad \xi \in \mathbb{R}^+$$

(because binomial coefficients are ≥ 1), and (69) shows that

$$|\hat{A}_n(\zeta)| \leq \text{const}\, \tau_1^n \frac{|\zeta|^{n-1}}{(n-1)!} \mathrm{e}^{(\tau+2)|\zeta|}, \qquad n \geq 1,$$

as desired. □

The statement relative to $\tilde{\Phi}_0(z,\varepsilon)$ in Theorem 1.4 follows from Proposition 3.2 (with $C_0 = 1 + C\varepsilon_0'$ and $\tau = 1 + \tau_0 + \tau_1\varepsilon_0'$ for instance).

3.3 The analytic continuation of $\hat{\Phi}_0(\zeta,\varepsilon)$ through the cuts $\pm 2\pi \mathrm{i}[1,+\infty)$

Before going on with the study of the Borel transforms of the formal solutions of the secondary inner equations, we build on the previous arguments to improve our knowledge of the analytic continuation of $\hat{\Phi}_0$, with a view to the study of its singularities in Section 4.3.

To deal with multivalued analytic continuation, it is convenient to define a Riemann surface $\mathcal{R}^{(1)}$ over $\mathbb{C}$, in which $\mathcal{R}^{(0)}$ will appear as the principal sheet and which is itself a part of a larger Riemann surface $\mathcal{R}$.

Definition 3.8. Let $\mathcal{R}$ be the set of all homotopy classes[4] of paths issuing from the origin and lying inside $\mathbb{C} \setminus 2\pi\mathrm{i}\,\mathbb{Z}$ (except for their initial point), and let $\pi : \mathcal{R} \to (\mathbb{C} \setminus 2\pi\mathrm{i}\,\mathbb{Z}) \cup \{0\}$ be the map, which associates with any class c the extremity $\gamma(1)$ of any path $\gamma : [0,1] \to \mathbb{C}$ which represents c. We consider $\mathcal{R}$ as a Riemann surface by pulling back by π the complex structure of $(\mathbb{C} \setminus 2\pi\mathrm{i}\,\mathbb{Z}) \cup \{0\}$.

Observe that $\pi^{-1}(0)$ consists of only one point (the homotopy class of the constant path), which we may call the origin of $\mathcal{R}$. We define the "principal sheet" of $\mathcal{R}$ as the set of all the classes of segments $[0,\zeta]$, $\zeta \in \mathcal{R}^{(0)}$; equivalently, it is the connected component of $\pi^{-1}(\mathcal{R}^{(0)})$ which contains the origin; we identify it with the cut plane $\mathcal{R}^{(0)}$ itself. We define the "half-sheets" of $\mathcal{R}$ as the various connected components of $\pi^{-1}(\{\Re e\,\zeta \geq 0\})$ or of $\pi^{-1}(\{\Re e\,\zeta \leq 0\})$.

A holomorphic function of $\mathcal{R}$ can be viewed as a germ of holomorphic function at the origin of $\mathbb{C}$ which admits analytic continuation along any path avoiding $2\pi\mathrm{i}\,\mathbb{Z}$; we then say that this germ "extends holomorphically to $\mathcal{R}$" (see Section 1.3 of [Sau05]). This definition a priori does not authorize analytic continuation along a path which leads to the origin, unless this path stays in $\mathcal{R}^{(0)}$.

It turns out that the Borel transform $\hat{\Phi}_0$ of the formal solution of the first inner equation extends holomorphically to $\mathcal{R}$; however, in this section, we content ourselves with explaining why $\hat{\Phi}_0$ extends holomorphically to a subspace $\mathcal{R}^{(1)}$ of $\mathcal{R}$.

Definition 3.9. We define $\mathcal{R}^{(1)} \subset \mathcal{R}$ as the union of the principal sheet $\mathcal{R}^{(0)}$ and the "contiguous" half-sheets, i.e. a point ζ in $\mathcal{R}^{(1)}$ can be represented by a path γ_ζ which issues from 0 and lies in $\mathbb{C} \setminus 2\pi\mathrm{i}\,\mathbb{Z}$ but crosses at most once the imaginary axis (no crossing at all means we stay in $\mathcal{R}^{(0)}$, but we arrive to a new half-sheet when we cross between two consecutive singular points $2\pi\mathrm{i}m$ and $2\pi\mathrm{i}(m+1)$, or $-2\pi\mathrm{i}(m+1)$ and $-2\pi\mathrm{i}m$, with $m \geq 1$).

We follow Sections 2.1.2 and 2.3.3 of [OSS03] and use auxiliary subsets $\mathcal{R}^{(1)}_\rho$, the points of which can be represented by paths γ_ζ which stay in $\mathcal{R}^{(0)}$ or pass between two discs $D(\pm 2\pi\mathrm{i}m, m\rho)$ and $D\big(\pm 2\pi\mathrm{i}(m+1), (m+1)\rho\big)$ with $1 \leq m < \frac{1}{2}(\frac{2\pi}{\rho} - 1)$ and cross the imaginary axis at most once—see the left part of Figure 3 and the precise definition in Section 2.3.3 of [OSS03] (which deals with the same situation but without the factor 2π). Observe that

$$\mathcal{R}^{(1)} = \bigcup_{0<\rho<\frac{2\pi}{3}} \mathcal{R}^{(1)}_\rho.$$

The right part of Figure 3 illustrates the possibility of defining, for each $\zeta \in \mathcal{R}^{(1)}_\rho$, a path Γ_ζ which represents ζ, is contained in $\mathcal{R}^{(1)}_\rho$ and is "symmetrically contractile". The meaning of this property and the definition of Γ_ζ are given in Section 2.3.3 of [OSS03]; here, we only mention the existence of a constant $K_\rho > 0$ such that

$$|\pi(\zeta)| \leq \ell(\zeta) \leq K_\rho |\pi(\zeta)|, \qquad \zeta \in \mathcal{R}^{(1)}_\rho, \tag{70}$$

where $\ell(\zeta)$ denotes the length of Γ_ζ, and a lemma which extends to $\mathcal{R}^{(1)}_\rho$ the Lemma 3.1 that we used to control convolution products in $\mathcal{R}^{(0)}_\rho$:

[4]When mentioning homotopy of paths, we always refer to homotopy with fixed extremities.

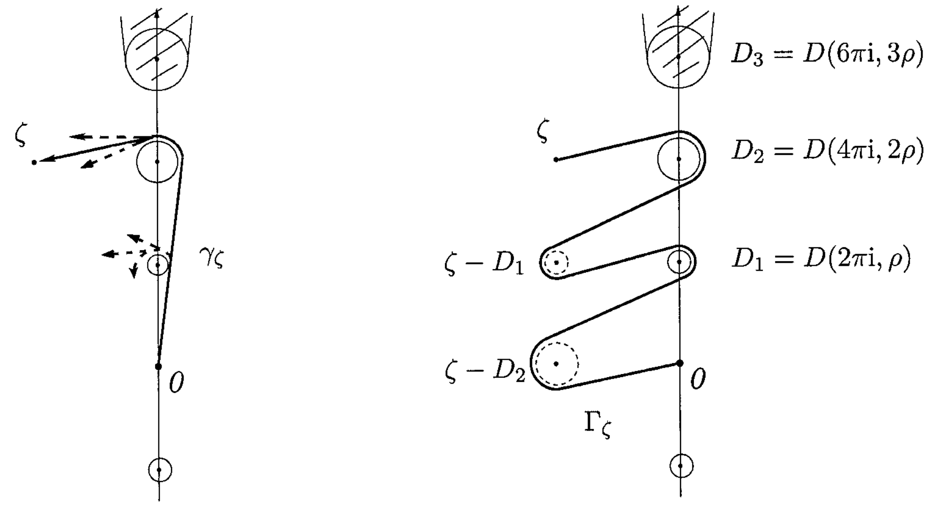

Figure 3: Left: One path among the ones which define points of $\mathcal{R}_\rho^{(1)}$. Right: The path Γ_ζ defines the same point $\zeta \in \mathcal{R}_\rho^{(1)}$ as γ_ζ.

Lemma 3.10. *Suppose $\hat\varphi$ and $\hat\psi$ extend holomorphically to $\mathcal{R}^{(1)}$. Then $\hat\varphi * \hat\psi$ extends holomorphically to $\mathcal{R}^{(1)}$.*

Suppose moreover $\rho \in (0, \frac{2\pi}{3})$, $\tau_1, \tau_2 \geq 0$ and $\hat\Phi$ and $\hat\Psi$ are non-negative continuous monotonic non-decreasing functions on $\mathbb{R}^+$ such that

$$|\hat\varphi(\zeta)| \leq \hat\Phi\big(\ell(\zeta)\big)\, e^{\tau_1 \ell(\zeta)}, \quad |\hat\psi(\zeta)| \leq \hat\Psi\big(\ell(\zeta)\big)\, e^{\tau_2 \ell(\zeta)}, \qquad \zeta \in \mathcal{R}_\rho^{(1)}.$$

Then

$$|\hat\varphi * \hat\psi(\zeta)| \leq \hat\Phi * \hat\Psi\big(\ell(\zeta)\big)\, e^{\tau \ell(\zeta)}, \qquad \zeta \in \mathcal{R}_\rho^{(1)}, \tag{71}$$

where $\tau = \max(\tau_1, \tau_2)$.

The proof is given in [GS01], p. 539. The idea is that the analytic continuation of $\hat\varphi * \hat\psi$ at a point ζ represented by a path γ_ζ is given by $\int_{\Gamma_\zeta} \hat\varphi(\zeta_1)\hat\psi(\zeta - \zeta_1)\, d\zeta_1$.

We leave it to the reader to adapt the computations of the previous section so as to prove that $\hat\Phi_0$ extends holomorphically to $\mathcal{R}_\rho^{(1)}$ with a bound $C_0'\, e^{\tau_0' \ell(\zeta)}$ (follow the same steps, replacing $|\zeta|$ by $\ell(\zeta)$; it is essentially the proof of Proposition 3.3 that needs to be adapted, it will involve a lemma analogous to Lemma 3.7 for which one must use (70)). Inequality (25) for $\hat\Phi_0$ follows from this (choose $\rho > 0$ less than $\pi\lambda\cos\beta$ so that the path $\Gamma_{\lambda,\beta}$ pass between the discs $D(2\pi i, \rho)$ and $D(4\pi i, 2\rho)$).

We thus obtain that $\tilde\Phi_0 \in \widetilde{\mathrm{RES}}^{(1)}$, with the notation of

Definition 3.11. We define $\widetilde{\mathrm{RES}}^{(1)}$ as $\mathcal{B}^{-1}\,\widehat{\mathrm{RES}}^{(1)}$, where $\widehat{\mathrm{RES}}^{(1)}$ is the set of all $\hat\varphi \in \mathbb{C}\{\zeta\}$ such that

(i) $\hat\varphi(\zeta)$ extends analytically to $\mathcal{R}^{(1)}$,

(ii) for each $\rho \in (0, \frac{2\pi}{3})$, there exist $\tau, C > 0$ such that $|\hat\varphi(\zeta)| \leq C\, e^{\tau \ell(\zeta)}$ for $\zeta \in \mathcal{R}_\rho^{(1)}$.

