Enumerating super edge-magic labelings for some types of path-like trees

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Abstract

The main goal of this paper is to use a variation of the Kronecker product of matrices in order to obtain lower bounds for the number of non isomorphic super edge-magic labelings of some types of path-like trees. As a corollary of the results obtained here we also obtain lower bounds for the number of harmonious labelings of the same type of trees.

Keywords: (Special) super edge-magic, harmonious, graceful, α-labeling, path-like tree.

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1 Introduction

For the undefined concepts and notation used in this paper, we refer the reader to either [9] or [13]. All graphs considered in this paper are simple, that is to say, they contain no loops or multiple edges. In 1998 Enomoto et al. [5] defined the concept of super edge-magic labeling as follows: a graph $G = (V, E)$ of order p and size q is super edge-magic if there is a bijective function $f : V \cup E \longrightarrow \{i\}_{i=1}^{p+q}$ such that (1) $f(V) = \{i\}_{i=1}^{p}$ and (2) $f(x) + f(xy) + f(y) = k \ \forall xy \in E$. The function f is called a super edge-magic labeling of G and k is called either the magic sum or the valence of f.

It is worthwhile mentioning that an equivalent labeling had already appeared in the literature in 1991 under the name of strongly indexable labeling [1], however the most popular term used nowadays is super edge-magic and we will keep this terminology through the rest of the paper.

The following lemma found in [7] characterizes super edge-magic labelings in terms of the labels of the vertices and provides us with an alternative definition of super edge-magic labelings that will prove to be very useful.

Lemma 1.1 *A graph $G = (V, E)$ of order p and size q is super edge-magic if and only if there is a bijective function $g : V \longrightarrow \{i\}_{i=1}^{p}$ such that $S = \{g(x) + g(y) : \ xy \in E\}$ is a set of exactly q consecutive integers. In such a case g can be uniquely extended to a super edge-magic labeling of G, namely f_g.*

In what follows, when we talk about super edge-magic labelings we mean a function as the function described in the above lemma, rather than the function as described in the original definition of Enomoto et al.

In [11] Muntaner-Batle introduced the concept of special super edge-magic labelings for bipartite graphs as follows: let $G = (V_1 \cup V_2, E)$ be a bipartite graph of order $p = p_1 + p_2$ where $p_i = |V_i|$ for $i \in \{1, 2\}$ and size q. A super edge-magic labeling f of G is called special super edge-magic if it has the extra property that $f(V_1) = \{i\}_{i=1}^{p_1}$. If a graph G admits a special super edge-magic labeling, then G is called a special super edge-magic graph. The concept of (special) super edge-magic labeling was generalized to digraphs in [6] in the way that a digraph $\vec{G} = (V, E)$ is called (special) super edge-magic if its underlying graph $und(\vec{G})$ is (special) super edge-magic. If we assume that each vertex of a (special) super edge-magic digraph $\vec{G} = (V, E)$ of order p takes the name of the label that some (special) super edge-magic labeling assignes to it, then we define the adjacency matrix of $\vec{G}$, denoted by $A(\vec{G}) = (a_{ij})$, to be the $p \times p$ matrix where

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{if } (i,j) \notin E. \end{cases}$$

In 1980, Graham and Sloan [8] introduced the concept of harmonious labelings and harmonious graphs as follows: a graph G with q edges is harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y)$ $(mod\ q)$ the resulting edge labels are distinct. When G is a tree, exactly one label may be used on two vertices.

Another modification of super edge-magic labeling, that is only applied to either paths or linear forests is the strong super edge-magic labeling [3].

Let $G = (V, E)$ be either a path or a linear forest, of order p and size q and assume that $f : V \longrightarrow \{1, 2, \ldots, p\}$ is a super edge-magic labeling of G with the extra property that if $xy \in E$ and $d_G(x, x') = d_G(y, y') < +\infty$, then we have that $f(x)+f(y) = f(x')+f(y')$. From now on, we will call this property strong. Then, we call f a strong super edge-magic labeling of G, and we call G a strong super edge-magic graph.

The type of graph labeling that has interested to the larger number of researchers is probably the graceful labeling. This concept was first defined by Rosa [12] and appeared under the name of β-valuation. The name graceful labeling appeared first in a paper of Golomb [10] and it is now become the most popular term to denote this labeling.

A function f is a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{i\}_{i=0}^{q}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.

