Identification of one-parameter bifurcations giving rise to periodic orbits, from their period function

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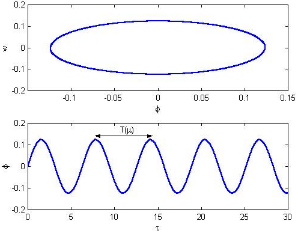
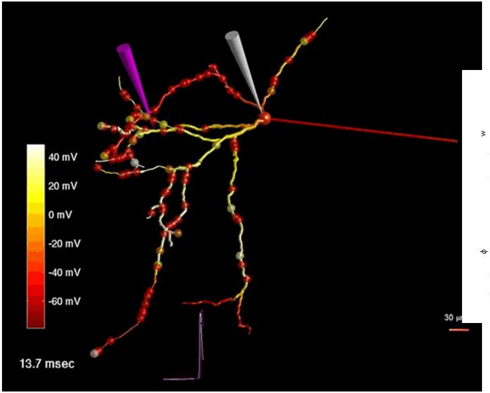
If it is possible to measure the *period* $T(\mu)$ of some *periodic orbits* observed experimentally, and its evolution as μ varies.

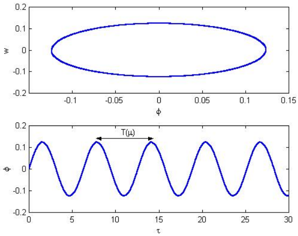


Then perhaps it is possible to extract information on the uncertain parameters λ from $T(\mu)$?

This problem arises when studying neuron activities in the brain with the aim of determining the *synaptic conductances* λ that it receives.

By injecting external currents μ, in the neurons it is possible to extract information about the period of the oscillations of the voltage of the cell $T(\mu)$.





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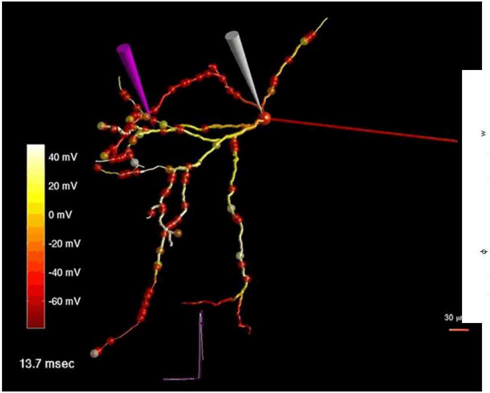

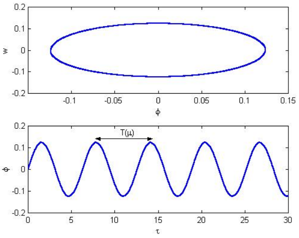

From measurements $T(\mu_i, \lambda)$ for $i = 1, \ldots, q$ some kind of regression is needed to estimate λ.

The analytical knowledge of $T(\mu, \lambda)$ is an advantage to do this regression.

2. STARTING POINT

From the knowledge $T(\mu)$ of a *one parameter* family of P.O. it is possible to identify the type of *bifurcation* which has originated the P.O.?

YES $\Rightarrow$ important restrictions on the uncertainties $\lambda \in \mathbb{R}^p$ to be estimated
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Our results are restricted to the *planar analytic* case where the dependence of the differential equation on μ is also *analytic*.

$$\begin{cases} \dot{x} = P(x, y; \mu), \\ \dot{y} = Q(x, y; \mu). \end{cases} \quad \text{or equivalently} \quad X(x, y; \mu) = P(x, y; \mu)\frac{\partial}{\partial x} + Q(x, y; \mu)\frac{\partial}{\partial y} \tag{1}$$

where $(x, y) \in \mathbb{R}^2$ and $\mu \in \Lambda \subset \mathbb{R}$ an open interval containing zero.

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where $(x, y) \in \mathbb{R}^2$ and $\mu \in \Lambda \subset \mathbb{R}$ an open interval containing zero.

Our objective is to relate the form of $T(\mu)$ with the type of *bifurcation* of *limit cycles*.

2. "MOST ELEMENTARY" BIFURCATIONS

The most elementary bifurcations of planar vector fields of the above form occur when the the vector field

$$X(x,y;\mu) = P(x,y;\mu)\frac{\partial}{\partial x} + Q(x,y;\mu)\frac{\partial}{\partial y}$$

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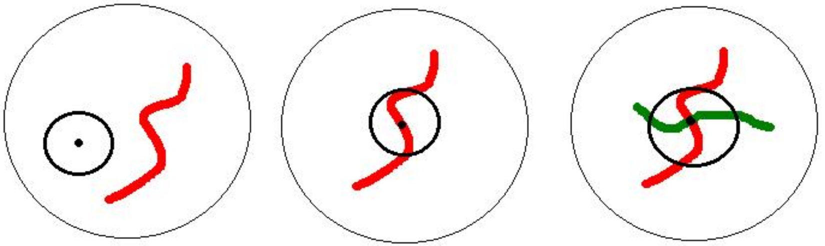
Intuitively



STRUCTURAL STABLE VECTOR FIELD

1st DEGREE OF STRUCTURAL INSTABILITY

2nd DEGREE OF STRUCTURAL INSTABILITY

All the possible bifurcations that can occur for a vector field with a 1st degree of structural instability are known (Andronov et al. 1973 and Sotomayor 1974)

Among them we only study the *isolated* ones that give rise to *isolated periodic orbits* (P.O.).

