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A CANONICAL REDUCED FORM FOR SINGULAR TIME INVARIANT LINEAR SYSTEMS

M. Isabel García-Planas
Departament Matemàtica Aplicada I
Universitat Politècnica
Spain
maria.isabel.garcia@upc.edu

Adolfo Díaz
Departament Matemàtica Aplicada I
Universitat Politècnica de Catalunya
Spain
adiaz_pr@yahoo.com

Abstract

We consider quadruples of matrices (E, A, B, C), representing singular linear time invariant systems in the form

$$\left.\begin{aligned} E\dot{x}(t) &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \tag{1}$$

with $E, A \in M_{p\times n}(C)$, $B \in M_{p\times m}(C)$ and $C \in M_{q\times n}(C)$ under proportional and derivative feedback and proportional and derivative output injection.

In this paper we present a canonical reduced form preserving the structure of the system and provides a decomposition of the system into two independent systems, one being a maximal regular system and the second one a minimal completely singular one.

Key words

Singular Systems, equivalence relation, canonical form.

1 Introduction

We denote by $M_{p\times q}(C)$ the space of complex matrices having p rows and q columns, and in the case which $p = q$ we write $M_n(C)$. We consider the set $\mathcal{M}$ of quadruples of matrices (E, A, B, C) representing families of singular linear time invariant systems in the form $E\dot{x} = Ax+Bu$, $y = Cx$ with $E, A \in M_{p\times n}(C)$, $B \in M_{p\times m}(C)$ and $C \in M_{q\times n}(C)$, and we define an equivalence relation that permit us to decompose the system into two independent systems one being a maximal regular system and the second a minimal completely singular one. Each one of the subsystems decompose into independent systems in the form.

$$\begin{cases} \dot{x}_1 = N_2x_1 + B_1u \\ y_1 = C_1x_1 \end{cases} \tag{2}$$

$$\begin{cases} \dot{x}_2 = N_3x_2 + B_2u \end{cases} \tag{3}$$

$$\begin{cases} \dot{x}_3 = N_4x_3 \\ y_3 = C_2x_3 \end{cases} \tag{4}$$

$$\begin{cases} \dot{x}_4 = Jx_4 \end{cases} \tag{5}$$

$$\begin{cases} N_1\dot{x}_5 = x_5 \end{cases} \tag{6}$$

$$\begin{cases} L_1\dot{x}_6 = R_1x_6 \end{cases} \tag{7}$$

$$\begin{cases} L_2^t\dot{x}_7 = R_2^tx_7 \end{cases} \tag{8}$$

$$\{B_3u_3 = 0 \quad \text{or} \quad \{C_3x_8 = 0. \tag{9}$$

System (2) is a maximal controllable and observable subsystem, system (3) is the maximal controllable no observable subsystem, system (4) is the maximal observable no controllable subsystem , system (5) is a standard system with no modifiable finite zeros, system (6) is a system containing all no transferable infinite zeros, and finally systems (7) and (8) are completely singular systems.

The equivalence relation considered is the one that accept one or more, of the following transformations: basis change in the state space, input space, output space, feedback, derivative feedback, output injection, derivative output injection and premultiplication by an invertible matrix.

In the sequel we will use the following notations.
- I_n denotes the n-order identity matrix,
- N denotes a nilpotent matrix in its reduced form $N = \operatorname{diag}(N_1, \ldots, N_\ell)$, $N_i = \begin{pmatrix} 0 & I_{n_i - 1} \\ 0 & 0 \end{pmatrix} \in M_{n_i}(C)$,
- J denotes the Jordan matrix $J = \operatorname{diag}(J_1, \ldots, J_t)$, $J_i = \operatorname{diag}(J_{i_1}, \ldots, J_{i_s})$, $J_{i_j} = \lambda_i I_{i_j} + N$,
- L denotes the diagonal matrix $L = \operatorname{diag}(L_1, \ldots, L_q)$, where $L_j = \begin{pmatrix} I_{n_j} & 0 \end{pmatrix} \in M_{n_j \times (n_j+1)}(C)$,
- R denotes the diagonal matrix $R = \operatorname{diag}(R_1, \ldots, R_p)$, where $R_j = \begin{pmatrix} 0 & I_{n_j} \end{pmatrix} \in M_{n_j \times (n_j+1)}(C)$,
- t, r_c y r_o determine the quantity of controllable and observable, controllable no observable and observable no controllable blocks that appear in an standard triple of matrices.e