Obviously, $\widehat{\mathrm{RES}}^{(1)} \subset \widehat{\mathrm{RES}}^{(0)}$ and $\widetilde{\mathrm{RES}}^{(1)} \subset \widetilde{\mathrm{RES}}^{(0)}$.

What will be used in Section 4 is the fact that the analytic continuation of $\hat\Phi_0$ can be followed around the points $2\pi i$ and $-2\pi i$.

3.4 The Borel transform of $\tilde{\Phi}_n(z,\varepsilon;b_1,\dots,b_n)$

The analysis to control $\hat{\Phi}_n(\zeta,\varepsilon;b_1,\dots,b_n)$ is easier than for $\hat{\Phi}_0(\zeta,\varepsilon)$, and we shall not give details about the dependence on ε, since it is easily seen to be analytic with uniform bounds for $|\varepsilon|\le\varepsilon_0'$, nor on the dependence on $b_1,\dots,b_n$, which is clearly polynomial.

According to (36) and (7), we have

$$\tilde{\Phi}_n=-\tilde{\psi}_1\Delta^{-1}(\tilde{\psi}_2\tilde{f}_n)+\tilde{\psi}_2\Delta^{-1}(\tilde{\psi}_1\tilde{f}_n)+b_n\tilde{\psi}_2,\qquad \tilde{f}_n=\tilde{C}_{n,0}+\sum\tilde{C}_{n_0,r}\tilde{\Phi}_{n_1}\dots\tilde{\Phi}_{n_r},\tag{72}$$

where the sum is taken over all $n_0\ge 0, r\ge 1$ such that $n_0+r\ge 2$ and $n_1,\dots,n_r\ge 1$ such that $n_0+n_1+\dots+n_r=n$, and with

$$\tilde{C}_{n,r}=\frac{1}{r!}\partial_y^r\mathcal{F}_n\big(\tilde{\Phi}_0(z,\varepsilon),\varepsilon\big).$$

The part of Theorem 1.4 concerning $\hat{\Phi}_n$, $n\ge 1$, follows from (72) and the following stability properties of the space $\widetilde{\mathrm{RES}}^{(0)}$ of Definition 1.3:

Lemma 3.12.

(i) The space $\widetilde{\mathrm{RES}}^{(0)}$ is stable under multiplication, differentiation and the shift operator T.

(ii) If $F(y)\in\mathbb{C}\{y\}$ and $\tilde{\varphi}(z)\in\widetilde{\mathrm{RES}}^{(0)}\cap z^{-1}\mathbb{C}[[z^{-1}]]$, then $F\big(\tilde{\varphi}(z)\big)\in\widetilde{\mathrm{RES}}^{(0)}$.

(iii) If $\tilde{\varphi}\in\widetilde{\mathrm{RES}}^{(0)}\cap\mathbb{C}((z^{-1}))_{(1)}$ (using the notation (32) of Lemma 2.3), then $\Delta^{-1}\tilde{\varphi}$ and $\Delta^{-1}_{(0)}\tilde{\varphi}\in\widetilde{\mathrm{RES}}^{(0)}$.

Lemma 3.13. *The formal series $\tilde{\psi}_1,\tilde{\psi}_2,\tilde{C}_{n,r}$ all belong to $\widetilde{\mathrm{RES}}^{(0)}$ (with uniform estimates for $|\varepsilon|\le\varepsilon_0'$).*

Indeed, in view of (72), these lemmas imply that $\tilde{\Phi}_n\in\widetilde{\mathrm{RES}}^{(0)}$ by induction on $n\ge 1$.

Proof of Lemma 3.12: Let $\tilde{\varphi},\tilde{\psi}\in\widetilde{\mathrm{RES}}^{(0)}$. We denote by $P(z)=\sum a_k z^k$ and $Q(z)=\sum b_k z^k$ the polynomial parts of $\tilde{\varphi}(z)$ and $\tilde{\psi}(z)$, and by $\hat{\varphi}(\zeta)$ and $\hat{\psi}(\zeta)$ their Borel transforms (thus the extended Borel transforms, as defined at the end of Section 3.1, are $\mathcal{B}_{\mathrm{ext}}\tilde{\varphi}=\sum a_k\delta^{(k)}+\hat{\varphi}(\zeta)$ and $\mathcal{B}_{\mathrm{ext}}\tilde{\psi}=\sum b_k\delta^{(k)}+\hat{\psi}(\zeta)$; both sums over k are finite). Let $\rho\in(0,2\pi)$ and $\tau,c>0$ such that $|\hat{\varphi}(\zeta)|,|\hat{\psi}(\zeta)|\le c\,\mathrm{e}^{\tau|\zeta|}$ for $\zeta\in\mathcal{R}^{(0)}_{\rho/2}$.

(i) According to (46), we have

$$\mathcal{B}\big(\tilde{\varphi}\cdot\tilde{\psi}\big)=\sum a_k\hat{\psi}^{(k)}+\sum b_k\hat{\varphi}^{(k)}+\hat{\varphi}*\hat{\psi}.$$

The Cauchy inequalities imply that

$$|\hat{\varphi}^{(k)}(\zeta)|,\ |\hat{\psi}^{(k)}(\zeta)|\le k!(\rho/2)^{-k}c\,\mathrm{e}^{\tau(|\zeta|+\rho/2)},\qquad \zeta\in\mathcal{R}^{(0)}_{\rho}\tag{73}$$

(because the disc of center ζ and radius $\rho/2$ is included in $\mathcal{R}^{(0)}_{\rho}$ whenever $\zeta\in\mathcal{R}^{(0)}_{\rho}$). On the other hand, Lemma 3.1 implies that

$$\big|\hat{\varphi}*\hat{\psi}(\zeta)\big|\le c^2|\zeta|\,\mathrm{e}^{\tau|\zeta|}\le c^2\,\mathrm{e}^{(\tau+1)|\zeta|}.$$

Hence $\tilde\varphi\cdot\tilde\psi\in\widetilde{\mathrm{RES}}^{(0)}$. For the stability of $\widetilde{\mathrm{RES}}^{(0)}$ under ∂_z and T, use (43).

(ii) We now suppose $\tilde\varphi\in\widetilde{\mathrm{RES}}^{(0)}\cap z^{-1}\mathbb{C}[[z^{-1}]]$, i.e. $P\equiv 0$. Let $F(y)=\sum_{r\ge0}a_r y^r\in\mathbb{C}\{y\}$. Substitution gives rise to the formally convergent series

$$\tilde\psi(z)=F\big(\tilde\varphi(z)\big)=\sum_{r\ge0}a_r\tilde\varphi(z)^r\in\mathbb{C}[[z^{-1}]].$$

Its Borel transform is obtained by discarding the constant term: $\hat\psi=\sum_{r\ge1}a_r\hat\varphi^{*r}$. Lemma 3.1 yields $\big|\hat\varphi^{*r}(\zeta)\big|\le c^r\frac{|\zeta|^{r-1}}{(r-1)!}\,\mathrm{e}^{\tau|\zeta|}$ in $\mathcal{R}^{(0)}_\rho$ (and even in $\mathcal{R}^{(0)}_{\rho/2}$) and there exist $C,\kappa>0$ such that $|a_r|\le C\kappa^r$, hence $\big|\hat\psi(\zeta)\big|\le cC\kappa\,\mathrm{e}^{(c\kappa+\tau)|\zeta|}$, and $\tilde\psi\in\widetilde{\mathrm{RES}}^{(0)}$.

(iii) We now suppose $\tilde\varphi\in\mathbb{C}((z^{-1}))_{(1)}$, i.e. $\hat\varphi(0)=0$, hence $\hat\varphi=1*\hat\varphi'$. We have

$$\mathcal{B}\big(\Delta^{-1}\tilde\varphi\big)=\mathcal{B}\big(\Delta^{-1}_{(0)}\tilde\varphi\big)=J\cdot\hat\varphi.$$

Indeed, $\Delta^{-1}P$ and $\Delta^{-1}_{(0)}P$ are polynomials in z because Δ^{-1} and $\Delta^{-1}_{(0)}$ leave $\mathbb{C}[z]$ invariant, as was mentioned in Remark 2.4, and we can apply (59) to $\tilde\varphi(z)-P(z)\in z^{-2}\mathbb{C}[[z^{-1}]]$. The function $J(\zeta)$ was defined by (61) and, in $\mathcal{R}^{(0)}_\rho$, we can use the bound provided by (68), together with the bounds $|\hat\varphi(\zeta)|\le c\,\mathrm{e}^{\tau|\zeta|}$ or

$$|\hat\varphi(\zeta)|\le c'|\zeta|\,\mathrm{e}^{\tau|\zeta|},\qquad c'=c(\rho/2)^{-1}\,\mathrm{e}^{\tau\rho/2}$$

(the last one results from the Cauchy inequalities (73) and Lemma 3.1 applied to $\hat\varphi=1*\hat\varphi'$). Hence

$$\big|J(\zeta)\cdot\hat\varphi(\zeta)\big|\le K\Big(\big|\hat\varphi(\zeta)\big|+\frac{1}{|\zeta|}\big|\hat\varphi(\zeta)\big|\Big)\le K(c+c')\,\mathrm{e}^{\tau|\zeta|},\qquad \zeta\in\mathcal{R}^{(0)}_\rho,$$

which shows that $\Delta^{-1}\tilde\varphi,\Delta^{-1}_{(0)}\tilde\varphi\in\widetilde{\mathrm{RES}}^{(0)}$. □

Proof of Lemma 3.13: Let $\rho\in(0,2\pi)$. We know from Section 3.2 that there exists $\tau,c>0$ such that the Borel transform of $\tilde\Phi_0(z)=-\mathrm{i}z^{-1}+O(z^{-3})$ satisfies

$$\big|\hat\Phi_0(\zeta,\varepsilon)\big|\le c\,\mathrm{e}^{\tau|\zeta|},\qquad \zeta\in\mathcal{R}^{(0)}_{\rho/2}.$$

The formal series $\tilde\psi_1=\partial_z\tilde\Phi_0=\mathrm{i}z^{-2}+O(z^{-4})$ thus has a Borel Transform $\hat\psi_1=-\zeta\hat\Phi_0$ which satisfies

$$\big|\hat\psi_1(\zeta)\big|\le c|\zeta|\,\mathrm{e}^{\tau|\zeta|}\le c\,\mathrm{e}^{(\tau+1)|\zeta|},\qquad \zeta\in\mathcal{R}^{(0)}_{\rho/2}.$$

In particular $\tilde\psi_1\in\widetilde{\mathrm{RES}}^{(0)}$.

