A canonical reduced form for singular time invariant linear systems. Part I

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Abstract:- We consider quadruples of matrices (E, A, B, C), representing singular linear time invariant systems in the form $E\dot{x} = Ax + Bu$, $y = Cx$ with $E, A \in M_{p\times n}(C)$, $B \in M_{p\times m}(C)$ and $C \in M_{q\times n}(C)$under proportional and derivative feedback and proportional and derivative output injection.

In this paper we present a canonical reduced form preserving the structure of the system and we obtain a collection of invariants that they permit us to deduce the canonical reduced form.

Key-Words:- Singular linear systems,proportional and derivative feedback, proportional and derivative output injection, canonical reduced form, structural invariants.

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1 Introduction

We denote by $M_{p\times n}(C)$ the space of complex matrices having p rows and n columns, and in the case which $p = n$ we write $M_n(C)$.

We consider the set $\mathcal{M}$ of quadruples of matrices (E, A, B, C) representing families of singular linear time invariant systems in the form

$$\left.\begin{aligned} E\dot{x}(t) &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \qquad (1)$$

with $E, A \in M_{p\times n}(C)$, $B \in M_{p\times m}(C)$ and $C \in M_{q\times n}(C)$. These equations arise in theoretical areas as differential equations on manifolds as well as in applied areas as systems theory and control, [8], [10].

Different useful and interesting equivalence relations between singular systems have been defined. We deal with the equivalence relation that accepting one or more, of the following transformations: basis change in the state space, input space, output space, feedback, derivative feedback, output injection, derivative output injection and premultiplication by an invertible matrix.

The obtention of canonical forms for this equivalence relation defined in the space of quadruples of matrices is an open problem. For regularizable systems A. Díaz, Mª I. García-Planas and S. Tarragona [4] a canonical reduced form was proposed.

We recall that L. Dai [2], studied canonical forms for singular systems but they do not consider feedback and derivative feedback nor output injection and derivative output injection in the equivalence relation, only consider basis change in the state space and premultiplication by an invertible matrix.

2 Notations

In the sequel we will use the following notations.

- I_n denotes the n-order identity matrix,
- N denotes a nilpotent matrix in its reduced form $N = \text{diag}(N_1, \dots, N_\ell)$, $N_i = \begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix} \in M_{n_i}(C)$,
- J denotes the Jordan matrix $J = \text{diag}(J_1, \dots, J_t)$, $J_i = \text{diag}(J_{i_1}, \dots, J_{i_s})$, $J_{i_j} = \lambda_i I + N$,
- $L = \text{diag} = (L_1, \dots, L_q)$, $L_j = \begin{pmatrix} I_{n_j} & 0 \end{pmatrix} \in M_{n_j \times (n_j+1)}(C)$,
- $R = \text{diag}(R_1, \dots, R_p)$, $R_{n_j} = \begin{pmatrix} 0 & I_{n_j} \end{pmatrix} \in M_{n_j \times (n_j+1)}(C)$.

3 Equivalence of singular linear systems

We consider singular linear as in (1), many interesting and useful equivalence relations between singular systems have been defined. We deal with the equivalence relation accepting one or more, of the following transformations: basis change in the state space, input space, output space, operations of state and derivative feedback, state and derivative output injection and to pre-multiply the first equation in (1) by an invertible matrix. That is to say.

Definition 1 *Two quadruples $(E_i, A_i, B_i, C_i) \in \mathcal{M}$, $i = 1, 2$, are equivalent if and only if there exist matrices $P \in Gl(n;\mathbb{C})$,$Q \in Gl(p;\mathbb{C})$, $R \in Gl(m;\mathbb{C})$, $S \in Gl(q;\mathbb{C})$, $F_E^B, F_A^B \in M_{m\times n}(\mathbb{C})$, $F_E^C, F_A^C \in M_{p\times q}(\mathbb{C})$ such that*

$$\begin{aligned} E_2 &= QE_1P + QB_1F_E^B + F_EC_1P, \\ A_2 &= QA_1P + QB_1F_A^B + F_AC_1P, \\ B_2 &= QB_1R, \\ C_2 &= SC_1P, \end{aligned}$$