The concept of α-valuation (also called α-labeling) was also introduced by Rosa in [12] and it is a restriction of graceful labelings for bipartite graphs.

An α-labeling (or α-valuation) of a graph G is a graceful labeling with the additional property that there exists an integer k so that for each edge xy of G either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$.

Next we introduce the concept of negative strong α-valuation for paths as follows: let $G = (V, E)$ be a path of order p and assume that $f : V \longrightarrow \{0, 1, \ldots, p - 1\}$ is an α-labeling of G with the extra property that if $xy \in E$ and $d_G(x, x') = d_G(y, y')$ then we have that $|f(x) - f(y)| = |f(x') - f(y')|$. From now on, we will call this property negative strong and f a negative strong α-labeling of G.

At this point all labelings that will appear in the rest of this paper have already been defined. Thus, we are now ready to introduce the other concepts and results that will be necessary in order to properly understand the results obtained this paper. Let us start with path-like trees, which were first introduced in Barrientos Ph.D. thesis [4] as follows: given an embedding of P_n in the 2-dimensional grid $P_n \times P_n$, we consider the ordered set of subpaths $L_1, L_2, \ldots, L_m$ which are maximal straight segments in the embedding, and such that the end of L_i is the beginning of L_{i+1}. Assume that $L_i \cong P_2$ for some i and that some vertex x of L_{i-1} is at distance 1 in the grid of some vertex y of L_{i+1}. An elementary transformation of the path consists in replacing the edge of

L_i by the new edge xy. We say that a tree T of order n is a path-like tree when it can be obtained from some embedding of P_n in the grid by a sequence of elementary transformations.

In [4] Barrientos proved that all path-like trees admit an α-valuation, and hence all path-like trees are graceful. He did this using the fact that the path P_n admits a negative strong α-labeling. From the proof provided by Barrientos it is very easy to obtain that all path-like trees are special super edge-magic by using a strong super edge-magic labeling of the path P_n, and hence they are also super edge-magic. Furthermore in [7] Figueroa-Centeno et al. proved that if a tree is super edge-magic, then it is also harmonious. Therefore all path-like trees are also harmonious.

Let l_1, l_2 be two labelings of the vertices of a graph G. We say that l_1 is isomorphic to l_2, namely $l_1 \cong l_2$, if and only if there exists an automorphism φ of G, such that $l_1(x) = l_2(\varphi(x))$ for all $x \in V(G)$.

Next let us define the following concept, that will be of help in order to find lower bounds for the number of non isomorphic harmonious labelings. Given a tree of order p, let l be a bijective labeling of the vertices with the numbers in the set $\{0, 1, \ldots, p-1\}$. The reduction of l, denoted by $red(l)$, is a new labeling of the tree in which each vertex takes the same label assigned by l reduced modulo $p-1$.

Note that two non-isomorphic labelings may have isomorphic reductions. For instance, take the star $K_{1,n}$ and consider two labelings of the star l_1 and l_2 defined as follows: l_1 assigns 0 to the central vertex of the tree and the remaining labels are assigned to the leaves randomly, while l_2 assigns n to the center of the star and the remaining labels are assigned to the leaves randomly. It is obvious that l_1 and l_2 are non isomorphic. However $red(l_1) \cong red(l_2)$. The next result caracteritzes when two labelings of a tree have isomorphic reductions.

Proposition 1.1 *Let T be a tree of order p and let l_1 and l_2 be two bijective labelings of the vertices of T onto the set $\{0, 1, \ldots, p-1\}$ such that $red(l_1) \cong red(l_2)$. Then either $l_1 \cong l_2$ or it is possible to obtain one labeling from the other by interchanging the labels 0 and $p-1$.*

Proof.
It is clear that if $l_1 \cong l_2$ then $red(l_1) \cong red(l_2)$. Assume that $red(l_1) \cong red(l_2)$ and let $\varphi \in Aut(T)$ be such that

$$red(l_1)(x) = red(l_2)(\varphi(x)), \text{ for all } x \in V(T).$$

Let $y, z \in V(T)$ such that $l_1(y) = 0$ and $l_1(z) = p-1$. Note that, since $l_i(x) = red(l_i)(x)$ for each $l(x) \notin \{0, p-1\}$, we only have two possibilities. Either $l_1(y) = l_2(\varphi(y))$ and $l_1(z) = l_2(\varphi(z))$, thus $l_1 \cong l_2$; or $l_1(y) = l_2(\varphi(z))$ and $l_1(z) = l_2(\varphi(y))$. In that case, the labeling that we obtain by interchanging the labels 0 and $p-1$ in l_1 is isomorphic to l_2. Indeed, let us define l_1' as follows:

$$l_1'(x) = \begin{cases} l_1(x) & x \notin \{y, z\}, \\ l_1(z) & x = y, \\ l_1(y) & x = z. \end{cases}$$