Let us now consider $\tilde\psi_2=\tilde\psi_1\Delta^{-1}_{(0)}\Big(\frac{1}{\tilde\psi_1 T\tilde\psi_1}\Big)$ (according to Corollary 2.7). We have $\tilde\psi_1T\tilde\psi_1=-z^{-4}(1-\tilde A)$ with the Borel transform of $\tilde A\in z^{-1}\mathbb{C}[[z^{-1}]]$ defined by $\hat A=\partial^4_\zeta\big(\hat\psi_1*(\mathrm{e}^{-\zeta}\hat\psi_1)\big)$; Lemma 3.1 and the Cauchy inequalities yield $\big|\hat\psi_1*(\mathrm{e}^{-\zeta}\hat\psi_1)\big|\le c^2\frac{|\zeta|^3}{3!}\,\mathrm{e}^{(\tau+1)|\zeta|}\le c^2\,\mathrm{e}^{(\tau+2)|\zeta|}$ in $\mathcal{R}^{(0)}_{\rho/2}$ and

$$\big|\hat A(\zeta)\big|\le c'\,\mathrm{e}^{(\tau+2)|\zeta|},\qquad \zeta\in\mathcal{R}^{(0)}_\rho,$$

with $c'=4!(\rho/2)^{-4}c^2\,\mathrm{e}^{(\tau+2)\rho/2}$. Thus $\tilde A\in\widetilde{\mathrm{RES}}^{(0)}$ and Lemma 3.12 (ii) implies that $(1-\tilde A)^{-1}\in\widetilde{\mathrm{RES}}^{(0)}$. Point (i) of this lemma then implies that $-z^4(1-\tilde A)^{-1}\in$

$\widetilde{\mathrm{RES}}^{(0)}$, and point (iii) yields $\Delta^{-1}\left(\frac{1}{\tilde\psi_1 T\tilde\psi_1}\right) = \Delta^{-1}\big(-z^4(1-\tilde A)^{-1}\big) \in \widetilde{\mathrm{RES}}^{(0)}$, whence $\tilde\psi_2 \in \widetilde{\mathrm{RES}}^{(0)}$.

The case of the $\tilde C_{n,r}$'s is treated by applying Lemma 3.12 (ii) to the Taylor expansion of $\mathcal{F}_n(y,\varepsilon) = \sum_{p\geq 1}\mathcal{F}_{n,p}(\varepsilon)y^p$ (in fact, since $\mathcal{F}$ is odd in y, the coefficients with even p vanish) and its derivatives (the uniformity of the estimates for $|\varepsilon| \leq \varepsilon_0'$ stems from the inequalities $|\mathcal{F}_{n,p}(\varepsilon)| \leq (h_0/2)^{-n}(y_0/2)^{-p}\max\big\{|\mathcal{F}(y,h,\varepsilon)|;\ |y| \leq y_0/2,\ |h| \leq h_0/2,\ |\varepsilon| \leq \varepsilon_0'\big\}$). $\square$

3.5 The analytic continuation of the $\hat\Phi_n$'s through the cuts $\pm 2\pi\mathrm{i}[1,+\infty)$

Lemmas 3.12 and 3.13 are also valid for the space $\widetilde{\mathrm{RES}}^{(1)}$ introduced in Section 3.3, as can be checked by means of Lemma 3.10. By adapting the above arguments, one can thus deduce that all the $\tilde\Phi_n$'s belong to $\widetilde{\mathrm{RES}}^{(1)}$: their Borel transforms extend holomorphically to $\mathcal{R}^{(1)}$, with bounds of the form $C_n' \,\mathrm{e}^{\tau_n' \ell(\zeta)}$ in each $\mathcal{R}^{(1)}_\rho$, and they satisfy inequalities of the form (25) (choose $\rho > 0$ less than $\pi\lambda\cos\beta$ so that the path $\Gamma_{\lambda,\beta}$ pass between the discs $D(2\pi\mathrm{i},\rho)$ and $D(4\pi\mathrm{i},4\rho)$).

What will be used in Section 4 is the fact that the analytic continuation of $\hat\Phi_n$ can be followed around the points $2\pi\mathrm{i}$ and $-2\pi\mathrm{i}$ for any $n \geq 1$.

4 The bridge equation

The goal of this section is to analyze the singularities of the $\hat\Phi_n$'s, so as to prove Theorem 1.10. In Section 4.1, we first describe a normalized fundamental system of formal solutions of the linear equation $(\mathbf{FL})_b$ introduced in Section 1.3, then we return to the theory of the analytic functions of the complex variable ζ in the "Borel plane": Écalle's formalism of singularities (Section 4.2) will allow us to obtain information on $\Delta_{\pm 2\pi\mathrm{i}}\tilde\Phi_n$ almost "automatically" by considering the counterpart of equation $(\mathbf{FL})_b$ in the space of singularities (Section 4.3).

4.1 A normalized fundamental system of formal solutions for the full variational equation

We now prove Proposition 1.8.

We first define $\tilde\Psi_1(z,h;b) = \partial_z\tilde\Phi(z,h;b)$: this is clearly an even solution of $(\mathbf{FL})_b$, which is of the form (17) with $\tilde\Psi_{1,0}(z,\varepsilon) = \mathrm{i}z^{-2} + O(z^{-4})$ and $\tilde\Psi_{1,n} \in z^{4n-2}\mathbb{C}[[z^{-1}]]$ (in fact, $\tilde\Psi_{1,0}$ was already used under the name $\tilde\psi_1$ in Sections 2.1 and 2.2—see (35)). In view of Appendix A.1, the last property can be written $\tilde\Psi_1 \in z^{-2}\mathbb{C}[[z^{-1}, h^2z^4]]$; as a consequence of (9), we also have $b_1 = 0 \;\Rightarrow\; \tilde\Psi_1 \in z^{-2}\mathbb{C}[[z^{-1},(hz)^2]]$, which amounts to (18). As mentioned in Section 1.2, the space $\widetilde{\mathrm{RES}}^{(0)}$ is stable under differentiation, thus each $\tilde\Psi_{1,n} = \partial_z\tilde\Phi_{1,n}$ belongs to $\widetilde{\mathrm{RES}}^{(0)}$.

In order to define $\tilde\Psi_2$, we write $\partial_y\mathcal{F}\big(\tilde\Phi(z,h,\varepsilon;b),h,\varepsilon\big) = 2 + A(z,h,\varepsilon) = 2 + A_0 + \sum_{n\geq 1}h^{2n}A_n$, with $A_0 = A_0(z,\varepsilon)$ as in (30) and $A_n = A_n(z,\varepsilon;b_1,\dots,b_n)$ for $n \geq 1$. Thus, equation $(\mathbf{FL})_b$ reads $\mathcal{L}\Psi = 0$ with $\mathcal{L}\Psi := P\Psi - A(z,h,\varepsilon)\Psi$ (still using the operator P defined by (31)). We shall proceed as in Section 2.1, adapting to the case of $\mathbb{C}((z^{-1}))[[h^2]]$ the theory of linear difference operators and particularly point (iv): the independent solution $\tilde\Psi_2$ will be defined as $\tilde\Psi_1\Delta^{-1}_{(0)}\big(\frac{1}{\tilde\Psi_1 T\tilde\Psi_1}\big)$, we just need to check that this definition makes sense.

Let $\mathscr{A}_4 = \mathbb{C}[[z^{-1}, h^2 z^4]]$. The formal series $\frac{1}{\tilde{\Psi}_1}$ is a well-defined even element of $z^2\mathscr{A}_4 \subset \mathbb{C}((z^{-1}))[[h^2]]$. The argument of Lemma 2.6 shows that $\frac{1}{\tilde{\Psi}_1} T\big(\frac{1}{\tilde{\Psi}_1}\big) \in z^4\mathscr{A}_4$ has no residuum with respect to z and that $\Delta_{(0)}^{-1}\big(\frac{1}{\tilde{\Psi}_1 T\tilde{\Psi}_1}\big)$ is a well-defined odd element of $z^5\mathscr{A}_4$ (the operator $\Delta_{(0)}^{-1} = \partial_z^{-1} \circ \big(I - \frac{1}{2}\partial_z + \sum_{\ell\geq 1}\beta_\ell \partial_z^{2\ell}\big)$ is well-defined in $\{\sum h^{2n}\varphi_n(z) \in \mathbb{C}((z^{-1}))[[h^2]] \mid \forall n \geq 0,\ \varphi_n \in \mathbb{C}((z^{-1}))_{(1)}\}$). We thus get a normalized fundamental system of solutions $(\tilde{\Psi}_1, \tilde{\Psi}_2)$ with $\tilde{\Psi}_2 = \tilde{\Psi}_1 \Delta_{(0)}^{-1}\big(\frac{1}{\tilde{\Psi}_1 T\tilde{\Psi}_1}\big)$ odd and of the form (17). Moreover $\tilde{\Psi}_{2,0}(z,\varepsilon) = -\frac{\mathrm{i}}{5}z^3 + O(z)$ (already used under the name $\tilde{\psi}_2$ in Sections 2.1 and 2.2) and each $\tilde{\Psi}_{2,n}$ belongs to $z^{4n+3}\mathbb{C}[[z^{-1}]]$, and also to $z^{2n+3}\mathbb{C}[[z^{-1}]]$ when $b_1 = 0$ (using $\mathscr{A}_2 = \mathbb{C}[[z^{-1}, (hz)^2]]$ in that case).

Let us check that each $\tilde{\Psi}_{2,n}$ belongs to $\widetilde{\mathrm{RES}}^{(0)}$. Let $\tilde{\chi} = \tilde{\Psi}_1 T\tilde{\Psi}_1$. Since $\tilde{\Psi}_1 \in \widetilde{\mathrm{RES}}^{(0)}[[h^2]]$, point (i) of Lemma 3.12 yields $\tilde{\chi} = \sum_{n\geq 0} h^{2n}\tilde{\chi}_n(z) \in \widetilde{\mathrm{RES}}^{(0)}[[h^2]]$. Writing the first term as $\tilde{\chi}_0(z) = -z^{-4}\big(1 + \tilde{\varphi}(z)\big)$ with $\tilde{\varphi} \in \widetilde{\mathrm{RES}}^{(0)} \cap z^{-1}\mathbb{C}[[z^{-1}]]$, we see that $1/\tilde{\chi}_0$ belongs to $\widetilde{\mathrm{RES}}^{(0)}$ by point (ii) of Lemma 3.12 applied to $F(y) = (1+y)^{-1}$. It follows that $\frac{1}{\tilde{\chi}} = \frac{1}{\tilde{\chi}_0}\big(1 + \sum_{r\geq 1}(-1)^r\big(\sum_{n\geq 1} h^{2n}\frac{\tilde{\chi}_n}{\tilde{\chi}_0}\big)^r\big)$ belongs to $\widetilde{\mathrm{RES}}^{(0)}[[h^2]]$ (each term is a polynomial in $\frac{1}{\tilde{\chi}_0}, \tilde{\chi}_1, \tilde{\chi}_2, \dots$) and $\Delta_{(0)}^{-1}\big(\frac{1}{\tilde{\chi}}\big) \in \widetilde{\mathrm{RES}}^{(0)}[[h^2]]$ by point (iii) of Lemma 3.12. Hence $\tilde{\Psi}_2 \in \widetilde{\mathrm{RES}}^{(0)}[[h^2]]$.

