

## Darboux theory of integrability for a class of nonautonomous vector fields

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The goal of this paper is to extend the classical Darboux theory of integrability from autonomous polynomial vector fields to a class of nonautonomous vector fields. We also provide sufficient conditions for applying this theory of integrability and we illustrate this theory in several examples. © 2009 American Institute of Physics. [doi:10.1063/1.3205450]

### I. INTRODUCTION

To decide when a differential system is integrable or not is one of the hardest problems of the theory of differential equations. The existence and the calculus of first integrals are in general a difficult problem. Many techniques have been applied in order to construct first integrals, such as Lie symmetries,<sup>27</sup> Noether symmetries,<sup>8</sup> the Painlevé analysis,<sup>4</sup> the use of Lax pairs,<sup>17</sup> the Darboux method,<sup>9</sup> and the direct method.<sup>12,14</sup>

In 1878 Darboux in Ref. 9 presented a simple method to construct first integrals and integrating factors for planar polynomial vector fields using their invariant algebraic curves. This theory has been useful for studying different relevant problems of planar polynomial differential systems such as problems related to centers, limit cycles, and bifurcation problems, see, for instance, Refs. 13, 19, and 28.

Also Darboux in Ref. 10 extended his method to polynomial vector fields in  $\mathbb{C}^n$  where the existence of invariant algebraic surfaces is the key point to build up first integrals (see also Ref. 18 and for some applications see, for instance, Refs. 20–23). Nowadays Darboux's method has been improved for polynomial vector fields basically taking into account the exponential factors<sup>6</sup> and the multiplicity of the invariant algebraic hypersurfaces, see, for instance, Refs. 6, 7, and 22–26.

There are works such as Ref. 11 which generalize the Darboux theory of integrability using the concept of generalized cofactors. In this paper we extend the Darboux theory of integrability from the polynomial vector fields to a class of nonautonomous vector fields. More precisely we deal with differential vector fields in the plane that are polynomials in the variables  $x$  and  $y$  and their coefficients are convenient  $C^1$  functions in the time, i.e., in the independent variable. As far as we know it is the first time in the literature that such generalization is considered.

The main results of this paper are Theorems 1 and 2 where the ideas of Darboux to the mentioned class of nonautonomous vector fields are generalized. We prove that a sufficient number of invariant surfaces and exponential factors generate a linearly dependent set of cofactors over the field of the coefficients of the system. In particular, in the case where a property  $\mathcal{W}^*$  (related to a kind of Wronskian of the cofactors) holds then a subset of the cofactors is linearly dependent over  $\mathbb{C}$ . In this case we can construct one or even two invariants (a first integral depending on time), see Theorem 2. These invariants are very special because they are generalized Darboux functions, see relation (6) and Theorem 1.

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Jouanolou in Ref. 16 managed to connect in a very sophisticated way the Darboux theory of integrability with the existence of a rational first integral. Recently Jouanolou's work has been reproved using simple arguments of linear algebra, see Refs. 6 and 24. In this work we also provide the generalization of Jouanolou's result on rational integrability for our class of nonautonomous vector fields, see Theorem 2 statements (f) and (g).

We note that in the Darboux theory of integrability for autonomous polynomial vector fields only a sufficient number of invariant objects guarantee the construction of first integrals given by Darboux functions. In the case of nonautonomous systems we additionally need a certain property  $\mathcal{W}^*$  to extend to these vector fields the previous results. Only in this case are we able to construct invariants (first integrals depending on the time) given by generalized Darboux functions.

The structure of the paper is the following. In Sec. II we present the basic definitions and the statements of Theorems 1 and 2. The most important properties of the basic concepts such as invariant surfaces, exponential factors, invariants, and Jacobi multiplier are presented in Sec. III. In Sec. IV we provide the proof of Theorem 1. Theorem 2 is proved in Sec. V. Some illustrative examples are given in Sec. VI. We see that sometimes for the same vector field we can have a subset of cofactors linearly dependent on  $\mathbb{C}$  and another subset of cofactors linearly dependent on  $\mathcal{C}^1(U, \mathbb{F})$  and not in  $\mathbb{C}$  (see Example 6). Finally, in Sec. VII we apply Theorem 2 to the Higgs system and we see that the elevated number of invariant surfaces and exponential factors does not guarantee the existence of a linearly dependent subset over  $\mathbb{C}$  because property  $\mathcal{W}^*$  does not hold. Additionally, we see that the Higgs system has a Jacobi multiplier given by a generalized Darboux function.

## II. BASIC DEFINITIONS AND STATEMENT OF THE MAIN RESULTS

Let  $\mathbb{F}$  be either  $\mathbb{R}$  or  $\mathbb{C}$ , and let  $U$  be an open subset of  $\mathbb{F}$ . We denote by  $\mathcal{C}^1(U, \mathbb{F})$  the set of  $\mathcal{C}^1$  functions from  $U \rightarrow \mathbb{F}$  such that they do not vanish in  $U$  except in a subset of Lebesgue measure zero and such that the closure of their domain of definition is  $U$ . Additionally, we denote by  $\mathcal{C}^1(U, \mathbb{F})[x, y]$  the ring of polynomials in the variables  $x$  and  $y$  with coefficients in  $\mathcal{C}^1(U, \mathbb{F})$ . Note that  $\mathcal{C}^1(U, \mathbb{F})$  is a field with the addition and the product of functions. In particular,  $\mathcal{C}^1(U, \mathbb{F}) \times [x, y]$  is a domain of unique factorization. In the following we denote by  $\delta A$  the *degree* of the polynomial  $A \in \mathcal{C}^1(U, \mathbb{F})[x, y]$ . We also denote by  $\mathcal{C}^1(U, \mathbb{F})(x, y)$  the ring of rational functions in the variables  $x$  and  $y$  and coefficients in  $\mathcal{C}^1(U, \mathbb{F})$ .

We deal with the differential systems of the form

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = P(x, y, t) = \sum_{0 \leq i+j \leq m} a_{ij}(t) x^i y^j, \\ \dot{y} &= \frac{dy}{dt} = Q(x, y, t) = \sum_{0 \leq i+j \leq m} b_{ij}(t) x^i y^j, \end{aligned} \quad (1)$$

with  $P, Q \in \mathcal{C}^1(U, \mathbb{F})[x, y]$ . In what follows these systems will be called *nonautonomous polynomial differential systems*. Note that  $t$  is the independent variable of system (1) called the *time*. If the coefficients of the system are in  $\mathcal{C}^1(U, \mathbb{R})$ , then the time will be real; if they are in  $\mathcal{C}^1(U, \mathbb{C})$  then the time can be real or complex. In fact we shall work with three classes of differential systems (1) When  $(x, y, t) \in \mathbb{R}^2 \times U \subset \mathbb{R}^2 \times \mathbb{R}$  we say that (1) is a *real differential system*. If  $(x, y, t) \in \mathbb{C}^2 \times U \subset \mathbb{C}^2 \times \mathbb{C}$  we say that (1) is a *complex differential system*. Finally, if  $(x, y, t) \in \mathbb{C}^2 \times U \subset \mathbb{C}^2 \times \mathbb{R}$  we say that (1) is a *mixed differential system*. When we only say that (1) is a differential system this will mean that the system can be either a real or a complex differential system. The results of this paper can be extended to mixed differential systems, but in order to simplify their expression we only present them for the first two classes of differential systems. Note that in the particular case that the coefficients  $a_{ij}(t)$  and  $b_{ij}(t)$  are polynomials in the variable  $t$  then adding to the differential system  $\dot{x}$  and  $\dot{y}$  the equation  $\dot{t}=1$  we can apply the classical Darboux theory of integrability in  $\mathbb{F}^3$ .

Clearly if all the coefficients of system (1) are nonconstant system (1) is a nonautonomous differential system. Then we associate with the differential system (1) the vector field

$$X = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + \frac{\partial}{\partial t}. \quad (2)$$

Note that system (1) is a polynomial differential system in the variables  $x$  and  $y$  of degree  $m = \delta X = \max\{\delta P, \delta Q\}$ . The solutions of system (1) or of its associated vector field (2) will be denoted by  $(x(t), y(t))$  with  $t \in U$  or by  $(x(t), y(t), t)$  depending on the context.

Let  $f \in C^1(U, \mathbb{F})[x, y]$ . We said that  $f=0$  is an *invariant surface* in  $\mathbb{C}^2 \times U$  of system (1) if it satisfies

$$X(f) = P \frac{\partial f}{\partial x} + Q \frac{\partial f}{\partial y} + \frac{\partial f}{\partial t} = Kf, \quad (3)$$

with  $K \in C^1(U, \mathbb{F})[x, y]$ . Note that  $\delta K \leq \delta X - 1$ . We call  $K$  the *cofactor* of the invariant surface  $f=0$  for the vector field  $X$ . Note that on the zero set of  $f$  the gradient of  $f$  is orthogonal to the components of the vector field (2). This explains why we say that the surface  $f=0$  is an invariant surface for the vector field (2). For the computation of explicit invariant surfaces see the examples in Secs. VI and VII.

We consider  $K_1, \dots, K_r \in C^1(U, \mathbb{F})[x, y]$ . In what follows we denote by  $\mathcal{W}_r$  the determinant

$$\mathcal{W}_r = \mathcal{W}_r[K_1, \dots, K_r] = \begin{vmatrix} K_1 & \cdots & K_r \\ K_1' & \cdots & K_r' \\ \dots & & \dots \\ K_1^{(r-1)} & \cdots & K_r^{(r-1)} \end{vmatrix}, \quad (4)$$

where we have denoted by  $K_i' = \partial K_i / \partial t$ . Usually  $\mathcal{W}_r$  is called the *Wronskian* of  $K_1, \dots, K_r$  with respect to the variable  $t \in U$ .