Then, $l_1'(x) = l_2(\varphi(x))$ for each $x \in V(T)$. □

Lemma 1.2 *Let T be a tree and denote by $Sem(T)$ and by $Harm(T)$ the sets of all non isomorphic super edge-magic labelings and the set of all non isomorphic harmonious labelings respectively. Then*

$$\frac{|Sem(T)|}{2} \leq |Harm(T)|$$

Proof.
Indeed, if M is a super edge-magic labeling of a tree T of order p then the labeling obtained by subtracting one unit to each label of M and reducing the resulting labels modulo $p-1$, provides a harmonious labeling of T. □

From the previous comments, we observe that path-like trees have nice labeling properties, since they admit many different types of labelings. However, as far as we know it has not been studied how many non isomorphic labelings of different types they admit. The goal in this paper is to find lower bounds for the number of non isomorphic super edge-magic labelings of path-like trees. We must say that we have not been able to apply the techniques that we use to all path-like trees, but only to a certain subset of them. However, when the techniques apply, we are able to find an exponential number of such labelings. As a corollary, we also obtain exponential lower bounds for the number of non isomorphic harmonious labelings. The way we do it uses the following operation on digraphs that was first introduced in [6].

Let D be a digraph and let $\Gamma = \{F_1, F_2, \ldots, F_s\}$ be a family of digraphs such that $V(F_i) = V$ for every $i \in \{1, \ldots, s\}$. Consider a function $h : E(D) \longrightarrow \Gamma$, then the product $D \otimes_h \Gamma$ is a digraph with vertex set $V(D) \times V$ and $((a, b), (c, d)) \in E(D \otimes_h \Gamma) \iff (a, c) \in E(D) \wedge (b, d) \in E(h(a, c))$. The adjacency matrix of $D \otimes_h \Gamma$, $A(D \otimes_h \Gamma)$, is obtained by multiplying every 0 entry of $A(D)$ by the $|V| \times |V|$ nul matrix and every 1 entry of $A(D)$ by $A(h(a, c))$. Notice that when h is constant, this operation coincides with the classical Kronecker product of matrices. From now on, let S_p denote the set of all super edge-magic 1-regular labeled digraphs of odd order p where each vertex takes the name of the label that has been assigned to it.

The following result was stablished in [6]:

Lemma 1.3 *Let D be a super edge-magic digraph and let $h : E(D) \longrightarrow S_p$ be any function. Then $und(D \otimes_h S_p)$ is super edge-magic.*

For the rest of the paper let $\overrightarrow{C}_n$ denote a cycle of order n where the vertices take the name of the labels of a super edge-magic labeling with a strong orientation. Also when we write the set $\{\overrightarrow{C}_n, \overleftarrow{C}_n\}$ we mean the set of cycles where the vertices take the labels of the same super edge-magic labeling and the edges of each cycle are oriented with a different strong orientation. Then since $\{\overrightarrow{C}_n, \overleftarrow{C}_n\} \subseteq S_n$ the product defined in the statement of the following lemma found in [2] makes sense.

Lemma 1.4 *Let $m, n \in \mathbb{N}$ and consider the product $\overrightarrow{C}_m \otimes_h \{\overrightarrow{C}_n, \overleftarrow{C}_n\}$ where $h : E(\overrightarrow{C}_m) \longrightarrow \{\overrightarrow{C}_n, \overleftarrow{C}_n\}$. Let g be a generator of a cyclic subgroup of $\mathbb{Z}_n$, namely $< g >$, such that $| < g > | = k$. Also let $N_g(h^-) < m$ be a natural number that satisfies the following congruence relation*

$$m - 2N_g(h^-) \equiv g \ (mod \ \ n).$$

If the function h assigns $\overleftarrow{C}_n$ to exactly $N_g(h^-)$ arcs of $\overrightarrow{C}_m$ then the product

$$\overrightarrow{C}_m \otimes_h \{\overrightarrow{C}_n, \overleftarrow{C}_n\}$$

consists of exactly $\frac{n}{k}$ disjoint copies of a strongly oriented cycle $\overrightarrow{C}_{mk}$. In particular if $gcd(g, n) = 1$, then $< g >= \mathbb{Z}_n$ and if the function h assigns $\overleftarrow{C}_n$ to exactly $N_g(h^-)$ arcs of $\overrightarrow{C}_m$ then

$$\overrightarrow{C}_m \otimes_h \{\overrightarrow{C}_n, \overleftarrow{C}_n\} \cong \overrightarrow{C}_{mn}.$$