In fact, in view of Sections 3.3 and 3.5, we can strengthen the statements on the analytic continuation of the Borel transforms: by an easy adaptation of the above arguments, one can check that the $\hat{\Psi}_{j,n}$'s belong to $\widehat{\mathrm{RES}}^{(1)}$, thus

$$\tilde{\Psi}_1, \tilde{\Psi}_2 \in \widetilde{\mathrm{RES}}^{(1)}[[h^2]]. \tag{74}$$

This will be used in Section 4.3.

We end this section with the proof of equation (28) of Proposition 1.13. Let $n \geq 1$; the formal series $\partial_{b_n}\tilde{\Phi}$ is clearly an odd solution of $(\mathbf{FL})_b$ and is thus proportional to $\tilde{\Psi}_2$:

$$\partial_{b_n}\tilde{\Phi} = \beta(h,\varepsilon;b)\tilde{\Psi}_2(z,h,\varepsilon;b), \qquad \beta \in \mathbb{C}[[h^2]].$$

We have $\partial_{b_n}\tilde{\Phi} = \sum_{p\geq n} h^{2p}\,\partial_{b_n}\tilde{\Phi}_p$ and, according to (38), $\partial_{b_n}\tilde{\Phi}_n = \tilde{\Psi}_{2,0}$, hence $\beta = h^{2n} + O(h^{2(n+1)})$.

4.2 Écalle's theory of singularities

Let $\varpi \in (0, \frac{\pi}{2})$. The results contained in Sections 3.3 and 3.5 allow us to define multivalued analytic functions $\overset{\vee}{\chi}_n$ by the formulas

$$\overset{\vee}{\chi}_n(\zeta) = \hat{\Phi}_n(2\pi\mathrm{i} + \zeta), \qquad |\zeta| < 2\pi,\ -\tfrac{3\pi}{2} - \varpi < \arg\zeta < \tfrac{\pi}{2} + \varpi, \tag{75}$$

i.e. functions holomorphic in a part of the Riemann surface of the logarithm (which also depend analytically on ε, and on $b_1, \dots, b_n$ if $n \geq 1$)—see Figure 4.

Similarly, we can consider $\hat{\Phi}_n(-2\pi\mathrm{i} + \zeta)$ for $-\frac{\pi}{2} - \varpi < \arg\zeta < \frac{3\pi}{2} + \varpi$. These analytic functions are examples of "majors of singularities". After moding out by the regular germs, the equivalence classes that we obtain can be considered as the "singularities" of $\hat{\Phi}_n$ at $\pm 2\pi\mathrm{i}$; these are examples of singularities in the direction $\theta = \pm\frac{\pi}{2}$ in the following sense:

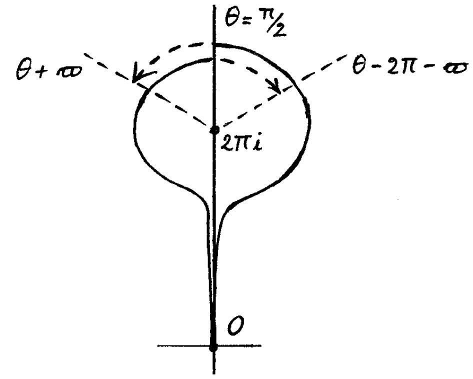

Figure 4: The sector in which the $\overset{\vee}{\chi}_n$'s are defined.

Definition 4.1. Let $\theta \in \mathbb{R}$ and $\varpi > 0$. Consider the space $\mathscr{M}_{\theta,\varpi}$ of germs of holomorphic functions $\overset{\vee}{\varphi}(\zeta)$ defined for $\theta - 2\pi - \varpi < \arg\zeta < \theta + \varpi$ and $|\zeta|$ small enough. The quotient space $\mathrm{SING}_{\theta,\varpi} = \mathscr{M}_{\theta,\varpi}/\mathbb{C}\{\zeta\}$ is called the space of singularities in the direction θ with aperture 2ϖ.

A germ $\overset{\vee}{\varphi} \in \mathscr{M}_{\theta,\varpi}$ is called a *major*, its class in $\mathrm{SING}_{\theta,\varpi}$ is called the *singularity* of $\overset{\vee}{\varphi}$ and is denoted by $\mathrm{sing}(\overset{\vee}{\varphi})$ or $\overset{\triangledown}{\varphi}$.

With any singularity $\overset{\triangledown}{\varphi} \in \mathrm{SING}_{\theta,\varpi}$ is associated a germ $\hat{\varphi}$, which is obtained from any major of $\overset{\triangledown}{\varphi}$ by the formula

$$\hat{\varphi}(\zeta) = \overset{\vee}{\varphi}(\zeta) - \overset{\vee}{\varphi}(\zeta\,\mathrm{e}^{-2\pi\mathrm{i}}), \qquad |\zeta| < 2\pi, \;\; \theta - \varpi < \arg\zeta < \theta + \varpi$$

and which is called the *minor*, or the variation, of $\overset{\triangledown}{\varphi}$, and denoted by

$$\hat{\varphi} = \mathrm{var}\,\overset{\triangledown}{\varphi}.$$

Elementary examples of singularities are

$$\delta^{(n)} = \mathrm{sing}\Big(\frac{(-1)^n n!}{2\pi\mathrm{i}\zeta^{n+1}}\Big), \qquad n \in \mathbb{N}, \qquad\qquad {}^\flat\hat{\varphi} = \mathrm{sing}\Big(\hat{\varphi}(\zeta)\frac{\log\zeta}{2\pi\mathrm{i}}\Big), \qquad \hat{\varphi} \in \mathbb{C}\{\zeta\}, \tag{76}$$

with $\mathrm{var}\,\delta^{(n)} = 0$ and $\mathrm{var}({}^\flat\hat{\varphi}) = \hat{\varphi}$. Although all the singularities we shall encounter in this article will be combinations of such elementary singularities, it is worth to have at one's disposal a general theory which does not even require, for instance, that the minor of a singularity be a regular germ. Observe that the kernel of var consists of the singularities represented by convergent Laurent expansions, which can thus be written $\sum_{n\ge0} a_n\delta^{(n)}$ with $\limsup(n!|a_n|)^{1/n} = 0$ (because the corresponding majors $\overset{\vee}{\varphi}$ must be single-valued and holomorphic in a punctured disc).

In the space of general singularities $\mathrm{SING}_{\theta,\varpi}$, one can define a *convolution product* which makes it a commutative algebra and which is an extension of the convolution product discussed in Section 3.1 in the sense that

$${}^\flat\hat{\varphi} * {}^\flat\hat{\psi} = {}^\flat(\hat{\varphi} * \hat{\psi}), \qquad \hat{\varphi}, \hat{\psi} \in \mathbb{C}\{\zeta\}.$$

The reader is referred e.g. to Section 2.4.1 of [OSS03] or Section 3 of [Sau05] for the definition of this convolution of singularities. Let us simply mention here that a major of $\overset{\triangledown}{\varphi} * \overset{\triangledown}{\psi}$ is obtained by considering, for any majors $\overset{\vee}{\varphi}$ and $\overset{\vee}{\psi}$, the analytic continuation of an integral of the form $\int_u^{\zeta + u\,e^{-i\pi}} \overset{\vee}{\varphi}(\zeta_1)\overset{\vee}{\psi}(\zeta-\zeta_1)\,d\zeta_1$ with a well-chosen auxiliary point u.

The notation $\delta^{(n)}$ in (76) is coherent with the notation $\delta^{(n)} = \mathcal{B}_{\text{ext}} z^n$ used in Section 3.1: the definition of the convolution of singularities is such that $\delta := \delta^{(0)}$ is the unit for convolution and

$$\delta^{(n)} * {}^\flat\hat{\varphi} = \hat{\varphi}(0)\delta^{(n-1)} + \hat{\varphi}'(0)\delta^{(n-2)} + \cdots + \hat{\varphi}^{(n-1)}(0)\delta + {}^\flat(\hat{\varphi}^{(n)}), \qquad n \in \mathbb{N}^*,\ \hat{\varphi} \in \mathbb{C}\{\zeta\},$$

which is the proper rewriting of formula (46) in the formalism of singularities. This means that, from now on, the extended Borel transform which was defined at the end of Section 3.1 will be better interpreted as the algebra isomorphism

$$\mathcal{B}_{\text{ext}} : \mathbb{C}((z^{-1}))_{\text{Gev}} \to \text{DP} \oplus {}^\flat\big(\mathbb{C}\{\zeta\}\big) \subset \text{SING}_{\theta,\varpi},$$
$$\mathcal{B}_{\text{ext}}\Big(\sum_{n\ge -v} a_n z^{-n}\Big) = a_0\delta + a_{-1}\delta' + \cdots + a_{-v}\delta^{(v)} + {}^\flat\Big(\sum_{n\ge 1} a_n \frac{\zeta^{n-1}}{(n-1)!}\Big)$$

(in fact, the definition of the Laplace transform too can be extended to certain singularities, so that $\delta^{(n)}$ and z^{-n} correspond to each other—see Section 3.2 of [Sau05]). More generally, $\delta^{(n)} * \text{sing}(\overset{\vee}{\varphi}) = \text{sing}(\overset{\vee}{\varphi}{}^{(n)})$ for any $\overset{\vee}{\varphi} \in \mathscr{M}_{\theta,\varpi}$.

The definition of the alien derivations $\Delta_{2\pi i}$ and $\Delta_{-2\pi i}$ given in Section 1.3 can also be extended. Given any $\overset{\triangledown}{\varphi} \in \text{SING}_{\pi/2,\varpi}$ such that $\hat{\varphi} = \text{var}\,\overset{\triangledown}{\varphi}$ extends analytically along $(0, 2\pi i)$ and $\hat{\varphi}(2\pi i + \zeta)$ defines an element of $\mathscr{M}_{\pi/2,\varpi}$, we set

$$\Delta_{2\pi i}\overset{\triangledown}{\varphi} = \text{sing}\,\big(\hat{\varphi}(2\pi i + \zeta)\big). \tag{77}$$

This is a generalization of Definition 1.7, which can be rephrased as follows: let $\tilde{\varphi} \in \widetilde{\text{RES}}^{(0)}$ and $\overset{\triangledown}{\varphi} = \mathcal{B}_{\text{ext}}\tilde{\varphi}$, then $\hat{\varphi} = \text{var}\,\overset{\triangledown}{\varphi}$ has a simply ramified singularity at $2\pi i$ if and only if $\Delta_{2\pi i}\overset{\triangledown}{\varphi} \in \text{DP} \oplus {}^\flat\big(\mathbb{C}\{\zeta\}\big)$ (and the connection between (16) and (77) is $\Delta_{2\pi i}\overset{\triangledown}{\varphi} = \mathcal{B}_{\text{ext}}\Delta_{2\pi i}\tilde{\varphi}$).