Assume that  $h, g \in C^1(U, \mathbb{F})[x, y]$  and are relatively prime polynomials in the variables  $x$  and  $y$ . The function  $F(x, y, t) = \exp(g/h)$  is called an *exponential factor* of the differential system (1) if for some  $L \in C^1(U, \mathbb{F})[x, y]$  of degree at most  $m-1$  in  $x$  and  $y$  it satisfies

$$X(F) = P \frac{\partial F}{\partial x} + Q \frac{\partial F}{\partial y} + \frac{\partial F}{\partial t} = LF, \quad (5)$$

and we say that  $L$  is the cofactor of the exponential factor  $F$ .

Let  $W$  be an open subset of  $\mathbb{F}^2 \times U$  such that its Lebesgue measure is the Lebesgue measure of  $\mathbb{F}^2 \times U$ . An *invariant* of the differential system (1) is a nonconstant  $C^1$  function  $I: W \rightarrow \mathbb{F}$  in the variables  $(x, y, t)$  such that  $(\partial I / \partial x, \partial I / \partial y) \neq (0, 0)$  for all  $(x, y, t) \in W$  with  $(x, y, t) \neq (0, 0, t)$ , and  $I(x(t), y(t), t)$  is constant on all solution curves  $(x(t), y(t), t)$  of system (1) contained in  $W$ , or equivalently  $X(I) \equiv 0$  on  $W$ .

Let  $I(x, y, t)$  be an invariant of system (1) defined in  $W$ , and let  $(x(t), y(t))$  be a solution of (1) contained in  $W$ . If  $I(x(t_0), y(t_0), t_0) = I_0 \in \mathbb{F}$  then the surface  $I(x, y, t) = I_0$  in  $W$  contains the solution  $(x(t), y(t), t)$ . Of course an invariant is a first integral depending on the time.

Consider  $I_1$  and  $I_2$  two invariants of system (1) defined in  $W_1$  and  $W_2$ , respectively. As usual we denote by  $I_x$  the partial derivative of  $I$  with respect to the variable  $x$ . We say that the two functions of the forms  $I_1(x, y, t)$  and  $I_2(x, y, t)$  are *independent* in their common domain of definition  $W_1 \cap W_2$  (except possibly a subset of Lebesgue measure zero in  $W_1 \cap W_2$ ) if the matrix

$$\begin{pmatrix} I_{1x} & I_{1y} & I_{1t} \\ I_{2x} & I_{2y} & I_{2t} \end{pmatrix}$$

has rank equal to 2. So if system (1) has two independent invariants  $I_1$  and  $I_2$  defined on an open subset  $W$  of  $\mathbb{F}^2 \times U$ , then the curves  $\{I_1 = \text{const}\} \cap \{I_2 = \text{const}\}$  contained in  $W$  are formed by solu-

tions of system (1). In this case we shall say that system (1) is *completely integrable*.

Let  $W$  be an open subset of  $\mathbb{F}^2 \times U$ . A  $C^1$  function  $M: W \rightarrow \mathbb{F}$  which is not identically zero on  $W$  is called the *Jacobi last multiplier* or simply *Jacobi multiplier* of system (1) on  $W$  if for its associated vector field

$$X = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + \frac{\partial}{\partial t},$$

it satisfies

$$XM = -\operatorname{div}(X)M \quad \text{or} \quad \operatorname{div}(MP, MQ, M) = 0$$

in  $W$ . Hence the Jacobi multiplier is a solution of the linear partial differential equation

$$P \frac{\partial M}{\partial x} + Q \frac{\partial M}{\partial y} + \frac{\partial M}{\partial t} + \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) M = 0.$$

In this article we are interested in constructing invariants and Jacobi multipliers of system (1) using the invariant surfaces and the exponential factors of the differential system (1). More precisely, consider  $f_1, \dots, f_p \in C^1(U, \mathbb{F})[x, y]$  irreducible and coprimes and  $F_i = \exp(g_i/h_i)$  with  $g_i, h_i \in C^1(U, \mathbb{F})[x, y]$  coprimes for  $i=1, \dots, q$ . Let  $G \in C^1(U, \mathbb{F})$  and  $\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q \in C^1(U, \mathbb{F})$ . Any function of the form

$$f_1^{\lambda_1(t)} \dots f_p^{\lambda_p(t)} F_1^{\mu_1(t)} \dots F_q^{\mu_q(t)} e^G \tag{6}$$

will be called a *generalized Darboux function*.

**Theorem 1:** *If a nonautonomous differential system (1) has an invariant or a Jacobi multiplier given by a generalized Darboux function of the form (6) then  $\lambda_i(t) = \lambda_i \in \mathbb{F}$  for all  $i=1, \dots, p$  and  $\mu_j(t) = \mu_j \in \mathbb{F}$  except if  $h_j = h_j(t, x, y) \in C^1(U, \mathbb{F})$  and  $\delta g_j \leq \delta X - 1$ .*

Theorem 1 is proved in Sec. IV. Let  $S = \{K_1, \dots, K_p, L_1, \dots, L_q\}$  be a set of polynomials of  $C^1(U, \mathbb{F})_{m-1}[x, y]$  such that there is no more than one polynomial with constant coefficients. Let  $S_r$  be a subset of  $S$  of  $r$  elements. We denote by  $\mathcal{W}(S_r)$  the Wronskian of these  $r$  elements defined in (4). We write  $\mathcal{W}_r = 0$  if for all subsets  $S_r$  of  $S$  we have that  $\mathcal{W}(S_r) = 0$  for all  $t \in U$ . If for some subset  $S_r$  we have that  $\mathcal{W}(S_r) \neq 0$  for some  $t \in U$ , then we write  $\mathcal{W}_r \neq 0$ . We say that the set of the polynomials of  $S$  satisfies *condition  $\mathcal{W}^*$*  if there exists  $s \in \{2, 3, \dots, p+q\}$  such that  $\mathcal{W}_j \neq 0$  for  $j=2, 3, \dots, s-1$  and  $\mathcal{W}_s = 0$ .

**Theorem 2:** *We assume that the differential system (1) of degree  $m$  admits the invariant surfaces  $f_i = 0$  with cofactors  $K_i \neq 0$  for  $i=1, \dots, p$ ;  $q$  exponential factors  $F_j = \exp(g_j/h_j)$  with cofactors  $L_j \neq 0$  for  $j=1, \dots, q$ . Let  $G \in C^1(U, \mathbb{F})$  with  $\dot{G} = g$ . We assume that  $f_1, \dots, f_p$  and  $F_1, \dots, F_q$  are pairwise independent and that  $f_1, \dots, f_p$  are coprimes in the ring  $C^1(U, \mathbb{F})[x, y]$ . Then the following statements hold.*

(a) *There exist  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero such that*

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j + g = 0, \tag{7}$$

*if and only if the (multivalued) function*

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q} e^G \tag{8}$$

*is an invariant of system (1). Moreover if (1) is a real system, then the function (8) is real.*

(b) *There exist  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero such that*

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j + \operatorname{div}(X) + g = 0, \tag{9}$$

if and only if the function (8) is a Jacobi multiplier of system (1). Moreover if (1) is a real system, then the function (8) is real.

- (c) If  $p+q=m(m+1)/2$ , then there exist  $\lambda_i(t), \mu_j(t) \in C^1(U, \mathbb{F})$  not all zero satisfying either

$$\sum_{i=1}^p \lambda_i(t) K_i + \sum_{j=1}^q \mu_j(t) L_j + \operatorname{div}(X) = 0 \quad (10)$$

or

$$\sum_{i=1}^p \lambda_i(t) K_i + \sum_{j=1}^q \mu_j(t) L_j = 0. \quad (11)$$

- (d) If  $p+q=m(m+1)/2+1$ , then there exist  $\lambda_i(t), \mu_j(t) \in C^1(U, \mathbb{F})$  not all zero satisfying (11). In particular, if property  $\mathcal{W}^*$  holds for the cofactors  $K_i$  with  $\lambda_i(t) \neq 0$  and  $L_j$  with  $\mu_j(t) \neq 0$ , we have that  $\lambda_i(t)$  and  $\mu_j(t)$  are constants of  $\mathbb{F}$ , and so system (1) has an invariant of the form (8).
- (e) If  $p+q=m(m+1)/2+2$  then there exist  $\lambda_i^1(t), \mu_j^1(t) \in C^1(U, \mathbb{F})$  and  $\lambda_i^2(t), \mu_j^2(t) \in C^1(U, \mathbb{F})$  not all zero satisfying condition (11). In particular, if property  $\mathcal{W}^*$  holds for the cofactors  $K_i^1$  with  $\lambda_i^1(t) \neq 0$  and  $L_j^1$  with  $\mu_j^1(t) \neq 0$ , then system (1) has an invariant of the form (8). In a similar way if property  $\mathcal{W}^*$  holds for the cofactors associated with  $\lambda_i^2(t) \neq 0$  and  $\mu_j^2(t) \neq 0$ , then the system (1) has a second invariant of the form (8).

In the following we assume that  $f_i, g_j, h_j \in C^2(U, \mathbb{F})[x, y]$ .