Elementary bifurcations giving rise to P.O.

(a) Hopf bifurcation.
(b) Bifurcation from semi–stable periodic orbit.
(c) Saddle-node bifurcation
(d) Saddle loop bifurcation.

We will characterize the asymptotic expansion of the period of the emerging P.O.

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Under the conditions that give rise to a bifurcation of P.O. we have

- Hopf bifurcation: $T(\mu) = T_0 + T_1\mu + O(|\mu|^{3/2})$, with $T_0 > 0$ but T_1 can be 0.

- Bifurcation from semi–stable periodic orbit: $T^{\pm}(\mu) = T_0 \pm T_1\sqrt{|\mu|} + O(\mu)$, with $T_0 > 0$ but T_1 can be 0.

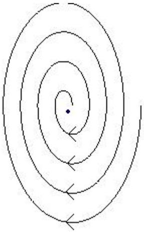
- Saddle-node bifurcation: $T(\mu) \sim T_0/\sqrt{\mu}^{(*)}$

- Saddle loop bifurcation: $T(\mu) = c\ln|\mu| + O(1)$, with $c \neq 0$.

$^{(*)}$ Where $T(\mu) \sim a + f(\mu)$ means that $\lim_{\mu\to 0} \frac{T(\mu) - a}{f(\mu)} = 1$.

HOPF BIFURCATION

It is originated by the change of the stability of the equilibrium



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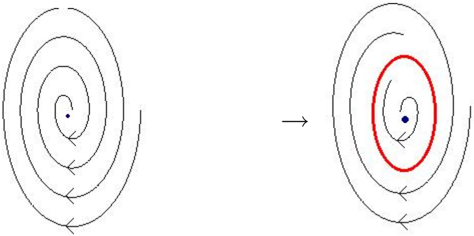

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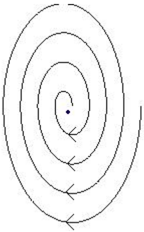

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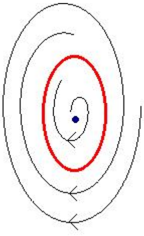

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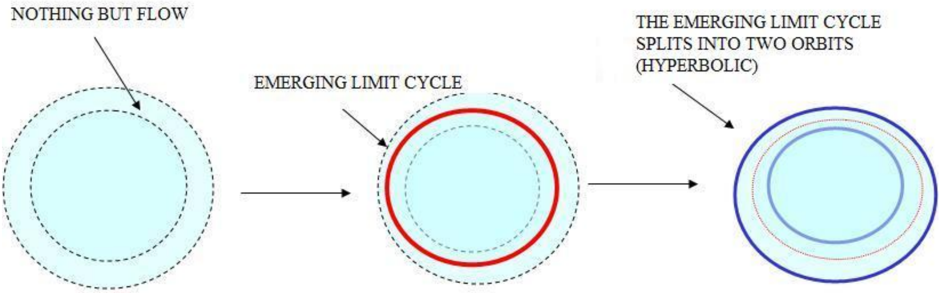

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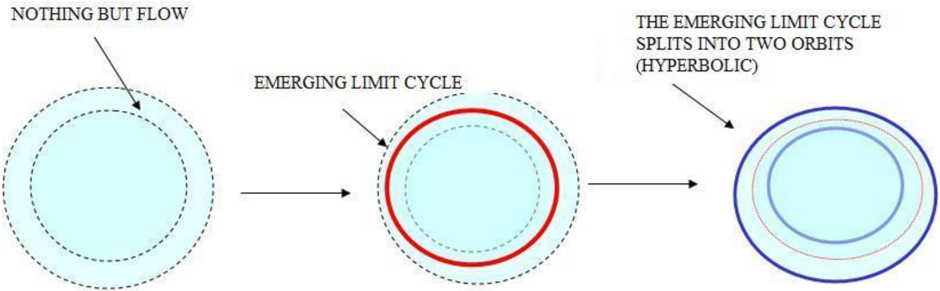

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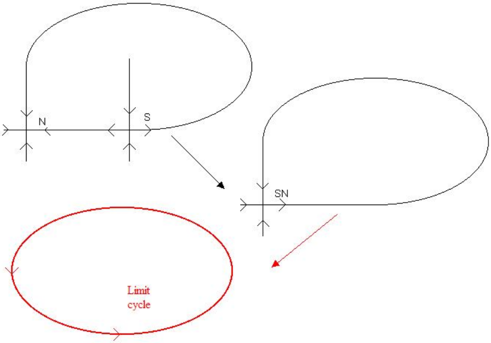
SADDLE–NODE BIFURCATION

The bifurcation takes place when a connected saddle and a node collapse.