In [3] Bača et al. proved the following result:

Lemma 1.5 *Let C_n be a cycle on n vertices, $n \geq 11$ odd. The number of non isomorphic super edge-magic labelings of C_n is at least $\dfrac{5}{4}\, 2^{\lfloor \frac{n}{3} \rfloor} + 1$.*

2 Counting strong super edge-magic labelings of P_n

This section is devoted to count the number of strong super edge-magic labelings of P_n. We denote by $V(P_n) = \{u_i\}_{i=1}^n$ and $E(P_n) = \{u_i u_{i+1}\}_{i=1}^{n-1}$. Next let us introduce the following lemma.

Lemma 2.1 *Let $f : V(P_n) \longrightarrow \{i\}_{i=1}^{n}$ be a vertex labeling of P_n such that*

$$f(u_j) + f(u_{j+1}) = f(u_{j-k}) + f(u_{j+1+k}),$$

for each $k \leq \min\{j-1, n-j-1\}$. If $f(u_1) = a, f(u_2) = b$ and $f(u_3) = c$ then

$$f(u_{2i-1}) = a + (i-1)(c-a) \text{ and } f(u_{2i}) = b + (i-1)(c-a)$$

for each i such that $1 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$.

Proof.
Let us prove by induction that $f(u_{2i-1}) = a + (i-1)(c-a)$, $f(u_{2i}) = b + (i-1)(c-a)$ and $f(u_{2i+1}) = c + (i-1)(c-a)$.

The result is clearly true for $i = 1$. Suppose that the result holds for $i = l$ and let us prove it for $i = l + 1$.

$$f(u_{2(l+1)-1}) = f(u_{2l+1}) = a + (c-a) + (l-1)(c-a) = a + l(c-a).$$

Also,

$$f(u_{2(l+1)}) = f(u_{2l}) + f(u_{2l+1}) - f(u_{2l-1}) = b + l(c-a).$$

Finally,

$$f(u_{2(l+1)+1}) = f(u_{2l+1}) + f(u_{2l+2}) - f(u_{2l}) = c + l(c-a).$$

□

Corollary 2.1 *Let $f : V(P_n) \longrightarrow \{i\}_{i=1}^{n}$, be a super edge–magic labeling of P_n such that $f(u_j) + f(u_{j+1}) = f(u_{j-k}) + f(u_{j+1+k})$, for each $k \leq \min\{j-1, n-j-1\}$. If $f(u_1) = a, f(u_2) = b$ and $f(u_3) = c$ then*

- *either $c - a = 1$ and $(a, b) \in \{(1, \lceil \frac{n}{2} \rceil + 1), (\lfloor \frac{n}{2} \rfloor + 1, 1)\}$, or*
- *$c - a = -1$ and $(a, b) \in \{(\lceil \frac{n}{2} \rceil, n), (n, \lfloor \frac{n}{2} \rfloor)\}$.*

Proof.
Let us consider the sums $e_j = f(u_j) + f(u_{j+1})$, for each $j = 1, \ldots, n-1$. By Lemma 2.1, we have:

$$e_{2i} = f(u_{2i}) + f(u_{2i+1}) = b + (i-1)(c-a) + a + i(c-a) = a + b + (2i-1)(c-a).$$

Similarly,

$$e_{2i-1} = f(u_{2i-1}) + f(u_{2i}) = a + (i-1)(c-a) + b + (i-1)(c-a) = a + b + (2i-2)(c-a).$$

Hence, $e_j = a + b + (j-1)(c-a)$, $j = 1, \ldots, n-1$. Therefore, by Lemma 1.1 either $c - a = 1$ or $c - a = -1$. Moreover, since the labels belong to the set $\{1, 2, \ldots, n\}$ it is easy to check that we only have four possibilities for the pair (a, b) and the result follows. □

Thus, we have that any strong super edge-magic labeling of P_n is absolutely determined by the labels a, b, c and there are only four possibilities for the labels a, b and c. Each one of which provides a special super edge-magic labeling of P_n. Therefore we have proven that:

Theorem 2.1 *P_n admits exactly four strong super edge-magic labelings, but only two are non isomorphic. Moreover, each one of them is special.*

Next we will show that P_n admits exactly four negative strong α-labelings two of which are non isomorphic. Let $\mathfrak{S}_n$ be the set of strong super edge-magic labelings of P_n and let $\mathfrak{S}_n^-$ be the set of negative strong α-labelings of P_n.