Given a quadruple of matrices $(E, A, B, C) \in \mathcal{M}$, we can associate the following matrix pencil

$$H(\lambda) = \begin{pmatrix} \lambda E + A & \lambda B & B \\ \lambda C & 0 & 0 \\ C & 0 & 0 \end{pmatrix},$$

and we have

Proposition 1 *Two quadruples are equivalent under equivalent relation considered if and only if the associates matrix pencils are strictly equivalent.* So, we can apply kronecker's theory of singular pencils as presented in [5].

Corollary 1 *Let $H(\lambda)$ be a matrix pencil associated to the quadruple $(E, A, B, C) \in \mathcal{M}$. Then $H(\lambda)$ its is equivalent to the pencil $\lambda F + G$ with*

$$F = \begin{pmatrix} L & & & \\ & L^T & & \\ & & I_1 & \\ & & & N \end{pmatrix}, \; G = \begin{pmatrix} R & & & \\ & R^T & & \\ & & J & \\ & & & I_2 \end{pmatrix}$$

Remember that the kronecker canonical form of a pencil characterized by two sets of minimal indices and sets of finite and infinite elementary divisors.

Remark 1 Given a quadruple of matrices (E, A, B, C), will call eigenvalues of the quadruple to the eigenvalues of the associate pencil $H(\lambda)$. Obviously, the collection of eigenvalues of a quadruple are invariant under equivalence relation considered.

5 A new reduced form

The Kronecker canonical form of a pencil does not preserve the inner partitioning of the matrix pencil $H(\lambda)$ in matrices E, A, B, C desirable when studying dynamical systems. In this section, a new reduced form to respect the equivalence relation considered and maintaining the structure of the four matrices defining the system is obtained.

Theorem 1 *let $(E, A, B, C) \in \mathcal{M}$ be a quadruple of matrices. Then it is equivalent under equivalence relation considered to the following quadruple.*

$$\left(\begin{pmatrix} E_k & \\ & 0 \end{pmatrix}, \begin{pmatrix} A_k & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & I_b \end{pmatrix}, \begin{pmatrix} 0 & \\ & I_c \end{pmatrix} \right).$$

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where (E_k, A_k) is a pair of matrices in its Kronecker reduced form (see [3]).

If confusion it is not possible, we denote this quadruple by: (E_k, A_k, I_b, I_c).

Proof.

There exist matrices $Q_0 \in Gl(p;\mathbb{C})$, $R_0 \in Gl(m;\mathbb{C})$ such that

$$Q_0 B R_0 = \begin{pmatrix} 0 & \\ & I_b \end{pmatrix} = B_b, \quad b = \operatorname{rank}(B)$$

matrices $P_0 \in Gl(n;\mathbb{C})$, $S_0 \in Gl(q;\mathbb{C})$ such that

$$S_0 C P_0 = \begin{pmatrix} 0 & \\ & I_c \end{pmatrix} = C_c, \quad c = \operatorname{rank}(C)$$

So, taking $Q = Q_0$, $P = P_0$, $R = R_0$, $S = S_0$, $F_E^B = F_A^B = 0$, and $F_E^C = F_A^C = 0$, the quadruple is equivalent to (E', A', B_b, C_c).

Partitioning matrices E' and A' following the blocks in B_b and C_c $E' = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$, $A' = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ where $E_1, A_1 \in M_{(p-b)\times(n-c)}(\mathbb{C})$, $E_2, A_2 \in M_{(p-b)\times c}(\mathbb{C})$, $E_3, A_3 \in M_{b\times(n-c)}(\mathbb{C})$ and $E_4, A_4 \in M_{b\times c}(\mathbb{C})$.