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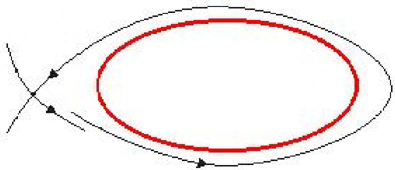
SADDLE LOOP BIFURCATION

Occurs when a saddle loop breaks and the *separatrices* change their relative position



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Under the conditions that give rise to a bifurcation of P.O. we have

$$T(\mu) = c \ln |\mu| + O(1)$$

with $c \neq 0$.

(a) Hopf bifurcation. Usual arguments using polar coordinates, Taylor developments, and variational equations.

(b) Bifurcation from semi–stable periodic orbit. Try to reduce the problem to the Hopf bifurcation framework, but instead of using polar coordinates using specific local coordinates (Ye et al. 1983)

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4. WHAT ABOUT THE PROOFS?

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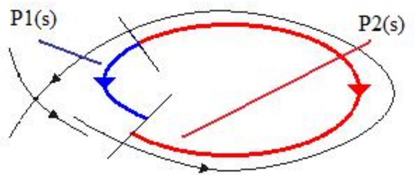
(d) Saddle loop bifurcation. Usual arguments+Normal forms but the framework is very different because the structure of the transition maps of the flow, and the transit time functions are **NON–DIFFERENTIABLE!**

SKETCH OF THE PROOF IN THE SADDLE LOOP CASE

(A) LOCATION. A periodic orbit can be seen as a zero of the *displacement function*

$$\mathrm{D}(s;\mu) = P_1(s;\mu) - P_2(s;\mu)$$



So the P.O. are located by curve $s_l(\mu)$ such that

$$\mathrm{D}(s_l(\mu);\mu) = 0.$$

We would like to apply the implicit function theorem, but... $P_1(s)$ is not differentiable. (Roussarie, 1998)

(B) TRANSIT TIME ANALYSIS.

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But... $T_1(s)$ is also non–differentiable. (Broer, Roussarie, Simó, 1996)

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Lemma.

Consider the family $\{X_\mu\}_{\mu\in\Lambda}$, such that for $\mu_0 \in \Lambda$, X_{μ_0} has a saddle connection at p_{μ_0}.

Set $r(\mu) = \lambda_1(\mu)/\lambda_2(\mu)$, the ratio of the eigenvalues of the saddle.

For any $k \in \mathbb{N}$ there exists a $\mathcal{C}^k$ diffeomorphism Φ (also depending $\mathcal{C}^k$ on μ) such that, in some neighbourhood of $(p_{\mu_0}, \mu_0) \in \mathbb{R}^2 \times \Lambda$:

(a) If $r(\mu_0) \notin \mathbb{Q}$,

$$X_\mu = \Phi_*\left(\lambda_1(\mu)\, x\, \frac{\partial}{\partial_x} + \lambda_2(\mu)\, y\, \frac{\partial}{\partial_y}\right)$$

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(b) If $r(\mu_0) = p/q$

$$X_\mu = \Phi_*\left(\frac{1}{f(u;\mu)}\left(x\frac{\partial}{\partial_x} + yg(u;\mu)\frac{\partial}{\partial_y}\right)\right),$$

where $f(u;\mu)$ and $g(u;\mu)$ are polynomials in $u := x^p y^q$ with coefficients $\mathcal{C}^\infty$ functions in μ.

Proposition.

(a) If $r(0) > 1$ then

$$P_1(s;\mu) = s^{r(\mu)}\left(1 + \psi_1(s;\mu)\right) \quad \text{and}$$

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(b) If $r(0) = 1$ then,

$$P_1(s;\mu) = s^{r(\mu)}\Big(1+\alpha_2(\mu)s\,\omega\big(s;\alpha_1(\mu)\big)+\psi_1(s;\mu)\Big), \quad \text{and}$$

$$T_1(s;\mu) = \tfrac{-1}{\lambda_1(\mu)} \ln s + \beta_1(\mu)s\,\omega\big(s;\alpha_1(\mu)\big)+\psi_2(s;\mu).$$

Where $\alpha_1(\mu) = 1 - r(\mu)$, and α_2 and β_1 are $\mathcal{C}^\infty$.

Where ψ_i belong to a class of functions $\mathcal{B}$ "*with good behaviour*" at $(0,\mu_0)$, and ω is the Roussarie–Ecalle compensator.

It is a function which captures the non–regular behaviour of the so called *Dulac maps*.

Definition.

The function defined for $s > 0$ and $\alpha \in \mathbb{R}$ by means of

$$\omega(s;\alpha) = \begin{cases} \frac{s^{-\alpha}-1}{\alpha} & \text{if } \alpha \neq 0, \\ -\ln s & \text{if } \alpha = 0, \end{cases}$$

is called the *Roussarie-Ecalle compensator*.

See (Roussarie, 1998).

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• The picture in Slide 4 comes from Edgerton *Simulating in vivo-like Synaptic Input Patterns in Multicompartmental Models*.
`http://www.brains-minds-media.org/archive/225`

THANK YOU!