SAKARYA UNIVERSITY
INSTITUTE OF SCIENCE AND TECHNOLOGY

GENERALIZED FIBONACCI AND LUCAS NUMBERS OF THE FORM kx^2

Ph.D. THESIS

Olcay KARAATLI

Department	:	MATHEMATICS
Field of Science	:	ALGEBRA AND NUMBER THEORY
Supervisor	:	Prof. Dr. Refik KESKİN

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LIST OF SYMBOLS AND ABBREVIATIONS

$\mathbb{N}$	: the set of natural numbers
$\mathbb{Z}$	: the set of integers
$\mathbb{Z}^+$	: the set of positive integers
$a \mid b$	: a is a factor of b
$a \nmid b$	: a is not a factor of b
(a,b)	: the greatest common divisor of a and b
$a \bmod b$	: the remainder when a is divided by b
$\equiv$	: is congruent to
$\square$	: perfect square
$\left(\frac{*}{*}\right)$	: Jacobi symbol
(U_n)	: Generalized Fibonacci sequence
(V_n)	: Generalized Lucas sequence
(F_n)	: Fibonacci sequence
(L_n)	: Lucas sequence
(P_n)	: Pell sequence
(Q_n)	: Pell-Lucas sequence

SUMMARY

Keywords: Fibonacci Numbers, Lucas Numbers, Generalized Fibonacci Numbers, Generalized Lucas Numbers, Diophantine Equations, Pell Equations, Congruences, Jacobi Symbol

Investigations of the properties of generalized Fibonacci and Lucas sequences have been able to hold mathematician's interest over time. These investigations have given rise to questions in when the terms of generalized Fibonacci and Lucas sequences are perfect square ($=\square$).

In this thesis, it is dealt with generalized Fibonacci numbers $U_n(P,Q)$ and generalized Lucas numbers $V_n(P,Q)$ of the form kx^2 with the special consideration that $Q=\pm 1$ and $k=5$ or $k=7$.

In Chapter 1, the historical information about Fibonacci's life and Fibonacci and Lucas sequences are briefly mentioned. Then, the definitions of generalized Fibonacci and Lucas sequences are given. Since there is a close relation between the terms of these sequences and the solutions of certain Diophantine equations, it is mentioned about Diophantine equations and Pell equations, which are the special cases of Diophantine equations. Furthermore, the literature concerning generalized Fibonacci and Lucas numbers of the form kx^2 are given.

In Chapter 2, the most important properties of generalized Fibonacci and Lucas numbers are listed. In the succeeding subchapters, generalized Fibonacci and Lucas numbers of the form $5x^2$ are considered with special consideration that $Q=\pm 1$ and some results are obtained. By the help of these results, it is observed the close relation between the terms of generalized Fibonacci and Lucas sequences and the solutions of certain Diophantine equations. Also, the equations $U_n(P,1)=5U_m(P,1)\square$, $U_n(P,-1)=5U_m(P,-1)\square$, $V_n(P,1)=5V_m(P,1)\square$, and $V_n(P,-1)=5V_m(P,-1)\square$ are solved.

In Chapter 3, the equations $U_n(P,1)=7\square$, $U_n(P,1)=7U_m(P,1)\square$, $V_n(P,1)=7\square$, and $V_n(P,1)=7V_m(P,1)\square$ are solved.

kx^2 BİÇİMİNDEKİ GENELLEŞTİRİLMİŞ FİBONACCİ VE LUCAS SAYILARI

ÖZET

Anahtar kelimeler: Fibonacci Sayıları, Lucas Sayıları, Genelleştirilmiş Fibonacci Sayıları, Genelleştirilmiş Lucas Sayıları, Diyofant Denklemleri, Pell Denklemleri, Kongrüanslar, Jacobi Sembolü

Genelleştirilmiş Fibonacci ve Lucas dizilerinin özelliklerini içeren araştırmalar zamanla matematikçilerin ilgisini çekmiştir. Bu araştırmalar hangi durumlarda genelleştirilmiş Fibonacci ve Lucas dizilerinin terimlerinin tamkare ($=\square$) oldukları sorusunu akıllara getirmiştir.

Bu tezde kx^2 biçimindeki genelleştirilmiş Fibonacci sayıları $U_n(P,Q)$ ve genelleştirilmiş Lucas sayıları $V_n(P,Q)$, $Q=\pm 1$ ve $k=5$ veya $k=7$ özel şartları altında incelendi.

Birinci bölümde, Fibonacci'nin hayatı ve Fibonacci ve Lucas dizileri hakkında tarihsel bilgiler verildi. Ardından, genelleştirilmiş Fibonacci ve Lucas dizilerinin tanımları verildi. Bu dizilerin terimleri ile bazı Diyofant denklemlerinin çözümleri arasındaki yakın ilişkiden dolayı Diyofant denklemleri ve Diyofant denklemlerinin özel durumları olan Pell denklemlerinden bahsedildi. Ayrıca, kx^2 biçimindeki genelleştirilmiş Fibonacci ve Lucas sayılarını içeren literatür bilgisi verildi.

İkinci bölümde, genelleştirilmiş Fibonacci ve Lucas sayılarının en önemli özellikleri listelendi. İkinci bölümün alt bölümlerinde, $5x^2$ biçimindeki genelleştirilmiş Fibonacci ve Lucas sayıları, $Q=\pm 1$ özel şartları altında ele alındı ve bazı sonuçlar elde edildi. Elde edilen bu sonuçlar yardımıyla, genelleştirilmiş Fibonacci ve Lucas dizilerinin terimleri ile bazı Diyofant denklemlerinin çözümleri arasındaki yakın ilişki gözlemlendi. Ayrıca, $U_n(P,1)=5U_m(P,1)\square$, $U_n(P,-1)=5U_m(P,-1)\square$, $V_n(P,1)=5V_m(P,1)\square$, ve $V_n(P,-1)=5V_m(P,-1)\square$ denklemleri çözüldü.

Üçüncü bölümde, $U_n(P,1)=7\square$, $U_n(P,1)=7U_m(P,1)\square$, $V_n(P,1)=7\square$, ve $V_n(P,1)=7V_m(P,1)\square$ denklemleri çözüldü.

CHAPTER 1. INTRODUCTION

Leonardo Fibonacci, also called Leonardo Pisano or Leonard of Pisa, is the greatest mathematician of the European Middle Ages and has a significant impact on mathematics. Although his work is quite well known, little is known about his life. Leonard of Pisa $(1175-1250)$ was born in Pisa, Italy.

Fibonacci's father Guglielmo Bonacci was a kind of merchant at Bugia, a town on the Northern Africa, located in present day Algeria. He wanted his son Fibonacci to follow his trade. So, he brought Fibonacci to Bugia and encouraged him to learn arithmetic and the skill of calculation. Fibonacci was educated by a Muslim schoolmaster, who introduced him Hindu-Arabic numeration system and some computational techniques.

While most of Europe at that time were using Romen numerials, Fibonacci realised the many advantages of Hindu-Arabic system which was much more efficient and easier to work with.

Fibonacci then travelled around the Mediterrenean visiting Egypt, Syria, Greece, South France, and Constantinople. During these visits, he became familiar with languages Latin, Arabic, and Greek. He came in contact with early works, especially with arithmetic, algebra, and geometry. After his extended visits to different countries of the world, Fibonacci made an extensive study of Greek, Babylonian, Indian, and Arabic mathematics.

Fibonacci returned to Italy around 1200 and in 1202, he published his work Liber Abaci (Book of Counting), which was a major famous book in the Middle Ages provided a good deal of interest in mathematics for further study and research in arithmetic, algebra, and geometry.

Liber Abaci contained not only rules and algorithms for computing with Hindu-Arabic numeration system, but also a large collection of interesting problem of various kinds. A second edition of Liber Abaci was published in 1228.

Fibonacci produced other books such as Practica Geometriae (Practice of Geometry) in 1220 and Liber Quadratorum (Book of Square Numbers) in 1225.

In spite of his many influential contributions to mathematics, Fibonacci is not most remembered for any of these reasons, but rather for a single sequence of numbers that bears his name, which comes from a problem he poses in Liber Abaci.

The result of the problem generates the sequence of numbers, for which Fibonacci is the most famous:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

The sequence of numbers above is known as Fibonacci sequence, in which each new number is the sum of the two numbers preceeding it.

The terms of the Fiboancci sequence are referred to as Fibonacci numbers and the n th term of Fibonacci numbers is denoted by F_n. The first and the second Fibonacci numbers are given as $F_1 = F_2 = 1$. All the other terms are defined by the relation

$$F_{n+1} = F_n + F_{n-1} \tag{1.1}$$

for $n \geq 2$.

Sequences defined in this manner, in which each term is defined as a certain function of previous terms, are called recursive sequences. The process of assigning a numerical value to the individual term is called a recurrence process, and a specific equation that describes a recurrence process, like equation (1.1) above, is called as a recurrence relation.

It was the French mathematician François Edouard Anatole Lucas who gave the name Fibonacci sequence in May of 1876. He found many other important applications as well as having the series of numbers that are closely related to Fibonacci numbers, called Lucas numbers. And Lucas numbers are given as the following:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$$

The terms of Lucas sequence are referred to as Lucas numbers and the nth Lucas number is denoted by L_n. As it is seen from the sequence of numbers above, the first and the second Lucas numbers are given as $L_1 = 2$, $L_2 = 1$ and therefore these numbers satisfy the recurrence relation

$$L_{n+1} = L_n + L_{n-1}$$

for $n \geq 2$.

Fibonacci and Lucas numbers appear in almost every branch of mathematics, obviously in number theory, but also in differantial equations, probability, statistics, numerical analysis, and lineer algebra. They also occur in physics, biology, chemistry, and electrical engineering. For more detailed information about how Fibonacci and Lucas numbers appear in the branch of mathematics and also in nature, we refer the reader to [1].

If we look at ratios of consecutive Fibonacci numbers or Lucas numbers, we see that these ratios appear to approach a number close to 1.618..., which is known as golden ratio. This property was first discovered by astronomer mathematician Johannes Kepler.

Discovering the value of a Fibonacci number or a Lucas number can be sometimes tedious and difficult. For instance, finding the fifth Fibonacci number or Lucas number is not difficult but finding the twentieth Fiboancci number or Lucas number is much more difficult since the process involves finding and summing the previous nineteenth terms.

In 1843, the French mathematician Jacques Marie Binet $(1786-1856)$ discovered a closed formula, called as Binet's formula, which can find any Fibonacci number or Lucas number without having to find any of the previous numbers in the sequences. The Binet formulas are as follows:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ [2].

Actually, these formulas were first discovered in 1718 by the French mathematician Abraham De Moivre $(1667-1754)$ using generating functions, and also independently in 1844 by the French engineer mathematician Gabriel Lamé $(1795-1870)$.

After people began to pay more analytical attention to the nature and surrounding them, they noticed that Fibonacci and Lucas numbers are everywhere. So that reason, many mathematicians started to deal with these numbers.

In fact, both Fibonacci numbers and Lucas numbers have many beautiful, interesting and useful properties. Especially, congruences, divisibility properties, and many identities concerning these numbers are only a few of them and many studies have been made related to them. We can refer the reader to [3] to see the following congruences concerning Fibonacci and Lucas numbers.

$$F_{2mn+r} \equiv (-1)^{mn} F_r \pmod{F_m},$$

$$L_{2mn+r} \equiv (-1)^{mn} L_r \pmod{F_m},$$

$$L_{2mn+r} \equiv (-1)^{(m+1)n} L_r \pmod{L_m},$$

and

$$F_{2mn+r} \equiv (-1)^{(m+1)n} F_r (\mathrm{mod}\, L_m),$$

for all $n \in \mathbb{N} \cup \{0\}$ and $m, r \in \mathbb{Z},$ where m is a nonzero integer.

It was shown by using Binet's formula that $F_{2n} = F_n L_n.$ So, $F_n \mid F_{2n}.$ In order to generalize this, mathematicians thought about under what conditions does $F_m \mid F_n$? It was proven that if $m \mid n,$ then, $F_m \mid F_n.$ The converse of this statement was proven by L. Carlitz in 1964. According to Carlitz, if $F_m \mid F_n,$ then, $m \mid n.$ This divisibility property was also given by the same author [4] for Lucas numbers. The property is as follows:

$$L_m \mid L_n \quad \text{if and only if } m \mid n \quad \text{and } n = mk \quad \text{for some odd } k > 0,$$

where $m \geq 2.$

We now turn our attention to the generalizations of these sequences.

It was the work of Lucas $(1842-1891)$ [5] that generalized such sequences as follows:

If P and Q are nonzero integers, then, the roots of the characteristic equation $X^2 - PX + Q = 0$ are

$$\alpha = \frac{P + \sqrt{P^2 - 4Q}}{2} \text{ and } \beta = \frac{P - \sqrt{P^2 - 4Q}}{2}.$$

Hence,

$$\alpha + \beta = P, \ \ \alpha\beta = Q, \text{ and } \alpha - \beta = \sqrt{P^2 - 4Q}.$$

Assuming $P^2 - 4Q \neq 0,$ the terms of the sequences $\left(U_n(P,Q)\right)$ and $\left(V_n(P,Q)\right)$ were defined by Binet's formula, namely

$$U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n(P,Q) = \alpha^n + \beta^n$$

for $n \geq 0.$ The sequences $\left(U_n(P,Q)\right)$ and $\left(V_n(P,Q)\right)$ are known as generalized Fibonacci and Lucas sequences, respectively.

In 1965, A. F. Horadam [6, 7] introduced the recurrence sequence $\left(W_n(a,b;P,Q)\right),$ or briefly $(W_n),$ defined by

$$W_{n+1} = PW_n - QW_{n-1},\ W_0 = a,\ W_1 = b,$$

and it generalizes many important sequences (see [8, 9]), for instance:

a) The generalized Fibonacci sequence $(U_n),$ where

$$U_n = W_n(0,1;P,-Q).$$

b) The generalized Lucas sequence $(V_n),$ where

$$V_n = W_n(2,P;P,-Q).$$

c) The Fibonacci sequence $(F_n),$ where

$$F_n = W_n(0,1;1,-1).$$

d) The Lucas sequence $(L_n),$ where

$$L_n = W_n(2,1;1,-1).$$

e) The Pell sequence (P_n), where

$$P_n = W_n(0,1;2,-1).$$

f) The Pell-Lucas sequence (Q_n), where

$$Q_n = W_n(2,2;1,-1).$$

Hence, we define the generalized Fibonacci sequence and generalized Lucas sequence by the following recursions:

$$U_0(P,Q)=0,\ U_1(P,Q)=1,\ U_{n+1}(P,Q)=PU_n(P,Q)+QU_{n-1}(P,Q),\ n\geq 1$$

and

$$V_0(P,Q)=2,\ V_1(P,Q)=P,\ V_{n+1}(P,Q)=PV_n(P,Q)+QV_{n-1}(P,Q),\ n\geq 1.$$

$U_n(P,Q)$ is called the nth generalized Fibonacci number and $V_n(P,Q)$ is called the nth generalized Lucas number. Also generalized Fibonacci and Lucas numbers for negative subscripts are defined as

$$U_{-n}(P,Q)=\frac{-U_n(P,Q)}{(-Q)^n} \text{ and } V_{-n}(P,Q)=\frac{V_n(P,Q)}{(-Q)^n}$$

for $n\geq 1$, respectively. For $P^2+4Q\neq 0$, if $\alpha=(P+\sqrt{P^2+4Q})/2$ and $\beta=(P-\sqrt{P^2+4Q})/2$ are the roots of the characteristic equation $x^2-Px-Q=0$, then, the Binet formulas, which give the terms of the sequences (U_n) and (V_n), have the forms

$$U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n(P,Q) = \alpha^n + \beta^n$$

for all $n \in \mathbb{Z}$.