Now, it suffices to apply the following proportional and derivative feedbacks and proportional and derivative output injections $F_{E'}^{B'} = \begin{pmatrix} 0 & 0 \\ -E_3 & -E_4 \end{pmatrix}$, $F_{A'}^{B'} = \begin{pmatrix} 0 & 0 \\ -A_3 & -A_4 \end{pmatrix}$, $F_{E'}^{C'} = \begin{pmatrix} 0 & -E_2 \\ 0 & 0 \end{pmatrix}$, $F_{A'}^{C'} = \begin{pmatrix} 0 & -A_2 \\ 0 & 0 \end{pmatrix}$ and $S = I_q$, $R = I_m$ $Q = \begin{pmatrix} Q_1 & \\ & I_c \end{pmatrix}$, $P = \begin{pmatrix} P_1 & \\ & I_b \end{pmatrix}$, and Q_1 and P_1 in such a way that $Q_1(\lambda E_1 + A_1)P_1 = \lambda E_k + A_k$ with (E_k, A_k) in its Kronecker reduced form. □

Proposition 5 *Let (E, A, B, C) be a quadruple and (E_k, A_k, I_b, I_c) its reduced form. The eigenvalues of the pencil $\lambda E_k + A_k$ coincide with eigenvalues of the quadruple (E, A, B, C).*

Proof.

Taking into account that the eigenvalues are invariant under equivalence relation we can use the equivalent reduced form

$$\begin{array}{l} \operatorname{rank}\begin{pmatrix} \lambda E + A & \lambda B & B \\ \lambda C & 0 & 0 \\ C & 0 & 0 \end{pmatrix} = \\ \operatorname{rank}\begin{pmatrix} \lambda E + A & B \\ C & 0 \end{pmatrix} = \\ \operatorname{rank}(\lambda E_k + A_k) + b + c \end{array}$$

□

Definition 2 *We define the following matrices*

$$\mathcal{H}_i \in M_{i(p+q)\times i(n+m)}(\mathbb{C})$$

$$\begin{aligned}
\mathcal{H}_1 &= \begin{pmatrix} E & B \\ C & 0 \end{pmatrix}, \\
\mathcal{H}_2 &= \begin{pmatrix} E & B & & \\ C & 0 & & \\ A & 0 & E & B \\ & & C & 0 \end{pmatrix}, \\
&\vdots \\
\mathcal{H}_\ell &= \begin{pmatrix} E & B & & & & & \\ C & 0 & & & & & \\ A & 0 & E & B & & & \\ & & C & 0 & & & \\ & & A & 0 & . & & \\ & & & & \ddots & & \\ & & & & & E & B \\ & & & & & C & 0 \end{pmatrix},
\end{aligned}$$

$\ell = 1, 2, \dots$

if we need specify the quadruple we write $\mathcal{H}_i(E, A, B, C)$.

Proposition 6 *Let (E, A, B, C) be a quadruple of matrices. Then, the numbers $r_\ell^{\mathcal{H}} = \operatorname{rank}\mathcal{H}_\ell$ are invariant under equivalence relation considered.*

Definition 3 *For all $\lambda \in \mathbb{C}$ we define the following matrices*

$$\mathcal{J}_i(\lambda) \in M_{i(p+q)\times i(n+m)}(\mathbb{C})$$

$$\mathcal{J}_1(\lambda) = \begin{pmatrix} \lambda E + A & B \\ C & 0 \end{pmatrix},$$

$$\mathcal{J}_2(\lambda) = \begin{pmatrix} \lambda E + A & B & & \\ C & 0 & & \\ E & 0 & \lambda E + A & B \\ & & C & 0 \end{pmatrix},$$

$$\vdots$$

$$\mathcal{J}_\ell(\lambda) = \begin{pmatrix} \lambda E+A & B & & & & & \\ C & 0 & & & & & \\ E & 0 & \lambda E+A & B & & & \\ & & C & 0 & & & \\ & & E & 0 & . & & \\ & & & & & \ddots & \\ & & & & & & \lambda E+A \quad B \\ & & & & & & C \quad 0 \end{pmatrix},$$

$\ell = 1, 2, \ldots$
If we need to specify the quadruple we will write $\mathcal{J}_i(\lambda, E, A, B, C)$.