It turns out that the operator $\Delta_{2\pi i}$ thus extended still satisfies the Leibniz rule. One can extend similarly $\Delta_{-2\pi i}$ to a subspace of $\text{SING}_{-\pi/2,\varpi}$.

We shall content ourselves with a particular case of the previous situation: $\overset{\triangledown}{\text{RES}}{}^{(1)} = \text{var}^{-1}\big(\widehat{\text{RES}}^{(1)}\big)$ is a subalgebra of $\text{SING}_{\pm\pi/2,\varpi}$ and $\Delta_{\pm 2\pi i}$ is a well-defined operator from $\overset{\triangledown}{\text{RES}}{}^{(1)}$ to $\text{SING}_{\pm\pi/2,\varpi}$ which satisfies the Leibniz rule.

4.3 Proof of Theorem 1.10

a) Let us consider $\hat{\Phi}_n = \mathcal{B}\tilde{\Phi}_n$ as the minor of

$$\overset{\triangledown}{\Phi}_n = \mathcal{B}_{\text{ext}}\tilde{\Phi}_n \in \overset{\triangledown}{\text{RES}}{}^{(1)}$$

(thus $\overset{\triangledown}{\Phi}_0 = {}^\flat\hat{\Phi}_0$, but $\overset{\triangledown}{\Phi}_n$ contains a Dirac polynomial for $n \ge 1$). Let

$$\overset{\triangledown}{\chi}{}_n^{\pm} = \Delta_{\pm 2\pi i}\overset{\triangledown}{\Phi}_n \in \text{SING}_{\pm\pi/2,\varpi}\,.$$

The statement in Theorem 1.10 amounts to

(i) $\overset{\triangledown}{\chi}{}_n^{\pm} \in \mathrm{DP} \oplus {}^\flat\big(\mathbb{C}\{\zeta\}\big)$;

(ii) $\overset{\triangledown}{\chi}{}_n^{\pm} = \sum_{n_1+n_2=n} \big(A_{n_1}^{\pm} \overset{\triangledown}{\psi}_{1,n_2} + \mathrm{i}B_{n_1}^{\pm} \overset{\triangledown}{\psi}_{2,n_2}\big)$, where $\overset{\triangledown}{\psi}_{j,n} = \mathcal{B}_{\mathrm{ext}}\tilde{\psi}_{j,n}$, with certain families of constants $(A_n^{\pm}), (B_n^{\pm})$.

In fact, we shall prove directly (ii), which entails (i). The idea consists in studying equations satisfied by the $\overset{\triangledown}{\chi}{}_n^{\pm}$'s, which are derived from equations satisfied by the $\overset{\triangledown}{\Phi}_n$'s.

b) The equations satisfied by the $\overset{\triangledown}{\Phi}_n$'s will be obtained by applying $\mathcal{B}_{\mathrm{ext}}$ to the inner equations $(\mathbf{IE})_n$. The case $n = 0$ deserves special attention because the first inner equation is non-linear.

Suppose $\tilde{\varphi} \in z^{-1}\mathbb{C}[[z^{-1}]]$ with $\hat{\varphi} = \mathcal{B}\tilde{\varphi} \in \widehat{\mathrm{RES}}^{(1)}$ and let $\overset{\triangledown}{\varphi} = {}^\flat\hat{\varphi}$ (thus $\overset{\triangledown}{\varphi} = \mathcal{B}_{\mathrm{ext}}\hat{\varphi}$). Let $v(y) = \sum_{n\geq 1} v_n y^n \in \mathbb{C}\{y\}$ and $\tilde{\psi}(z) = v\big(\tilde{\varphi}(z)\big)$. The arguments of Section 3.3 show that $\hat{\psi} = \mathcal{B}\tilde{\psi} = \sum_{n\geq 1} v_n \hat{\varphi}^{*n} \in \widehat{\mathrm{RES}}^{(1)}$. Let us denote by $v^*(\overset{\triangledown}{\varphi})$ the singularity which is thus defined by ${}^\flat\hat{\psi} = \sum_{n\geq 1} v_n \overset{\triangledown}{\varphi}{}^{*n}$. One can check that

$$\Delta_{\pm 2\pi\mathrm{i}}\big(v^*(\overset{\triangledown}{\varphi})\big) = \big(\partial_y v^*(\overset{\triangledown}{\varphi})\big) * \Delta_{\pm 2\pi\mathrm{i}}\overset{\triangledown}{\varphi}. \tag{78}$$

Suppose now that $\tilde{\varphi} \in \mathbb{C}((z^{-1}))$ with $\hat{\varphi} = \mathcal{B}\tilde{\varphi} \in \widehat{\mathrm{RES}}^{(1)}$ and let $\overset{\triangledown}{\varphi} = \mathcal{B}_{\mathrm{ext}}\tilde{\varphi}$ (thus $\overset{\triangledown}{\varphi}$ may differ from ${}^\flat\hat{\varphi}$ by a Dirac polynomial). Then one can check that

$$\mathcal{B}_{\mathrm{ext}}\big(\tilde{\varphi}(z+1)\big) = \mathrm{e}^{-\zeta}\,\overset{\triangledown}{\varphi}, \qquad \Delta_{\pm 2\pi\mathrm{i}}\mathcal{B}_{\mathrm{ext}}\big(\tilde{\varphi}(z+1)\big) = \mathrm{e}^{-\zeta}\,\Delta_{\pm 2\pi\mathrm{i}}\overset{\triangledown}{\varphi}. \tag{79}$$

Here we use the fact that $\mathrm{SING}_{\theta,\varpi}$ is a $\mathbb{C}\{\zeta\}$-module: the multiplication of a singularity by a regular germ $\alpha(\zeta)$ is defined from the product of any major representing the singularity with $\alpha(\zeta)$; the second identity in (79) follows from the fact that $\alpha(\zeta) = \mathrm{e}^{-\zeta}$ is entire and satisfies $\alpha(\pm 2\pi\mathrm{i}) = 1$. Of course, we have $\mathcal{B}_{\mathrm{ext}}\big(\tilde{\varphi}(z-1)\big) = \mathrm{e}^{\zeta}\,\overset{\triangledown}{\varphi}$ and $\Delta_{\pm 2\pi\mathrm{i}}(\mathrm{e}^{\zeta}\,\overset{\triangledown}{\varphi}) = \mathrm{e}^{\zeta}\,\Delta_{\pm 2\pi\mathrm{i}}\overset{\triangledown}{\varphi}$ in a similar way.

After these preliminaries, we can apply $\mathcal{B}_{\mathrm{ext}}$ to the first inner equation: $(\mathbf{IE})_0$ yields

$$(\mathrm{e}^{-\zeta} + \mathrm{e}^{\zeta})\overset{\triangledown}{\Phi}_0 = \mathcal{F}_0^*\big(\overset{\triangledown}{\Phi}_0\big), \tag*{$(\overset{\triangledown}{\mathbf{IE}})_0$}$$

to which we apply $\Delta_{\pm 2\pi\mathrm{i}}$, getting

$$(\mathrm{e}^{-\zeta} + \mathrm{e}^{\zeta})\overset{\triangledown}{\chi}{}_0^{\pm} = \partial_y\mathcal{F}_0^*\big(\overset{\triangledown}{\Phi}_0\big) * \overset{\triangledown}{\chi}{}_0^{\pm} \tag{80}$$

by virtue of (78) and (79). Notice that the equation (80) satisfied by $\overset{\triangledown}{\chi}{}_0^{\pm}$ is the counterpart in $\mathrm{SING}_{\pm\pi/2,\varpi}$ via $\mathcal{B}_{\mathrm{ext}}$ of the variational equation associated with $(\mathbf{IE})_0$, namely

$$\psi(z+1) + \psi(z-1) = \partial_y\mathcal{F}_0\big(\tilde{\Phi}_0(z)\big)\psi(z). \tag{81}$$

Equation (81) is a priori given for an unknown $\psi \in \mathbb{C}((z^{-1}))$, while equation (80) makes sense for an unknown in $\mathrm{SING}_{\pm\pi/2,\varpi}$. It is easy to see that the set of solutions of (81) is the linear span of $(\tilde{\Psi}_{1,0}, \tilde{\Psi}_{2,0})$ and is thus contained in $\mathbb{C}((z^{-1}))_{\mathrm{Gev}}$. In the rest of this section, we shall see that, although $\mathrm{SING}_{\pm\pi/2,\varpi}$ is a much larger space than $\mathcal{B}_{\mathrm{ext}}\big(\mathbb{C}((z^{-1}))_{\mathrm{Gev}}\big)$, the set of solutions of (80) is the linear span of $(\mathcal{B}_{\mathrm{ext}}\tilde{\Psi}_{1,0}, \mathcal{B}_{\mathrm{ext}}\tilde{\Psi}_{2,0})$ and is thus contained in $\mathrm{DP} \oplus {}^\flat\big(\mathbb{C}\{\zeta\}\big)$. This will be done by mimicking the arguments of Appendix A.2 for the theory of linear second-order difference equations.

c) Similarly to $(\overset{\triangledown}{\mathbf{IE}})_0$, which is the counterpart in $\overset{\triangledown}{\mathrm{RES}}{}^{(1)}$ of $(\mathbf{IE})_0$, there are equations $(\overset{\triangledown}{\mathbf{IE}})_n$ which are the counterparts of equations $(\mathbf{IE})_n$ and are satisfied by the $\overset{\triangledown}{\Phi}_n$'s. We shall content ourselves with writing one equation in $\overset{\triangledown}{\mathrm{RES}}{}^{(1)}[[h^2]]$; the equations $(\overset{\triangledown}{\mathbf{IE}})_n$ can be obtained by expanding it in powers of h^2.

Let $\overset{\triangledown}{\Phi} = \sum_{n\geq 0} h^{2n}\overset{\triangledown}{\Phi}_n \in \overset{\triangledown}{\mathrm{RES}}{}^{(1)}[[h^2]]$. According to (5), we had $\mathcal{F}(y,h) = \sum_{n\geq 0} h^{2n}\mathcal{F}_n(y) \in y\mathbb{C}\{y\}[[h^2]]$, we can thus define

$$\mathcal{F}^*\big(\overset{\triangledown}{\Phi},h\big) = \sum_{\substack{n_0\geq 0, r\geq 0\\ n_1,\dots,n_r\geq 1}} \frac{1}{r!} h^{2(n_0+n_1+\cdots+n_r)} \partial_y^r \mathcal{F}^*_{n_0}\big(\overset{\triangledown}{\Phi}_0\big) * \overset{\triangledown}{\Phi}_{n_1} * \cdots * \overset{\triangledown}{\Phi}_{n_r}.$$

By applying $\mathcal{B}_{\mathrm{ext}}$ to $(\mathbf{FIE})$, we get

$$(\mathrm{e}^{-\zeta}+\mathrm{e}^{\zeta})\overset{\triangledown}{\Phi} = \mathcal{F}^*\big(\overset{\triangledown}{\Phi},h\big). \qquad (\overset{\triangledown}{\mathbf{FIE}})$$

We now apply $\Delta_{\pm 2\pi\mathrm{i}}$ to equation $(\overset{\triangledown}{\mathbf{FIE}})$ and get an equation for $\overset{\triangledown}{\chi}{}^{\pm} = \sum_{n\geq 0} h^{2n}\overset{\triangledown}{\chi}{}^{\pm}_n \in \mathrm{SING}_{\pm\pi/2,\varpi}[[h^2]]$:

$$(\mathrm{e}^{-\zeta}+\mathrm{e}^{\zeta})\overset{\triangledown}{\chi}{}^{\pm} = \partial_y\mathcal{F}^*\big(\overset{\triangledown}{\Phi},h\big) * \overset{\triangledown}{\chi}{}^{\pm}. \qquad (\overset{\triangledown}{\mathbf{FL}})_b$$

By expanding $(\overset{\triangledown}{\mathbf{FL}})_b$ in powers of h^2,we would get a system of equations satisfied by the $\overset{\triangledown}{\chi}_n$'s, the first of which is (80).