Since $U_n = U_n(-P,Q) = (-1)^n U_n(P,Q)$ and $V_n = V_n(-P,Q) = (-1)^n V_n(P,Q)$, it will be assumed that $P \geq 1$. Moreover, we assume that $P^2 + 4Q > 0$. Instead of $U_n(P,Q)$ and $V_n(P,Q)$, we will sometimes use U_n and V_n, respectively.

As is seen from the definition of the generalized Fibonacci sequence (U_n) and generalized Lucas sequence (V_n), Fibonacci sequence (F_n), Lucas sequence (L_n), Pell sequence (P_n), and Pell-Lucas sequence (Q_n) are the special cases of the generalized Fibonacci sequence (U_n) and generalized Lucas sequence (V_n). Moreover, for $Q = -1$, we represent (U_n) and (V_n) by $(U_n(P,-1))$ and $(V_n(P,-1))$, respectively. For more information about generalized Fibonacci and Lucas numbers, one can consult [10, 11, 12, 13].

Generalized Fibonacci and Lucas numbers have many useful properties. The following properties are connected with the greatest common divisor of them.

Let m and n be positive integers, and $d = (m,n)$. Then,

g) $(U_m, U_n) = U_d$,

h) If $\frac{m}{d}$ and $\frac{n}{d}$ are odd, then, $(V_m, V_n) = V_d$,

i) If $m = n$, then, $(U_m, V_n) = 1$ or 2,

E. Lucas [5, 14], using only elementary identities, proved the parts of the statements above (see also Carmichael [15]). Furthermore, these can be found in [16, 17, 18, 19].

The divisibility properties of generalized Fibonacci and Lucas numbers are as follows: [10, 17, 18, 19, 20].

j) If $U_m \neq 1,$ then, $U_m \mid U_n$ if and only if $m \mid n.$

k) If $V_m \neq 1,$ then, $V_m \mid V_n$ if and only if $m \mid n$ and $\frac{n}{m}$ is odd.

l) If $V_m \neq 1,$ then, $V_m \mid U_n$ if and only if $m \mid n$ and $\frac{n}{m}$ is even.

Since there is a close relation between these numbers and certain Diophantine equations, we mention about Diophantine equations.

A Diophantine equation is an equation in which only integer solutions are allowed. The name "Diophantine" comes from Diophantus, an Alexandrian mathematician of the third century A. D., but such equations have a very long history, extending back to ancient Egypt, Babylonia, and Greece. In general, a quadratic Diophantine equation is an equation of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{1.2}$$

where $a, b, c, d, e,$ and f are fixed integers. The principal question when studying a given Diophantine equation is whether a solution exists, and in the case they exist, how many solutions there are and whether there is a general form for the solutions.

Any Diophantine equation of the form $x^2 - dy^2 = N$ is known as Pell equation, where d is not a perfect square and N is any nonzero fixed integer. Pell equation is a special case of (1.2). For $N = \pm 1$, the equations $x^2 - dy^2 = \pm 1$ are known as classical Pell equations. The Pell equation is perhaps the oldest Diophantine equation that has interested mathematicians all over the world for probably more than a 1000 years now. The name of this equation arose from Leonhard Euler's mistakenly attributing its study to John Pell, who searched for integer solutions of the equations of this type in 17 th century. The notations (x, y) and $x + y\sqrt{d}$ are used interchangeably to denote solutions of the equation

$$x^2 - dy^2 = N. \tag{1.3}$$

If $x = u$ and $y = v$ are integers which satisfy the equation (1.3), then, we say that the number $u + v\sqrt{d}$ is a solution of (1.3).

Let us consider all the solutions $x + y\sqrt{d}$ of the equation

$$x^2 - dy^2 = 1 \tag{1.4}$$

with positive integers x and y. Among these solutions there is a least solution $x_1 + y_1\sqrt{d}$, in which x_1 and y_1 have their least positive values. The number $x_1 + y_1\sqrt{d}$ is called the fundamental solution of (1.4). If $x_1 + y_1\sqrt{d}$ is the fundamental solution of (1.4), then, all positive integer solutions of (1.4) are obtained by the formula

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

with $n \geq 1$. While the equation (1.4) is always solvable if the positive number d is not a perfect square, the equation

$$x^2 - dy^2 = -1 \tag{1.5}$$

is solvable only for certain values of d. If the equation (1.5) is solvable for a given integer d and if $x_1 + y_1\sqrt{d}$ is the least solution with positive integers x_1 and y_1, then we say that $x_1 + y_1\sqrt{d}$ is the fundamental solution of (1.5). If $x_1 + y_1\sqrt{d}$ is the fundamental solution of (1.5), then, $(x_1 + y_1\sqrt{d})^2$ is the fundamental solution of (1.4). So, the square of any solution of (1.5) is obviously a solution of (1.4).

We now turn to the equation

$$u^2 - dv^2 = N, \tag{1.6}$$

where d is a positive integer which is not a perfect square and N is a nonzero integer. If $\alpha = u + v\sqrt{d}$ is a solution of (1.6) and $\varepsilon = x + y\sqrt{d}$ is a solution of (1.4), then also

$$\alpha\varepsilon = (u + v\sqrt{d})(x + y\sqrt{d}) = (ux + vyd) + (uy + vx)\sqrt{d}$$

is a solution of (1.6). Let $\alpha_1 = u_1 + v_1\sqrt{d}$ and $\alpha_2 = u_2 + v_2\sqrt{d}$ be any two given solutions of (1.6). Then, α_1 and α_2 are called associated solutions if there exists a solution $\varepsilon = x + y\sqrt{d}$ of (1.4) such that

$$\alpha_1 = \varepsilon\alpha_2.$$

The set of all solutions associated with each other forms a class of solutions of (1.6). The necessary and sufficient condition for the two given solutions $\alpha_1 = u_1 + v_1\sqrt{d}$ and $\alpha_2 = u_2 + v_2\sqrt{d}$ belong to the same class is that the numbers

$$\frac{u_1u_2 - v_1v_2d}{N} \text{ and } \frac{v_1u_2 - u_1v_2}{N}$$

are integers.

If K is a class, then, $\overline{K} = \{u - v\sqrt{d} \mid u + v\sqrt{d} \in K\}$ is also a class. The class K and $\overline{K}$ are said to be conjugates of each other. Conjugate classes are in general distinct, but may sometimes coincide. If $K = \overline{K}$, then, we say that the class K is ambiguous.

Nagell [21] gives the fundamental solution in a given class K as follows:

Among all the solutions $u + v\sqrt{d}$ in a given class K, we choose a solution $u^* + v^*\sqrt{d}$ in the following way: Let v^* be the least nonnegative value of v occuring in K. If K is not ambiguous, then, u^* is uniquely determined since $-u^* + v^*\sqrt{d}$ belongs to the conjugate class $\overline{K}$. If K is ambiguous, we determine u^* by $u^* \geq 0$. The solution $u^* + v^*\sqrt{d}$ defined in this way is said to be the fundamental solution of the class K. For the fundamental solution note that $|u^*|$ is the least value of $|u|$ which is possible for $u + v\sqrt{d}$ belongs to the class K. Finally note that $u^* = 0$ or $v^* = 0$ if and only if K is ambiguous. If $N = \pm 1$, clearly there is only one class, and then, it is ambiguous. If $u^* + v^*\sqrt{d}$ is the fundamental solution of the class K, then, all positive integer solutions $u + v\sqrt{d}$ of the class K are given by

$$u + v\sqrt{d} = (u^* + v^*\sqrt{d})(x + y\sqrt{d}),$$

where $x + y\sqrt{d}$ runs through all the solutions of (1.4).

We now give criteria for finding the fundamental solutions of the various classes of solutions when (1.6) is solvable. Here are the statements as stated by Nagell [21, pp. $204-208$].

Let the number N in (1.6) be positive. If $u_0 + v_0\sqrt{d}$ is the fundamental solution of the class K of (1.6) and if $x_1 + y_1\sqrt{d}$ is the fundamental solution of (1.4), we have the inequalities

$$0 \le v_0 \le \frac{y_1\sqrt{N}}{\sqrt{2(x_1+1)}} \text{ and } 0 < |u_0| \le \sqrt{\frac{1}{2}(x_1+1)N}. \qquad (1.7)$$

Let the number N be positive in (1.6) and consider the equation

$$u^2 - dv^2 = -N. \qquad (1.8)$$

If $u_0 + v_0\sqrt{d}$ is the fundamental solution of the class K of (1.8) and if $x_1 + y_1\sqrt{d}$ is the fundamental solution of (1.4), we have the inequalities

$$0 < v_0 < \frac{y_1\sqrt{N}}{\sqrt{2(x_1-1)}} \text{ and } 0 \le |u_0| \le \sqrt{\frac{1}{2}(x_1-1)N}. \qquad (1.9)$$

Furthermore, if p is prime, then, the Pell equation

$$u^2 - dv^2 = \pm p \qquad (1.10)$$

has at most one solution $u + v\sqrt{d}$ in which u and v satisfy the inequalities (1.7) or (1.9), respectively, provided $u \ge 0$. If the equation (1.10) is solvable, it has one or two classes of solutions, according as the prime p divides $2d$ or not.

Further details on Diophantine equations and Pell equations can be found in [21, 22, 23, 24, 25, 26, 27, 28, 29].

In order to see how Fibonacci and Lucas numbers are related to Diophantine equations, one can see the following:

It is well known that all positive integer solutions of the Diophantine equations

$$x^2 - 5y^2 = \pm 4$$

and

$$x^2 - xy - y^2 = \pm 1$$

are given by $(x, y) = (L_n, F_n)$ and (F_{n+1}, F_n) with $n \geq 1$, respectively.

Despite the elementary properties of Fibonacci and Lucas numbers are easily established, see [8], there are a number of more interesting and difficult questions connected with these numbers. One of them is about that under what conditions Fibonacci and Lucas numbers are perfect square? Although historical information is going to be done about this subject later, we only want to mention about that shortly.

Many studies about Fibonacci and Lucas numbers which are perfect square have been done by mathematicians. And the results of these studies are used to solve certain Diophantine equations. For instance, after determining the Fibonacci and Lucas numbers which are perfect square, the equations $x^4 - 5y^2 = \pm 4$, $x^4 - x^2y - y^2 = \pm 1$, $x^2 - 5y^4 = \pm 4$, and $x^2 - xy^2 - y^4 = \pm 1$ are easily solvable. In order to see the relations between these sequences and the equations above, we refer the reader to [1], [10], [30], and [31].

Moreover, it is possible to see the generalized Fibonacci and Lucas numbers as solutions of certain Diophantine equations. For instance, all positive integer solutions of the equations

$$x^2 - (P^2 + 4)y^2 = \pm 4 \text{ and } x^2 - (P^2 - 4)y^2 = 4$$

are given by $(x, y) = \left(V_n(P,1), U_n(P,1)\right)$ and $(x, y) = \left(V_n(P,-1), U_n(P,-1)\right)$ with $n \geq 1$, respectively. And all positive integer solutions of the equations

$$x^2 - Pxy - y^2 = \pm 1 \text{ and } x^2 - Pxy + y^2 = 1$$

are given by $(x, y) = \left(U_{n+1}(P,1), U_n(P,1)\right)$ and $(x, y) = \left(U_{n+1}(P,-1), U_n(P,-1)\right)$ with $n \geq 1$, respectively.

Interested readers can see [32, 33, 34, 35] for the solutions of the equations above.

It is obvious that replacing x by x^2 or y by y^2 into the equations above give some other Diophantine equations which can be easily solved if the generalized Fibonacci and Lucas numbers which are perfect square are known.

We now collect here the studies containing the generalized Fibonacci and Lucas numbers of the form kx^2.

Investigations of the properties of second order linear recurrence sequences have given rise to questions concerning whether, for certain pairs (P,Q), U_n or V_n is a perfect square $(=\square)$. In particular, the squares in sequences (U_n) and (V_n) were investigated by many authors.

From a result of Ljunggren [36], it was shown that if $P=2$, $Q=1$, and $n\geq 2$, then, $P_n=\square$ precisely for $n=7$, and Pethő [46] showed that these are the only perfect powers in the Pell sequence (see also Cohn [47]). And it was also shown that $P_n=2\square$ precisely for $n=2$. In 1964, Cohn [37] proved that if $P=Q=1$, then, the only perfect square greater than 1 in the sequence (F_n) is $F_{12}=12^2$ (see also Alfred [38], Burr [39], and Wyler [40]). Cohn [41] applied this result and a related result [42] to determine all solutions of several Diophantine equations. He [42], [43] also solved the equations $F_n=2\square$ and $L_n=\square$, $2\square$. Robbins [44], under the conditions that $P=Q=1$, found all solutions of the equation $F_n=px^2$ such that p is prime and either $p\equiv 3 \pmod 4$ or $p<10000$ and then, in 1991 the same author [45], using elementary techniques, found all solutions of the equation $L_n=px^2$, where p is prime and $p<1000$. Cohn [41], [48] determined the squares and twice the squares in $\left(U_n(P,\pm 1)\right)$ and $\left(V_n(P,\pm 1)\right)$ when P is odd. Ribenboim and McDaniel [17] determined all indices n such that $U_n=\square$, $2U_n=\square$, $V_n=\square$, or $2V_n=\square$ for all odd relatively prime integers P and Q. Bremner and Tzanakis [49] extend the result of the equation $U_n=\square$ by determining all generalized Fibonacci sequence (U_n) with

$U_{12} = \square$, subject only to the restriction that $(P,Q)=1$. In a latter paper, the same authors [50] show that for $n=2,...,7$, then, U_n is a square for infinitely many coprime P,Q and determine all sequences (U_n) with $U_n = \square$, $n=8,10,11$. And also in [51], they discuss the more general problem of finding all integers n,P,Q for which $U_n = k\square$ for a given integer k.

Although the problem for even values of P seem to be harder, in 1998, Kagawa and Terai [52] considered a similar problem, such as the problem considered by Ribenboim and McDaniel [17], for the case when P is even and $Q=1$. Using elementary properties of elliptic curves, they showed that if $P=2t$ with t even, $U_n(P,1) = \square$, $2U_n(P,1) = \square$, $V_n(P,1) = \square$, or $2V_n(P,1) = \square$ implies $n \le 3$ under some assumptions. Applying these results, the authors solved some Diophantine equations of the forms $4x^4 - (P^2+4)y^2 = \pm 1$, $x^4 - (P^2+4)y^2 = -1$, $x^2 - 4(P^2+4)y^4 = \pm 1$, and $x^2 - (P^2+4)y^4 = 1$.