- (f) If  $p+q \geq m(m+1)/2+3$ , then there exist  $\lambda_i^1(t), \mu_j^1(t) \in C^1(U, \mathbb{F})$  and  $\lambda_i^2(t), \mu_j^2(t) \in C^1(U, \mathbb{F})$  not all zero satisfying condition (11). If the property  $\mathcal{W}^*$  holds for the cofactors associated with  $\lambda_i^k \neq 0$  and  $\mu_j^k \neq 0$  for  $k=1, 2$ , then system (1) has two independent invariants of the form (8) and at least one of them is a rational function of  $C^1(U, \mathbb{F})(x, y)$ .
- (g) If  $p+q \geq m(m+1)/2+4$ , then there exist  $\lambda_i^1(t), \mu_j^1(t) \in C^1(U, \mathbb{F})$  and  $\lambda_i^2(t), \mu_j^2(t) \in C^1(U, \mathbb{F})$  not all zero satisfying condition (11). If the property  $\mathcal{W}^*$  holds for the cofactors associated with  $\lambda_i^k \neq 0$  and  $\mu_j^k \neq 0$  for  $k=1, 2$ , then system (1) has two independent invariants of the form (8) being both rational functions of  $C^1(U, \mathbb{F})(x, y)$ .

In Sec. V we present the proof of Theorem 2.

### III. GENERAL PROPERTIES

In this section we present some general properties of the invariant surfaces, the invariants, the Jacobi multipliers, and the exponential factors.

#### A. Invariant surfaces

We recall that an invariant surface of the vector field (2) satisfies relation (3).

*Lemma 3:* Let  $f, g \in C^1(U, \mathbb{F})[x, y]$ . We assume that  $f$  and  $g$  are relatively prime over  $C^1(U, \mathbb{F})[x, y]$ . Then for the differential system (1)  $fg=0$  is an invariant surface with cofactor  $K_{fg}$  if and only if  $f=0$  and  $g=0$  are invariant surfaces with cofactors  $K_f$  and  $K_g$ , respectively. Moreover  $K_{fg}=K_f+K_g$ .

*Proof:* If  $f=0$  and  $g=0$  are invariant surfaces with cofactors  $K_f$  and  $K_g$  then  $\delta K_f, \delta K_g \leq \delta X - 1$  and

$$X(fg) = (Xf)g + f(Xg) = K_f f g + f K_g g = (K_f + K_g) f g. \quad (12)$$

Obviously  $\delta(K_f + K_g) \leq \delta X - 1$ . Hence  $fg=0$  is an invariant surface with cofactor  $K_f + K_g$ .

If now we assume that  $fg=0$  is an invariant surface with cofactor  $K_{fg}$ , then we have that  $\delta K_{fg} \leq \delta X - 1$  and

$$gX(f) + fX(g) = X(fg) = K_{fg}fg. \tag{13}$$

Since  $f$  and  $g$  are relative prime over  $\mathcal{C}^1(U, \mathbb{F})[x, y]$  from (13) we obtain that  $f$  divides  $X(f)$  and  $g$  divides  $X(g)$ . We denote by  $K_f = X(f)/f$  and  $K_g = X(g)/g$  and note that  $\delta K_f, \delta K_g \leq \delta X - 1$ . Hence  $f$  and  $g$  are invariant surfaces of system (1) with cofactors  $K_f$  and  $K_g$ , respectively. Additionally, from (13) and (12) we have that  $K_{fg} = K_f + K_g$ .  $\square$

*Proposition 4:* Suppose that  $f \in \mathcal{C}^1(U, \mathbb{F})[x, y]$  and let  $f = f_1^{n_1} \cdots f_r^{n_r}$  be the factorization of  $f$  over  $\mathcal{C}^1(U, \mathbb{F})[x, y]$ . Then for the system (1),  $f=0$  is an invariant surface with cofactor  $K_f$  if and only if  $f_i=0$  is an invariant surface for each  $i=1, \dots, r$  with cofactor  $K_{f_i}$ . Moreover  $K_f = n_1 K_{f_1} + \cdots + n_r K_{f_r}$ .

*Proof:* First we assume that  $f_i=0$  is an invariant surface with cofactor  $K_{f_i}$  for  $i=1, \dots, r$ . Then we have

$$X(f) = X(f_1^{n_1} \cdots f_r^{n_r}) = (f_1^{n_1} \cdots f_r^{n_r}) \sum_{i=1}^r n_i \frac{X(f_i)}{f_i} = f \sum_{i=1}^r n_i \frac{X(f_i)}{f_i} = f \sum_{i=1}^r n_i K_{f_i}.$$

So, taking  $K_f = \sum_{i=1}^r n_i K_{f_i}$ , it follows that  $f=0$  is an invariant surface with cofactor  $K_f$ .

Now we assume that  $f=0$  is an invariant surface with cofactor  $K_f$ . From Lemma 3, it follows easily that  $f=0$  is an invariant surface with cofactor  $K_f$  if and only if  $f_i^{n_i}=0$  is an invariant surface with cofactor  $K_{f_i^{n_i}}$  for  $i=1, \dots, r$  and  $K_f = K_{f_1^{n_1}} + \cdots + K_{f_r^{n_r}}$ . Since  $f_i^{n_i}=0$  is an invariant surface with cofactor  $K_{f_i^{n_i}}$ , we have that  $K_{f_i^{n_i}} f_i^{n_i} = X(f_i^{n_i}) = n_i f_i^{n_i-1} X(f_i)$ . Hence  $X(f_i) = (K_{f_i^{n_i}} f_i) / n_i = K_{f_i} f_i$ , where we have taken  $K_{f_i} = K_{f_i^{n_i}} / n_i$ . So, the surface  $f_i=0$  is an invariant surface with cofactor  $K_{f_i}$ . Moreover, we have  $K_f = \sum_{i=1}^r K_{f_i^{n_i}} = \sum_{i=1}^r n_i K_{f_i}$ , and the proof is completed.  $\square$

The real vector fields are special because whenever they have a complex invariant surface they also have another complex invariant surface, the conjugate one as we note in the following proposition.

*Proposition 5:* For a real differential system (1), if  $U \subset \mathbb{R}$  and  $f \in \mathcal{C}^1(U, \mathbb{C})[x, y]$  then  $f=0$  is a complex invariant surface with cofactor  $K \in \mathcal{C}^1(U, \mathbb{C})[x, y]$  if and only if  $\bar{f}=0$  is another invariant surface with cofactor  $\bar{K}$ . Here conjugation of  $f$  means conjugation of the coefficients of the polynomial in  $x$  and  $y$  defined by  $f$ .

*Proof:* We assume that  $f=0$  is an invariant surface with cofactor  $K$  of the real differential system (1). Then equality (3) holds. Hence we also have that

$$\bar{P} \frac{\partial \bar{f}}{\partial x} + \bar{Q} \frac{\partial \bar{f}}{\partial y} + \frac{\partial \bar{f}}{\partial t} = \bar{K} \bar{f}.$$

Since  $x$  and  $y$  and their coefficients in  $P$  and  $Q$  are real this equality becomes

$$P \frac{\partial \bar{f}}{\partial x} + Q \frac{\partial \bar{f}}{\partial y} + \frac{\partial \bar{f}}{\partial t} = \bar{K} \bar{f}.$$

So  $\bar{f}=0$  is an invariant surface with cofactor  $\bar{K}$  of the real differential system (1). In a similar way the converse can be proved.  $\square$

**B. Invariants**

We note that the existence of two independent invariants of the differential system (1) yields to the complete description of the orbits of system (1) whenever these invariants are both defined. Any other invariant must be a function of the two independent invariants as we show in Lemma 6.

*Lemma 6:* Let  $W$  be an open subset of  $\mathbb{F}^2 \times U$ . Let  $I_1, I_2 \in \mathcal{C}^2(W, \mathbb{F})$  be two independent invariants of system (1).

- (a) If  $I_3 \in C^2(W, \mathbb{F})$  is another invariant of system (1), then there exists functions  $C_1, C_2 \in C^2(W, \mathbb{F})$  such that

$$\nabla I_3 = C_1 \nabla I_1 + C_2 \nabla I_2. \quad (14)$$

- (b) The functions  $C_1$  and  $C_2$  (if nonconstants) are also invariants of system (1).

*Proof:* For the proof of statement (a) since  $I_1$  and  $I_2$  are independent invariants, for each  $(x, y, t) \in \mathbb{F}^2 \times U \subset \mathbb{F}^3$  (except possibly a subset of Lebesgue measure zero) the vectors  $\nabla I_1$  and  $\nabla I_2$  form a basis of the vector subspace  $S$  of  $\mathbb{F}^3$  orthogonal to the vector field  $X$  associated with system (1). Since  $I_3$  is an invariant of  $X$  we have that  $\nabla I_3 \in S$ . Then, for each  $(x, y, t) \in \mathbb{F}^2 \times U \subset \mathbb{F}^3$  we have that  $\nabla I_3$  will be a combination of  $\nabla I_1$  and  $\nabla I_2$  (except possibly a subset of Lebesgue measure zero). So for every  $(x, y, t) \in \mathbb{F}^2 \times U \subset \mathbb{F}^3$  expression (14) is proved.

Now we prove statement (b). For each  $(x, y, t) \in \mathbb{F}^2 \times U \subset \mathbb{F}^3$  (except possibly a subset of Lebesgue zero measure) the following holds.