Theorem 2.2 $|\mathfrak{S}_n| = |\mathfrak{S}_n^-|$.

Proof.
Let V_1 and V_2 be the stable sets of P_n and let $f : \mathfrak{S}_n \longrightarrow \mathfrak{S}_n^-$ be the function defined by the rule $f(h) = \bar{h}$ where

$$\bar{h}(u) = \begin{cases} h(u) & \text{whenever } u \in V_1, \\ (n+1) + \lceil \frac{n}{2} \rceil - h(u) & \text{otherwise.} \end{cases}$$

Next we have to show that f is a bijective function. It was stablished in [7] that the function f transforms special super edge-magic labelings of trees into α-labelings. Since all strong super edge-magic labelings of P_n are special super edge-magic, it follows that the images of these labelings under the function f are α-labelings. Hence, we assume that h is a strong super edge-magic labeling. We will show that $\bar{h}$ is a negative strong α-labeling. Let $xy \in E(P_n)$ and assume that $d(x, x') = d(y, y')$ where $\{x', y'\} \subseteq V(P_n)$. Since h is a strong super edge-magic labeling, it follows that $h(x) + h(y) = h(x') + h(y')$. Without loss of generality assume that $x \in V_1$ and $y \in V_2$. We consider two cases.

Case 1: $x' \in V_1$ and $y' \in V_2$.

$$\begin{aligned} &\Longrightarrow\ h(x) + h(y) = h(x') + h(y') \\ &\Longrightarrow\ h(x) - (n+1) - \lceil \tfrac{n}{2} \rceil + h(y) = h(x') - (n+1) - \lceil \tfrac{n}{2} \rceil + h(y') \\ &\Longrightarrow\ h(x) - [(n+1) + \lceil \tfrac{n}{2} \rceil - h(y)] = h(x') - [(n+1) + \lceil \tfrac{n}{2} \rceil - h(y')] \\ &\Longrightarrow\ \bar{h}(x) - \bar{h}(y) = \bar{h}(x') - \bar{h}(y'). \end{aligned}$$

Case 2: $x' \in V_2$ and $y' \in V_1$. This case is similar to case 1.

Let us see now that the preimage of any strong negative α-labeling is a strong super edge-magic labeling.

Assume that $\bar{h}$ is a strong negative α-labeling of P_n. Then the preimage of $\bar{h}$ under f, namely h, is a special super edge-magic labeling. We will show that h is strong.

Let $xy \in E(P_n)$ and assume that $d(x, x') = d(y, y')$ where $\{x', y'\} \subseteq V(P_n)$. Since h is a negative strong α-labeling, it follows that $|h(x) - h(y)| = |h(x') - h(y')|$. Without loss of generality assume that $x \in V_1$ and $y \in V_2$. We consider two cases:

Case 1: $x' \in V_1$ and $y' \in V_2$.

$$\begin{aligned}
&\Longrightarrow \quad |\bar{h}(x) - \bar{h}(y)| = |\bar{h}(x') - \bar{h}(y')| \\
&\Longrightarrow \quad \bar{h}(x) - \bar{h}(y) = \bar{h}(x') - \bar{h}(y') \\
&\Longrightarrow \quad h(x) - [(n+1) + \lceil \tfrac{n}{2} \rceil - h(y)] = h(x') - [(n+1) + \lceil \tfrac{n}{2} \rceil - h(y')] \\
&\Longrightarrow \quad h(x) + h(y) = h(x') + h(y')
\end{aligned}$$

Case 2: $x' \in V_2$ and $y' \in V_1$. This case is basically similar to case 1.