Besides, Mignotte and Pethő [53] proved that if $n > 4$, then, $U_n(P,-1) = wx^2$ is impossible when $w \in \{1,2,3,6\}$, moreover these equations have solutions for $n=4$ only if $P=338$. Extending the method of Mignotte and Pethő, Nakamula and Pethő [54] gave the solutions of the equations $U_n(P,1) = w\square$ where $w \in \{1,2,3,6\}$. In 1998, Ribenboim and McDaniel [18] showed that if P is even, $Q \equiv 3 (\mathrm{mod}\, 4)$, and $U_n = \square$, then, n is a square or twice an odd square and all prime factors of n divides $P^2 + 4Q$. In a latter paper, for all odd values of P and Q, the same authors [19] determined all indices n such that $U_n = kx^2$ under the assumptions that for all integer $u \ge 1$, k is such that, for each odd divisor h of k, the Jacobi symbol $\left(\frac{-V_{2^u}}{h}\right)$ is defined and equals to 1. Afterwards, they solved the equation $V_n = 3\square$ for $P \equiv 1,3 (\mathrm{mod}\, 8)$, $Q \equiv 3 (\mathrm{mod}\, 4)$, $(P,Q)=1$ and solved the equation $U_n = 3\square$ for all odd relatively prime integers P and Q. Moreover, Cohn [55] solved the equations $U_n(P,\pm 1) = U_m(P,\pm 1)x^2$, $U_n(P,\pm 1) = 2U_m(P,\pm 1)x^2$, $V_n(P,\pm 1) = V_m(P,\pm 1)x^2$, and

$V_n(P,\pm 1) = 2V_m(P,\pm 1)x^2$ when P is odd. Keskin and Yosma [56] gave the solutions of the equations $F_n = 2F_m x^2$, $L_n = 2L_m x^2$, $F_n = 3F_m x^2$, $F_n = 6F_m x^2$, $L_n = 6L_m x^2$. Also, Keskin and Şiar proved in [57] that there is no integer x such that $F_n = 5F_m x^2$ for $m \geq 3$. In [58], Şiar and Keskin, assuming $Q = 1$, solved the equation $V_n = 2V_m x^2$ when P is even. They determined all indices n such that $V_n = kx^2$ when $k \mid P$ and P is odd. They show that there is no integer solution of the equations $V_n = 3x^2$ and $V_n = 6x^2$ for the case when P is odd and also they give the solutions of the equations $V_n = 3V_m x^2$ and $V_n = 6V_m x^2$. More generally, a main theorem was proved by Shorey and Stewart [59]:

Given $A \geq 1$, there exists an effectively computable number $C \geq 1$, which depends on A, such that if $n > 0$ and $U_n = A\square$ or $V_n = A\square$, then, $n < C$.

This thesis deals with Fibonacci and Lucas numbers of the form $U_n(P,Q)$ and $V_n(P,Q)$ with the special consideration that $Q = \pm 1$.

In Chapter 2, we list the most important properties of the generalized Fibonacci and Lucas numbers U_n and V_n; most of these are well known and the others are new. In the succeeding subchapters, we consider the generalized Fibonacci and Lucas numbers of the form $5\square$ and determine all indices n such that $U_n(P,1) = 5\square$, $U_n(P,-1) = 5\square$, $U_n(P,1) = 5U_m(P,1)\square$, and $U_n(P,-1) = 5U_m(P,-1)\square$ under some assumptions on P. We solve the equations $V_n(P,1) = 5\square$ and $V_n(P,-1) = 5\square$ when P is odd. Moreover, we prove that the equations $V_n(P,1) = 5V_m(P,1)\square$ and $V_n(P,-1) = 5V_m(P,-1)\square$ have no solutions.

In Chapter 3, the equations $U_n(P,1) = 7\square$, $U_n(P,1) = 7U_m(P,1)\square$, $V_n(P,1) = 7\square$, and $V_n(P,1) = 7V_m(P,1)\square$ are solved under some assumptions on P.

Our method used in this thesis is elementary and the main tools that we employ are the Jacobi symbol $\left(\frac{*}{*}\right)$ that we make extensive use of it, divisibility properties, and congruence properties concerning generalized Fibonacci and Lucas numbers.

CHAPTER 2. GENERALIZED FIBONACCI AND LUCAS NUMBERS OF THE FORM $5x^2$

In this chapter, we first list the most important properties of the generalized Fibonacci and Lucas numbers U_n and V_n. Then, we solve the equations $U_n(P,1)=5\square$, $U_n(P,-1)=5\square$, $U_n(P,1)=5U_m(P,1)\square$, and $U_n(P,-1)=5U_m(P,-1)\square$ under some assumptions on P. And we solve the equations $V_n(P,1)=5\square$ and $V_n(P,-1)=5\square$ when P is odd. Moreover, we prove that the equations $V_n(P,1)=5V_m(P,1)\square$ and $V_n(P,-1)=5V_m(P,-1)\square$ have no solutions.

2.1. Some Theorems and Identities

In this subsection, we give some theorems, lemmas, and well known identities about generalized Fibonacci and Lucas numbers, which will be needed in the proofs of the theorems related to the title of this chapter.

Definition 2.1.1. Let a and b be integers, at least one of which is not zero. The greatest common divisor of a and b, denoted by (a,b), is the largest integer which divides both a and b.

The first two theorems of the following four theorems are given for $Q=1$ and the others for $Q=-1$. The proofs of them can be found in [60].

Theorem 2.1.1. Let $n\in\mathbb{N}\cup\{0\}$, $m,r\in\mathbb{Z}$ and m be a nonzero integer. Then,

$$U_{2mn+r}\equiv(-1)^{mn}U_r(\mathrm{mod}\,U_m) \tag{2.1}$$

and

$$V_{2mn+r} \equiv (-1)^{mn} V_r (\operatorname{mod} U_m). \tag{2.2}$$

Theorem 2.1.2. Let $n \in \mathbb{N} \cup \{0\}$ and $m,\ r \in \mathbb{Z}$. Then,

$$U_{2mn+r} \equiv (-1)^{(m+1)n} U_r (\operatorname{mod} V_m) \tag{2.3}$$

and

$$V_{2mn+r} \equiv (-1)^{(m+1)n} V_r (\operatorname{mod} V_m). \tag{2.4}$$

Theorem 2.1.3. Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$ and m be a nonzero integer. Then,

$$U_{2mn+r} \equiv U_r (\operatorname{mod} U_m) \tag{2.5}$$

and

$$V_{2mn+r} \equiv V_r (\operatorname{mod} U_m). \tag{2.6}$$

Theorem 2.1.4. Let $n \in \mathbb{N} \cup \{0\}$ and $m,\ r \in \mathbb{Z}$. Then,

$$U_{2mn+r} \equiv (-1)^n U_r (\operatorname{mod} V_m) \tag{2.7}$$

and

$$V_{2mn+r} \equiv (-1)^n V_r (\operatorname{mod} V_m). \tag{2.8}$$

We omit the proofs of the following two lemmas, as they are based on mathematical induction.

Lemma 2.1.1. If n is a positive even integer, then, $V_n \equiv 2Q^{\frac{n}{2}} (\operatorname{mod} P^2)$ and if n is an odd positive integer, then, $V_n \equiv nPQ^{\frac{n-1}{2}} (\operatorname{mod} P^2)$.

Lemma 2.1.2. If n is a positive even integer, then, $U_n \equiv \frac{n}{2} P Q^{\frac{n-2}{2}} \pmod{P^2}$ and if n is an odd positive integer, then, $U_n \equiv Q^{\frac{n-1}{2}} \pmod{P^2}$.

The following lemma can be found in [17] and [19].

Lemma 2.1.3. Let P, Q, and m be odd positive integers, and $r \geq 1$. Then,

(**l**) If $3 \nmid m$, $V_{2^r m} \equiv \begin{cases} 3 \pmod 8), & \text{if } r=1 \text{ and } Q \equiv 1 \pmod 4 \\ 7 \pmod 8, & \text{otherwise.} \end{cases}$

(**m**) If $3 \mid m$, $V_{2^r m} \equiv 2 \pmod 8$.

When P and Q are odd, it follows from the lemma above

$$\left(\frac{-1}{V_{2^r}} \right) = -1 \tag{2.9}$$

for $r \geq 1$.

Before coming to the main results of this chapter several properties concerning generalized Fibonacci and Lucas numbers are needed.

$$U_{-n} = -(-Q)^n U_n \text{ and } V_{-n} = (-Q)^n V_n, \tag{2.10}$$

$$U_{2n} = U_n V_n, \tag{2.11}$$

$$V_{2n} = V_n^2 - 2(-Q)^n, \tag{2.12}$$

$$V_n^2 - (P^2 + 4Q) U_n^2 = 4(-Q)^n, \tag{2.13}$$

$$U_{3n} = U_n \left((P^2 + 4Q) U_n^2 + 3(-Q)^n \right), \tag{2.14}$$

$$V_{3n} = V_n (V_n^2 - 3(-Q)^n), \tag{2.15}$$

$$U_{5n} = U_n \left((P^2 + 4Q)^2 U_n^4 + 5(-Q)^n (P^2 + 4Q) U_n^2 + 5Q^{2n} \right). \tag{2.16}$$

If $5 \mid U_n$ or $5 \mid P^2+4Q$, then, from (2.16), we have

$$U_{5n} = 5U_n(5a+Q^{2n}) \tag{2.17}$$

for some $a \geq 0$.

Moreover,

$$V_{5n} = V_n(V_n^{\ 4} - 5(-Q)^n V_n^{\ 2} + 5Q^{2n}). \tag{2.18}$$

We immediately have from (2.18) that

$$V_{5n}(P,1) = \begin{cases} V_n(P,1)\left(V_n^4(P,1) - 5V_n^2(P,1)+5\right), \text{ if } n \text{ is even} \\ V_n(P,1)\left(V_n^4(P,1) + 5V_n^2(P,1)+5\right), \text{ if } n \text{ is odd.} \end{cases} \tag{2.19}$$

If $5 \mid P$ and n is odd, then, from Lemma 2.1.1, it is seen that $5 \mid V_n$. Therefore (2.19) implies that

$$V_{5n}(P,1) = 5V_n(P,1)(5a+1) \tag{2.20}$$

for some positive integer a.

Lemma 2.1.1 and the identity (2.13) give

$$5 \mid V_n(P,\pm 1) \text{ if and only if } 5 \mid P \text{ and } n \text{ is odd.} \tag{2.21}$$

Moreover,

$$V_{7n} = V_n\left(V_{2n}^{\ 3} - (-Q)^n V_{2n}^{\ 2} - 2Q^{2n}V_{2n} + (-Q)^{3n}\right). \tag{2.22}$$

By using (2.12), we readily obtain from (2.22) that

$$V_{7n} = V_n(V_n^6 - 7(-Q)^n V_n^4 + 14Q^{2n}V_n^2 - 7(-Q)^{3n}). \tag{2.23}$$

Then, we readily obtain from (2.23) that

$$V_{7n}(P,1) = \begin{cases} V_n(P,1)\left(V_n^6(P,1) - 7V_n^4(P,1) + 14V_n^2(P,1) - 7\right), \text{ if } n \text{ is even} \\ V_n(P,1)\left(V_n^6(P,1) + 7V_n^4(P,1) + 14V_n^2(P,1) + 7\right), \text{ if } n \text{ is odd.} \end{cases} \tag{2.24}$$

If $7 \mid P$ and n is odd, then, $7 \mid V_n$ from Lemma 2.1.1 and therefore from (2.24), it follows that

$$V_{7n}(P,1) = 7V_n(P,1)(7a+1) \tag{2.25}$$

for some positive integer a. Moreover, we have

$$\text{If } P \text{ is odd and } n \geq 1, \text{ then } 2 \mid V_n \Leftrightarrow 2 \mid U_n \Leftrightarrow 3 \mid n, \tag{2.26}$$

$$\text{If } V_m \neq 1, \text{ then } V_m \mid V_n \text{ iff } m \mid n \text{ and } n/m \text{ is odd,} \tag{2.27}$$

$$\text{If } U_m \neq 1, \text{ then } U_m \mid U_n \text{ iff } m \mid n. \tag{2.28}$$

Let $m = 2^a k,\ n = 2^b l,\ k$ and l are odd, $a, b \geq 0,$ and $d = (m,n).$ Then,

$$(U_m, V_n) = \begin{cases} V_d, \text{ if } a > b, \\ 1 \text{ or } 2, \text{ if } a \leq b. \end{cases} \tag{2.29}$$

If P is odd, then,

$$\left(U_n(P,1),\ V_n(P,1)\right) = \begin{cases} 1, \text{ if } 3 \nmid n, \\ 2, \text{ if } 3 \mid n, \end{cases} \tag{2.30}$$

$$\left(\frac{U_3(P,1)}{V_{2^r}(P,1)}\right)=1 \qquad (2.31)$$

for $r \geq 2$,

$$\left(\frac{5}{V_{2^r}(P,1)}\right)=\begin{cases}-1, \text{ if } 5 \mid P \text{ or } P^2 \equiv 1 \pmod 5, \\ 1, \text{ if } P^2 \equiv -1 \pmod 5,\end{cases} \qquad (2.32)$$

for $r \geq 1$.

Moreover,

$$\left(\frac{5}{V_{2^r}(P,-1)}\right)=\begin{cases}-1, \text{ if } 5 \mid P \text{ or } P^2 \equiv -1 \pmod 5, \\ 1, \text{ if } P^2 \equiv 1 \pmod 5,\end{cases} \qquad (2.33)$$

for $r \geq 1$.

If $3 \mid P$, then, from (2.12), we have

$$V_{2^r}(P,1) \equiv 2 \pmod 3 \qquad (2.34)$$

for all positive integer r.

If $3 \nmid P$, then, from (2.12), we get $V_{2^r}(P,-1) \equiv 2 \pmod 3$ for $r \geq 1$ and therefore

$$\left(\frac{3}{V_{2^r}(P,-1)}\right)=1. \qquad (2.35)$$

If $3 \mid P$, then, again from (2.12), we get $V_{2^r}(P,-1) \equiv 2 \pmod 3$ for $r \geq 2$ and therefore

$$\left(\frac{3}{V_{2^r}(P,-1)}\right)=1. \tag{2.36}$$

If $r \geq 2,$ then, we immediately have from (2.12) that $V_{2^r}(P,-1) \equiv -1\left(\bmod \frac{P^2-3}{2}\right).$

Under the condition that P is odd, the congruence above gives

$$\left(\frac{(P^2-3)/2}{V_{2^r}(P,-1)}\right)=\left(\frac{P^2-3}{V_{2^r}(P,-1)}\right)=1. \tag{2.37}$$

If $r=1,$ then,

$$V_{2^r}(P,-1)=V_2(P,-1) \equiv -2(\bmod P) \tag{2.38}$$

and if $r \geq 2,$ then, from (2.12), we have

$$V_{2^r}(P,-1) \equiv 2(\bmod P). \tag{2.39}$$

Also,

$$V_n(P,-1)=U_{n+1}(P,-1)-U_{n-1},(P,-1) \tag{2.40}$$

for all $n \in \mathbb{Z}.$

In addition to the identities above, if P is even, then, it is seen that

$$\begin{aligned} &U_n \text{ is even} \Leftrightarrow n \text{ is even,}\\ &U_n \text{ is odd} \Leftrightarrow n \text{ is odd,}\\ &V_n \text{ is even for all } n \in \mathbb{N}. \end{aligned} \tag{2.41}$$

Most of the properties above are well-known (see, for example [61], Ch. 2); properties between $(2.10)-(2.15)$ can be found in [41], [17], [19], and [10]; properties between $(2.26)-(2.29)$ can be found in [41], [17], [19], and [16]; properties (2.30) and (2.31) can be found in [41], and [17], [19], respectively. Finally, property (2.41) can be found in [18]. The other properties are straightforward and the proofs of them are easy. So, we omit their proofs.