From relation (14) we have that

$$\begin{aligned} I_{3x} &= C_1 I_{1x} + C_2 I_{2x}, \\ I_{3y} &= C_1 I_{1y} + C_2 I_{2y}, \\ I_{3t} &= C_1 I_{1t} + C_2 I_{2t}, \end{aligned} \quad (15)$$

and derivating the first equation of (15) with respect to  $y$  and subtracting the derivating of the second equation of (15) with respect to  $x$  we obtain

$$C_{1y} I_{1x} - C_{1x} I_{1y} + C_{2y} I_{2x} - C_{2x} I_{2y} = 0. \quad (16)$$

In a similar way we get

$$\begin{aligned} C_{1t} I_{1x} - C_{1x} I_{1t} + C_{2t} I_{2x} - C_{2x} I_{2t} &= 0, \\ C_{1t} I_{1y} - C_{1y} I_{1t} + C_{2t} I_{2y} - C_{2y} I_{2t} &= 0. \end{aligned} \quad (17)$$

Now multiplying the first equation of (17) by  $I_{2y}$  and subtracting Eq. (16) multiplied by  $I_{2t}$  we obtain

$$(I_{2y} C_{1t} - I_{2t} C_{1y}) I_{1x} + (I_{1y} I_{2t} - I_{2y} I_{1t}) C_{1x} + (I_{2y} C_{2t} - I_{2t} C_{2y}) I_{2x} = 0. \quad (18)$$

By similar arguments we also have

$$\begin{aligned} (I_{2x} C_{1t} - I_{2t} C_{1x}) I_{1y} + (I_{2t} I_{1x} - I_{1t} I_{2x}) C_{1y} + (I_{2x} C_{2t} - I_{2t} C_{2x}) I_{2y} &= 0, \\ (I_{2x} C_{1y} - I_{2y} C_{1x}) I_{1t} + (I_{2y} I_{1x} - I_{1y} I_{2x}) C_{1t} + (I_{2x} C_{2y} - I_{2y} C_{2x}) I_{2t} &= 0. \end{aligned} \quad (19)$$

Now adding the Eq. (18) and the second of (19) and subtracting the first of (19) we get that

$$\begin{vmatrix} I_{1x} & I_{1y} & I_{1t} \\ I_{2x} & I_{2y} & I_{2t} \\ C_{1x} & C_{1y} & C_{1t} \end{vmatrix} = 0.$$

So for each  $(x, y, t) \in \mathbb{F}^2 \times U \subset \mathbb{F}^3$  (except perhaps a subset of Lebesgue zero measure)  $\nabla C_1$  belongs to the two-dimensional vector space generated by  $\{\nabla I_1, \nabla I_2\}$  and (if not a constant) also satisfies  $(C_{1x}, C_{1y}) \neq (0, 0)$ . Hence  $C_1$  (if not a constant) is an invariant of system (1). Similarly  $C_2$  (if not a constant) is also an invariant of system (1).  $\square$

### C. Jacobi multiplier

For a planar vector field of the form

$$X = p(x,y) \frac{\partial}{\partial x} + q(x,y) \frac{\partial}{\partial y},$$

where  $p$  and  $q$  are  $C^1$  functions we define an *integrating factor*  $R=R(x,y)$  to be a  $C^1$  function satisfying

$$X(R) = -\operatorname{div}(X)R \quad \text{or} \quad \operatorname{div}(Rp, Rq) = 0,$$

with  $\operatorname{div}(X)=p_x+q_y$ . If the integrating factor  $R$  is defined in an open and simply connected set then there is a first integral  $H=H(x,y)$  of the planar system associated with this integrating factor. Hence, the existence of an integrating factor for a planar vector field means integrability. For higher dimensions the concept of the integrating factor is generalized via the Jacobi multiplier (see Ref. 15).

*Remark 7: The following holds.*

- (i) *The existence of a Jacobi multiplier  $M$  of the nonautonomous differential system (1), written also as*

$$\dot{x} = dx/dt = P, \quad \dot{y} = dy/dt = Q, \quad \dot{t} = dt/dt = 1,$$

*is associated with the existence of the divergence-free nonautonomous differential system*

$$\dot{x} = dx/dt = MP, \quad \dot{y} = dy/dt = MQ, \quad \dot{t} = dt/dt = M.$$

*Hence the Jacobi multiplier  $M$  represents a change in time and yields to a divergence-free system.*

- (ii) *Due to the relation  $X(M)=-\operatorname{div}(X)M$  the set  $M=0$  is an invariant surface in  $\mathbb{F}^2 \times U$  (maybe not polynomial in the variables  $x$  and  $y$ ) with cofactor  $-\operatorname{div}(X)$  for the differential system (1), which means that the set  $M=0$  is formed by orbits of the system.*

The following proposition follows from ideas of Ref. 3.

*Proposition 8: The following statements hold.*

- (a) *If the differential system (1) has two independent Jacobi multipliers  $M_1$  and  $M_2$  on the open subset  $W$  of  $\mathbb{F}^2 \times U$ , then on the open set  $W \setminus \{M_2=0\}$  the function  $M_1/M_2$  is an invariant.*  
 (b) *The existence of an invariant  $I \in C^2(W, \mathbb{F})$  and of a Jacobi multiplier  $M$  implies that system (1) restricted to every surface  $I(x,y,t)=C$ , with  $C \in \mathbb{F}$ , is integrable in the sense that such restricted system has an integrating factor of the form  $R=R(x, F(x,t;C), t)=(M/I_y) \times (x, F(x,t;C), t)$  if  $y=F(x,t,C)$  is the solution of  $I(x,y,t)=C$ .*

*Proof:* Statement (a) follows directly by the definitions of the Jacobi multiplier and the invariant. Now we prove statement (b). Let  $I=I(x,y,t)$  be an invariant of system (1) and  $C \in \mathbb{F}$ . Then, by the definition of invariant, without loss of generality we may assume that  $I_y \neq 0$ . From the implicit function theorem we can solve locally the relation  $I(x,y,t)=C$  with respect to the variable  $y$ ,  $y=F(x,t;C)$ . Then on the level set  $I(x,y,t)=C$  system (1) goes over to

$$\begin{aligned} \dot{x} &= P(x, F(x,t;C), t), \\ \dot{t} &= 1. \end{aligned} \tag{20}$$

*Claim:* The planar system (20) has the integrating factor

$$R = R(x, F(x,t;C), t) = \frac{M}{I_y}(x, F(x,t;C), t). \tag{21}$$



Now we prove the claim. Since  $M=M(x, y, t)$  is a Jacobi multiplier of system (1) we have that  $(PM)_x + (QM)_y + M_t = 0$ , or equivalently

$$\left(PI_y \frac{M}{I_y}\right)_x + \left(QI_y \frac{M}{I_y}\right)_y + \left(I_y \frac{M}{I_y}\right)_t = 0.$$

Taking into account that  $I=I(x, y, t)$  is an invariant of system (1) the above relation becomes

$$\left(PI_y \frac{M}{I_y}\right)_x - \left(\frac{M}{I_y}(PI_x + I_t)\right)_y + \left(I_y \frac{M}{I_y}\right)_t = 0,$$

or equivalently,

$$I_y \left(\frac{PM}{I_y}\right)_x - I_x \left(\frac{PM}{I_y}\right)_y + I_y \left(\frac{M}{I_y}\right)_t - I_t \left(\frac{M}{I_y}\right)_y = 0.$$

Now since  $I_y \neq 0$  (locally) we can divide the above relation by  $I_y$  and we obtain

$$\left(\frac{PM}{I_y}\right)_x - \frac{I_x}{I_y} \left(\frac{PM}{I_y}\right)_y + \left(\frac{M}{I_y}\right)_t - \frac{I_t}{I_y} \left(\frac{M}{I_y}\right)_y = 0.$$

But on the level set  $I(x, y, t) = C$  when  $I_y \neq 0$  we have that  $y = F(x, t; C)$ , and so we obtain

$$(PR)_x - \frac{I_x}{I_y} (PR)_y + R_t - \frac{I_t}{I_y} R_y = 0,$$

or equivalently

$$(PR)_x - F_x (PR)_y + R_t - F_t R_y = 0. \quad (22)$$

Since  $(PR)_y = R_y = 0$  [see (20) and (21)], we get

$$(PR)_x + R_t = 0.$$

Hence  $R$  is an integrating factor of system (20). In short, we have proved the claim. So the planar system (20) is integrable and this completes the proof of the proposition.  $\square$

## D. Exponential factors

We recall that an exponential factor of system (1) satisfies relation (5).

*Proposition 9:* If  $\exp(g/h)$  is an exponential factor with cofactor  $L$  for the differential system (1) and if  $h$  is not a constant, then  $h=0$  is an invariant surface with cofactor  $K_h$ , and  $g$  satisfies the equation  $X(g) = gK_h + hL$ .

*Proof:* Let  $F(x, y, t) = \exp(g/h)$  be an exponential factor with cofactor  $L$  for the differential system (1). Then we have

$$LF = X(F) = FX\left(\frac{g}{h}\right) = F \frac{hX(g) - gX(h)}{h^2}.$$

So we obtain  $hX(g) - gX(h) = h^2L$ . From this equation and since  $h$  and  $g$  are relative prime over  $C^1(U, \mathbb{F})[x, y]$ , we obtain that  $h$  divides  $X(h)$ . Let  $K_h = X(h)/h$  and note that  $\delta K_h \leq \delta X - 1$ . Then  $h=0$  is an invariant surface for the differential system (1) with cofactor  $K_h$ , and  $g$  satisfies  $X(g) = gK_h + hL$ .  $\square$

*Proposition 10:* For the real differential system (1) the complex function  $\exp(g/h)$  with  $h, g \in C^1(U, \mathbb{C})[x, y]$  is an exponential factor with cofactor  $L \in C^1(U, \mathbb{C})[x, y]$  if and only if the complex function  $\exp(\bar{g}/\bar{h})$  is an exponential factor with cofactor  $\bar{L}$ .