Proposition 7 *Let (E, A, B, C) be a quadruple of matrices, for all $\lambda \in \mathbb{C}$ the numbers $r_\ell(\lambda) = \operatorname{rank} \mathcal{J}_\ell(\lambda)$ are invariants under equivalence relation considered.*

Definition 4 *We define the following matrices*

$$\mathcal{C}_i \in M_{(i+1)p+iq \times in+(i+1)m}(\mathbb{C})$$

$$\mathcal{C}_0 = (B),$$

$$\mathcal{C}_1 = \begin{pmatrix} E & B & 0 \\ C & 0 & 0 \\ A & 0 & B \end{pmatrix},$$

$$\mathcal{C}_2 = \begin{pmatrix} E & B & & & \\ C & 0 & & & \\ A & 0 & E & B & 0 \\ & & C & 0 & 0 \\ & & A & 0 & B \end{pmatrix},$$

$$\vdots$$

$$\mathcal{C}_\ell = \begin{pmatrix} E & B & & & & & & \\ C & 0 & & & & & & \\ A & 0 & E & B & & & & \\ & & C & 0 & & & & \\ & & A & 0 & . & & & \\ & & & & & \ddots & & \\ & & & & & & E & B & 0 \\ & & & & & & C & 0 & 0 \\ & & & & & & A & 0 & B \end{pmatrix},$$

$\ell = 1, 2, \ldots$.
If we need to specify the quadruple we will write $\mathcal{C}_i(E, A, B, C)$.

Proposition 8 *Let (E, A, B, C) be a quadruple of matrices. Then, the numbers $r_\ell^{\mathcal{C}} = \operatorname{rank} \mathcal{C}_\ell$ are invariant under equivalence relation considered.*

Definition 5 *We define the following matrices*

$$\mathcal{O}_i \in M_{ip+(i+1)q \times (i+1)n+im}(\mathbb{C})$$

$$\mathcal{O}_0 = (C),$$

$$\mathcal{O}_1 = \begin{pmatrix} A & B & E \\ C & 0 & 0 \\ 0 & 0 & C \end{pmatrix},$$

$$\mathcal{O}_2 = \begin{pmatrix} A & B & E & & \\ C & 0 & 0 & & \\ & & A & B & E \\ & & C & 0 & 0 \\ & & 0 & 0 & C \end{pmatrix},$$

$$\vdots$$

$$\mathcal{O}_\ell = \begin{pmatrix} A & B & E & & & & & \\ C & 0 & 0 & & & & & \\ & & A & B & E & & & \\ & & C & 0 & 0 & & & \\ & & & & . & & & \\ & & & & & \ddots & & \\ & & & & & & A & B & E \\ & & & & & & C & 0 & 0 \\ & & & & & & 0 & 0 & C \end{pmatrix},$$

$\ell = 1, 2, \ldots$
If we need to specify the quadruple we will write $\mathcal{O}_i(E, A, B, C)$

Proposition 9 *Let (E, A, B, C) be a quadruple of matrices. Then, the numbers $r_\ell^{O} = \operatorname{rank} \mathcal{O}_\ell$ are invariant under equivalence relation considered.*

Now, we are going to relate the r-numbers defined with the r-numbers of the pencil $\lambda E_k + A_k$ associated to the pair (E_k, A_k) (see [3] for definition and properties.)

We denote by (1) the quadruple (E, A, B, C) and by (2) the pair (E_k, A_k).

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With these notations we have the following proposition.