2.2. Generalized Fibonacci and Lucas Numbers of the form $5x^2$

In this subsection, we assume that $Q=1$. For brevity, let $U_n=U_n(P,1)$ and $V_n=V_n(P,1)$. We determine all indices n such that $U_n=5\square$ and $U_n=5U_m\square$ under some assumptions on P. We show that the equation $V_n=5\square$ has a solution only if $n=1$ for the case when P is odd. Moreover, we prove that the equation $V_n=5V_m\square$ has no solutions.

It is convenient to gather here the theorems, lemmas, and some results which will be used in the proofs of the main theorems of this subsection.

We state the following theorem from [54].

Theorem 2.2.1. Let $P>0$. If $U_n=wx^2$ with $w\in\{1,2,3,6\}$, then, $n\le 2$ except when $(P,n,w)=(2,4,3)$, $(2,7,1)$, $(4,4,2)$, $(1,12,1)$, $(1,3,2)$, $(1,4,3)$, $(1,6,2)$, and $(24,4,3)$.

We have the following two theorems from [41], [48], and [17].

Theorem 2.2.2. If P is odd, then, the equation $V_n=x^2$ has the solutions $n=1$, $P=\square$, and $P\neq 1$ or $n=1,3$ and $P=1$ or $n=3$ and $P=3$.

Theorem 2.2.3. If P is odd, then, the equation $V_n=2x^2$ has the solutions $n=0$ or $n=6$ and $P=1,5$.

The first one of the following three theorems can be obtained from Theorem 6 and the others from Theorems 11 and 12 given in [55].

Theorem 2.2.4. Let P be an odd integer, $m \geq 2$ be an integer, and $U_n = 2U_m x^2$ for some integer x. Then, $P=1$ with $n=3$, $m=2$; $n=6$, $m=2$; $n=12$, $m=3$; $n=12$, $m=6$; or $P=5$ with $n=12$, $m=6$.

Theorem 2.2.5. Let P be an odd integer, $m \geq 1$ be an integer, and $V_n = V_m x^2$ for some integer x. Then, $n=m$ or $n=3$, $m=1$, $P=1$.

Theorem 2.2.6. Let P be an odd integer, $m \geq 1$ be an integer, and $V_n = 2V_m x^2$ for some integer x. Then $n=6$, $m=1$, $P=1$.

We can give the following theorem from [58].

Theorem 2.2.7. Let $k>1$ be a squarefree positive divisor of the odd integer P. If $V_n = kx^2$ for some integer x, then, $n=1$ or $n=3$.

Now we give some well known theorems in number theory. For more detailed information, see [29] or [62].

Theorem 2.2.8. Let m be an odd integer. Suppose that $x^2 \equiv -a^2 (\bmod m)$ for some nonzero integers x and a. Then, $m \equiv 1 (\bmod 4)$.

We omit the proof of the following theorem since it can be easily seen by induction.

Theorem 2.2.9. Let k be an integer with $k \geq 1$. Then, $L_{2^k} \equiv 3 (\bmod 4)$.

By using Theorems 2.1.9 and 2.1.10, we readily obtain,

Corollary 2.2.1. Let a be any nonzero integer. If $k \geq 1$, then, there is no integer x such that $x^2 \equiv -a^2 (\bmod L_{2^k})$.

We omit the proof of the following theorem due to Keskin and Demirtürk [63].

Theorem 2.2.10. All nonnegative integer solutions of the equation $u^2 - 5v^2 = 1$ are given by $(u,v) = (L_{3z}/2, F_{3z}/2)$ with $z(\geq 0)$ even and all nonnegative integer solutions of the equation $u^2 - 5v^2 = -1$ are given by $(u,v) = (L_{3z}/2, F_{3z}/2)$ with $z(\geq 1)$ odd.

By using the theorem above, we can give the following theorem without proof.

Theorem 2.2.11. All nonnegative integer solutions of the equation $x^2 - 4xy - y^2 = -5$ are given by $(x,y) = (L_{3z+3}/2, L_{3z}/2)$ with $z(\geq 0)$ even and all nonnegative integer solutions of the equation $x^2 - 4xy - y^2 = -1$ are given by $(x,y) = (F_{3z+3}/2, F_{3z}/2)$ with $z(\geq 1)$ odd.

For the proofs of the following four theorems, one can consult [32, 33, 34, 35].

Theorem 2.2.12. All positive integer solutions of the equations $x^2 - (P^2 + 4)y^2 = \pm 4$ are given by $(x,y) = \left(V_n(P,1),\ U_n(P,1)\right)$ with $n \geq 1$.

Theorem 2.2.13. All positive integer solutions of the equation $x^2 - (P^2 - 4)y^2 = 4$ are given by $(x,y) = \left(V_n(P,-1),\ U_n(P,-1)\right)$ with $n \geq 1$.

Theorem 2.2.14. All positive integer solutions of the equations $x^2 - Pxy - y^2 = \pm 1$ are given by $(x,y) = \left(U_{n+1}(P,1),\ U_n(P,1)\right)$ with $n \geq 1$.

Theorem 2.2.15. All positive integer solutions of the equation $x^2 - Pxy + y^2 = 1$ are given by $(x, y) = \left(U_{n+1}(P, -1),\ U_n(P, -1)\right)$ with $n \geq 1$.

Now we give the following results involving Fibonacci and Lucas numbers with nonnegative integers a and m.

$$F_m = a^2 \text{ iff } m = 0, 1, 2, 12, \tag{2.42}$$

$$F_m = 2a^2 \text{ iff } m = 0, 3, 6, \tag{2.43}$$

$$F_m = 5a^2 \text{ iff } m = 0, 5, \tag{2.44}$$

$$F_m = 10a^2 \text{ iff } m = 0, \tag{2.45}$$

$$L_m = a^2 \text{ iff } m = 1, 3, \tag{2.46}$$

$$L_m = 2a^2 \text{ iff } m = 0, 6. \tag{2.47}$$

The equations (2.42) and (2.43) are Theorems 3 and 4 in [43]; (2.44) follows from Theorem 3 in [44]; (2.45) follows from Theorem 3 in [64]; (2.46) and (2.47) are Theorems 1 and 2 in [43].

The following lemma can be proved by using Theorem 2.1.1.

Lemma 2.2.1.

$$5 \mid U_n \Leftrightarrow \begin{cases} 2 \mid n, \text{ if } 5 \mid P, \\ 3 \mid n, \text{ if } P^2 \equiv -1 \pmod{5}, \\ 5 \mid n, \text{ if } P^2 \equiv 1 \pmod{5}, \end{cases}$$

and

$$3 \mid U_n \Leftrightarrow \begin{cases} 2 \mid n, \text{ if } 3 \mid P, \\ 4 \mid n, \text{ if } 3 \nmid P. \end{cases}$$

From this point on, we assume that $m,n \geq 1$. Now we prove two theorems which help us to determine for which values of n, the equation $U_n = 5x^2$ has solutions and for which values of m,n, the equations $V_n = 5V_m x^2$ and $U_n = 5U_m x^2$ have solutions.

Although the solutions of the equations given in the following first two theorems can be get by using computer programme MAGMA [65], we will solve them by using only elementary methods.

Theorem 2.2.16. The only positive integer solution of the equation $x^4 + 3x^2 + 1 = 5y^2$ is given by $(x,y) = (1,1)$ and the only positive integer solution of the equation $x^4 - 3x^2 + 1 = 5y^2$ is given by $(x,y) = (2,1)$.

Proof: Assume that $x^4 \pm 3x^2 + 1 = 5y^2$ for some positive integers x and y. Multiplying both sides of the equations by 4 and completing the square give

$$(2x \pm 3)^2 - 5 = 5(2y)^2.$$

Then, it follows that

$$(2y)^2 - 5\left((2x \pm 3)/5\right)^2 = -1.$$

By Theorem 2.2.10, we get $2y = L_{3z}/2$ and $(2x^2 \pm 3)/5 = F_{3z}/2$ for some odd positive integer z. Assume that $z > 1$. Then, we can write $z = 4q \pm 1$ for some $q > 0$ and therefore $z = 2.2^k a \pm 1$ with $2 \nmid a$ and $k \geq 1$. Thus by (2.3), we get

$$F_{3z} = F_{3(4q\pm1)} = F_{12q\pm3} = F_{2.2^k 3a \pm 3} \equiv -F_{\pm 3} \equiv -F_3 (\bmod L_{2^k}),$$

i.e.,

$$F_{3z} \equiv -2 (\bmod L_{2^k}).$$

Substituting the value of F_{3z} and rewriting the above congruence give

$$4x^2 \pm 6 \equiv -10 (\mathrm{mod}\, L_{2^k}).$$

This shows that

$$4x^2 + 6 \equiv -10 (\mathrm{mod}\, L_{2^k}) \text{ or } 4x^2 - 6 \equiv -10 (\mathrm{mod}\, L_{2^k})$$

Then, it follows that

$$x^2 \equiv -4 (\mathrm{mod}\, L_{2^k})$$

or

$$x^2 \equiv -1 (\mathrm{mod}\, L_{2^k}),$$

which is a contradiction by Corollary 2.2.1. Thus $z=1$ and therefore $2x^2 \pm 3 = 5F_3/2$ and $2y = L_3/2$. A simple computation shows that $y=1$ and $x=1$ or $x=2$. This means that the equation $x^4+3x^2+1=5y^2$ has only the positive integer solution $(x,y)=(1,1)$ and the equation $x^4-3x^2+1=5y^2$ has only the positive integer solution $(x,y)=(2,1)$.

Theorem 2.2.17. The equation $x^4+5x^2+5=5y^2$ has no solutions x and y in positive integers.

Proof: Assume that $x^4+5x^2+5=5y^2$ for some positive integers x and y. Since $(2y+2)^2+(4y-1)^2=20y^2+5$, it follows that

$$(2y+2)^2+(4y-1)^2=(2x^2+5)^2.$$

Clearly, $d=(2y+2,\ 4y-1)=1$ or 5. Assume that $d=1$. By the Pythagorean theorem, there exist positive integers a and b with $(a,b)=1$, a and b are opposite parity, such that

$$2x^2+5=a^2+b^2,\ 2y+2=2ab,\ 4y-1=a^2-b^2.$$

The latter two equations imply that

$$-5 = a^2 - 4ab - b^2. \tag{2.48}$$

Thus by Theorem 2.2.11, we get $a = L_{3z+3}/2$, $b = L_{3z}/2$ for some nonnegative even integer z. On the other hand, from the equations $-5 = a^2 - 4ab - b^2$ and $2x^2 + 5 = a^2 + b^2$, we readily obtain $x^2 = a(a-2b)$. Since $(a,b) = 1$, it follows that, $r = (a, a-2b) = 1$ or 2. If $r = 1$, then, there exist coprime positive integers u and v such that $a = u^2$, $a - 2b = v^2$. Thus $L_{3z+3} = 2a = 2u^2$ and therefore $3z+3 = 6$ by (2.47), which is impossible since z is even. If $r = 2$, then, $a = 2u^2$, $a - 2b = 2v^2$. Thus $L_{3z+3} = 4u^2 = (2u)^2$ and therefore $3z+3 = 1$ or 3 by (2.46). The first of these is impossible. And the second implies that $z = 0$. Thus $a = 2$, $b = 1$. Since $2x^2 + 5 = a^2 + b^2$, it follows that $x = 0$, which is impossible since x is positive. Assume that $d = 5$. Then, there exist positive integers a and b with $(a,b) = 1$, a and b are opposite parity, such that

$$2x^2 + 5 = 5a^2 + 5b^2,\ 2y + 2 = 10ab,\ 4y - 1 = 5a^2 - 5b^2.$$

The above first equation implies that $5 \mid x$ and therefore $x = 5t$ for some positive integer t. And the latter two equations imply that $-5 = 5a^2 - 20ab - 5b^2$, i.e., $-1 = a^2 - 4ab - b^2$. Completing the square gives $(a-2b)^2 - 5b^2 = -1$. Thus by Theorem 2.2.10, we get $a = F_{3z+3}/2$, $b = F_{3z}/2$ for some odd positive integer z. On the other hand, by using $x = 5t$, from the equations $-5 = 5a^2 - 20ab - 5b^2$ and $2x^2 + 5 = 5a^2 + 5b^2$, we obtain $5t^2 = a(a-2b)$. Since $(a,b) = 1$, clearly, $(a, a-2b) = 1$ or 2. Assume that $(a, a-2b) = 1$. This implies that either $a = 5u^2$, $a - 2b = v^2$ or $a = u^2$, $a - 2b = 5v^2$. If the first of these is satisfied, then, it is seen that $F_{3z+3} = 10u^2$ and therefore $3z+3 = 0$ by (2.45), which is impossible in positive integers. If the second is satisfied, then, it is seen that $F_{3z+3} = 2u^2$ and therefore $3z+3 = 0, 3$ or 6 by (2.43). But it is obvious that the cases $3z+3 = 0$ and

$3z+3=3$ are impossible in positive integers. If $3z+3=6$, then, $z=1$ and therefore $a=2$, $b=1$. Since $2x^2+5=5a^2+5b^2$, it follows that $x^2=10$, which is impossible. Assume that $(a,a-2b)=2$. Then, either $a=10u^2$, $a-2b=2v^2$ or $a=2u^2$, $a-2b=10v^2$. If the first of these is satisfied, then, $F_{3z+3}=20u^2=5(2u)^2$ and therefore $3z+3=0$ or 5 by (2.44), which are impossible in positive integers. If the second is satisfied, then, $F_{3z+3}=4u^2=(2u)^2$ and therefore $3z+3=0,1,2$ or 12 by (2.42). But there does not exist any positive integer z such that $3z+3=0,1$ or 2. If $3z+3=12$, then, we get $z=3$ and therefore $a=72$, $b=17$. Since $2x^2+5=5a^2+5b^2$, it follows that $x^2=13680$, which is impossible.

Theorem 2.2.18. If P is odd, then, the equation $V_n=5x^2$ has a solution only if $n=1$.