*Proof:* We assume that the function  $F(x, y, t) = \exp(g/h)$  is an exponential factor with cofactor  $L$  of the real differential system (1). Hence  $X(F) = LF$ , and conjugating it we obtain

$$P \frac{\partial \exp(\bar{g}/\bar{h})}{\partial x} + Q \frac{\partial \exp(\bar{g}/\bar{h})}{\partial y} + \frac{\partial \exp(\bar{g}/\bar{h})}{\partial t} = \bar{L} \exp(\bar{g}/\bar{h}),$$

where we have taken into account that  $P$  and  $Q$  are real polynomials in the variables  $x$  and  $y$ . So from this equation we obtain that  $\exp(\bar{g}/\bar{h})$  is an exponential factor with cofactor  $\bar{L}$  for the real differential system (1). In a similar way we prove the converse.  $\square$

**IV. PROOF OF THEOREM 1**

First we prove the following three lemmas.

*Lemma 11:* If  $F=e^g$  with  $g \in C^1_{m-1}(U, \mathbb{F})[x, y]$  is an exponential factor of (1) with cofactor  $L$ , then  $\tilde{F}=e^{\mu(t)g}$  is also an exponential factor of (1) with cofactor  $\dot{\mu}g + \mu L$ .

*Proof:* The proof is trivial and follows by the definition of the exponential factor (5).  $\square$

*Lemma 12:* Suppose that the differential system (1) of degree  $m$  admits the invariant surfaces  $f_i=0$  with cofactors  $K_i \neq 0$  for  $i=1, \dots, p$ ;  $q$  exponential factors  $F_j=\exp(g_j/h_j)$  with cofactors  $L_j \neq 0$  and  $h_j$  nonconstant polynomial for  $j=1, \dots, q$ . Let  $G \in C^1(U, \mathbb{F})$  with  $\dot{G}=g$ . If system (1) has an invariant or a Jacobi multiplier of the form

$$I = f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q} e^G = f_1(x, y, t)^{\lambda_1(t)} \dots f_p(x, y, t)^{\lambda_p(t)} F_1(x, y, t)^{\mu_1(t)} \dots F_q(x, y, t)^{\mu_q(t)} e^{G(t)}, \tag{23}$$

then  $\lambda_i, \mu_j \in \mathbb{F}$  for all  $i=1, \dots, p$  and  $j=1, \dots, q$ .

*Proof:* First we note that

$$\begin{aligned} X(I) &= X(f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q} e^G) = \left( \sum_{i=1}^p \lambda_i \frac{X(f_i)}{f_i} + \sum_{j=1}^q \mu_j \frac{X(F_j)}{F_j} + g + \sum_{i=1}^p \dot{\lambda}_i \log f_i + \sum_{j=1}^q \dot{\mu}_j \log F_j \right) I \\ &= \left( \sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j + g + \sum_{i=1}^p \dot{\lambda}_i \log f_i + \sum_{j=1}^q \dot{\mu}_j \log F_j \right) I. \end{aligned}$$

Note that if  $\dot{\lambda}_i=0$  and  $\dot{\mu}_j=0$  for all  $i=1, \dots, p$  and all  $j=1, \dots, q$ , then  $\lambda_i, \mu_j \in \mathbb{F}$ . Now we assume that for some  $i$  or some  $j$  we have that  $\dot{\lambda}_i \neq 0$  or  $\dot{\mu}_j \neq 0$ .

If the function  $I$  is an invariant then we have that  $X(I)=0$ , or equivalently

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j + g = - \sum_{i=1}^p \dot{\lambda}_i \log f_i - \sum_{j=1}^q \dot{\mu}_j \log F_j = - \sum_{i=1}^p \log(\prod f_i^{\dot{\lambda}_i}) - \sum_{j=1}^q \dot{\mu}_j \frac{g_j}{h_j}. \tag{24}$$

Note that the left hand side of (24) is a polynomial in  $C^1(U, \mathbb{F})[x, y]$ . If some  $\dot{\lambda}_i \neq 0$  then  $\sum_{i=1}^p \log(\prod f_i^{\dot{\lambda}_i})$  is a series in  $x$  and  $y$  with coefficients in  $C^1(U, \mathbb{F})$ . Hence,  $\dot{\lambda}_i=0$  for all  $\lambda_i = 1, \dots, p$ . If some  $\dot{\mu}_k \neq 0$  then from relation (24) we can write

$$\frac{h_k}{\dot{\mu}_k} \left( \prod_{\substack{j=1 \\ j \neq k}}^q h_j \right) \left( \sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j + g + \sum_{\substack{j=1 \\ j \neq k}}^q \dot{\mu}_j \frac{g_j}{h_j} \right) = g_k \tag{25}$$

and this equality is a contradiction because by the definition of the exponential factor (5) we have that  $g_k$  and  $h_k$  are coprimes in  $C^1(U, \mathbb{F})[x, y]$ . Hence  $\dot{\mu}_k \neq 0$  for all  $k=1, \dots, q$ .

If the function  $I$  is a Jacobi multiplier then in relation (24) we just add on the left hand side the divergence of the system which is a polynomial of  $C^1_{m-1}(U, \mathbb{F})[x, y]$  and the proof follows using the same arguments.  $\square$

*Lemma 13:* Suppose that the differential system (1) of degree  $m$  admits the invariant surfaces  $f_i=0$  with cofactors  $K_i \neq 0$  for  $i=1, \dots, p$ ;  $q+r$  exponential factors  $F_j=\exp(g_j/h_j)$  with cofactors  $L_j \neq 0$  for  $j=1, \dots, q+r$  and such that  $h_j \in C^1(U, \mathbb{F})$  and  $\delta g_j \leq m-1$  for  $j=q+1, \dots, q+r$ . Let  $G$

$\in C^1(U, \mathbb{F})$  with  $\dot{G}=g$ . If system (1) has an invariant or a Jacobi multiplier of the form (23), then

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} \tilde{F}_{q+1} \cdots \tilde{F}_{q+r} e^G \tag{26}$$

is also an invariant or a Jacobi multiplier with  $\lambda_i, \mu_j \in \mathbb{F}$  for all  $i=1, \dots, p$  and  $j=1, \dots, q$  where  $\tilde{F}$  is defined as in the statement of Lemma 11.

*Proof:* We repeat the proof of Lemma 12 for the invariant or the Jacobi multiplier (26).  $\square$

*Proof of Theorem 1:* The proof of Theorem 1 follows directly by Lemmas 12 and 13.  $\square$

**V. PROOF OF THEOREM 2**

We denote by  $C_{m-1}^1(U, \mathbb{F})[x, y]$  the vector subspace of  $C^1(U, \mathbb{F})[x, y]$  formed by the polynomials in the variables  $x$  and  $y$  of degree at most  $m-1$  and coefficients in  $C^1(U, \mathbb{F})$ . We note that  $\dim_{C^1(U, \mathbb{F})} C_{m-1}^1(U, \mathbb{F})[x, y] = m(m+1)/2$ . Let  $K = K(x, y, t) = \sum_{i+j=0}^{m-1} k_{ij}(t)x^i y^j \in C_{m-1}^1(U, \mathbb{F})[x, y]$ . We consider the isomorphism

$$C_{m-1}^1(U, \mathbb{F})[x, y] \rightarrow C^1(U, \mathbb{F})^{m(m+1)/2}$$

given by

$$K \mapsto (k_{00}(t), k_{10}(t), k_{01}(t), \dots, k_{m-1,0}(t), k_{m-2,1}(t), \dots, k_{0,m-1}(t)),$$

i.e., we identify the linear vector spaces  $C_{m-1}^1(U, \mathbb{F})[x, y]$  and  $C^1(U, \mathbb{F})^{m(m+1)/2}$ .

*Lemma 14:* Assume that  $K_1, \dots, K_r \in C^1(U, \mathbb{F})[x, y] \setminus \{0\}$ .

- (a) If there is  $(x_0, y_0, t_0) \in U$  such that  $\mathcal{W}_r(x_0, y_0, t_0) \neq 0$ , then  $K_1, \dots, K_r$  are linearly independent over  $\mathbb{F}$ .
- (b) If the set  $\{K_1, \dots, K_r\}$  satisfies the condition  $\mathcal{W}^*$  for some  $s \in \{2, 3, \dots, r\}$ , then there exists a subset of  $s$  elements linearly dependent over  $\mathbb{F}$ .

*Proof:* We consider  $c_1, \dots, c_r \in \mathbb{F}$  such that  $c_1 K_1 + \dots + c_r K_r = 0$ . Then the system

$$c_1 K_1 + \dots + c_r K_r = 0,$$

$$c_1 K'_1 + \dots + c_r K'_r = 0,$$

...

$$c_1 K_1^{(r-1)} + \dots + c_r K_r^{(r-1)} = 0,$$

is a linear system in the variables  $c_1, \dots, c_r \in \mathbb{F}$ . The determinant of this linear system is

$$\mathcal{W}_r = \mathcal{W}_r[K_1, \dots, K_r] = \begin{vmatrix} K_1 & \cdots & K_r \\ K'_1 & \cdots & K'_r \\ \dots & \dots & \dots \\ K_1^{(r-1)} & \cdots & K_r^{(r-1)} \end{vmatrix},$$

where  $K' = \partial K / \partial t$ . Note that  $\mathcal{W}_r$  belongs to  $C^1(U, \mathbb{F})[x, y]$ .