The above point (ii) will follow from

Lemma 4.2. *Let*

$$\overset{\triangledown}{\Psi}_j = \sum_{n\geq 0} h^{2n}\mathcal{B}_{ext}\tilde{\Psi}_{j,n} \in \overset{\triangledown}{\mathrm{RES}}{}^{(1)}[[h^2]], \qquad j=1,2$$

and suppose that $\overset{\triangledown}{\chi} \in \mathrm{SING}_{\pm\pi/2,\varpi}[[h^2]]$ *satisfies*

$$(\mathrm{e}^{-\zeta}+\mathrm{e}^{\zeta})\overset{\triangledown}{\chi} = \partial_y\mathcal{F}^*\big(\overset{\triangledown}{\Phi},h\big) * \overset{\triangledown}{\chi}. \tag{82}$$

Then there exist $A,B\in\mathbb{C}[[h^2]]$ *such that* $\overset{\triangledown}{\chi} = A\overset{\triangledown}{\Psi}_1 + \mathrm{i}B\overset{\triangledown}{\Psi}_2$.

Proof. Observe that $\overset{\triangledown}{\Psi}_1, \overset{\triangledown}{\Psi}_2 \in \overset{\triangledown}{\mathrm{RES}}{}^{(1)}[[h^2]]$ because of (74). Since $(\tilde{\Psi}_1,\tilde{\Psi}_2)$ is a normalized fundamental system of solutions of $(\mathbf{FL})_b$, we obtain (by applying $\mathcal{B}_{\mathrm{ext}}$) that $\overset{\triangledown}{\Psi}_1$ and $\overset{\triangledown}{\Psi}_2$ are particular solutions of (82) and that

$$\overset{\triangledown}{\Psi}_1 * T\overset{\triangledown}{\Psi}_2 - \big(T\overset{\triangledown}{\Psi}_1\big) * \overset{\triangledown}{\Psi}_2 = \delta,$$

if we denote by T the operator of $\mathrm{SING}_{\pm\pi/2,\varpi}[[h^2]]$ defined by $T\overset{\triangledown}{\chi} = \mathrm{e}^{-\zeta}\,\overset{\triangledown}{\chi}$. One can check that T is an automorphism of the convolution algebra $\mathrm{SING}_{\pm\pi/2,\varpi}[[h^2]]$ (because of the definition of the convolution and of the property $\mathrm{e}^{-\zeta_1-\zeta_2} = \mathrm{e}^{-\zeta_1}\,\mathrm{e}^{-\zeta_2}$); this implies that the proof of Lemma A.2 is valid in this space. As a consequence, if we set

$$\overset{\triangledown}{A} = \overset{\triangledown}{\chi} * T\overset{\triangledown}{\Psi}_2 - (T\overset{\triangledown}{\chi}) * \overset{\triangledown}{\Psi}_2, \quad \mathrm{i}\overset{\triangledown}{B} = \overset{\triangledown}{\Psi}_1 * T\overset{\triangledown}{\chi} - (T\overset{\triangledown}{\Psi}_1) * \overset{\triangledown}{\chi}, \tag{83}$$

then we get

$$\overset{\triangledown}{\chi} = \overset{\triangledown}{A} * \overset{\triangledown}{\Psi}_1 + \mathrm{i}\overset{\triangledown}{B} * \overset{\triangledown}{\Psi}_2, \qquad \overset{\triangledown}{A} - T^{-1}\overset{\triangledown}{A} = \mathrm{i}\overset{\triangledown}{B} - T^{-1}(\mathrm{i}\overset{\triangledown}{B}) = 0. \tag{84}$$

But the last equation means that any major $\overset{\vee}{B}$ of $\overset{\triangledown}{B}$ satisfies $(1-\mathrm{e}^{\zeta})\overset{\vee}{B}(\zeta,h) = R(\zeta,h) \in \mathbb{C}\{\zeta\}[[h^2]]$, hence $\overset{\triangledown}{B} = -2\pi\mathrm{i}R(0,h)\delta$; similarly, $\overset{\triangledown}{A}$ too is proportional to δ. □

Applying this lemma to $\overset{\triangledown}{\chi}{}^{\pm}$, we get $A^{\pm} = \sum_{n\geq 0} A_n^{\pm} h^{2n}$ and $B^{\pm} = \sum_{n\geq 0} B_n^{\pm} h^{2n}$ defined by (83) and such that $\overset{\triangledown}{\chi}{}^{\pm} = A^{\pm}\overset{\triangledown}{\Psi}_1 + \mathrm{i}B^{\pm}\overset{\triangledown}{\Psi}_2$. By expanding this relation in powers of h^2, we get the desired formula for the $\overset{\triangledown}{\chi}{}_n^{\pm}$'s.

d) According to Theorem 1.4, the minors $\hat{\Phi}_n(\zeta)$ depend polynomially on $b_1,\dots,b_n$ and analytically on ε for $|\varepsilon| < \varepsilon_0$; one can thus choose majors $\overset{\vee}{\chi}{}_n^{\pm}$ with the same dependence on ε and b. The same is true for the singularities $\overset{\triangledown}{\Psi}_{1,n}, \overset{\triangledown}{\Psi}_{2,n}$, as is easily checked from the construction of $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ in Section 4.1. The definition of the coefficients $A_n^{\pm}, B_n^{\pm}$ via formula (83) then shows that they are polynomials in b which depend analytically on ε (by regularity of a convolution product with respect to parameters). Their vanishing for $\varepsilon = 0$ follows from Proposition 1.2.

Only formulas (21)–(22) remain to be proved. For this, we use again the notations of Section 3.2 and consider expansions at first order in ε: the proof of Proposition 3.2 shows that

$$\tilde{\Phi}_0 = \Phi_{0,0}(z) + \varepsilon\tilde{\varphi}(z) + O(\varepsilon^2), \qquad \Phi_{0,0}(z) = -\mathrm{i}z^{-1}, \quad \tilde{\varphi} = \mathcal{L}_{0,0}^{-1}W_{0,0},$$
$$W_{0,0}(z) = V'\big(\Phi_{0,0}(z),0,0\big)$$

with the operator $\mathcal{L}_{0,0}^{-1}$ of (51) (the convergent series $W_{0,0}(z)$ is the series $w_0(z,0)$ of (48)).

We can thus write

$$\tilde{\varphi} = \tilde{\alpha}\,\psi_{1,0} + \tilde{\beta}\,\psi_{2,0}, \qquad \tilde{\alpha} = -\Delta^{-1}(W_{0,0}\psi_{2,0}), \quad \tilde{\beta} = \Delta^{-1}(W_{0,0}\psi_{1,0}),$$

where $\psi_{1,0} = \partial_z\Phi_{0,0} = \mathrm{i}z^{-2}$ is nothing but $\big(\tilde{\Psi}_{1,0}\big)_{|\varepsilon=0}$ and

$$\psi_{2,0}(z) = \left(-\frac{1}{5}z^5 + \frac{1}{3}z^3 - \frac{2}{15}z\right)\psi_{1,0}(z)$$

coincides with $\big(\tilde{\Psi}_{2,0}\big)_{|\varepsilon=0}$. Since the alien derivations are derivations which annihilate the convergent series, we get

$$\Delta_{\pm 2\pi\mathrm{i}}\tilde{\varphi} = (\Delta_{\pm 2\pi\mathrm{i}}\tilde{\alpha})\psi_{1,0} + (\Delta_{\pm 2\pi\mathrm{i}}\tilde{\beta})\psi_{2,0},$$

and we know in advance that the alien derivatives $\Delta_{\pm 2\pi\mathrm{i}}\tilde{\alpha}$ and $\Delta_{\pm 2\pi\mathrm{i}}\tilde{\beta}$ boil down to complex numbers

$$\Delta_{\pm 2\pi\mathrm{i}}\tilde{\alpha} = A_{0,1}^{\pm}, \quad \Delta_{\pm 2\pi\mathrm{i}}\tilde{\beta} = \mathrm{i}B_{0,1}^{\pm},$$

according to (20), since $\Delta_{\pm 2\pi\mathrm{i}}\tilde{\Phi}_0 = A_0^{\pm}\tilde{\Psi}_{1,0} + \mathrm{i}B_0^{\pm}\tilde{\Psi}_{2,0} = \varepsilon\Delta_{\pm 2\pi\mathrm{i}}\tilde{\varphi} + O(\varepsilon^2)$.

Indeed, in view of (59), the minors $\hat{\alpha}(\zeta)$ and $\hat{\beta}(\zeta)$ are meromorphic in $\mathbb{C}$, with simple poles only:

$$\hat{\alpha} = -J\mathcal{B}(W_{0,0}\psi_{2,0}), \quad \hat{\beta} = J\mathcal{B}(W_{0,0}\psi_{1,0}),$$

with $J(\zeta) = (\mathrm{e}^{-\zeta} - 1)^{-1}$. Therefore,

$$A^{\pm}_{0,1} = 2\pi \mathrm{i}\mathcal{B}(W_{0,0}\psi_{2,0})_{|\zeta=\pm 2\pi\mathrm{i}}, \quad \mathrm{i}B^{\pm}_{0,1} = -2\pi\mathrm{i}\mathcal{B}(W_{0,0}\psi_{1,0})_{|\zeta=\pm 2\pi\mathrm{i}}.$$

We compute $W_{0,0}\psi_{1,0} = \Phi'_{0,0}(z)V'\big(\Phi_{0,0}(z),0,0\big) = \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}z}$ with $\mathcal{V}(z) = V_0(-\mathrm{i}z^{-1})$, hence $\beta\mathcal{V}(\zeta) = -\mathrm{i}\hat{V}_0(-\mathrm{i}\zeta)$ and $\beta(W_{0,0}\psi_{1,0}) = \mathrm{i}\zeta\hat{V}_0(-\mathrm{i}\zeta)$, which gives the value

$$\frac{1}{2\pi}B^{\pm}_{0,1} = \pm 2\pi\hat{V}_0(\pm 2\pi).$$

On the other hand, $W_{0,0}\psi_{2,0} = \big(-\frac{1}{5}z^5 + \frac{1}{3}z^3 - \frac{2}{15}z\big)\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}z}$ yields

$$\beta(W_{0,0}\psi_{2,0}) = \left(-\frac{1}{5}\partial_\zeta^5 + \frac{1}{3}\partial_\zeta^3 - \frac{2}{15}\partial_\zeta\right)\big(-\zeta\beta\mathcal{V}(\zeta)\big),$$

whence $\frac{1}{2\pi}A^{\pm}_{0,1} = D\hat{V}_0(\pm 2\pi)$.