Proof: Assume that $V_n=5x^2$. Then, by (2.21), it follows that $5\,|\,P$ and n is odd. Hence, by Theorem 2.2.7, we have $n=1$ or $n=3$. If $n=3$, then, $V_3=P(P^2+3)=5x^2$. Since $5\,|\,P$, it follows that $(P/5)(P^2+3)=x^2$. Clearly, $d=(P/5,P^2+3)=1$ or 3. Assume that $d=1$. Then, $P=5a^2$ and $P^2+3=b^2$ for some positive integers a and b. This implies that $b^2\equiv 3(\bmod 5)$, which is impossible. Assume that $d=3$. Then, we get $P=15a^2$ and $P^2+3=3b^2$ for some positive integers a and b. It is seen from $P^2+3=3b^2$ that $3\,|\,P$ and therefore $P=3c$ for some positive integer c. Hence, we obtain the Pell equation $b^2-3c^2=1$. It is well known that all positive integer solutions of this equation are given by

$$(b,c)=\left(V_m(4,-1)/2,U_m(4,-1)\right)$$

with $m\geq 1$. On the other hand, substituting $P=15a^2$ into $P=3c$, we get $c=5a^2$. So, we are interested in finding whether the equation $U_m(4,-1)=5\square$ has a solution. Assume that the equation $U_m(4,-1)=5\square$ has a solution. Since $5\,|\,U_3(4,-1)$, it can be seen that if $5 | U_m(4,-1)$, then, $3 | m$ and therefore $m = 3r$ for some positive integer r. Thus from (2.14), we get

$$U_m(4,-1) = U_{3r}(4,-1) = U_r(4,-1)\left((P^2-4)U_r^2(4,-1)+3\right) = U_r(4,-1)(12U_r^2(4,-1)+3).$$

Clearly, $d = \left(U_r(4,-1), 12U_r^2(4,-1)+3\right) = 1$ or 3. Assume that $d = 1$. Then, either $U_r(4,-1) = a^2$, $12U_r^2(4,-1)+3 = 5b^2$ or $U_r(4,-1) = 5a^2$, $12U_r^2(4,-1)+3 = b^2$ for some positive integers a and b. But both of them are impossible since $b^2 \equiv 3 \pmod 4$ in these two cases. Assume that $d = 3$. Then, either

$$U_r(4,-1) = 3a^2,\ 12U_r^2(4,-1)+3 = 15b^2 \tag{2.49}$$

or

$$U_r(4,-1) = 15a^2,\ 12U_r^2(4,-1)+3 = 3b^2 \tag{2.50}$$

for some positive integers a and b. Assume that (2.49) is satisfied. A simple computation shows that $\left(2(U_r(4,-1)\right)^2 - 5b^2 = -1$. Thus by Theorem 2.2.10, we obtain $2U_r(4,-1) = L_{3z}/2$ for some odd positive integer z. Substituting $U_r(4,-1) = 3a^2$ into the previous equation gives $3a^2 = L_{3z}/4$, i.e., $L_2 a^2 = L_{3z}/4$. This implies that $L_2 | L_{3z}$. Then, by (2.27), we get $2 | 3z$, which is impossible since z is odd. Assume that (2.50) is satisfied. It is easily seen that $\left(2U_r(4,-1)\right)^2 + 1 = b^2$, that is, $b^2 - \left(2U_r(4,-1)\right)^2 = 1$, implying that $U_r(4,-1) = 0$. This is impossible since r is a positive integer. So $n = 3$ cannot be a solution. If $n = 1$, then, $V_1 = P = 5\Box$. It is obvious that this is a solution.

By using Theorem 2.2.12, the immediate corollary follows.

Corollary 2.2.2. The equations $25x^4 - (P^2+4)y^2 = \pm 4$ have positive integer solutions only when $P = 5a^2$ with a odd.

Theorem 2.2.19. Let $V_m \neq 1$. There is no integer x such that $V_n = 5V_m x^2$.

Proof: Assume that $V_n = 5V_m x^2$ for some $x > 0$. Then, by (2.21), it follows that $5 \mid P$ and n is odd. Moreover, since $V_m \mid V_n$, there exists an odd integer t such that $n = mt$ by (2.27). Thus m is odd. Therefore we have $V_n \equiv nP (\bmod P^2)$ and $V_m \equiv mP (\bmod P^2)$ by Lemma 2.1.1. This shows that $nP \equiv 5mPx^2 (\bmod P^2)$, i.e., $n \equiv 5mx^2 (\bmod P)$. Since $5 \mid P$, it follows that $5 \mid n$. Also since $n = mt$, first, assume that $5 \mid t$. Then, $t = 5s$ for some odd positive integer s and therefore $n = mt = 5ms$. By (2.19), we readily obtain $V_n = V_{5ms} = V_{ms}(V_{ms}^4 + 5V_{ms}^2 + 5)$. Since ms is odd and $5 \mid P$, it follows that $5 \mid V_{ms}$ by (2.21). Therefore $(V_{ms} / V_m)\left((V_{ms}^4 + 5V_{ms}^2 + 5)/5\right) = x^2$. Clearly, $\left(V_{ms} / V_m, (V_{ms}^4 + 5V_{ms}^2 + 5)/5\right) = 1$. This implies that $V_{ms}^4 + 5V_{ms}^2 + 5 = 5b^2$ for some positive integer a and b. But this is impossible by Theorem 2.2.17. Now assume that $5 \nmid t$. Since $n = mt$ and $5 \mid n$, it is seen that $5 \mid m$. Then, we can write $m = 5^r a$ with $5 \nmid a$ and $r \geq 1$. By (2.20), we readily obtain $V_m = V_{5^r a} = 5V_{5^{r-1} a}(5a_1 + 1)$ for some positive integer a_1. Thus we conclude that $V_m = V_{5^r a} = 5^r V_a (5a_1 + 1)(5a_2 + 1) \cdots (5a_r + 1)$ for some positive integers a_i with $1 \leq i \leq r$. Let $A = (5a_1 + 1)(5a_2 + 1) \cdots (5a_r + 1)$. It is obvious that $5 \nmid A$. Thus we have $V_m = 5^r V_a A$. Similarly, we see that $V_n = V_{5^r at} = 5^r V_{at}(5b_1 + 1)(5b_2 + 1) \cdots (5b_r + 1)$ for some positive integers b_j with $1 \leq j \leq r$. Let $B = (5b_1 + 1)(5b_2 + 1) \cdots (5b_r + 1)$. It is obvious that $5 \nmid B$. Thus we have $V_n = 5^r V_{at} B$. This shows that $5^r V_{at} B = 5 \cdot 5^r V_a A x^2$, i.e., $V_{at} B = 5V_a A x^2$. By Lemma 2.1.1 and the identity (2.21), it is seen that $atPB \equiv 5aPAx^2 (\bmod P^2)$ and therefore we get $atB \equiv 5aAx^2 (\bmod P)$. Using the fact that $5 \mid P$, we get $5 \mid atB$. But this is impossible since $5 \nmid a$, $5 \nmid t$, and $5 \nmid B$.

Theorem 2.2.20. If P is odd and $5 \mid P$, then, the equation $U_n = 5x^2$ has a solution $n = 2$, $P = 5\square$. If $P^2 \equiv 1 (\mathrm{mod} 5)$, then, the equation $U_n = 5x^2$ has a solution $n = 5$,

$P=1$. If P is odd and $P^2 \equiv -1(\bmod 5)$, then, the equation $U_n = 5x^2$ has no solutions.

Proof: Assume that P is odd and $5 \mid P$. Since $5 \mid U_n$, it follows that n is even by Lemma 2.2.1. Then, $n = 2t$ for some positive integer t. By (2.11), we get $U_n = U_{2t} = U_t V_t = 5x^2$. Clearly, $(U_t, V_t) = 1$ or 2 by (2.30). Let $(U_t, V_t) = 1$. Then, either

$$U_t = a^2,\ V_t = 5b^2 \tag{2.51}$$

or

$$U_t = 5a^2,\ V_t = b^2 \tag{2.52}$$

for some positive integers a and b. Assume that (2.51) is satisfied. By Theorem 2.2.18, we get $t = 1$ and therefore $n = 2$. Then, $P = 5\square$ is a solution. Assume that (2.52) is satisfied. Since $5 \mid U_t$, it follows that t is even by Lemma 2.2.1. Thus $t = 2r$ for some positive integer r. By using (2.12), we get $V_{2r} = V_r^2 \pm 2 = b^2$, which is impossible. Let $(U_t, V_t) = 2$. Then, either

$$U_t = 10a^2,\ V_t = 2b^2 \tag{2.53}$$

or

$$U_t = 2a^2,\ V_t = 10b^2 \tag{2.54}$$

for some positive integers a and b. Equation (2.53) has no solutions, because the values of t and P for which $V_t = 2b^2$ are $t = 6$ and $P = 5$ by Theorem 2.2.3, which gives $U_6 = 3640 \neq 10a^2$. Assume that (2.54) is satisfied. Since $5 \mid V_t$, it follows that t is odd by (2.21). If $t = 1$, then, $U_1 = 1 = 2a^2$, which is impossible. Assume that $t > 1$. Then, $t = 4q \pm 1$ for some $q > 1$. And so, by (2.1), we get $U_t = U_{2.2q\pm 1} \equiv U_{\pm 1}(\bmod U_2)$, implying that $2a^2 \equiv 1(\bmod P)$. Since $5 \mid P$, the previous congruence becomes $2a^2 \equiv 1 (\bmod 5)$, which is impossible since $\left(\frac{2}{5}\right) = -1$. The proof is completed for the case when P is odd and $5 \mid P$.

Assume that $P^2 \equiv 1 (\bmod 5)$. Since $5 \mid U_n$, it follows that $5 \mid n$ by Lemma 2.2.1. Thus $n = 5t$ for some positive integer t. Since $P^2 \equiv 1 (\bmod 5)$, it is obvious that $5 \mid P^2 + 4$ and therefore there exists a positive integer A such that $P^2 + 4 = 5A$. By (2.16), we get $U_n = U_{5t} = U_t\left((P^2+4)^2 U_t^4 \pm 5(P^2+4)U_t^2 + 5\right)$. Substituting $P^2 + 4 = 5A$ into the previous equation gives $U_n = U_{5t} = 5U_t(5A^2U_t^4 \pm 5AU_t^2 + 1)$. Let $B = A^2U_t^4 \pm AU_t^2$. Then, we get $U_n = U_{5t} = 5U_t(5B+1) = 5x^2$, i.e., $U_t(5B+1) = x^2$. It can be seen that $(U_t, 5B+1) = 1$. This shows that $U_t = a^2$ and $5B + 1 = b^2$ for some positive integers a and b. By Theorem 2.2.1, we get $t \le 2$ or $t = 12$ and $P = 1$. If $t = 1$, then, $n = 5$ and therefore we get $U_5 = P^4 + 3P^2 + 1 = 5x^2$. By Theorem 2.2.16, it follows that $P = 1$. So the equation $U_n = 5x^2$ has a solution $n = 5$ and $P = 1$. If $t = 2$, then, $n = 10$ and therefore we obtain $U_{10} = 5x^2$, implying that $U_5V_5 = 5x^2$ by (2.11). Since $5 \mid U_5$, it follows that $(U_5/5)V_5 = x^2$. By (2.30), clearly, $(U_5/5, V_5) = 1$. This implies that $U_5 = 5a^2$, $V_5 = b^2$. Since $U_5 = P^4 + 3P^2 + 1$, it follows that $P = 1$ by Theorem 2.2.16. But then, $V_5 = 11 = b^2$, which is impossible. If $t = 12$ and $P = 1$, then, it follows that $n = 60$. Thus we obtain $U_{60} = 5x^2$, which is impossible by (2.44). The proof is completed for the case when $P^2 \equiv 1 (\bmod 5)$.

Assume that P is odd and $P^2 \equiv -1 (\bmod 5)$. Since $5 \mid U_n$, it follows that $3 \mid n$ by Lemma 2.2.1 and therefore $n = 3m$ for some positive integer m. Assume that m is even. Then, $m = 2s$ for some positive integer s and therefore $n = 6s$. Thus by (2.11), we get $U_n = U_{6s} = U_{3s}V_{3s} = 5x^2$. By (2.30), clearly, $(U_{3s}, V_{3s}) = 2$. Then, either

$$U_{3s} = 10a^2, V_{3s} = 2b^2 \tag{2.55}$$

or

$$U_{3s} = 2a^2, V_{3s} = 10b^2 \tag{2.56}$$

for some positive integers a and b. Assume that (2.55) is satisfied. By Theorem 2.2.3, it follows that $3s = 6$ and $P = 1,5$. But this is impossible since $P^2 \equiv -1 (\mathrm{mod}\, 5)$. Assume that (2.56) is satisfied. Since $5 \mid V_{3s}$, it follows that $5 \mid P$ by (2.21). But this contradicts the fact that $P^2 \equiv -1 (\mathrm{mod}\, 5)$. Now assume that m is odd. Then, by (2.14), we get $U_n = U_{3m} = U_m\left((P^2+4)U_m^2 - 3\right)$. Clearly, $\left(U_m, (P^2+4)U_m^2 - 3\right) = 1$ or 3. Since m is odd, it follows that $3 \nmid U_m$ by Lemma 2.2.1 and therefore $\left(U_m, (P^2+4)U_m^2 - 3\right) = 1$. Then,

$$U_m = 5a^2, (P^2+4)U_m^2 - 3 = b^2 \tag{2.57}$$

or

$$U_m = a^2, (P^2+4)U_m^2 - 3 = 5b^2 \tag{2.58}$$

for some positive integers a and b. Assume that (2.57) is satisfied. Since m is odd, we obtain $V_m^2 + 1 = b^2$ by (2.13). This shows that $V_m = 0$, which is impossible. Assume that (2.58) is satisfied. Since both m and P are odd, it follows that $m = 1$ by Theorem 2.2.1. If $m = 1$, then, $n = 3$ and therefore $P^2 + 1 = 5y^2$, which is impossible since we get $y^2 \equiv 2 (\mathrm{mod}\, 8)$ in this case.

By using Theorems 2.2.12 and 2.2.14, we give the following corollary.

Corollary 2.2.3. The equations $25x^4 - 5Px^2y - y^2 = \pm 1$ and $x^2 - 25(P^2+4)y^4 = \pm 4$ have positive integer solutions only when $P = 1$ or $P = 5a^2$ with a odd.

In [57], the authors show that the equation $F_n = 5F_m x^2$ has no solutions when $m \geq 3$. Now, we give the following theorem.

Theorem 2.2.21. Let $P>1$ and $m>1$. The equation $U_n = 5U_m x^2$ has no solutions in any of the following cases:

(i): $P^2 \equiv 1 (\bmod 5)$;

(ii): $P^2 \equiv -1 (\bmod 5)$, n is odd, and P is odd or $4 \mid P$;

(iii): $P^2 \equiv -1 (\bmod 5)$, n is even, and P is odd;

(iv): P is odd and $5 \mid P$.

Proof: Assume that $U_n = 5U_m x^2$ for some positive integer x. Since $U_m \mid U_n$, it follows that $m \mid n$ by (2.28). Thus $n = mt$ for some positive integer t. Since $n \neq m$, we have $t > 1$.

Case I: Let $P^2 \equiv 1 (\bmod 5)$. It is obvious that $5 \mid P^2 + 4$. Since $5 \mid U_n$, it follows that $5 \mid n$ by Lemma 2.2.1. Now we divide the proof into two subcases.