2 Equivalence of singular systems

We consider singular linear as in (1), many interesting and useful equivalence relations between singular systems have been defined. As pointed in the introduction, we deal with the equivalence relation accepting one or more, of the following transformations: basis change in the state space, input space, output space, operations of state and derivative feedback, state and derivative output injection and to premultiply the first equation in (1), by an invertible matrix. That is to say.

Definition 1. *Two quadruples $(E_i, A_i, B_i, C_i) \in \mathcal{M}, i = 1, 2$, are equivalent if and only if there exist matrices $P \in Gl(n; C)$, $Q \in Gl(p; C)$, $R \in Gl(m; C)$, $S \in Gl(q; C)$, $F_E^B, F_A^B \in M_{m\times n}(C)$, $F_E^C, F_A^C \in M_{p\times q}(C)$ such that*

$$\begin{aligned} E_2 &= QE_1P + QB_1F_E^B + F_E^C C_1 P, \\ A_2 &= QA_1P + QB_1F_A^B + F_A^C C_1 P, \\ B_2 &= QB_1R, \\ C_2 &= SC_1P. \end{aligned}$$

Given a quadruple of matrices $(E, A, B, C) \in \mathcal{M}$, we can associate the following matrix pencil

$$H(\lambda) = \begin{pmatrix} \lambda E + A & \lambda B & B \\ \lambda C & 0 & 0 \\ C & 0 & 0 \end{pmatrix},$$

and we have

Proposition 1. *Two quadruples are equivalent under equivalent relation considered if and only if the associates matrix pencils are strictly equivalent.*

So, we can apply kroneckers theory of singular pencils (see [5]).

Corollary 1. *Let $H(\lambda)$ be a matrix pencil associated to the quadruple $(E, A, B, C) \in \mathcal{M}$. Then $H(\lambda)$ it is equivalent to the pencil $\lambda F + G$ with*

$$F = \begin{pmatrix} L & & & \\ & L^t & & \\ & & I_1 & \\ & & & N \end{pmatrix}, G = \begin{pmatrix} R & & & \\ & R^t & & \\ & & J & \\ & & & I_2 \end{pmatrix}$$

Then, we can reduce any quadruple to a simpler form. This reduced form does not preserve the structure of the system, so, we propose the following reduced form as starting point to obtain the desired reduction.

Theorem 1. *Let $(E, A, B, C) \in \mathcal{M}$ be a quadruple of matrices. Then it is equivalent under equivalence relation considered, to a quadruple $(E_\omega, A_\omega, I_\omega, I_\omega)$ in the following form:*

$$\left(\begin{pmatrix} \overline{E} & \\ & 0 \end{pmatrix}, \begin{pmatrix} \overline{A} & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & I_b \end{pmatrix}, \begin{pmatrix} 0 & \\ & I_c \end{pmatrix} \right)$$

where $(\overline{E}, \overline{A})$ is a pair in its Kronecker reduced form.