Finally, it is clear that the image of two non isomorphic labelings under f are non isomorphic labelings. □

3 m-labelings of P_n

The fact that all path-like trees are super edge-magic depends, in a way, on the fact that the path P_n is strong super edge-magic, since any strong super edge-magic labeling of P_n can be used in order to obtain super edge-magic labelings of path-like trees. In the previous section we have proven that there exist exactly two non isomorphic strong super edge-magic labelings of the path P_n, for every $n \geq 4$. Unfortunately, in order to obtain a non trivial number of non isomorphic super edge-magic labelings of path-like trees, this number seems to be not enough. The goal in this section is to introduce a new type of super edge-magic labeling for paths, that although it is not strong, it is close to be strong and also serves to our purpose of obtaining super edge-magic labelings of path-like trees from them. This type of labeling that we refer to, we call it m-labeling and it is inspired in the strong super edge-magic labelings for paths. Let us introduce the following example of a super edge-magic labeling of P_{25}:

$1 - 19 - 6 - 24 - 11 - 4 - 17 - 9 - 22 - 14 - 2 - 20 - 7 - 25 - 12 - 5 - 18 - 10 - 23 - 15 - 3 - 16 - 8 - 21 - 13$.

Notice that when we read the set of labels from left to right and divide it into disjoint groups of five consecutive labels we obtain:

$$1, 19, 6, 24, 11.\sharp \quad 4, 17, 9, 22, 14.\sharp \quad 2, 20, 7, 25, 12.\sharp \quad 5, 18, 10, 23, 15.\sharp \quad 3, 16, 8, 21, 13.$$

Then for each edge xy and two vertices α, β of the same group of x, y such that $d(\alpha, x) = d(y, \beta)$ then $x + y = \alpha + \beta$. From now on a super edge-magic labelings of P_n such that the labels of P_n can be divided into k groups of length m (so that $km = n$) with the property shown in the previous example, will be called a m-labeling.

In [2] it was established an algorithm that allows us to create strong super edge-magic labelings of linear forests with an odd number of components where each component has the same order.

Next, we will modify it, so that the new resulting algorithm will allow us to create m-labelings for paths of certain lengths.

Algorithm

Input:

1. Let m be an odd number. Oriented cycle $\overrightarrow{C}_m$ with:
   - Vertex set $V(\overrightarrow{C}_m) = \{v_i\}_{i=1}^m$ and $E(\overrightarrow{C}_m) = \{(v_i, v_{i+1})\}_{i=1}^{m-1} \cup \{(v_m, v_1)\}$
   - Consider a function $f : V(\overrightarrow{C}_m) \longrightarrow \{i\}_{i=1}^m$ defined by the rule

$$f(v_i) = \begin{cases} \dfrac{i+1}{2}, & \text{if } i \text{ is odd,} \\ \lceil \dfrac{m}{2} \rceil + \dfrac{i}{2}, & \text{if } i \text{ is even.} \end{cases}$$

2. The set $\Gamma_n = \{F_1, F_1', \ldots, F_{\frac{s}{2}}, F_{\frac{s}{2}}'\}$ is the family of all connected 1-regular digraphs of order $n = 2K + 1$ where each digraph is labeled in a super edge-magic way and each vertex takes the name of its label. Each couple (F_j, F_j') comes from the same underlying 2-regular graph but it has been oriented in opposite way.

3. Let (F, F') be a fixed cuple from Γ_n. Define a function $h : E(\overrightarrow{C}_m) \longrightarrow \Gamma_n$ with

$$h(v_{i-1}v_i) = \begin{cases} F & \text{whenever } i \text{ is even,} \\ F' & \text{otherwise,} \end{cases}$$

   and $h(v_m v_1) \in \{F, F'\}$.

Algorithm

1. Rename each vertex of $\overrightarrow{C}_m$ with the name of its label, creating a new super edge-magic digraph $\overrightarrow{C}^l_m$, with adjacency matrix $A(\overrightarrow{C}^l_m)$.
2. Compute $\overrightarrow{C}^l_m \bigotimes_h \Gamma_n \cong \overrightarrow{C}_{mn}$.
3. Let $(x_i, y_i) \in V(\overrightarrow{C}^l_m \bigotimes_h \Gamma_n)$. Remove the directions and relabel the vertex (x_i, y_i) with z_i where $z_i = n(x_i-1)+y_i$ creating a new graph C^l_{mn}. Remove the edge with the minimum label, namely e, and consider the labeled graph $Q = C^l_{mn} \setminus \{e\}$.

Output

The labeling we obtain of P_{mn} is a m-labeling.