Subcase (i): Assume that $5 \mid t$. Then, $t = 5s$ for some positive integer s and therefore $n = mt = 5ms$. By (2.16), we obtain

$$U_n = U_{5ms} = U_{ms}\left((P^2+4)^2 U_{ms}^4 \pm 5(P^2+4)U_{ms}^2 + 5\right) = 5U_m x^2. \tag{2.59}$$

It is easily seen that $5 \mid (P^2+4)^2 U_{ms}^4 \pm 5(P^2+4)U_{ms}^2 + 5$. Also we have $(P^2+4)^2 U_{ms}^4 \pm 5(P^2+4)U_{ms}^2 + 5 = V_{ms}^4 \pm 3V_{ms}^2 + 1$ by (2.13). So rearranging the equation (2.59) gives

$$x^2 = (U_{ms}/U_m)\left((V_{ms}^4 \pm 3V_{ms}^2 + 1)/5\right).$$

Clearly, $\left(U_{ms}/U_m,\ (V_{ms}^4 \pm 3V_{ms}^2 + 1)/5\right) = 1$. This implies that $V_{ms}^4 \pm 3V_{ms}^2 + 1 = 5b^2$ for some $b > 0$. Thus by Theorem 2.2.16, we get $V_{ms} = 1$ or $V_{ms} = 2$. The first of these is impossible. If the second is satisfied, then, $ms=0,$ which contradicts the fact that $m>1.$

Subcase (ii): Assume that $5\nmid t.$ Since $5\mid n,$ it follows that $5\mid m.$ Then, we can write $m=5^r a$ with $5\nmid a$ and $r\geq 1.$ Since $5\mid P^2+4,$ it can be seen by (2.17) that $U_m=U_{5^r a}=5U_{5^{r-1}a}(5a_1+1)$ for some positive integer a_1. And thus we conclude that $U_m=U_{5^r a}=5^r U_a(5a_1+1)(5a_2+1)\cdots(5a_r+1)$ for some positive integers a_i with $1\leq i\leq r.$ Let $A=(5a_1+1)(5a_2+1)\cdots(5a_r+1).$ It is obvious that $5\nmid A$ and we have $U_m=5^r U_a A.$ Similarly, we get $U_n=U_{5^r at}=5^r U_{at}(5b_1+1)(5b_2+1)\cdots(5b_r+1)$ for some positive integers b_j with $1\leq j\leq r.$ Let $B=(5b_1+1)(5b_2+1)\cdots(5b_r+1).$ It is obvious that $5\nmid B.$ Thus we have $U_n=5^r U_{at}B.$ Substituting the new values of U_n and U_m into $U_n=5U_m x^2$ gives $5^r U_{at}B=5\cdot 5^r U_a Ax^2.$ This shows that $U_{at}B=5U_a Ax^2.$ Since $5\nmid B,$ it follows that $5\mid U_{at},$ implying that $5\mid at$ by Lemma 2.2.1. This contradicts the fact that $5\nmid a$ and $5\nmid t.$

Case II: Let $P^2\equiv -1(\bmod 5)$ and n is odd. Then, both m and t are odd. Thus we can write $t=4q\pm 1$ for some $q\geq 1.$ And so, by (2.1), we get

$$5U_m x^2=U_n=U_{(4q\pm 1)m}=U_{2.2mq\pm m}\equiv U_m(\bmod U_{2m}).$$

Using (2.11) gives $5x^2\equiv 1(\bmod V_m).$ Since m is odd, it follows that $P\mid V_m$ by Lemma 2.1.1. Then, we have

$$5x^2\equiv 1(\bmod P). \tag{2.60}$$

Assume that P is odd. Then, (2.60) implies that $1=\left(\dfrac{5}{P}\right).$ Since $P^2\equiv -1(\bmod 5),$ it can be seen that $P\equiv \pm 2(\bmod 5).$ Hence we get

$$1=\left(\frac{5}{P}\right)=\left(\frac{P}{5}\right)=\left(\frac{\pm 2}{5}\right)=-1,$$

a contradiction. Now assume that P is even. If $8 \mid P,$ then, it follows from (2.60) that $5x^2 \equiv 1(\bmod 8),$ which is impossible since we get $x^2 \equiv 5(\bmod 8)$ in this case. If $4 \mid P$ and $8 \nmid P,$ then, from (2.60), we get

$$5x^2 \equiv 1(\bmod P/4).$$

This shows that $\left(\frac{5}{P/4}\right)=1.$ Since $P^2 \equiv -1(\bmod 5),$ it can be seen that $P/4 \equiv \pm 2(\bmod 5).$ Hence we get

$$1=\left(\frac{5}{P/4}\right)=\left(\frac{P/4}{5}\right)=\left(\frac{\pm 2}{5}\right)=-1,$$

a contradiction.

Case III: Let $P^2 \equiv -1(\bmod 5),$ n is even, and P is odd. Since $n=mt,$ we divide the proof into two subcases.

Subcase (i): Assume that t is even. Then, $t=2s$ for some positive integer $s.$ Thus we get $5x^2=U_n/U_m=U_{2ms}/U_m=(U_{ms}/U_m)V_{ms}.$ Clearly, $d=(U_{ms}/U_m,\ V_{ms})=1$ or 2 by (2.30). Let $d=1.$ Then, either

$$U_{ms}=U_m a^2 \text{ and } V_{ms}=5b^2 \tag{2.61}$$

or

$$U_{ms}=5U_m a^2 \text{ and } V_{ms}=b^2 \tag{2.62}$$

for some positive integers a and $b.$ Assume that (2.61) is satisfied. Since $5 \mid V_{ms},$ it follows that $5 \mid P$ by (2.21). This contradicts the fact that $P^2 \equiv -1(\bmod 5).$ Assume that (2.62) is satisfied. By Theorem 2.2.2, we get $ms=3$ and $P=3$. Since $m>1$, it follows that $m=3$. This is impossible since we get $1=5a^2$ in this case. Let $d=2$. This implies that either

$$U_{ms}=2U_m a^2 \text{ and } V_{ms}=10b^2 \tag{2.63}$$

or

$$U_{ms}=10U_m a^2 \text{ and } V_{ms}=2b^2 \tag{2.64}$$

for some positive integers a and b. Assume that (2.63) is satisfied. Since $5 \mid V_{ms}$, it follows that $5 \mid P$ by (2.21). This contradicts the fact that $P^2 \equiv -1 (\bmod 5)$. Assume that (2.64) is satisfied. By Theorem 2.2.3, we get $ms=6$ and $P=1,5$. But this is impossible since $P^2 \equiv -1 (\bmod 5)$.

Subcase (ii): Assume that t is odd. Since $t>1$, we can write $t=4q+1$ or $t=4q+3$ for some $q>0$. On the other hand, since n is even and $n=mt$, it follows that m is even. Therefore we can write $m=2^r a$ with a odd and $r>0$. Assume that $t=4q+1$. Then, $n=mt=4qm+m=2\cdot 2^{r+k}b+m$ with b odd. Hence, we get

$$5U_m x^2=U_n=U_{2\cdot 2^{r+k}b+m} \equiv -U_m (\bmod V_{2^{r+k}})$$

by (2.3). Since $(U_m, V_{2^{r+k}})=(U_{2^r a}, V_{2^{r+k}})=1$ by (2.29), it follows that $5x^2 \equiv -1 (\bmod V_{2^{r+k}})$. But this is impossible. Because $\left(\dfrac{5}{V_{2^{r+k}}}\right)=1$ and $\left(\dfrac{-1}{V_{2^{r+k}}}\right)=-1$ by (2.32) and (2.9), respectively. Now assume that $t=4q+3$. Then, we have $n=mt=4qm+3m$. And so, by (2.1), we get

$$5U_m x^2=U_n=U_{4qm+3m} \equiv U_{3m} (\bmod U_{2m}).$$

By using (2.11) and (2.14), we readily obtain $5x^2 \equiv V_m^2 - 1 (\bmod V_m)$, which implies that $5x^2 \equiv -1 (\bmod V_m)$. Using the fact that $m = 2^r a$ with a odd, we have $5x^2 \equiv -1 (\bmod V_{2^r a})$, implying that $5x^2 \equiv -1 (\bmod V_{2^r})$ by (2.27). But this is impossible since $\left(\frac{5}{V_{2^r}}\right) = 1$ and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.32) and (2.9), respectively.

Case IV: Let P be odd and $5 \mid P$. Since $5 \mid U_n$, it follows that n is even by Lemma 2.2.1. Moreover, since $U_m \mid U_n$, there exists an integer t such that $n = mt$ by (2.28). Assume that t is even. Then, $t = 2s$ for some positive integer s. By (2.11), we get $U_n = U_{2ms} = U_{ms} V_{ms} = 5 U_m x^2$, implying that $(U_{ms} / U_m) V_{ms} = 5x^2$. Clearly, $(U_{ms} / U_m,\ V_{ms}) = 1$ or 2 by (2.30). If $(U_{ms} / U_m,\ V_{ms}) = 1$, then,

$$U_{ms} = U_m a^2,\ V_{ms} = 5b^2 \tag{2.65}$$

or

$$U_{ms} = 5 U_m a^2,\ V_{ms} = b^2 \tag{2.66}$$

for some positive integers a and b. Assume that (2.65) is satisfied. Then, by Theorem 2.2.18, we get $ms = 1$. This contradicts the fact that $m > 1$. Assume that (2.66) is satisfied. Then, by Theorem 2.2.2, we have $ms = 3$ and $P = 3$. But this is impossible since $5 \mid P$. If $(U_{ms} / U_m,\ V_{ms}) = 2$, then,

$$U_{ms} = 2 U_m a^2,\ V_{ms} = 10b^2 \tag{2.67}$$

or

$$U_{ms} = 10 U_m a^2,\ V_{ms} = 2b^2 \tag{2.68}$$

for some positive integers a and b. Assume that (2.67) is satisfied. Then, by Theorem 2.2.4, we get $ms = 12$, $m = 6$, $P = 5$. On the other hand, since $5 \mid V_{ms}$, it follows by (2.21) that $5 \mid P$ and ms is odd. This is a contradiction since $ms = 12$. Equation (2.68) has no solutions, since the possible values for which $V_{ms} = 2b^2$ are given by Theorem 2.2.3 and none of them gives a solution to $U_{ms} = 10U_m a^2$. Now assume that t is odd. Since $n = mt$ and n is even, it follows that m is even. Therefore we have $U_n \equiv (n/2)P(\operatorname{mod} P^2)$ and $U_m \equiv (m/2)P(\operatorname{mod} P^2)$ by Lemma 2.1.2. This shows that

$$(n/2)P \equiv 5(m/2)Px^2(\operatorname{mod} P^2),$$

i.e.,

$$(n/2) \equiv 5(m/2)x^2(\operatorname{mod} P).$$

Since $5 \mid P$, it is obvious that $5 \mid n$. Now we divide the remainder of the proof into two subcases.

Subcase (i): Assume that $5 \mid t$. Then, $t = 5s$ for some positive integer s and therefore $n = mt = 5ms$. By (2.16), we obtain

$$U_n = U_{5ms} = U_{ms}\left((P^2+4)^2 U_{ms}^4 + 5(P^2+4)U_{ms}^2 + 5\right) = 5U_m x^2. \tag{2.69}$$

Since ms is even and $5 \mid P$, it is seen that $5 \mid U_{ms}$ by Lemma 2.2.1. Also we have $(P^2+4)^2 U_{ms}^4 + 5(P^2+4)U_{ms}^2 + 5 = V_{ms}^4 - 3V_{ms}^2 + 1$ by (2.13). So rearranging the equation (2.69) gives

$$x^2 = (U_{ms}/U_m)\left((V_{ms}^4 - 3V_{ms}^2 + 1)/5\right).$$

Clearly, $\left(U_{ms}/U_m,\ (V_{ms}^4 - 3V_{ms}^2 + 1)/5\right) = 1$. This implies that $V_{ms}^4 - 3V_{ms}^2 + 1 = 5b^2$ for some $b > 0$. Thus by Theorem 2.2.16, we get $V_{ms} = 2$, implying that $ms = 0$, which is impossible.

Subcase (ii): Assume that $5 \nmid t$. Since $5 \mid n$, it follows that $5 \mid m$. Then, we can write $m = 5^r a$ with $5 \nmid a$, $2 \mid a$, and $r \geq 1$. It can be seen by (2.17) that

$U_m = U_{5^r a} = 5U_{5^{r-1}a}(5a_1+1)$ for some positive integer a_1. And thus we conclude that $U_m = U_{5^r a} = 5^r U_a (5a_1+1)(5a_2+1)\cdots(5a_r+1)$ for some positive integers a_i with $1 \le i \le r$. Let $A = (5a_1+1)(5a_2+1)\cdots(5a_r+1)$. Then, we have $U_m = 5^r U_a A$. In a similar manner, we get $U_n = U_{5^r at} = 5^r U_{at}(5b_1+1)(5b_2+1)\cdots(5b_r+1)$ for some positive integers b_j with $1 \le j \le r$. Let $B = (5b_1+1)(5b_2+1)\cdots(5b_r+1)$. It is obvious that $5 \nmid B$. Thus we have $U_n = 5^r U_{at} B$. Substituting the new values of U_n and U_m into $U_n = 5U_m x^2$ gives

$$5^r U_{at} B = 5 \cdot 5^r U_a A x^2. \tag{2.70}$$

On the other hand, since a and at are even, it follows from Lemma 2.1.2 that $U_{at} \equiv (at/2)P \pmod{P^2}$ and $U_a \equiv (a/2)P \pmod{P^2}$. So (2.70) becomes

$$5^r (at/2)PB \equiv 5 \cdot 5^r (a/2)PAx^2 \pmod{P^2}.$$

Rearranging the congruence above gives

$$(at/2)B \equiv 5(a/2)Ax^2 \pmod{P}.$$

Since $5 \mid P$, it follows that $5 \mid (at/2)B$, implying that $5 \mid atB$. This contradicts the fact that $5 \nmid a$, $5 \nmid t$, and $5 \nmid B$.

2.3. On the Equations $U_n = 5\square$ and $V_n = 5\square$

The purpose of this subchapter, assuming $P \ge 3$ is odd and $Q = -1$, is to determine the values of n such that $V_n = 5\square$ and $U_n = 5\square$. Moreover, we solve the equations $V_n = 5V_m\square$ and $U_n = 5U_m\square$.

One can see the proofs of the following two theorems in [66].

Theorem 2.3.1. Let $P \geq 3$ be odd. If $V_n = kx^2$ for some $k \mid P$ with $k > 1$, then, $n = 1$.

Theorem 2.3.2. Let $P \geq 3$ be odd. If $U_n = kx^2$ for some $k \mid P$ with $k > 1$, then, $n = 2$ or $n = 6$ and $3 \mid P$.

The following theorem is given in [17].

Theorem 2.3.3. Let $P \geq 3$ be odd. If $V_n = x^2$ for some integer x, then, $n = 1$. If $V_n = 2x^2$ for some integer x, then, $n = 3$, $P = 3, 27$.

We state the following theorem due to Ribenboim and McDaniel [17].

Theorem 2.3.4. Let $P \geq 3$ be odd. If $U_n = x^2$, then, $n = 1$ or $n = 6$ and $P = 3$.

The first one of the following three theorems can be obtained from Theorem 9 and the others from Theorems 14 and 15 given in [55].

Theorem 2.3.5. Let $P \geq 3$ be odd, $m, n > 1$ be integers. The equation $U_n = 2U_m x^2$ has no solutions except for the cases $n = 6$, $m = 3$, $P = 3, 27$.

Theorem 2.3.6. The equation $V_n = V_m x^2$, where $P \geq 3$, and P is odd, and $n > m > 0$ has only the trivial solution $n = m$.

Theorem 2.3.7. The equation $V_n = 2V_m x^2$, where $P \geq 3$, and P is odd, and $m, n > 0$ has no solutions.