A collection of structural invariants to the quadruple $(E_\omega, A_\omega, I_\omega, I_\omega)$ characterizing equivalent quadruples is the following collection of numbers

i) $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_z \geq 1$ nilpotency indices
ii) $k_1(\lambda) \geq k_2(\lambda) \geq \ldots \geq k_{j_\lambda}(\lambda) \geq 1$ Segre characteristic corresponding to eigenvalue λ
iii) $\epsilon_1 \geq \ldots \geq \epsilon_{r_\epsilon} > \epsilon_{r_\epsilon+1} = \ldots = \epsilon_{r_k} = 0$ column minimal indices
iv) $\eta_1 \geq \ldots \geq \eta_{l_\eta} > \eta_{l_\eta+1} = \ldots = \eta_{l_k} = 0$ row minimal indices

3 New canonical reduced form

Theorem 2. *Let $(E, A, B, C) \in \mathcal{M}$ be a quadruple of matrices. Then it can be reduced under equivalence relation considered, to the following reduced form (E_c, A_c, B_c, C_c):*

$$\left(\begin{pmatrix} I_1 & 0 \\ 0 & E_k \end{pmatrix}, \begin{pmatrix} A_e & \\ & A_k \end{pmatrix}, \begin{pmatrix} B_e & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} C_e & 0 \\ 0 & 0 \end{pmatrix} \right)$$

where (A_e, B_e, C_e) and (E_k, A_k) are in its Kronecker reduced form (see [6], [3] respectively).

Proof. Let (E, A, B, C) be a quadruple and $(E_\omega, A_\omega, I_\omega, I_\omega)$ its reduced form given in Theorem 1. So, the quadruple $(E_\omega, A_\omega, I_\omega, I_\omega)$ is partitioned in the following manner

$$\left(\begin{pmatrix} E_r & & \\ & E_s & \\ & & 0 \end{pmatrix}, \begin{pmatrix} A_r & & \\ & A_s & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & I_c \\ 0 & 0 & 0 \end{pmatrix} \right)$$

with

$$(E_r, A_r) = \left(\begin{pmatrix} I_1 & & \\ & I_2 & \\ & & N \end{pmatrix}, \begin{pmatrix} J & & \\ & J_0 & \\ & & I_3 \end{pmatrix} \right)$$

and

$$(E_s, A_s) = \left(\begin{pmatrix} L & \\ & L^t \end{pmatrix}, \begin{pmatrix} R & \\ & R^t \end{pmatrix} \right)$$

We proceed by performing the following steps.

Step 1: We consider the subquadruple (E', A', B', C') :

$$\left(\begin{pmatrix} L & & \\ & L^t & \\ & & 0 \end{pmatrix}, \begin{pmatrix} R & & \\ & R^t & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ I_b \end{pmatrix}, (0\ 0\ I_c) \right)$$

and we take the subtriple:

$$\left(\begin{pmatrix} L \\ 0 \end{pmatrix}, \begin{pmatrix} R \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_b \end{pmatrix} \right)$$

and we distinguish two cases depending on relation between r_k and b.

a) $r_k \leq b$

In this case the triple is written

$$\left(\begin{pmatrix} L & \\ 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} R & \\ 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ I_{r_c} & 0 \\ 0 & I_t \end{pmatrix} \right)$$

with $r_c = r_k$, $t = b - r_c \geq 0$. Then, it is controllable.

Detailing the subtriple $\left(\begin{pmatrix} L \\ 0 \end{pmatrix}, \begin{pmatrix} R \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_{r_c} \end{pmatrix} \right)$, we have

$$\begin{pmatrix} L \\ 0 \end{pmatrix} = \begin{pmatrix} L_1 & & & & 0 \\ 0 & & & & 0 \\ & L_2 & & & 0 \\ & 0 & & & 0 \\ & & \ddots & & \vdots \\ & & & L_{r_\epsilon} & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ I_{r_c} \end{pmatrix} = \begin{pmatrix} 0 & & & & 0 \\ 1 & & & & 0 \\ & 0 & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 0 & 0 \\ & & & 1 & 0 \\ 0\ 0 & & & 0 & I_{r_c - r_\epsilon} \end{pmatrix}$$

$$\begin{pmatrix} R \\ 0 \end{pmatrix} = \begin{pmatrix} R_1 & & & & 0 \\ 0 & & & & 0 \\ & R_2 & & & 0 \\ & 0 & & & 0 \\ & & \ddots & & \vdots \\ & & & R_{r_\epsilon} & 0 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

It is easy to verify that the controllability indices of each subsystem

$$\left(\begin{pmatrix} L_i \\ 0 \end{pmatrix}, \begin{pmatrix} R_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ B_i \end{pmatrix} \right) = \left(\begin{pmatrix} I_{\epsilon_i} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{\epsilon_i} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$i = 1, \dots, r_c$ are $k_i^{c\bar{o}} = \epsilon_i + 1$.