Proposition 10

$$\begin{aligned} r_\ell^{\mathcal{H}}(1) &= r_\ell^{\mathcal{H}}(2) + \ell(b+c) \\ r_\ell^{\mathcal{C}}(1) &= r_\ell^{\mathcal{C}}(2) + (\ell+1)b + \ell c \\ r_\ell^{\mathcal{O}}(1) &= r_\ell^{\mathcal{O}}(2) + \ell b + (\ell+1)c \\ r_\ell(\lambda)(1) &= r_\ell(\lambda)(2) + \ell(b+c) \end{aligned} \quad (2)$$

Proof. Taking into account propositions 6 to 9 Given a quadruple $(E, A, B, C) \in \mathcal{M}$, we can make use the equivalent reduced form quadruple (E_k, A_k, I_b, I_c).

Making row and column block elementary transformations operaciones to the matrices $\mathcal{H}_\ell$, $\mathcal{J}_\ell(\lambda_i)$, $\mathcal{C}_\ell$ and $\mathcal{O}_\ell$ for each $\ell = 1, 2, \ldots$, the result is obtained. □

Proposition 11

$$r_n(1) = r_n(2) + b + c.$$

Proof.

$$\begin{aligned} r_n(1) &= \operatorname{rank} \begin{pmatrix} \lambda E + A & B \\ C & 0 \end{pmatrix} \\ &= \operatorname{rank}(\lambda E_k + A_k) + b + c \\ &= r_n(2) + b + c. \end{aligned}$$

□

Corollary 3

$$r_{n+1}^{\mathcal{H}}(1) - r_n^{\mathcal{H}}(1) = r_n(1).$$

Proof.

$$\begin{aligned} & r_n(1) = r_n(2) + b + c = \\ & r_{n+1}^{\mathcal{H}}(2) - r_n^{\mathcal{H}}(2) + b + c = \\ & r_{n+1}^{\mathcal{H}}(1) - (n+1)(b+c) - r_n^{\mathcal{H}}(1) + \\ & \quad +n(b+c) + b + c = \\ & r_{n+1}^{\mathcal{H}}(1) - r_n^{\mathcal{H}}(1). \end{aligned}$$

□

Proposition 10 *Let $r_\ell^{\mathcal{CO}}(2)$ be the r-numbers corresponding to the infinite zeroes of the pencil $\lambda E_k + A_k$. Then*

$$\begin{aligned} r_1^{\mathcal{CO}}(2) &= r_n(1) - r_1^{\mathcal{H}}(1), \\ &\vdots \\ r_\ell^{\mathcal{CO}}(2) &= r_{\ell-1}^{\mathcal{H}}(1) - r_\ell^{\mathcal{H}}(1) + r_n(1), \\ \ell = 2, &\ \ldots \end{aligned}$$

Proof.

$$\begin{aligned} r_1^{\mathcal{CO}}(2) &= r_n(2) - r_1^{\mathcal{H}}(2) = \\ &= r_n(2) - r_1^{\mathcal{H}}(1) + b + c = \\ &= r_n(1) - r_1^{\mathcal{H}}(1). \\ \vdots \\ r_\ell^{\mathcal{CO}}(2) &= r_{\ell-1}^{\mathcal{H}}(2) - r_\ell^{\mathcal{H}}(2) + r_n(2) = \\ &= r_{\ell-1}^{\mathcal{H}}(1) - (\ell-1)(b+c) - r_\ell^{\mathcal{H}}(1) + \\ &\quad \ell(b+c) + r_n(2) = \\ &= r_{\ell-1}^{\mathcal{H}}(1) - r_\ell^{\mathcal{H}}(1) + r_n(1). \end{aligned}$$

□

Definition 6 *We will call infinite zeroes r-numbers of the quadruple (E, A, B, C) to $r_\ell^{\mathcal{CO}} = r_\ell^{\mathcal{CO}}(2)$, $\ell = 1, 2, \ldots$*

Corollary 4

$$\begin{aligned} r_1^{\mathcal{CO}} &= r_n - r_1^{\mathcal{H}}, \\ &\vdots \\ r_\ell^{\mathcal{CO}} &= r_{\ell-1}^{\mathcal{H}} - r_\ell^{\mathcal{H}} + r_n, \quad \ell = 2, \ldots \end{aligned}$$