A Appendix

A.1 The subspaces $z^{-v}\mathbb{C}[[z^{-1},(hz)^2]]$ of $\mathbb{C}((z^{-1}))[[h^2]]$

Lemma A.1. *For any $v \in \mathbb{Z}$, $z^{-v}\mathbb{C}[[z^{-1},(hz)^2]]$ can be identified with the subspace of $\mathbb{C}((z^{-1}))[[h^2]]$ which consists of all the series of the form $\sum_{n\geq 0} h^{2n}\varphi_n(z)$ with $\varphi_n(z) \in z^{2n-v}\mathbb{C}[[z^{-1}]]$.*

Proof. The general element of $z^{-v}\mathbb{C}[[z^{-1},(hz)^2]]$ is $z^{-v}\sum_{m,n\geq 0} a_{m,n}z^{-m}(hz)^{2n}$ with arbitrary coefficients $a_{m,n}$, and it can be rewritten $\sum_{n\geq 0} h^{2n}\varphi_n(z)$, where $\varphi_n(z) = \sum_{m\geq 0} a_{m,n}z^{2n-v-m}$ is the general element of $z^{2n-v}\mathbb{C}[[z^{-1}]]$. □

It follows that

$$\partial_z^p\left(z^{-v}\mathbb{C}[[z^{-1},(hz)^2]]\right) \subset z^{-v-p}\mathbb{C}[[z^{-1},(hz)^2]]$$

for any $p \in \mathbb{N}$. In particular $z^{-v}\mathbb{C}[[z^{-1},(hz)^2]]$ is stable under the operator

$$T \;:\; \phi(z,h) \mapsto \phi(z+1,h) = \sum_{p\geq 0}\frac{1}{p!}\partial_z^p\phi(z,h),$$

and, since $\mathbb{C}[[z^{-1},(hz)^2]]$ is a ring, it is easy to follow the value of v when we apply difference operators or multiplication to our formal series. For instance,

$$A \in \mathbb{C}[[h^2]],\ \phi \in z^{-v}\mathbb{C}[[z^{-1},(hz)^2]] \;\Rightarrow\; A(h)\cdot\phi(z,h) \in z^{-v}\mathbb{C}[[z^{-1},(hz)^2]],$$

since $\mathbb{C}[[h^2]] \subset \mathbb{C}[[z^{-1},(hz)^2]]$ (because $h^{2n} = z^{-2n}(hz)^{2n}$).

One has similar properties for the spaces $z^{-v}\mathbb{C}[[z^{-1},h^2z^4]]$, to be identified with $\{\sum h^{2n}\varphi_n(z) \in \mathbb{C}((z^{-1}))[[h^2]] \mid \forall n \geq 0,\ \varphi_n(z) \in z^{4n-v}\mathbb{C}[[z^{-1}]]\}$. This is used in Section 4.1.

A.2 Elementary theory of second-order linear difference equations

We gather here the proof of a few classical facts which were stated in Section 2.1. In addition to the operators T and T^{-1}, which are automorphisms of the field $\mathbb{C}((z^{-1}))$, and to the difference operators Δ and P defined by (31), we introduce

$$\nabla = I - T^{-1} = T^{-1}\Delta. \tag{85}$$

We consider a linear difference operator $\psi \mapsto \mathcal{L}_0\psi = P\psi - A\psi$ as in Section 2.1 and assume that ψ_1 and ψ_2 are two solutions of the homogeneous equation $\mathcal{L}_0\psi = 0$.

The verification of point (i) of Section 2.1 is immediate: $W = \psi_1(T\psi_2) - (T\psi_1)\psi_2$ satisfies

$$\nabla W = \psi_1(T\psi_2) - (T\psi_1)\psi_2 - (T^{-1}\psi_1)\psi_2 + \psi_1(T^{-1}\psi_2) = \psi_1(P\psi_2) - (P\psi_1)\psi_2,$$

hence $P\psi_j = A\psi_j$ implies $\nabla W = 0$, or $\Delta W = 0$. We have moreover

Lemma A.2. *Suppose ψ_1 and ψ_2 are solutions of $\mathcal{L}_0$ with $\mathcal{W}(\psi_1, \psi_2) \equiv 1$. Consider an arbitrary ψ and let $a = \mathcal{W}(\psi, \psi_2)$, $b = \mathcal{W}(\psi_1, \psi)$. Then*

$$\psi = a\psi_1 + b\psi_2 = (T^{-1}a)\psi_1 + (T^{-1}b)\psi_2, \tag{86}$$

$$\mathcal{L}_0\psi = -(\nabla a)T^{-1}\psi_1 - (\nabla b)T^{-1}\psi_2, \tag{87}$$

$$\nabla a = -\psi_2\mathcal{L}_0\psi, \quad \nabla b = \psi_1\mathcal{L}_0\psi. \tag{88}$$

Proof. The relations (86) are obtained by solving the linear system (which has determinant 1)

$$\begin{cases} (T\psi_2)\psi - \psi_2(T\psi) = a \\ -(T\psi_1)\psi + \psi_1(T\psi) = b. \end{cases}$$

We now compute $P\psi$ by using $T\psi = aT\psi_1 + bT\psi_2$ and $T^{-1}\psi = (T^{-1}a)T^{-1}\psi_1 + (T^{-1}b)T^{-1}\psi_2$:

$$P\psi = aT\psi_1 + bT\psi_2 - 2a\psi_1 - 2b\psi_2 + (T^{-1}a)T^{-1}\psi_1 + (T^{-1}b)T^{-1}\psi_2,$$

hence $P\psi = aP\psi_1 + bP\psi_2 - (\nabla a)T^{-1}\psi_1 - (\nabla b)T^{-1}\psi_2$ and (87) follows. Finally, the linear system with determinant 1

$$\begin{cases} (T^{-1}\psi_1)\nabla a + (T^{-1}\psi_2)\nabla b = -\mathcal{L}_0\psi \\ \psi_1\nabla a + \psi_2\nabla b = 0 \end{cases}$$

(where the first equation was just proved and the second is a consequence of (86)) implies (88). ∎

This lemma immediately yields the description of the solutions of $\mathcal{L}_0$ of point (ii) of Section 2.1.

In order to check point (iii), we give ourselves f, a^* and b^* such that $\Delta a^* = -\psi_2 f$ and $\Delta b^* = \psi_1 f$, and set $\psi = a^*\psi_1 + b^*\psi_2$. Let $a = \mathcal{W}(\psi, \psi_2)$ and $b = \mathcal{W}(\psi_1, \psi)$. We have

$$(Ta^*)\psi_1 + (Tb^*)\psi_2 = (\Delta a^*)\psi_1 + (\Delta b^*)\psi_2 + \psi = \psi.$$

We can thus write a linear system with determinant 1

$$\begin{cases} (T^{-1}\psi_1)a^* + (T^{-1}\psi_2)b^* = T^{-1}\psi \\ \psi_1 a^* + \psi_2 b^* = \psi \end{cases}$$

which shows that $a^* = T^{-1}a$ and $b^* = T^{-1}b$, whence $\nabla a = -\psi_2 f$ and $\nabla b = \psi_1 f$, and the conclusion follows from (87). (The variant of footnote 2 is obtained similarly by checking that $(T^{-1}a)\psi_1 + (T^{-1}b)\psi_2 = \psi$ in that case.)

As for point (iv), we suppose $P\psi_1 = A\psi_1$ with $\psi_1 T\psi_1$ invertible and consider an arbitrary c. As straightforward computation shows that $\mathcal{W}(\psi_1, c\psi_1) = (\Delta c)\psi_1 T\psi_1$. The conclusion follows from the computation

$$P(c\psi_1) = (Tc)T\psi_1 - 2c\psi_1 + (T^{-1}c)T^{-1}\psi_1 = (\Delta c)T\psi_1 + cP\psi_1 - (\nabla c)T^{-1}\psi_1$$

$$\Rightarrow\ \mathcal{L}_0(c\psi_1) = \frac{1}{\psi_1}\Delta\big((\Delta c)\psi_1 T\psi_1\big).$$

A.3 Proof of Proposition 1.6

In this appendix we fix a complex number z_0 in the domain $\mathcal{D}^u_{n_0}$ of Section 1.2 and a complex number ε with $|\varepsilon| < \varepsilon_0$. We shall treat only the question of the uniqueness of the solutions on $z_0 + \mathbb{R}^-$ (the case of $z_0 + \mathbb{R}^+$ is similar).

A.3.1 Preliminaries

For every $\sigma \geq 0$ and $\ell > 0$, the space $\mathbb{B}^u_{\sigma,\ell}$ will consist of all the complex-valued functions which are defined on the half-line $(z_0 - \ell^2) + \mathbb{R}^-$ and which are $O\big(|z|^{-\sigma}\big)$ (without any extra regularity requirement). This is a Banach space for the norm

$$\|\varphi\|_{\sigma,\ell} = \sup_{t \geq \ell^2} \big|(t+\ell)^\sigma \varphi(z_0 - t)\big|. \tag{89}$$

Observe that $z^{-\sigma'} \in \mathbb{B}^u_{\sigma,\ell}$ for every $\sigma' \geq \sigma$, because the condition $z_0 \in \mathcal{D}^u_{n_0}$ prevents z from vanishing on the half-line $z_0 + \mathbb{R}^-$, which contains $(z_0 - \ell^2) + \mathbb{R}^-$. Our motivation for the definition (89) is the fact that

$$\lim_{\ell\to\infty} \|z^{-\sigma}\|_{\sigma,\ell} = 1, \qquad \lim_{\ell\to\infty} \|z^{-\sigma'}\|_{\sigma,\ell} = 0 \ \text{ if } \ \sigma' > \sigma. \tag{90}$$

(*Proof:* We have

$$\|z^{-\sigma'}\|_{\sigma,\ell} = \sup_{t\geq \ell^2} \frac{(t+\ell)^\sigma}{|z_0 - t|^{\sigma'}}, \qquad \sigma' \geq \sigma.$$

It is thus sufficient to check (90) for $\sigma' = 2$ and $\sigma \leq 2$. We can write $|z_0 - t|^2 = (t - t_0)^2 + d^2$ with $d = |\Im m\, z_0|$ and $t_0 = \Re e\, z_0$. The function $t \mapsto \frac{(t+\ell)^\sigma}{(t-t_0)^2 + d^2}$ is easily seen to be decreasing for $t \geq T_+(\ell)$, where $T_+(\ell) \xrightarrow[\ell\to\infty]{} t_0$, thus $\|z^{-2}\|_{\sigma,\ell} = \frac{(\ell^2+\ell)^\sigma}{(\ell^2 - t_0)^2 + d^2}$ for ℓ large enough, whence the result follows.)