The following lemma can be proved by using (2.5).

Lemma 2.3.1.

$$5 \mid U_n \Leftrightarrow \begin{cases} 2 \mid n, \text{ if } 5 \mid P, \\ 3 \mid n, \text{ if } P^2 \equiv 1 (\operatorname{mod} 5), \\ 5 \mid n, \text{ if } P^2 \equiv -1 (\operatorname{mod} 5). \end{cases}$$

Throughout this subsection, we assume that m and n are positive integers.

Theorem 2.3.8. The equation $V_n = 5x^2$ has a solution only if $n = 1$.

Proof: Assume that $V_n = 5x^2$ for some integer x. Since $5 \mid V_n$, it follows from (2.21) that $5 \mid P$. This implies that $n = 1$ by Theorem 2.3.1.

By using Theorem 2.2.13, we have

Corollary 2.3.1. The equation $25x^4 - (P^2 - 4)y^2 = 4$ has positive integer solutions only when $P = 5a^2$ with a odd.

Theorem 2.3.9. There is no integer x such that $V_n = 5V_m x^2$.

Proof: Assume that $V_n = 5V_m x^2$. Then, by (2.21), it is seen that $5 \mid P$ and n is odd. Moreover, since $V_m \mid V_n$, there exists an odd integer t such that $n = mt$ by (2.27). Since n and t are odd and $n = mt$, m is also odd. Hence, we have from Lemma 2.1.1 that $V_n \equiv \pm nP (\operatorname{mod} P^2)$ and $V_m \equiv \pm mP (\operatorname{mod} P^2)$. This implies that $\pm nP \equiv \pm 5mPx^2 (\operatorname{mod} P^2)$, i.e., $n \equiv 5mx^2 (\operatorname{mod} P)$. Using the fact that $5 \mid P$, it follows that $5 \mid n$. Firstly, assume that $5 \mid t$. Then, $t = 5s$ for some odd positive integer s and therefore $n = mt = 5ms$. By (2.18), we immediately have $V_n = V_{5ms} = V_{ms}(V_{ms}^4 - 5V_{ms}^2 + 5)$. Since ms is odd and $5 \mid P$, it follows that $5 \mid V_{ms}$ by (2.21) and therefore

$$\frac{V_{ms}}{V_m}\left(\frac{V_{ms}^4-5V_{ms}^2+5}{5}\right)=x^2.$$

Clearly, $\left(V_{ms}/V_m,(V_{ms}^4-5V_{ms}^2+5)/5\right)=1.$ This implies that $V_{ms}=V_m u^2$ and $V_{ms}^4-5V_{ms}^2+5=5v^2$ for some positive integers u and v. Let $X=V_{ms}$. Now we consider the equation $X^4-5X^2+5=5v^2$. It is obvious that $5\,|\,X$. Assume that X is odd. Then, we readily obtain $5v^2\equiv 1(\mathrm{mod}\,8)$, which is impossible. Thus, X is even. Since $X^4-5X^2+5=(X^2-3)(X^2-2)-1$, we immediately have

$$5v^2\equiv -1\left(\mathrm{mod}(X^2-3)\right).$$

This means that

$$\left(\frac{5}{X^2-3}\right)=\left(\frac{-1}{X^2-3}\right).$$

Since X is even, it is easily seen that $\left(\frac{-1}{X^2-3}\right)=(-1)^{\frac{X^2-4}{2}}=1$. On the other hand, using the fact that $5\,|\,X$, we get

$$\left(\frac{5}{X^2-3}\right)=\left(\frac{X^2-3}{5}\right)=\left(\frac{-3}{5}\right)=\left(\frac{2}{5}\right)=-1,$$

a contradiction. Secondly, assume that $5\nmid t$. Since $n=mt$ and $5\,|\,n$, it is seen that $5\,|\,m$. Then, we can write $m=5^r a$ with $5\nmid a$ and $r\geq 1$. By (2.18), we obtain $V_m=V_{5^r a}=5V_{5^{r-1}a}(5a_1+1)$ for some positive integer a_1. Thus, we conclude that $V_m=V_{5^r a}=5^r V_a(5a_1+1)(5a_2+1)\cdots(5a_r+1)$ for some positive integers a_i with $1\leq i\leq r$. Let $A=(5a_1+1)(5a_2+1)\cdots(5a_r+1)$. Thus, we have $V_m=5^r V_a A$, where $5\nmid A$. In a similar manner, we see that $V_n=V_{5^r at}=5^r V_{at}(5b_1+1)(5b_2+2)\cdots(5b_r+1)$.

Thus, we have $V_n = 5^r V_{at} B$, where $5 \nmid B$. As a consequence, we get $5^r V_{at} B = 5 \cdot 5^r V_a A x^2$, implying that $V_{at} B = 5 V_a A x^2$. By Lemma 2.1.1, it is seen that $\pm atPB \equiv \pm 5aPAx^2 (\bmod P^2)$, i.e., $atB \equiv 5aAX^2 (\bmod P)$. Since $5 \mid P$, it follows that $5 \mid atB$. But this is impossible since $5 \nmid a$, $5 \nmid t$, and $5 \nmid B$.

Theorem 2.3.9. If $P \geq 3$ is odd, then, the equation $U_n = 5x^2$ has a solution $n = 2$ when $5 \mid P$ and $n = 3$ when $P^2 \equiv 1 (\bmod 5)$. The equation $U_n = 5x^2$ has no solutions when $P^2 \equiv -1 (\bmod 5)$.

Proof: Assume that $U_n = 5x^2$ for some integer x. Dividing the proof into three cases, we have

Case I: Let $5 \mid P$. Then, by Theorem 2.3.2, we see that $n = 2$ or $n = 6$ and $3 \mid P$. But, it can be easily shown that the equation $U_n = 5x^2$ has no solutions for the case when $n = 6$ and $3 \mid P$.

Case II: Let $P^2 \equiv 1 (\bmod 5)$. Since $5 \mid U_n$, it follows from Lemma 2.3.1 that $3 \mid n$. Hence, $n = 3m$ for some positive integer m. Assume that m is even. Then, $m = 2s$ for some positive integer s and therefore $n = 6s$. And so, by (2.11), we get $U_n = U_{6s} = U_{3s} V_{3s} = 5x^2$. Clearly, $(U_{3s}, V_{3s}) = 2$ by (2.26) and (2.29). Then, either

$$U_{3s} = 2a^2, \; V_{3s} = 10b^2 \tag{2.71}$$

or

$$U_{3s} = 10a^2, \; V_{3s} = 2b^2 \tag{2.72}$$

for some positive integers a and b. Assume that (2.71) is satisfied. Since $5 \mid V_{3s}$, it follows from (2.21) that $5 \mid P$. But this contradicts the fact that $P^2 \equiv 1 (\bmod 5)$. Now assume that (2.72) is satisfied. Then, by Theorem 2.3.3, we have $3s = 3$ and $P = 3, 27$. Therefore $s = 1$. If $P = 3$, then, $U_3 = P^2 - 1 = 8 = 10a^2$, which is impossible. If $P=27$, then, $U_3=P^2-1=27^2-1=10a^2$, which is also impossible. Now assume that m is odd. Then, by (2.14), we get $U_{3m}=U_m\left((P^2-4)U_m+3\right)$. Clearly, $\left(U_m,(P^2-4)U_m^2+3\right)=1$ or 3. Then, it follows that $(P^2-4)U_m^2+3=wa^2$ for some $w\in\{1,3,5,15\}$. Since $(P^2-4)U_m^2+3=V_{2m}+1$ by (2.12) and (2.13), it is seen that $V_{2m}+1=wa^2$. Assume that $m>1$. Then, $m=4q\pm1=2^r a\pm1$ with a odd and $r\geq 2$. Thus,

$$wa^2=V_{2m}+1\equiv 1-V_2\equiv -(P^2-3)(\operatorname{mod} V_{2^r})$$

by (2.8). This shows that

$$\left(\frac{w}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right).$$

By using (2.33), (2.35), and (2.36), it can be seen that $\left(\frac{w}{V_{2^r}}\right)=1$ for $w=3,5,15$. Moreover, $\left(\frac{-1}{V_{2^r}}\right)=-1$ and $\left(\frac{P^2-3}{V_{2^r}}\right)=1$ by (2.9) and (2.37), respectively. Thus, we get

$$1=\left(\frac{w}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right)=-1,$$

which is impossible. Therefore $m=1$ and thus $n=3$.

Case III: Let $P^2\equiv -1(\operatorname{mod}5)$. Since $5\mid U_n$, it follows that $5\mid n$ by Lemma 2.3.1. Thus $n=5t$ for some positive integer t. Since $P^2\equiv -1(\operatorname{mod}5)$, it is obvious that $5\mid P^2-4$ and therefore there exists a positive integer A such that $P^2-4=5A$. By

(2.16), we get $U_n = U_{5t} = U_t\left((P^2-4)^2U_t^4 + 5(P^2-4)U_t^2 + 5\right)$. Substituting $P^2 - 4 = 5A$ into the preceding equation gives $U_n = U_{5t} = 5U_t\left(5A^2U_t^4 + 5AU_t^2 + 1\right)$. Let $B = A^2U_t^4 + AU_t^2$. As a consequence, we have

$$U_n = U_{5t} = 5U_t(5B+1) = 5x^2,$$

implying that

$$U_t(5B+1) = x^2.$$

It can be easily seen that $(U_t, 5B+1) = 1$. This shows that $U_t = a^2$ and $5B+1 = b^2$ for some $a,\ b > 0$. By Theorem 2.3.4, we see that the only possible values of t and P for which $U_t = a^2$ are $t = 1$ or $t = 6$ and $P = 3$. If $t = 1$, then, $n = 5$ and therefore we get $U_n = U_{5t} = U_5 = P^4 - 3P^2 + 1 = 5x^2$. By Theorem 2.2.16, it follows that $P = 2$, which is impossible since P is odd. If $t = 6$, then, $n = 30$. A simple computation shows that there is no integer x such that $U_{30} = 5x^2$ for $P = 3$.

By using Theorems 2.2.13 and 2.2.15, we give the following corollary.

Corollary 2.3.2. The equations $x^2 - 25(P^2-4)y^4 = 4$ and $25x^4 - 5Px^2y + y^2 = 1$ have positive integer solutions only when $P = 5a^2$ with a odd or $P = V_{3z}(2,1)/2$ with $z > 0$ even.

Theorem 2.3.10. Let $P \geq 3$ and $m > 1$. The equation $U_n = 5U_mx^2$ has no solutions in any of the following cases:

(i): $P^2 \equiv -1 (\bmod 5)$;

(ii): P is odd and $5 \mid P$;

(iii): $P^2 \equiv 1 (\bmod 5)$, n is odd, and P is odd;

(iv): $P^2 \equiv 1 (\bmod 5)$, n is even, and P is odd.

Proof: Assume that $U_n = 5U_m x^2$ for some $x > 0$. Since $U_m \mid U_n$, it follows that $m \mid n$ by (2.28). Thus, $n = mt$ for some $t > 0$. Since $n \neq m$, we have $t > 1$.

Case I: Let $P^2 \equiv -1 (\bmod 5)$. It is obvious that $5 \mid P^2 - 4$. On the other hand, since $5 \mid U_n$, it follows that $5 \mid n$ by Lemma 2.3.1. Dividing the proof into two subcases, we have

Subcase (i): Assume that $5 \mid t$. Then, $t = 5s$ for some $s > 0$ and therefore $n = mt = 5ms$. By (2.16), we obtain

$$U_n = U_{5ms} = U_{ms}\left((P^2-4)^2 U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5\right) = 5U_m x^2. \qquad (2.73)$$

Since $5 \mid P^2 - 4$, it is seen that $5 \mid (P^2-4)^2 U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5$. Also, we have $(P^2-4)^2 U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5 = V_{ms}^4 - 3V_{ms}^2 + 1$ by (2.13). Rearranging the equation (2.73), we readily obtain

$$x^2 = (U_{ms} / U_m)\left((V_{ms}^4 - 3V_{ms}^2 + 1)/5\right),$$

where $\left(U_{ms} / U_m, (V_{ms}^4 - 3V_{ms}^2 + 1)/5\right) = 1$. Hence, $U_{ms} = U_m a^2$, $V_{ms}^4 - 3V_{ms}^2 + 1 = 5b^2$ for some $a, b > 0$. By Theorem 2.2.16, we get $V_{ms} = 2$, implying that $ms = 0$, which is impossible.

Subcase (ii): Assume that $5 \nmid t$. Since $5 \mid n$, it follows that $5 \mid m$. Then, we can write $m = 5^r a$ with $5 \nmid a$ and $r \geq 1$. By (2.17), it is seen that $U_m = U_{5^r a} = 5U_{5^{r-1} a}(5a_1 + 1)$ for some positive integer a_1. Thus, we conclude that $U_m = U_{5^r a} = 5^r U_a (5a_1 + 1)(5a_2 + 1) \cdots (5a_r + 1)$ for some positive integers a_i with

$1 \le i \le r$. Let $A = (5a_1+1)(5a_2+1)\cdots(5a_r+1)$. Then, we have $U_m = 5^r U_a A$, where $5 \nmid A$. In a similar manner, we get $U_n = U_{5^r at} = 5^r U_{at}(5b_1+1)(5b_2+1)\cdots(5b_r+1)$ for some positive integers b_i with $1 \le i \le r$. Let $B = (5b_1+1)(5b_2+1)\cdots(5b_r+1)$. Hence, we have $U_n = 5^r U_{at} B$, where $5 \nmid B$. As a consequence, we get

$$5^r U_{at} B = 5 \cdot 5^r U_a A x^2$$

i.e.,

$$U_{at} B = 5 U_a A x^2.$$

Since $5 \nmid B$, it follows that $5 \mid U_{at}$, implying that $5 \mid at$ by Lemma 2.3.1. This contradicts the fact that $5 \nmid a$ and $5 \nmid t$. This concludes the proof for the case when $P^2 \equiv -1 \pmod 5$.