Fixed t, we have that the maximal quantity of observable no controllable blocks in the pair (L^t, R^t) is $r_o = c - t$. Then if $l_k > r_o$, we have that the subtriple $((L^t\ 0), (R^t\ 0), (0\ I_c))$ is completely no observable.

We write this triple in the form

$$\left(\begin{pmatrix} L_1^t\ 0 & \\ & L_2^t\ 0 \end{pmatrix}, \begin{pmatrix} R_1^t\ 0 & \\ & R_2^t\ 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{r_0} & 0 & 0 \\ 0 & 0 & 0 & I_t \end{pmatrix} \right)$$

where the pair (L_1^t, R_1^t) contains the first r_o-blocks and the pair (L_2^t, R_2^t) contains the rest of $l = l_k - r_o$-blocks.

The subtriple $((L_1^t\ 0), (R_1^t\ 0), (0\ I_{r_o}))$, written in the form

$$\begin{pmatrix} (L_1^t & 0) \\ (0 & I_{ro}) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} L_{11}^t\ 0 & & & \\ & L_{12}^t\ 0 & & \\ & & \ddots & \\ & & & L_{1r_\eta}^t\ 0 \\ & & & 0 \end{pmatrix} \\ \begin{pmatrix} 0\ 1 & & & \\ & 0\ 1 & & \\ & & \ddots & \\ & & & 0\ 1 \\ 0\ 0\ 0\ 0 \dots 0\ 0\ I_{r_o - l_\eta} & & & \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} (R_1^t & 0) \\ (0 & I_{ro}) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} R_{11}^t\ 0 & & & \\ & R_{12}^t\ 0 & & \\ & & \ddots & \\ & & & R_{1r_\eta}^t\ 0 \\ & & & 0 \end{pmatrix} \\ \begin{pmatrix} 0\ 1 & & & \\ & 0\ 1 & & \\ & & \ddots & \\ & & & 0\ 1 \\ 0\ 0\ 0\ 0 \dots 0\ 0\ I_{r_o - l_\eta} & & & \end{pmatrix} \end{pmatrix},$$

is observable and observability indices of each subsystem

$$((L_{1i}^t\,0),(R_{1i}^t\,0),(0\,C_{1i}))=$$

$$\left(\begin{pmatrix}I_{\eta i} & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ I_{\eta i} & 0\end{pmatrix},\begin{pmatrix}0 & 1\end{pmatrix}\right)$$

$i=1,\dots,r_o$ are $k_i^{\overline{c}o}=\eta_i+1$.

We observe that, if $r_k\leq b$, then the pencil $\begin{pmatrix}\lambda E_k+A_k & I_b\\ I_c & 0\end{pmatrix}$ has column full rank. It have row full rank if and only if $l_k=r_o$, that is to say, $l_k\leq c$.

b) $r_k>b$

In this case, if $l_k>c$, the subquadruple (E',A',B',C') is no controllable and no observable and it can be decomposed into two independent subtriples: a triple no controllable

$$\left(\begin{pmatrix}L_1 & \\ 0 & \\ & L_2\end{pmatrix},\begin{pmatrix}R_1 & \\ 0 & \\ & R_2\end{pmatrix},\begin{pmatrix}0\\ I_b\\ 0\end{pmatrix}\right),$$

where the pair (L_1,R_1) contains the first r_c-b blocks and the pair (L_2,R_2) contains the rest $r=r_k-b$ blocks and the other no observable triple

$$\left(\begin{pmatrix}L_1^t & 0 & \\ & & L_2^t\end{pmatrix},\begin{pmatrix}R_1^t & 0 & \\ & & R_2^t\end{pmatrix},\begin{pmatrix}0 & I_c & 0\end{pmatrix}\right),$$

where the pair (L_1^t,R_1^t) contains the firsts r_o-c blocks and (L_2^t,R_2^t) contains the rest of the $l=l_k-c$ blocks. Obviously, $t=0$.