We now introduce right inverses for the operators Δ and ∇ defined by (31) and (85); by composition, we shall obtain a right inverse for $P = \Delta \circ \nabla = \nabla \circ \Delta$.

Lemma A.3. *Suppose $\sigma, \ell > 0$. Then the formulas*

$$\Delta_u^{-1}\varphi(z) = \sum_{n\geq 1}\varphi(z-n), \quad \nabla_u^{-1}\varphi(z) = \sum_{n\geq 0}\varphi(z-n)$$

define two linear operators $\Delta_u^{-1}, \nabla_u^{-1} : \mathbb{B}^u_{\sigma+1,\ell} \to \mathbb{B}^u_{\sigma,\ell}$ which are right inverses[5] of Δ and ∇ and which satisfy

$$\|\Delta_u^{-1}\varphi\|_{\sigma,\ell} \leq \tfrac{1}{\sigma}\|\varphi\|_{\sigma+1,\ell}, \quad \|\nabla_u^{-1}\varphi\|_{\sigma,\ell} \leq \left(\tfrac{1}{\sigma}+\tfrac{1}{\ell}\right)\|\varphi\|_{\sigma+1,\ell}$$

for every $\varphi \in \mathbb{B}^u_{\sigma+1,\ell}$.

Proof. Let $\varphi \in \mathbb{B}^u_{\sigma+1,\ell}$. The series which define $\Delta_u^{-1}\varphi(z)$ and $\nabla_u^{-1}\varphi(z)$ are absolutely convergent for $z = z_0 - t$ with $t \geq \ell^2$, because $|\varphi(z-n)| = |\varphi(z_0-n-t)| \leq \|\varphi\|_{\sigma+1,\ell}\,(n+t+\ell)^{-\sigma-1}$. Moreover,

$$\left|(t+\ell)^{\sigma}\,\Delta_u^{-1}\varphi(z_0-t)\right| \leq \|\varphi\|_{\sigma+1,\ell}\sum_{n\geq 1}\tfrac{x^{\sigma}}{(n+x)^{\sigma+1}}$$

with $x = t+\ell$, and there is a similar inequality for $\left|(t+\ell)^{\sigma}\,\nabla_u^{-1}\varphi(z_0-t)\right|$ but with a sum starting at $n=0$. Using $\sum_{n\geq 1}\frac{1}{(n+x)^{\sigma+1}} \leq \int_x^{+\infty}\frac{\mathrm{d}t}{t^{\sigma+1}} = \frac{1}{\sigma x^{\sigma}}$ and $\frac{1}{x} \leq \frac{1}{\ell}$, we get the desired inequality by passing to the supremum. ☐

Corollary A.4. *Suppose $\sigma, \ell > 0$. Then the formula*

$$P_u^{-1}\varphi(z) = \sum_{n\geq 1} n\,\varphi(z-n)$$

defines a linear operator $P_u^{-1} : \mathbb{B}^u_{\sigma+2,\ell} \to \mathbb{B}^u_{\sigma,\ell}$ such that, for every $\varphi \in \mathbb{B}^u_{\sigma+2,\ell}$,

$$\|P_u^{-1}\varphi\|_{\sigma,\ell} \leq \tfrac{1}{\sigma+1}\left(\tfrac{1}{\sigma}+\tfrac{1}{\ell}\right)\|\varphi\|_{\sigma+2,\ell}$$

and the function $\psi = P_u^{-1}\varphi$ is the only function defined on $z_0 - \ell^2 + \mathbb{R}^-$ which tends to 0 at infinity with $P\psi = \varphi$ on $z_0 - 1 - \ell^2 + \mathbb{R}^-$.

Proof. Observe that $P_u^{-1} = \nabla_u^{-1} \circ \Delta_u^{-1}$ with $\Delta_u^{-1} : \mathbb{B}^u_{\sigma+2,\ell} \to \mathbb{B}^u_{\sigma+1,\ell}$ and $\nabla_u^{-1} : \mathbb{B}^u_{\sigma+1,\ell} \to \mathbb{B}^u_{\sigma,\ell}$, and apply Lemma A.3. ☐

A.3.2 Case of the inner equation

We now begin the proof of Proposition 1.6 by considering the case $n_0 = 0$: we suppose that $\phi(z)$ and $\phi^*(z)$ are two solutions of $(\mathbf{IE})_0$ on $z_0 + \mathbb{R}^-$ of the form $-\mathrm{i}z^{-1} + O\big(|z|^{-\sigma}\big)$, with $2 < \sigma \leq 3$, and we wish to prove that $\phi = \phi^*$ on $z_0 + \mathbb{R}^-$.

Equation $(\mathbf{IE})_0$ can be written $P\phi = \mathcal{F}_1(\phi)$, with $\mathcal{F}_1(y) = \mathcal{F}(y,0,\varepsilon) - 2y = -2y^3 + O(y^5)$ and $\partial_y\mathcal{F}_1(y) = -6y^2 + O(y^4)$. Let

$$A^*(z) = \int_0^1 \partial_y\mathcal{F}_1\big((1-t)\phi(z) + t\phi^*(z)\big)\,\mathrm{d}t = 6z^{-2} + O\big(|z|^{-\sigma-1}\big). \tag{91}$$

[5] In the case of Δ_u^{-1}, we mean that $\psi = \Delta_u^{-1}\varphi$ satisfies $\Delta\psi = \varphi$ on $z_0 - 1 - \ell^2 + \mathbb{R}^-$ only (as a matter of fact, this identity holds also on the whole half-line $z_0 - \ell^2 + \mathbb{R}^-$, but this is because ψ has an extension to the segment $[z_0, z_0+1]$).

This function A^* is defined on $z_0 + \mathbb{R}^-$ and the function $\psi = \phi^* - \phi$ satisfies

$$P\psi = A^*\psi. \tag{92}$$

For every $\ell > 0$, we have $A^* \in \mathbb{B}_{2,\ell}$ and $\psi \in \mathbb{B}_{\sigma,\ell}$. Consequently, $A^*\psi \in \mathbb{B}_{\sigma+2,\ell}$ with $\|A^*\psi\|_{\sigma+2,\ell} \le \|A^*\|_{2,\ell}\,\|\psi\|_{\sigma,\ell}$, and Corollary A.4 shows that $\psi = P_u^{-1}(A^*\psi)$, hence

$$\|\psi\|_{\sigma,\ell} \le \tfrac{1}{\sigma+1}\left(\tfrac{1}{\sigma} + \tfrac{1}{\ell}\right)\|A^*\|_{2,\ell}\,\|\psi\|_{\sigma,\ell}.$$

By virtue of (90) and (91), we have $\|A^*\|_{2,\ell} \xrightarrow[\ell\to\infty]{} 6$, while the limit of $\frac{1}{\sigma+1}\left(\frac{1}{\sigma} + \frac{1}{\ell}\right)$ is $\frac{1}{\sigma(\sigma+1)} < \frac{1}{6}$, hence $\|\psi\|_{\sigma,\ell} = 0$ for ℓ large enough.

Knowing that $\psi = 0$ on $z_0 - \ell^2 + \mathbb{R}^-$ for a particular $\ell > 0$, we conclude that $\psi = 0$ on the whole of $z_0 + \mathbb{R}^-$ by iterating equation (92) in the form $\psi(z+1) = 2\psi(z) - \psi(z-1) + A^*(z)\psi(z)$.

Therefore $(\mathbf{IE})_0$ admits at most one solution of the form $-\mathrm{i}z^{-1} + O\big(|z|^{-\sigma}\big)$ and we know that $\Phi^u(z;\varepsilon)$ is such a solution (with $\sigma \le 3$).

A.3.3 Case of the secondary inner equations

Let $\sigma \in (2,3]$, $n_0 \ge 1$, $b_1, \dots, b_{n_0} \in \mathbb{C}$ and $\phi_n(z) = \Phi^u_n(z,\varepsilon;b_1,\dots,b_n)$ for $0 \le n \le n_0$. We already know by Corollary 1.5 that these functions solve $(\mathbf{IE})_0, (\mathbf{IE})_1, \dots, (\mathbf{IE})_{n_0}$ and that they are defined on $z_0 + \mathbb{R}^-$ and of the form $\big[\tilde\Phi_n(z,\varepsilon;b_1,\dots,b_n)\big]_{\le 2} + O\big(|z|^{-\sigma}\big)$. To conclude the proof of Proposition 1.6, it is thus sufficient to prove that any solution $\phi^*_{n_0}$ of $(\mathbf{IE})_{n_0}$ on $z_0+\mathbb{R}^-$ which is such that $\phi^*_{n_0} - \phi_{n_0} = O\big(|z|^{-\sigma}\big)$ must coincide with ϕ_{n_0}.

Equation $(\mathbf{IE})_{n_0}$ can be written $P\phi(z) - A^u(z)\phi(z) = f^u_{n_0}(z)$ with $A^u(z) = \partial_y\mathcal{F}\big(\Phi^u_0(z,\varepsilon),0,\varepsilon\big) - 2$ and $f^u_{n_0}$ determined according to formula (7). The function $\psi = \phi^*_{n_0} - \phi_{n_0}$ is thus a solution of the linear difference operator $\mathcal{L}^u_0$ defined by $\mathcal{L}^u_0\psi = P\psi - A^u\psi$.

Since $\Phi^u_0 = \mathcal{S}^-\tilde\Phi_0$, the classical properties of the Borel-Laplace summation operator $\mathcal{S}^-$ show that $A^u = \mathcal{S}^-A$, where the formal series A was defined by (30), and that $\psi^u_1 = \mathcal{S}^-\tilde\psi_1$ and $\psi^u_2 = \mathcal{S}^-\tilde\psi_2$ provide a normalized fundamental system of solutions of $\mathcal{L}^u_0$ on $z_0 + \mathbb{R}^-$ (in the sense of Section 2.1, point (i)), where $(\tilde\psi_1,\tilde\psi_2)$ is the formal fundamental system of $\mathcal{L}_0$ described in Corollary 2.7.

Now the general theory of Section 2.1, point (ii), shows that $\psi = a\psi^u_1 + b\psi^u_2$, with 1-periodic functions a and b defined on $z_0+\mathbb{R}$. Since $\psi^u_2(z) \sim -\frac{\mathrm{i}}{5}z^3$ and $\psi^u_1(z) \sim \mathrm{i}z^{-2}$, if the function b were not identically zero, this would contradict $\psi(z_0 - t) \xrightarrow[t\to+\infty]{} 0$, and also a must vanish identically to ensure $\psi(z) = O\big(|z|^{-\sigma}\big)$ with $\sigma > 2$.

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