If there is  $(x_0, y_0, t_0) \in U$  such that  $\mathcal{W}_r(x_0, y_0, t_0) \neq 0$  then  $c_1 = \dots = c_r = 0$  and therefore the polynomials  $K_1, \dots, K_r$  are linearly independent over  $\mathbb{F}$ . So statement (a) is proved.

Since  $\mathcal{W}_s = 0$  without loss of generality we can assume that

$$K_s = c_1(t)K_1 + \dots + c_{s-1}(t)K_{s-1},$$

$$K'_s = c_1(t)K'_1 + \dots + c_{s-1}(t)K'_{s-1},$$

...

$$K_s^{(s-1)} = c_1(t)K_1^{(s-1)} + \dots + c_{s-1}(t)K_{s-1}^{(s-1)}. \tag{27}$$

Derivating the row  $j$  and using the row  $j+1$ , for  $j=1, \dots, s-2$ , we get

$$c'_1(t)K_1 + \dots + c'_{s-1}(t)K_{s-1} = 0,$$

$$c'_1(t)K'_1 + \dots + c'_{s-1}(t)K'_{s-1} = 0,$$

...

$$c'_1(t)K_1^{(s-2)} + \dots + c'_{s-1}(t)K_{s-1}^{(s-2)} = 0. \tag{28}$$

Since  $\mathcal{W}_{s-1} \neq 0$ , from the last linear system in the variables  $c'_1(t), \dots, c'_s(t)$  we obtain that  $c'_1(t) = \dots = c'_{s-1}(t) = 0$ . So  $c_j(t) = c_j \in \mathbb{F}$  for  $j=1, \dots, s-1$ . But from the first equation of (27) not all the constants  $c_j$  are zero. Hence statement (b) is proved.  $\square$

*Proof of Theorem 2:* First we prove statement (a). Clearly the function (8), namely,  $I = I(x, y, t)$ , is an invariant of system (1) if and only if  $X(I) = 0$  where  $X$  is given by (2). Then from the equalities

$$\begin{aligned} X(I) &= X(f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q} e^{G}) = \left( \sum_{i=1}^p \lambda_i \frac{X(f_i)}{f_i} + \sum_{j=1}^q \mu_j \frac{X(F_j)}{F_j} + g \right) I = \left( \sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j + g \right) I \\ &= 0, \end{aligned}$$

the first part of statement (a) follows.

Suppose that  $X$  is a real vector field. Now if among the invariant surfaces of  $X$  a complex conjugate pair  $f=0$  and  $\bar{f}=0$  occurs (i.e.,  $\text{Im } f \neq 0$ ) then the invariant (8) has a real factor of the form  $f^\lambda \bar{f}^\lambda$ , which is the multivalued real function

$$[(\text{Re } f)^2 + (\text{Im } f)^2]^{\text{Re } \lambda} \exp\left(-2 \text{Im } \lambda \arctan\left(\frac{\text{Im } f}{\text{Re } f}\right)\right). \tag{29}$$

If among the exponential factors of  $X$  a complex conjugate pair  $F(x, y, t) = \exp(h/g)$  and  $\bar{F}(x, y, t) = \exp(\bar{h}/\bar{g})$  occurs, then  $(\exp(h/g))^\mu (\exp(\bar{h}/\bar{g}))^{\bar{\mu}}$  is a real factor of (8) of the form

$$\left(\exp\left(\frac{h}{g}\right)\right)^\mu \left(\exp\left(\frac{\bar{h}}{\bar{g}}\right)\right)^{\bar{\mu}} = \exp\left(2 \text{Re}\left(\mu \frac{h}{g}\right)\right). \tag{30}$$

In short the function (8) is real, and the proof of statement (a) is completed.

(b) Clearly the function (8), namely,  $M(x, y, t)$ , is a Jacobi multiplier of system (1) if and only if  $X(M) = -\text{div}(X)M$  where  $X$  is given by (2). Then we have

$$\begin{aligned} -M \text{div}(X) &= X(M) = X(f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q} e^{G(t)}) = \left( \sum_{i=1}^p \lambda_i \frac{X(f_i)}{f_i} + \sum_{j=1}^q \mu_j \frac{X(F_j)}{F_j} + g \right) M \\ &= \left( \sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j + g \right) M, \end{aligned}$$

the first assertion of statement (b) holds. The second assertion is similar to statement (a).

Now we assume that we are under the assumptions of statement (c). Let  $K$  be the divergence of system (1). All polynomials (in the variables  $x$  and  $y$ )  $K_i, L_j$ , and  $K$  belong to the vector space

$\mathcal{C}_{m-1}^1(U, \mathbb{F})[x, y]$  of dimension  $m(m+1)/2$  over  $\mathcal{C}^1(U, \mathbb{F})$ . Therefore we have  $p+q+1$  polynomials  $K_i, L_j$ , and  $K$  in  $\mathcal{C}_{m-1}^1(U, \mathbb{F})[x, y]$ . Since from our assumptions we have  $p+q+1=m(m+1)/2+1$ , either  $K$  is a linear combination of the polynomials  $K_i$  and  $L_j$  or a linear combination of those polynomials is zero. Hence, there exists  $\lambda_i, \mu_j \in \mathcal{C}^1(U, \mathbb{F})$  not all of them zero satisfying equality (10) in the first case, and in the second case we obtain the equality (11). Hence statement (c) is proved.

Now we assume that we are under the assumptions of statement (d). Note that the  $m(m+1)/2+1$  polynomials  $K_1, \dots, K_p, L_1, \dots, L_q \in \mathcal{C}_{m-1}^1(U, \mathbb{F})[x, y]$  must be linearly dependent over  $\mathcal{C}^1(U, \mathbb{F})$ , and so relation (11) holds. In particular, if property  $\mathcal{W}^*$  holds then due to Lemma 14(b) we have that a subset of the polynomials  $K_1, \dots, K_p, L_1, \dots, L_q$  must be linearly dependent on  $\mathbb{F}$ . Hence, due to statement (a) of Theorem 2 the proof of statement (d) is completed.

Statement (e) follows easily using similar arguments to statement (d). Moreover, the independence of the two invariants is due to their construction.

Now we prove statement (f). Let  $N=m(m+1)/2$ . We denote by  $\{v_1, \dots, v_{N+3}\} = \{f_1, \dots, f_p, F_1, \dots, F_q\}$ . Let  $s$  be the dimension of the vectorial subspace of  $\mathcal{C}_{m-1}^1(U, \mathbb{F})[x, y]$  generated by the cofactors  $K_1, \dots, K_{N+3}$  of  $v_1, \dots, v_{N+3}$ . Clearly  $s \leq N$ . Now in order to simplify the explanation of the proof and the notation we assume that  $s=N$  and that  $K_1, \dots, K_N$  are linearly independent in  $\mathcal{C}_{m-1}^1(U, \mathbb{F})[x, y]$  over  $\mathcal{C}^1(U, \mathbb{F})$ . If  $s < N$  the proof could follow using the same arguments.

For each  $r \in \{1, 2, 3\}$  there exists a vector  $(\sigma_1^r, \dots, \sigma_N^r, 1) \in \mathbb{C}^{N+1}(U, \mathbb{F})$  such that

$$\sigma_1^r K_1 + \dots + \sigma_N^r K_N + K_{N+r} = 0. \tag{31}$$

From the definition of  $v_j$  we get that  $K_j = X(v_j)/v_j$ . Hence, relation (11) holds.

In particular, if property  $\mathcal{W}^*$  holds for the subsets  $\{K_1, \dots, K_N, K_{N+r}\}$  for  $r=1, 2, 3$ , then we have that  $(\sigma_1^r, \dots, \sigma_N^r, 1) \in \mathbb{C}^{N+1}$  for  $r=1, 2, 3$ . Hence, from (31) and statement (a) we have that

$$X(\log(v_1^{\sigma_1^r} \dots v_N^{\sigma_N^r} v_{N+r})) = 0.$$

This means that the functions  $I_r = \log(v_1^{\sigma_1^r} \dots v_N^{\sigma_N^r} v_{N+r})$  for  $r=1, 2, 3$  are invariants of the vector field  $X$ .

Since  $f_i, g_j, h_k \in \mathcal{C}^2(U, \mathbb{F})[x, y]$  the invariants  $I_i \in \mathcal{C}^2(W, \mathbb{F})$  for  $i=1, 2, 3$ . By Lemma 6 statement (a) we have that there are  $C_1, C_2 \in \mathcal{C}^2(W, \mathbb{F})$  such that

$$\nabla I_3 = C_1 \nabla I_1 + C_2 \nabla I_2. \tag{32}$$

We claim that  $C_1$  and  $C_2$  are not both constants.