Proposition 13 *Let $r_\ell^{\overline{\mathcal{CO}}}(\lambda)(2)$ be the Jordan r-numbers corresponding to the eigenvalue λ to the pencil $\lambda E_k + A_k$. Then*

$$\begin{aligned} r_1^{\overline{\mathcal{CO}}}(\lambda)(2) &= r_n(1) - r_1(\lambda)(1), \\ &\vdots \\ r_\ell^{\overline{\mathcal{CO}}}(\lambda)(2) &= r_{\ell-1}(\lambda)(1) - r_\ell(\lambda)(1) + r_n(1), \\ &\ell = 2, \ldots \end{aligned}$$

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Proof.

$$
\begin{aligned}
r_1^{\overline{\mathcal{C}\mathcal{O}}}(\lambda)(2) &= r_n(2) - r_1(\lambda)(2)\\
&= r_n(2) - r_1(\lambda)(1) + b + c\\
&= r_n(1) - r_1(\lambda)(1).\\
&\vdots\\
r_\ell^{\overline{\mathcal{C}\mathcal{O}}}(\lambda)(2) &= r_{\ell-1}(\lambda)(2) - r_\ell(\lambda)(2) + r_n(2)\\
&= r_{\ell-1}(\lambda)(1) - (\ell-1)(b+c) -\\
&\quad r_\ell(\lambda)(1) + \ell(b+c) + r_n(2)\\
&= r_{\ell-1}(\lambda)(1) - r_\ell(\lambda)(1) + r_n(1).
\end{aligned}
$$

□

Definition 7 *We will call Jordan r-numbers corresponding to the eigenvalue* λ *of the quadruple* (E, A, B, C) *to* $r_\ell^{\overline{\mathcal{C}\mathcal{O}}}(\lambda) = r_\ell^{\overline{\mathcal{C}\mathcal{O}}}(\lambda)(2), \quad \ell = 1, 2, \ldots$

Corollary 5

$$
\begin{aligned}
r_1^{\overline{\mathcal{C}\mathcal{O}}}(\lambda) &= r_n - r_1(\lambda),\\
&\vdots\\
r_\ell^{\overline{\mathcal{C}\mathcal{O}}}(\lambda) &= r_{\ell-1}(\lambda) - r_\ell(\lambda) + r_n, \quad \ell = 2, \ldots
\end{aligned}
$$

Proposition 14 *Let* $r_\ell^{\mathcal{C}\overline{\mathcal{O}}}(2)$ *be the column minimal r-numbers of the pencil* $\lambda E_k + A_k$*. Then*

$$
\begin{aligned}
r_0^{\mathcal{C}\overline{\mathcal{O}}}(2) &= n - r_n(1) + r_0^{\mathcal{C}}(1),\\
r_1^{\mathcal{C}\overline{\mathcal{O}}}(2) &= r_1^{\mathcal{C}}(1) - r_0^{\mathcal{C}}(1) - r_n(1),\\
&\vdots\\
r_\ell^{\mathcal{C}\overline{\mathcal{O}}}(2) &= r_\ell^{\mathcal{C}}(1) - r_{\ell-1}^{\mathcal{C}}(1) - r_n(1), \;\; \ell = 2, \ldots
\end{aligned}
$$

Proof.