Theorem 3.1 *The graph $Q = C^l_{mn} \setminus \{e\}$, where e is the edge of C^l_{mn} such that the sum of the vertices incident with e is minimum, is a m-labeling of P_{mn}.*

Proof.
The relation $\overrightarrow{C}^l_m \bigotimes_h \Gamma_n \cong \overrightarrow{C}_{mn}$ holds by Lemma 1.4. Indeed, if we assign to (v_m, v_1), F then we are taking $g = 1$ in Lemma 1.4, meanwhile if we assign to (v_m, v_1), F' then we are taking $g = -1$. Moreover, as it was proven in [6], when we relabel each vertex (x_i, y_i) of $und(\overrightarrow{C}_{mn})$ with z_i where $z_i = n(x_i - 1) + y_i$, the result is a super edge-magic labeled graph that we denote by C^l_{mn}. Thus, $Q = C^l_{mn} \setminus \{e\}$ where e is the edge with minimum label, is a super edge-magic labeled path. Next, we read the labeling obtained from left to right, and we consider the linear forest of n components obtained from Q, where the first component consist of the subpath of Q induced by the first m vertices, the second component consist of the subpath obtained by the next m vertices of Q and so on. Notice that this linear forest is the linear forest that the algorithm in [3] would produce if it had been applied to the digraph $\overrightarrow{C}_m \setminus \{(v_m, v_1)\}$. Therefore, the linear forest has been labeled in a strong super edge-magic way. Thus, the labeling of Q is an m-labeling. □

4 Non isomorphic labelings of certain path-like trees

In this section we prove that the number of super edge-magic labelings of a fixed path-like tree that we obtain from a m-labelings of a path with the same order grows exponencially with respect to the order. The idea is the following. Let m, n be odd numbers. When we apply the algorithm of the previous section, we obtain a super edge-magic path of order mn, that can be partioned into m-subpaths: $P^1_m, \ldots, P^n_m$, each of them having the strong property ($\forall xy \in E(P^i_m)$ and $\forall x', y' \in V(P^i_m)$ such that $d_G(x, x') = d_G(y, y')$, then $f(x) + f(y) = f(x') + f(y')$). In each of these subpaths we can apply elementary transformations in order to obtain a super edge-magic path-like tree. In this construction a super edge-magic labeling l of $\vec{C}_m$ appears. Let $A(\vec{C}^l_m)$ be the adjacency matrix induced by this labeling.

Let us repeat the construction by replacing l by a new super edge-magic labeling l'. Then at least two entries of $A(\vec{C}^l_m)$ and $A(\vec{C}^{l'}_m)$ should be different. Thus, the corresponding adjacency matrices of $\vec{C}_{mn}$ induced by the two labelings should have at least $2m$ different entries.

Now we present a new description of path-like trees that will be useful in order to describe the families of path-like trees that have an exponencial number of non isomorphic super edge-magic labelings.

Embed in a horizontal line a linear forest with consecutive components $l_1, \ldots, l_n$ drawn from left to right. The end vertices of l_i ($i \in \{1, \ldots, n\}$) are a_i, b_i where vertex a_i is to the left of vertex b_i. Then, for every $i \in \{1, \ldots, n\}$ we join a vertex v_i of component l_i with a vertex v_{i+1} of component l_{i+1}, where $d(v_i, b_i) = d(a_{i+1}, v_{i+1})$. Each path-like tree can be obtained in this way. The set of path-like trees that we will consider have the following properties:

1. The edges that join vertices of two different components l_i and l_{i+1} of the linear forest, are never incident with a terminal vertex. Therefore the end vertices of these edges have always degree at least 3.

2. The resulting tree never contains vertices of degree 4.

3. There is no subpath in the resulting tree, that contains all vertices of degree 3 in the tree and has order greater than 2.

Suppose that we are considering a path-like tree in which the three properties hold. In the next lemma we prove that two labelings of such a path-like tree are isomorphic if and only if their restrictions to the forest that we obtain by removing the edges incident to the vertices of degree three are isomorphic.