If  $C_1$  and  $C_2$  are both constants then from relation (32) we have that

$$\nabla I_3 = \nabla(C_1 I_1 + C_2 I_2),$$

and therefore

$$I_3 = C_1 I_1 + C_2 I_2 + \log C_3$$

for some constant  $C_3$ . Hence

$$\log(v_1^{\sigma_1^3} \dots v_N^{\sigma_N^3} v_{N+3}) = C_1 \log(v_1^{\sigma_1^1} \dots v_N^{\sigma_N^1} v_{N+1}) + C_2 \log(v_1^{\sigma_1^2} \dots v_N^{\sigma_N^2} v_{N+2}) + \log C_3, \tag{33}$$

and so

$$v_1^{\sigma_1^3} \dots v_N^{\sigma_N^3} v_{N+3} = C_3 v_1^{C_1 \sigma_1^1} \dots v_N^{C_1 \sigma_N^1} v_{N+1}^{C_1},$$

$$v_1^{C_2\sigma_1^2} \dots v_N^{C_2\sigma_N^2} v_{N+2}^{C_2},$$

But from this relation we have that  $v_{N+3}$  depends on  $v_1, \dots, v_N, v_{N+1}, v_{N+2}$  which is a contradiction with the assumptions of Theorem 2. Hence, at least one of  $C_1$  and  $C_2$  is not a constant. Without loss of generality we may assume that  $C_1$  is not a constant. Then, from Lemma 6 statement (b) we have that  $C_1$  is also an invariant of system (1). Now, solving the linear system (32) we obtain that

$$C_1 = \frac{\begin{vmatrix} I_{3x} & I_{2x} \\ I_{3y} & I_{2y} \end{vmatrix}}{\begin{vmatrix} I_{1x} & I_{2x} \\ I_{1y} & I_{2y} \end{vmatrix}},$$

and using the expressions  $I_r = \log(v_1^{\sigma_1^r} \dots v_N^{\sigma_N^r} v_{N+r})$  for  $r = 1, 2, 3$  we get that  $C_1$  is a rational function. This completes the proof of statement (f).

Statement (g) follows in a similar way to the previous statements. In short, the proof of Theorem 2 is done. □

### VI. EXAMPLES

Here we present some illustrative examples.

*Example 1:* We consider the linear system

$$\dot{x} = x + \sin t, \quad \dot{y} = x + A(t), \tag{34}$$

where  $A \in C^1(U, \mathbb{F})$ . We are interested in finding the possible invariant surfaces of degree 1 of the form  $f = f_{10}(t)x + f_{01}(t)y + f_{00}(t)$  with cofactor  $K = k_{00}(t)$  where  $f_{00}, f_{10}, f_{01}, k_{00} \in C^1(U, \mathbb{F})$ . Note that relation  $Pf_x + Qf_y + f_t - Kf = 0$  yields to the following differential system:

$$\dot{f}_{01} - k_{00}f_{01} = 0,$$

$$\dot{f}_{10} + f_{10} + f_{01} - k_{00}f_{10} = 0,$$

$$\dot{f}_{00} + f_{10} \sin t + f_{01}A - k_{00}f_{00} = 0,$$

with general solution

$$f_{01}(t) = C_3 e^{\int k_{00}(t) dt},$$

$$f_{10}(t) = -C_3 e^{\int k_{00}(t) dt} + C_2 e^{\int (-1+k_{00}(t)) dt},$$

$$f_{00}(t) = \frac{e^{-t}}{2} \left( -2C_3 \cos te^t + C_2 \cos t + C_2 \sin t - 2C_3 \left( \int A(t) dt \right) e^t + 2C_1 e^t \right) e^{\int k_{00}(t) dt}.$$

In particular system (34) has the two invariant surfaces,

$$f_1 = -x + y - \cos t - \int A(t) dt, \quad f_2 = x + \frac{\cos t + \sin t}{2} + e^t,$$

with cofactors  $K_1 = 0$  and  $K_2 = 1$ . So, since  $K_1 = 0$  it follows that  $f_1$  is an invariant. Note that system (34) has divergence equal to  $\text{div} = 1 = K_2$  and so due to Theorem 2 statement (b) we obtain the Jacobi multiplier

$$M = \frac{1}{f_2} = \frac{2}{2x + \cos t + \sin t + 2e^t}.$$

Additionally, for  $g_2 = -1$  we have that  $K_2 + g_2 = 0$  and so according to Theorem 2 statement (a) system (34) has a second invariant

$$I_2 = f_2 e^{-t} = \left( x + \frac{\cos t + \sin t}{2} + e^t \right) e^{-t}.$$

*Remark 15:* In general, for the vector field (2) of degree  $m$  having an invariant surface  $f = 0$  of degree  $k$  in the variables  $x$  and  $y$ , with cofactor  $K$ , the relation  $Pf_x + Qf_y + f_t - Kf = 0$  yields to a system of  $\binom{m+k-1}{2}$  differential equations of first order with  $\binom{k}{2}$  variables and with  $\binom{m-1}{2}$  free coefficients coming from the cofactor  $K$ .

*Example 2:* We consider the differential system

$$\dot{x} = \frac{xy}{t^2} + x^2 + y^2 - 1, \quad \dot{y} = \frac{y^2 - 1}{t^2}, \quad (35)$$

for  $t \neq 0$  and let  $f_1 = x^2 + y^2 - 1$ ,  $f_2 = y - 1$ , and  $f_3 = y + 1$ . Note that system (35) has the three algebraic surfaces  $f_1 = 0$ ,  $f_2 = 0$ , and  $f_3 = 0$  invariants with cofactors  $K_1 = 2x + 2y/t^2$ ,  $K_2 = (y + 1)/t^2$ , and  $K_3 = (y - 1)/t^2$ , respectively. Let  $g_1 = -3/t^2$  and  $G_1 = 3/t$ . Direct computation shows that

$$-K_1 + K_2 - 2K_3 + \text{div}(X) + g_1 = 0.$$

So according to Theorem 2 statement (b) system (35) has the Jacobi multiplier

$$M = \frac{f_2}{f_1 f_3^2 e^{G_1}} = \frac{(y - 1)e^{3/t}}{(x^2 + y^2 - 1)(y + 1)^2}.$$

Additionally, for  $g_2 = 2/t^2$  and  $G_2 = -2/t$  note that

$$K_3 - K_2 + g_2 = 0,$$

and therefore by Theorem 2 statement (a) system (35) admits the invariant

$$I(x, y, t) = \frac{f_3}{f_2} e^{G_2} = \frac{(y + 1)}{(y - 1)} e^{-2/t}.$$

Note that  $I_y = -2e^{-2/t}/(y - 1)^2$ . Hence on the level set  $I(x, y, t) = C$  we have that  $y = (C + e^{-2/t})/(C - e^{-2/t}) = F(x, t; C)$ . If we restrict system (35) on the surface  $I(x, y, t) = C$  we obtain a planar differential system in the variables  $x$  and  $t$  (in general is not polynomial),

$$\dot{x} = \frac{-xe^{-4/t} + C^2x + x^2t^2e^{-4/t} - 2Cx^2t^2e^{-2/t} + C^2x^2t^2 + 4Ct^2e^{-2/t}}{(C - e^{-2/t})^2t^2},$$

$$\dot{t} = 1.$$

According to Proposition 8 statement (b) this system is integrable because it has the integrating factor

$$R = \frac{M}{I_y} \Big|_{y=F(x,t,C)} = \frac{-e^{-1/t}(C - e^{-2/t})}{C^2(x^2e^{-4/t} - 2Cx^2e^{-2/t} + C^2x^2 + 4Ce^{-2/t})}.$$

*Example 3:* We consider the differential system

$$\dot{x} = x, \quad \dot{y} = -t + y^2. \tag{36}$$

System (36) has the invariant surfaces  $f_1 = x$ ,  $f_2 = y \operatorname{Airy} \operatorname{Ai}(t) + \operatorname{Airy} \operatorname{Ai}(1, t)$  and  $f_3 = y \operatorname{Airy} \operatorname{Bi}(t) + \operatorname{Airy} \operatorname{Bi}(1, t)$  with cofactors  $K_1 = 1$ ,  $K_2 = y$ , and  $K_3 = y$  respectively. Here,  $\operatorname{Airy} \operatorname{Ai}$  and  $\operatorname{Airy} \operatorname{Bi}$  are the Airy functions  $\operatorname{Ai}$  and  $\operatorname{Bi}$ , see for more details Ref. 1. System (36) has divergence equal to  $2y + 1$ . We note that

$$-K_1 - K_2 - K_3 + \operatorname{div} = 0,$$

and so by Theorem 2 statement (b) system (36) admits the Jacobi multiplier

$$M = \frac{1}{f_1 f_2 f_3} = \frac{1}{x(y \operatorname{Airy} \operatorname{Ai}(t) + \operatorname{Airy} \operatorname{Ai}(1, t))(y \operatorname{Airy} \operatorname{Bi}(t) + \operatorname{Airy} \operatorname{Bi}(1, t))}.$$

Additionally, since

$$-K_1 + 1 = 0, \quad K_2 - K_3 = 0,$$

from Theorem 2 statement (a) we have that system (36) admits the two invariants

$$I_1 = \frac{e^t}{f_1} = \frac{e^t}{x} \quad \text{and} \quad I_2 = \frac{f_2}{f_3} = \frac{y \operatorname{Airy} \operatorname{Ai}(t) + \operatorname{Airy} \operatorname{Ai}(1, t)}{y \operatorname{Airy} \operatorname{Bi}(t) + \operatorname{Airy} \operatorname{Bi}(1, t)}.$$

Therefore, system (36) is a completely integrable system.