Now, we take the subtriple

$$((L^t\,0),(R^t\,0),(0\,I_c))$$

whose study is analogous to the previous one. If $l_k\leq c$, then the pencil $\begin{pmatrix}\lambda E_k+A_k & I_b\\ I_c & 0\end{pmatrix}$ has row full rank. It will have column full rank if and only if $r_k=b-t$, that is to say, $r_k\leq b$, where $t=c-l_k$.

Step 2: Now we consider the quadruple (E',A',B',C') obtained in step 1. If $t\neq 0$, we separate the subquadruple (E_1,A_1,B_1,C_1) :

$$\left(\begin{pmatrix}0 & \\ & N\end{pmatrix},\begin{pmatrix}0 & \\ & I_3\end{pmatrix},\begin{pmatrix}I_t\\ 0\end{pmatrix},\begin{pmatrix}I_t & 0\end{pmatrix}\right)$$

the nilpotent matrix N has z blocks, $N=\operatorname{diag}(N_1,\dots,N_z)$.

a) $z\leq t$

Making elementary transformations, the subquadruple (E_1,A_1,B_1,C_1) is reduced to

$$E_1=\begin{pmatrix}0 &&&&&\\ & N_1 &&&&\\ && \ddots &&&\\ &&& 0 &&\\ &&&& N_z &\\ &&&&& 0\end{pmatrix}$$

$$A_1=\begin{pmatrix}0 &&&&&\\ & I_{31} &&&&\\ && \ddots &&&\\ &&& 0 &&\\ &&&& I_{3z} &\\ &&&&& 0\end{pmatrix}$$

$$B_1=\begin{pmatrix}1 &&&&&\\ & 0 &&&&\\ && \ddots &&&\\ &&& 1 &&\\ &&&& 0 &\\ &&&&& I_{t-s}\end{pmatrix}$$

and

$$C_1=\begin{pmatrix}1 &&&&&\\ & 0 &&&&\\ && \ddots &&&\\ &&& 1 &&\\ &&&& 0 &\\ &&&&& I_{t-s}\end{pmatrix}$$

It is easy to verify that each subsystem $(E_{1i},A_{1i},B_{1i},C_{1i})$, $i=1,\dots,z$ in the form

$$E_{1i}=\begin{pmatrix}0 & \\ & N\end{pmatrix},A_{1i}=\begin{pmatrix}0 & \\ & I\end{pmatrix},$$

$$B_{1i}=\begin{pmatrix}1\\ 0\end{pmatrix},C_{1i}=\begin{pmatrix}1 & 0\end{pmatrix},$$

is controllable and observable and the controllable-observable indices are $k_i^{co}=\omega_i+1$.

b) $z>t$

In this case, the subquadruple (E_1,A_1,B_1,C_1) can be decomposed in the form

$$\left(\begin{pmatrix}0 && \\ & N_1 & \\ && N_2\end{pmatrix},\begin{pmatrix}0 && \\ & I_{31} & \\ && I_{32}\end{pmatrix},\begin{pmatrix}I_t\\ 0\\ 0\end{pmatrix},\begin{pmatrix}I_t & 0 & 0\end{pmatrix}\right)$$

where (N_1,I_{31}) contains the first t blocks of nilpotency and (N_2,I_{32}) the rest $s=z-t$-blocks.

4 Conclusion

In this paper we present a canonical reduced form preserving the structure of the system providing a decomposition of the system into independent subsystems where the structural properties of the system as controllability, observability for example, can be easily described.

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