$$
\begin{aligned}
r_0^{\mathcal{C}\overline{\mathcal{O}}}(2) &= (n-c) - r_n(2) = n - (r_n(2) + c)\\
&= n - (r_n(1) - b) = n - r_n(1) + r_0^{\mathcal{C}}(1).\\
r_1^{\mathcal{C}\overline{\mathcal{O}}}(2) &= r_1^{\mathcal{C}}(2) - r_n(2) =\\
&= r_1^{\mathcal{C}}(1) - 2b - c - r_n(2)\\
&= r_1^{\mathcal{C}}(1) - b - (r_n(2) + b + c)\\
&= r_1^{\mathcal{C}}(1) - r_0^{\mathcal{C}}(1) - r_n(1).\\
&\vdots\\
r_\ell^{\mathcal{C}\overline{\mathcal{O}}}(2) &= r_\ell^{\mathcal{C}}(2) - r_{\ell-1}^{\mathcal{C}}(2) - r_n(2)\\
&= r_\ell^{\mathcal{C}}(1) - \ell b - (\ell-1)c - r_{\ell-1}^{\mathcal{C}}(1) +\\
&\quad +(\ell-1)b + (\ell-2)c - r_n(2)\\
&= r_\ell^{\mathcal{C}}(1) - r_{\ell-1}^{\mathcal{C}}(1) - r_n(1).
\end{aligned}
$$

□

Definition 8 *We will call column minimal r-numbers of the quadruple* (E, A, B, C) *to* $r_\ell^{\mathcal{C}\overline{\mathcal{O}}} = r_\ell^{\mathcal{C}\overline{\mathcal{O}}}(2), \quad \ell = 0, 1, \ldots$

Corollary 6

$$
\begin{aligned}
r_0^{\mathcal{C}\overline{\mathcal{O}}} &= n - r_n + r_0^{\mathcal{C}},\\
r_1^{\mathcal{C}\overline{\mathcal{O}}} &= r_1^{\mathcal{C}} - r_0^{\mathcal{C}} - r_n,\\
&\vdots\\
r_\ell^{\mathcal{C}\overline{\mathcal{O}}} &= r_\ell^{\mathcal{C}} - r_{\ell-1}^{\mathcal{C}} - r_n, \;\; \ell = 2, \ldots
\end{aligned}
$$

Proposition 15 *Let* $r_\ell^{\overline{\mathcal{C}}\mathcal{O}}(2)$ *be the row minimal r-numbers of the pencil* $\lambda E_k + A_k$*. Then*

$$
\begin{aligned}
r_0^{\overline{\mathcal{C}}\mathcal{O}}(2) &= p - r_n(1) + r_0^{\mathcal{O}}(1),\\
r_1^{\overline{\mathcal{C}}\mathcal{O}}(2) &= r_1^{\mathcal{O}}(1) - r_0^{\mathcal{O}}(1) - r_n(1),\\
&\vdots\\
r_\ell^{\overline{\mathcal{C}}\mathcal{O}}(2) &= r_\ell^{\mathcal{O}}(1) - r_{\ell-1}^{\mathcal{O}}(1) - r_n(1), \;\; \ell = 2, \ldots
\end{aligned}
$$

Proof.

$$
\begin{aligned}
r_0^{\overline{\mathcal{C}}\mathcal{O}}(2) &= (p-b) - r_n(2) = p - (r_n(2) + b)\\
&= p - (r_n(1) - c) = p - r_n(1) + r_0^{\mathcal{O}}(1)\\
r_1^{\overline{\mathcal{C}}\mathcal{O}}(2) &= r_1^{\mathcal{O}}(2) - r_n(2)\\
&= r_1^{\mathcal{O}}(1) - b - 2c - r_n(2)\\
&= r_1^{\mathcal{O}}(1) - c - (r_n(2) + b + c)\\
&= r_1^{\mathcal{O}}(1) - r_0^{\mathcal{O}}(1) - r_n(1).\\
&\vdots\\
r_\ell^{\overline{\mathcal{C}}\mathcal{O}}(2) &= r_\ell^{\mathcal{O}}(2) - r_{\ell-1}^{\mathcal{O}}(2) - r_n(2)\\
&= r_\ell^{\mathcal{O}}(1) - (\ell-1)b - \ell c - r_{\ell-1}^{\mathcal{C}}(1) +\\
&\quad +(\ell-2)b + (\ell-1)c - r_n(2)\\
&= r_\ell^{\mathcal{O}}(1) - r_{\ell-1}^{\mathcal{O}}(1) - r_n(1).
\end{aligned}
$$