*Example 4:* For  $t \neq 0$  we consider the differential system

$$\dot{x} = x^2, \quad \dot{y} = -t + \frac{y^2}{t}. \tag{37}$$

System (37) has the four invariant surfaces  $f_1 = x$ ,  $f_2 = tx + 1$ ,  $f_3 = y \operatorname{Bessel} \operatorname{K}(0, -t) + \operatorname{Bessel} \operatorname{K}(1, -t)t$ , and  $f_4 = y \operatorname{Bessel} \operatorname{I}(0, t) + \operatorname{Bessel} \operatorname{I}(1, t)t$ . The functions  $\operatorname{Bessel} \operatorname{I}$  and  $\operatorname{Bessel} \operatorname{K}$  are the modified Bessel functions of the first and second kinds, see Ref. 1. The invariant surfaces have cofactors  $K_1 = K_2 = x$  and  $K_3 = K_4 = y/t$ , respectively. Hence, by Theorem 2 statement (a) system (37) has the two invariants

$$I_1 = \frac{f_2}{f_1} = \frac{tx + 1}{x},$$

$$I_2 = \frac{f_4}{f_3} = \frac{y \operatorname{Bessel} \operatorname{I}(0, t) + \operatorname{Bessel} \operatorname{I}(1, t)t}{y \operatorname{Bessel} \operatorname{K}(0, -t) + \operatorname{Bessel} \operatorname{K}(1, -t)t}.$$

Note that system (37) has divergence equal to  $2x + 2y/t$  and we have

$$-2K_1 - 2K_3 + \operatorname{div} = 0,$$

$$-K_1 - K_2 - K_3 - K_4 + \operatorname{div} = 0,$$

$$-K_1 - K_2 - 2K_3 + \operatorname{div} = 0,$$

...

Hence by Theorem 2 statement (b) system (37) admits the Jacobi multipliers

$$M_1 = \frac{1}{f_1^2 f_3}, \quad M_2 = \frac{1}{f_1 f_2 f_3 f_4}, \quad M_3 = \frac{1}{f_1 f_2 f_3}, \dots$$



*Example 5:* The one-parametric family

$$\dot{x} = x^3, \quad \dot{y} = -\frac{a}{e^t} + y, \quad (38)$$

with  $a \in \mathbb{F}$  has the invariant surfaces  $f_1 = x$ ,  $f_2 = a - 2e^t y$ , and  $f_3 = 2tx^2 + 1$  with cofactors  $K_1 = x^2$ ,  $K_2 = 2$ , and  $K_3 = 2x^2$ . Additionally, system (38) has the exponential factor  $F_2 = e^{1/x^2}$  with cofactor  $L_2 = -2$ . We note that

$$K_3 - 2K_1 = 0, \quad K_2 + L_2 = 0,$$

and so by Theorem 2 statement (a) we can construct the two independent invariants

$$I_1 = \frac{f_3}{f_1^2} = \frac{2tx^2 + 1}{x^2}, \quad I_2 = f_2 F = (a - 2e^t y)e^{1/x^2}.$$

Additionally, since the divergence of system (38) is  $\text{div} = 3x^2 + 1$  we have

$$-K_1 - K_3 + \frac{1}{2}L + \text{div} = 0, \quad -3K_1 - \frac{1}{2}K_2 + \text{div} = 0,$$

and therefore system (38) admits the Jacobi multipliers

$$M_1 = \frac{F^{1/2}}{f_1 f_3} = \frac{\sqrt{e^{1/x^2}}}{x(2tx^2 + 1)},$$

$$M_2 = \frac{1}{f_1^3 f_2^{1/2}} = \frac{1}{x^3 \sqrt{a - 2e^t y}}.$$

*Example 6:* The system

$$\dot{x} = (e^{at}y + t)x, \quad \dot{y} = (y + a)y, \quad (39)$$

has the algebraic surfaces  $f_1 = y$  and  $f_2 = y + a$  invariants with cofactors  $K_1 = y + a$  and  $K_2 = y$ . Additionally, it has the two exponential factors  $F_1 = e^{-at/y+a}$  and  $F_2 = e^{x+\cos t}$  with cofactors  $L_1 = -e^{-at}$  and  $L_2 = e^{at}xy + tx - \sin t$ , respectively. System (39) has divergence  $\text{div} = (e^{at} + 2)y + t + a$ . According to Theorem 2 statement (d) the cofactors  $K_1$ ,  $K_2$ ,  $L_1$ , and  $L_2$  are linearly dependent over  $\mathcal{C}^2(U, \mathbb{F})$ . In particular, they are linearly dependent over  $\mathbb{C}$ : take  $\lambda_1 = -1$ ,  $\lambda_2 = 1$ , and  $\mu_1 = \mu_2 = 0$ . Then we have

$$\lambda_1 K_1 + \lambda_2 K_2 + \mu_1 L_1 + \mu_2 L_2 + g_1 = 0,$$

with  $g_1 = a$  and so system (39) has the invariant

$$I_1(x, y, t) = \frac{(y + a)e^{at}}{y}.$$

Note that the cofactors  $K_2$ ,  $L_1$ , and  $L_2$  and the  $\text{div}$  are linearly dependent over  $\mathcal{C}^2(U, \mathbb{F})$ . Thus, for  $\lambda_2 = -2 - e^{at}$ ,  $\mu_1 = te^{at} + ae^{at}$ , and  $\mu_2 = 0$  we have

$$\lambda_2 K_2 + \mu_1 L_1 + \mu_2 L_2 + \text{div} = 0.$$

Since  $\mathcal{W}[K_2, L_1, L_2, \text{div}] \neq 0$  these four polynomials are independent over  $\mathbb{F}$ , see Lemma 14 statement (a).

## VII. THE HIGGS SYSTEM

In Ref. 5 the study of the black holes in the Higgs field is reduced to the study of differential polynomial Lotka–Volterra systems,

$$\dot{x} = x(y-1), \quad \dot{y} = y(1+y-2x^2-z^2), \quad \dot{z} = yz. \quad (40)$$

The flow of system (40) has been studied in Ref. 2 where the  $\alpha$  and the  $\omega$  limit set of the orbits of the system are described. Note that system (40) is an autonomous system and that  $H = H(x, y, z) = e^t x/z$  is an invariant of system (40) because

$$\dot{H} = \dot{x} \frac{\partial H}{\partial x} + \dot{y} \frac{\partial H}{\partial y} + \dot{z} \frac{\partial H}{\partial z} + \frac{\partial H}{\partial t} = 0.$$

We consider the level sets  $H=h$  or equivalently for every  $h$  we consider the surfaces  $e^t x/z=h$ . Then on every such surface system (40) can be written into the form

$$\dot{x} = x(y-1), \quad \dot{y} = y \left[ - \left( \frac{e^{2t}}{h^2} + 2 \right) x^2 + y + 1 \right]. \quad (41)$$

System (41) is a nonautonomous system of the form (1) and it has the two invariant surfaces  $f_1 = x$  and  $f_2 = y$  with cofactors  $K_1 = y-1$  and  $K_2 = -(e^{2t}/h^2 + 2)x^2 + y + 1$ , respectively. Additionally, it has the following exponential factors:

$$F_1 = e^{(e^{2t}/2h^2 + 1)x^2 + y + 1}, \quad L_1 = -2x^2 + y^2 + y,$$

$$F_2 = e^{(e^{2t}/2h^2 + 1)x^2 + x + y}, \quad L_2 = -2x^2 + y^2 + xy - x + y,$$

$$F_3 = e^{[(e^{-t}-1)y-1]e^t x}, \quad L_3 = - \frac{(e^{2t} + 2h^2)(e^{-t} - 1)}{h^2 e^t} xy,$$

$$F_4 = e^{-y+1/e^t x}, \quad L_4 = \frac{2h^2 + e^{2t}}{h^2 e^t} xy,$$

and

$$F_5 = e^{(1+e^t x + 2y + te^{2t} x^2 + y^2 - e^{-2t} y^2 + e^t xy)/x^2 e^{2t}},$$

$$F_6 = e^{[e^t x^2 + (e^{-3t} - e^{-4t})y^2 + (e^{-t} + e^{-2t})xy]/x^2} e^{(e^{-t} x + e^{-3t} y)/x^2},$$

with cofactors

$$L_5 = - \frac{(2 - 2e^{-2t} + 4h^2 e^{-2t} - 4e^{-4t} h^2) y^2}{h^2} + \frac{(e^t + 2e^{-t} h^2) xy + (4h^2 e^{-2t} + 2)y - h^2}{h^2},$$

$$L_6 = - \frac{(2e^{-t} - 2e^{-2t} - 4e^{-4t} h^2 + 4e^{-3t} h^2) y^2 - h^2 e^t}{h^2} + \frac{(1 + e^t + 2e^{-2t} h^2 + 2h^2 e^{-t}) xy - (e^{-t} + 2e^{-3t} h^2) y}{h^2}.$$

Moreover system (41) has divergence equal to  $\text{div} = -(e^{2t}/h^2 + 2)x^2 + 3y$ . We note that

$$-2K_1 - K_2 + \text{div} + g = 0,$$

with  $g(t) = -1$ . So system (41) has the Jacobi multiplier

$$M = \frac{e^{-t}}{x^2 y}.$$

System (41) has degree  $m=3$ . Therefore, according to Theorem 2 statement (d) the set  $S = \{K_1, K_2, L_1, L_3, L_4, L_5, L_6\}$  is linearly dependent over  $\mathcal{C}^2(U, \mathbb{F})$ . We note that condition  $\mathcal{W}^*$  does not hold for  $S$  because the Wronskian of all the elements of  $S$  is nonzero (see Lemma 14).

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