Lemma 4.1 *Assume that a path-like tree T has the three properties described before and let λ_1, λ_2 be two bijective labelings of $V(T)$. Let $e_1, \ldots, e_k$ be the set of edges incident with two vertices of degree 3. If λ_1 and λ_2 are isomorphic then the labelings that result from λ_1 and from λ_2 removing the edges $e_1, \ldots, e_k$, namely $\bar{\lambda}_1$ and $\bar{\lambda}_2$ respectively, are also isomorphic.*

Proof.
Let F be the forest that we obtain by removing the edges $e_1, \ldots, e_k$. For $i = 1, 2$ denote by $E(\bar{\lambda}_i)$ the set of edges of F in which each vertex is renamed by the label of λ_i. If $\lambda_1 \cong \lambda_2$ and $\bar{\lambda}_1 \ncong \bar{\lambda}_2$, then there is an edge $xy \in E(\bar{\lambda}_1) \setminus E(\bar{\lambda}_2)$. Since $\lambda_1 \cong \lambda_2$ it follows

that we have included the edge xy in order to obtain λ_2. The only way we have to do this is by adding a new edge xy. Hence x, y become vertices of degree 3 in λ_2. But since $xy \in E(\bar{\lambda}_1)$ it follows by Property 3 that $\min\{deg_{\lambda_1}(x), deg_{\lambda_2}(y)\} \leq 2$. Therefore $\lambda_1 \ncong \lambda_2$. □

Let m, n be positive odd integers. Let $\mathfrak{F}_{mn}$ be the set of path-like trees such that $T \in \mathfrak{F}_{mn}$ if it can be obtained by elementary transformations in each subpath of length m that belongs to a partition of P_{mn}, in such a way that, the three properties hold and the number of vertices incident with two vertices of degree 3 is less than $2m - 1$. Therefore we have that:

Theorem 4.1 *Let m, n be odd integers, $n \geq 11$. If $T \in \mathfrak{F}_{mn}$ then the number of non isomorphic super edge-magic labelings of T is at least*

$$\frac{5}{2}\, 2^{\lfloor \frac{n}{3} \rfloor} + 2.$$

Proof.
Let $h : E(\overrightarrow{C}_m) \longrightarrow \Gamma_n$ and $h' : E(\overrightarrow{C}_m) \longrightarrow \Gamma_n$ be two different functions, then:

Fact 1: The m-labelings induced in P_{mn} by the respective products $\overrightarrow{C}^l_m \bigotimes_h \Gamma_n$ and $\overrightarrow{C}^l_m \bigotimes_{h'} \Gamma_n$ are non isomorphic. Moreover they contain at least $2m$ different edges. Indeed, if the two induced m-labelings of P_{mn} were isomorphic, then the linear forest formed by the partition of each P_{mn} into m-subpaths would also be isomorphic. But, our algorithm coincides with the algorithm in [3] if it had been applied to the digraph $\overrightarrow{C}_m \setminus \{v_m, v_1\}$. Thus , by Theorem 2.1 in [3] they are non isomorphic. Let us see the second part. We know that the adjacency matrices of the product contain at least $2m$ different entries. Notice that, for $a_{ij} \in A(\overrightarrow{C}_m)$ if $a_{ij} = 1$ then $a_{ji} = 0$. Therefore, the underlying graph contains at least $2m$ different edges.

Fact 2: The labelings of T obtained by elementary transformations of $und(\overrightarrow{C}^l_m \bigotimes_h \Gamma_n)$ and $und(\overrightarrow{C}^l_m \bigotimes_{h'} \Gamma_n)$ are non isomorphic. Otherwise, by Lemma 4.1 if we delete the edges incident with the vertices of degree three of T the induced labelings in the linear forest would be isomorphic, but the number of these edges is by definition less than $2m - 1$, a contradiction.

Fact 3: By Lemma 1.5 the number of non isomorphic super edge-magic labelings of $\overrightarrow{C}_n$ is at least

$$\frac{5}{4}\, 2^{\lfloor \frac{n}{3} \rfloor} + 1.$$

Fact 4: Finally, the number of functions of the form $h : E(\overrightarrow{C}_m) \longrightarrow \Gamma_n$ is at least $\frac{5}{2}\, 2^{\lfloor \frac{n}{3} \rfloor} + 2$.

Indeed, each couple is related to a super edge-magic labeling of $\overrightarrow{C}_n$, and it has two possible orientations. □

Corollary 4.1 *Let m, n be odd integers, $n \geq 11$. If $T \in \mathfrak{F}_{mn}$ then the number of non isomorphic harmonius labelings of T is at least*

$$\frac{5}{4}\, 2^{\lfloor \frac{n}{3} \rfloor} + 1.$$

Proof.
It follows from the previous theorem and Lemma 1.2 □

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