□

Definition 9 *We will call row minimal r-numbers of the quadruple* (E, A, B, C) *to* $r_\ell^{\overline{\mathcal{C}}\mathcal{O}} = r_\ell^{\overline{\mathcal{C}}\mathcal{O}}(2), \; \ell = 0, 1, \ldots$

Corollary 7

$$
\begin{aligned}
r_0^{\overline{\mathcal{C}}\mathcal{O}} &= p - r_n + r_0^{\mathcal{O}},\\
r_1^{\overline{\mathcal{C}}\mathcal{O}} &= r_1^{\mathcal{O}} - r_0^{\mathcal{O}} - r_n,\\
&\vdots\\
r_\ell^{\overline{\mathcal{C}}\mathcal{O}} &= r_\ell^{\mathcal{O}} - r_{\ell-1}^{\mathcal{O}} - r_n, \quad \ell = 2, \ldots
\end{aligned}
$$

Collecting all these numbers information we obtain a complete system of invariants giving us the reduced form presented for systems.

Theorem 2 *For all quadruple of matrices (E, A, B, C), the following collection of números*

i) $(r_1^{\mathcal{CO}} \geq r_2^{\mathcal{CO}} \geq \cdots \geq r_{\ell_1}^{\mathcal{CO}} \geq r_{\ell_1+1}^{\mathcal{CO}} = \cdots = 0)$

ii) $(r_0^{\mathcal{C}} \geq 0; r_0^{\mathcal{C}\overline{\mathcal{O}}} \geq r_1^{\mathcal{C}\overline{\mathcal{O}}} \geq \cdots \geq r_{\ell_2-1}^{\mathcal{C}\overline{\mathcal{O}}} \geq r_{\ell_2}^{\mathcal{C}\overline{\mathcal{O}}} = \cdots = 0)$

iii) $(r_0^{\mathcal{O}} \geq 0; r_0^{\overline{\mathcal{C}}\mathcal{O}} \geq r_1^{\overline{\mathcal{C}}\mathcal{O}} \geq \cdots \geq r_{\ell_3-1}^{\overline{\mathcal{C}}\mathcal{O}} \geq r_{\ell_3}^{\overline{\mathcal{C}}\mathcal{O}} = \cdots = 0)$

iv) $(r_1^{\overline{\mathcal{CO}}}(\lambda) \geq r_2^{\overline{\mathcal{CO}}}(\lambda) \geq \cdots \geq r_{\ell(\lambda)}^{\overline{\mathcal{CO}}}(\lambda) \geq r_{\ell(\lambda)+1}^{\overline{\mathcal{CO}}}(\lambda) = \cdots = 0), \quad \lambda \in \mathbb{C}$

constitutes a complete system of invariants with respect equivalence relation considered.

Proof.

The non zero r-numbers permit us to deduce the collection of numbers

i) $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_s \geq 1$,

ii) $k_1(\lambda) \geq k_2(\lambda) \geq \cdots \geq k_{j(\lambda)}(\lambda) \geq 1, \quad \lambda \in \sigma(E, A, B, C)$,

iii) $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_{r_\epsilon} > \epsilon_{r_\epsilon+1} = \cdots = \epsilon_r = 0$,

iv) $\eta_1 \geq \eta_2 \geq \ldots \eta_{l_\eta} > \eta_{l_\eta+1} = \cdots = \eta_l = 0$.

that they are the structural invariants of the quadruple (E_k, A_k, I_b, I_c). □

5 Conclusion

We consider quadruples of matrices (E, A, B, C), representing singular linear time invariant systems in the form $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ with $E, A \in M_{p\times n}(C)$ no necessarily squares, under equivalence that accept basis change in the state space input and output spaces, feedback and derivative feedback as well output and derivative output injection and premultiplication by an invertible matrix. In this paper we obtain a canonical reduced form and a collection of invariants, that they permit us to deduce the canonical reduced form.

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