Case II: Let P be odd and $5 \mid P$. Since $5 \mid U_n$, it is seen from Lemma 2.3.1 that n is even. On the other hand, we have $n = mt$. So, we first assume that t is even. Then, $t = 2s$ for some $s > 0$. By (2.11), we get $U_n = U_{2ms} = U_{ms} V_{ms} = 5U_m x^2$, implying that $\left(U_{ms}/U_m\right)V_{ms} = 5x^2$. Clearly, $d = \left(U_{ms}/U_m,\ V_{ms}\right) = 1$ or 2 by (2.29). If $d = 1$, then,

$$U_{ms} = U_m a^2,\ V_{ms} = 5b^2 \tag{2.74}$$

or

$$U_{ms} = 5U_m a^2,\ V_{ms} = b^2 \tag{2.75}$$

for some $a, b > 0$. If (2.74) is satisfied, then, the only possible value of ms for which $V_{ms} = 5b^2$ is 1 by Theorem 2.3.1, which contradicts the fact that $m > 1$. If (2.75) is satisfied, then, by Theorem 2.3.3, we have $ms = 1$, which is impossible since $m > 1$. If $d = 2$, then,

$$U_{ms} = 2U_m a^2,\ V_{ms} = 10b^2 \tag{2.76}$$

or

$$U_{ms} = 10U_m a^2,\ V_{ms} = 2b^2 \tag{2.77}$$

for some $a, b > 0$. Suppose (2.76) holds. Then, by Theorem 2.3.5, we get $ms = 6$, $m = 3$, $P = 3, 27$. But there is no integer b such that $V_6 = 2b^2$ for the case when $P = 3$ or 27. Suppose (2.77) holds. Then, by Theorem 2.3.3, the only possible values of ms and P for which $V_{ms} = 2b^2$ are $ms = 3$ and $P = 3, 27$. Since $m > 1$, it follows that $m = 3$ and therefore we obtain $U_3 = 10U_3 a^2$, which is impossible. Now assume that t is odd. Since $n = mt$ and n is even, it follows that m is even. Hence, we have $U_n \equiv \pm(n/2)P(\bmod P^2)$ and $U_m \equiv \pm(m/2)P(\bmod P^2)$ by Lemma 2.1.2. This shows that $\pm\frac{n}{2}P \equiv \pm 5\frac{m}{2}Px^2(\bmod P^2)$, i.e., $\frac{n}{2} \equiv 5\frac{m}{2}x^2(\bmod P)$. Since $5 \mid P$, it is seen that $5 \mid n$. Dividing remainder of the proof into two subcases, we have

Subcase (i): Let $5 \mid t$. Then, $t = 5s$ for some $s > 0$ and therefore $n = mt = 5ms$. By (2.16), we obtain

$$U_n = U_{5ms} = U_{ms}\left((P^2-4)^2U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5\right). \tag{2.78}$$

Since ms is even and $5 \mid P$, it is seen that $5 \mid U_{ms}$ by Lemma 2.3.1. Also, we have $(P^2-4)^2U_{ms}^4 + 5(P^2-4)U_{ms}^2 + 5 = V_{ms}^4 - 3V_{ms}^2 + 1$ by (2.13). Hence, rearranging the equation (2.78) gives

$$x^2 = \left(U_{ms}/U_m\right)\left((V_{ms}^4 - 3V_{ms}^2 + 1)/5\right),$$

where $\left((U_{ms}/U_m), (V_{ms}^4 - 3V_{ms}^2 + 1)/5\right) = 1$. This implies that $U_{ms} = U_m a^2$ and $V_{ms}^4 - 3V_{ms}^2 + 1 = 5b^2$ for some $a, b > 0$. Thus, by Theorem 2.2.16, we get $V_{ms} = 2$, implying that $ms = 0$, which is impossible.

Subcase (ii): Let $5 \nmid t$. Since $5 \mid n$, it follows that $5 \mid m$. Then, we can write $m = 5^r a$ with $5 \nmid a$ and $r \geq 1$. By (2.17), it is seen that $U_m = U_{5^r a} = 5U_{5^{r-1}a}(5a_1 + 1)$ for some positive integer a_1. Thus, $U_m = U_{5^r a} = 5^r U_a (5a_1 + 1)(5a_2 + 1) \cdots (5a_r + 1)$ for some positive integers a_i with $1 \leq i \leq r$. Let $A = (5a_1 + 1)(5a_2 + 1) \cdots (5a_r + 1)$. Then, we have $U_m = 5^r U_a A$, where $5 \nmid A$. In a similar way, we get $U_n = U_{5^r at} = 5^r U_{at}(5b_1 + 1)(5b_2 + 1) \cdots (5b_r + 1)$ for some positive integers b_i with $1 \leq i \leq r$. Let $B = (5b_1 + 1)(5b_2 + 1) \cdots (5b_r + 1)$. Hence, we have $U_n = 5^r U_{at} B$, where $5 \nmid B$. Substituting the new values of U_n and U_m into $U_n = 5U_m x^2$ gives

$$5^r U_{at} B = 5 \cdot 5^r U_a A x^2$$

i.e.,

$$U_{at} B = 5U_a A x^2.$$

On the other hand, since a is even and at is even, it follows from Lemma 2.1.2 that $U_{at} \equiv \pm \frac{at}{2} P (\bmod P^2)$ and $U_a \equiv \pm \frac{a}{2} P (\bmod P^2)$. Hence, we have

$$\pm \frac{at}{2} PB \equiv \pm 5 \frac{a}{2} PAx^2 (\bmod P^2),$$

implying that $\frac{at}{2} B \equiv 5 \frac{a}{2} Ax^2 (\bmod P^2)$. Since $5 \mid P$, it follows that $5 \mid \frac{at}{2} B$, which shows that $5 \mid atB$. This contradicts the fact that $5 \nmid a$, $5 \nmid t$, and $5 \nmid B$. This concludes the proof for the case when $5 \mid P$.

Case III: Let $P^2 \equiv 1 (\bmod 5)$, n is odd, and P is odd. Then, both m and t are odd. Since $5 \mid U_n$, it follows immediately from Lemma 2.3.1 that $3 \mid n$. Using the fact that $n = mt$, we have

Subcase (i): Assume that $3 \mid m$. Since t is odd, we can write $t = 4q \pm 1$ for some $q > 0$. If $t = 4q+1$, then, $t = 2 \cdot 2^r a + 1$ with a odd and $r \geq 1$. And so, by (2.7), we get $U_n = U_{mt} = U_{2 \cdot 2^r am+m} \equiv -U_m (\bmod V_{2^r})$, implying that $5U_m x^2 \equiv -U_m (\bmod V_{2^r})$. Since $(U_m, V_{2^r}) = 1$ by (2.29), it follows that $5x^2 \equiv -1 (\bmod V_{2^r})$, which is impossible since $\left(\frac{5}{V_{2^r}}\right) = 1$ by (2.33) and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.9). If $t = 4q-1$, then, by (2.5), we get $U_n = U_{m(4q-1)} = U_{2 \cdot 2mq-m} \equiv -U_m (\bmod U_{2m})$. This shows that

$$5U_m x^2 \equiv -U_m (\bmod U_{2m}),$$

implying that

$$5x^2 \equiv -1 (\bmod V_m)$$

by (2.11). Since $3 \mid m$, it is seen by (2.27) that $V_3 \mid V_m$. Hence, we obtain $5x^2 \equiv -1 (\bmod V_3)$, i.e., $5x^2 \equiv -1 (\bmod P^2 - 3)$. But this is impossible since

$$\left(\frac{5}{(P^2-3)/2}\right) = \left(\frac{(P^2-3)/2}{5}\right) = \left(\frac{-1}{5}\right) = 1$$

and

$$\left(\frac{-1}{(P^2-3)/2}\right) = (-1)^{\frac{P^2-5}{4}} = -1.$$

Subcase (ii): Assume that $3 \nmid m$. Since $n = mt$ and $3 \mid n$, it follows that $3 \mid t$ and therefore $t = 3s$ for some $s > 0$. Then, by (2.14), we get

$$U_n = U_{3ms} = U_{ms}\left((P^2-4)U_{ms}^2 + 3\right) = 5U_m x^2,$$

implying that

$$(U_{ms} / U_m)\left((P^2-4)U_{ms}^2 + 3\right) = 5x^2.$$

Clearly, $d = \left(U_{ms}/U_m,\ \left((P^2-4)U_{ms}^2+3\right)\right) = 1$ or $3.$ If $d=1,$ then, either

$$U_{ms} = U_m a^2,\ (P^2-4)U_{ms}^2+3 = 5b^2 \tag{2.79}$$

or

$$U_{ms} = 5U_m a^2,\ (P^2-4)U_{ms}^2+3 = b^2 \tag{2.80}$$

for some $a,b>0.$ Suppose that (2.79) is satisfied. Then, by (2.13), we get $V_{ms}^2-1=5b^2$ and this gives by (2.12) that $V_{2ms}=5b^2-1.$ Since $ms>1$ is odd, $ms=4q\pm1$ for some $q>0.$ Thus $ms=2\cdot 2^r a\pm1$ with a odd and $r>0.$ By using (2.8), we get $5b^2-1=V_{2ms}\equiv -V_{\pm2}\equiv -V_2 (\bmod V_{2^r}).$ This shows that $5b^2-1\equiv -(P^2-2)(\bmod V_{2^r}),$ implying that $5b^2\equiv -(P^2-3)(\bmod V_{2^r}).$ By using (2.9), (2.23), and (2.37), it is seen that

$$1=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{5}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right)=-1,$$

a contradiction. Suppose that (2.80) is satisfied. By combining two equations, it can be easily seen that $b^2\equiv 3(\bmod 5),$ which is impossible. If $d=3,$ then, either

$$U_{ms} = 3U_m a^2,\ (P^2-4)U_{ms}^2+3 = 15b^2 \tag{2.81}$$

or

$$U_{ms} = 15U_m a^2,\ (P^2-4)U_{ms}^2+3 = 3b^2 \tag{2.82}$$

for some $a,b>0.$ If we combine two equations given in (2.81), we readily obtain $b^2\equiv 2(\bmod 3),$ which is impossible. Suppose (2.82) holds. Then, by (2.13), we get $V_{ms}^2-1=3b^2$ and this gives by (2.12) that $V_{2ms}=3b^2-1.$ Since $ms>1$ is odd, $ms=4q\pm1$ for some $q>0.$ Thus $ms=2\cdot 2^r a\pm1$ with a odd and $r>0.$ By using (2.8), we get $3b^2-1=V_{2ms}\equiv -V_{\pm2}\equiv -V_2(\bmod V_{2^r}).$ This shows that

$3b^2 - 1 \equiv -(P^2 - 2)(\bmod V_{2^r})$, implying that $3b^2 \equiv -(P^2 - 3)(\bmod V_{2^r})$. By (2.35), (2.36), (2.9), and (2.37), it is seen that

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{P^2 - 3}{V_{2^r}}\right) = -1,$$

a contradiction.

Case IV: Let $P^2 \equiv 1(\bmod 5)$, n is even, and P is odd. Since $n = mt$, we divide the proof into two subcases:

Subcase (i): Assume that t is even. Then, $t = 2s$ for some $s > 0$. Hence, we immediately have

$$U_n / U_m = U_{2ms} / U_m = (U_{ms} / U_m)V_{ms} = 5x^2.$$

Clearly, $d = (U_{ms} / U_m, V_{ms}) = 1$ or 2 by (2.29). If $d = 1$, then,

$$U_{ms} = U_m a^2,\ V_{ms} = 5b^2 \tag{2.83}$$

or

$$U_{ms} = 5U_m a^2,\ V_{ms} = b^2 \tag{2.84}$$

for some $a, b > 0$. Suppose (2.83) is satisfied. Since $5 \mid V_{ms}$, it follows from (2.21) that $5 \mid P$, which contradicts the fact that $P^2 \equiv 1(\bmod 5)$. Now suppose (2.84) is satisfied. By Theorem 2.3.3, the only possible value of ms for which $V_{ms} = b^2$ is 1, which is impossible since $m > 1$. If $d = 2$, then,

$$U_{ms} = 2U_m a^2,\ V_{ms} = 10b^2 \tag{2.85}$$

or

$$U_{ms} = 10U_m a^2,\ V_{ms} = 2b^2 \tag{2.86}$$

for some $a, b > 0$. Obviously, (2.85) is not satisfied because of the same reason given above for (2.83). If (2.86) holds, then, it is seen by Theorem 2.3.3 that the only possible values of ms and P for which $V_{ms} = 2b^2$ are $ms = 3$ and $P = 3, 27$. But these are impossible since $P^2 \equiv 1(\bmod 5)$.

Subcase (ii): Assume that t is odd. Since $t > 1$, we can write $t = 4q+1$ or $t = 4q+3$ for some $q > 0$. On the other hand, since n is even and $n = mt$, it follows that m is even. Therefore we can write $m = 2^r a$ with a odd and $r > 0$. Assume that $t = 4q+1$. Then, $n = mt = 4qm + m = 2 \cdot 2^{r+k} b + m$ with b odd. Hence, we get

$$5U_m x^2 = U_n = U_{2\cdot 2^{r+k} b+m} \equiv -U_m (\bmod V_{2^{r+k}})$$

by (2.7). Since $(U_m,\ V_{2^{r+k}}) = (U_{2^r a},\ V_{2^{r+k}}) = 1$ by (2.29), it follows that

$$5x^2 \equiv -1(\bmod V_{2^{r+k}}).$$

This is impossible. Because $\left(\dfrac{5}{V_{2^{r+k}}}\right) = 1$ and $\left(\dfrac{-1}{V_{2^{r+k}}}\right) = -1$ by (2.33) and (2.9), respectively. Now assume that $t = 4q+3$. Then, we have $n = mt = 4qm + 3m$. And so, by (2.5), we get

$$5U_m x^2 = U_n = U_{4qm+3m} \equiv U_{3m} (\bmod U_{2m}).$$

By using (2.11) and (2.14), we readily obtain

$$5x^2 \equiv V_m^2 - 1(\bmod V_m),$$

which implies that

$$5x^2 \equiv -1(\bmod V_m).$$

Using the fact that $m = 2^r a$ with a odd, we have

$$5x^2 \equiv -1 (\bmod V_{2^r a}),$$

implying that

$$5x^2 \equiv -1 (\bmod V_{2^r})$$

by (2.27). But this is impossible since $\left(\frac{5}{V_{2^r}}\right) = 1$ and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.23) and (2.9), respectively.

CHAPTER 3. ON THE LUCAS SEQUENCE EQUATIONS $V_n = 7\square$ AND $V_n = 7V_m\square$

In this section, using congruences, with extensive reliance upon the Jacobi symbol, and by the help of the methods of solutions of Pell equations, we will solve the equations $U_n = 7\square$, $V_n = 7\square$, $U_n = 7U_m\square$, and $V_n = 7V_m\square$. For all odd values of P and $Q = 1$, the equation $U_n = 7\square$ has only the solution $(n, P) = (2, 7\square)$ when $7 \mid P$ and the equation $V_n = 7x^2$ has only the solution $(n, P) = (1, 7\square)$ when $7 \mid P$ or $(n, P) = (4, 1)$ when $P^2 \equiv 1 (\mathrm{mod}\, 7)$. We show that the equation $V_n = 7V_m\square$ is solvable if and only if $P^2 \equiv 4 (\mathrm{mod}\, 7)$ and $(n, m) = (3, 1)$. Moreover, we show that the equation $U_n = 7U_m\square$ has only the solution $(n, m, P, \square) = (8, 4, 1, 1)$ when P is odd.

Now, we shall establish some theorems and lemmas which will be required later.

The following two theorems can be found in [58].

Theorem 3.1. If P is odd, then, the equation $V_n = 3x^2$ has a solution $n = 1$ or $n = 2$ and if P is even and $3 \nmid P$, then, the equation $V_n = 3x^2$ has no solutions.

Theorem 3.2. If P is odd, $m \geq 1$, and $V_n = 3V_m x^2$, then, $m = 1$ and $n = 3$.

The following three lemmas can be proved by using Theorems 2.1.1 and 2.1.2.

Lemma 3.1.

$$3 \mid V_n \Leftrightarrow \begin{cases} n \equiv 1 \pmod 2, & \text{if } 3 \mid P, \\ n \equiv 2 \pmod 4, & \text{if } 3 \nmid P. \end{cases}$$

Lemma 3.2.

$$7 \mid V_n \Leftrightarrow \begin{cases} 2 \nmid n, \text{ if } 7 \mid P, \\ 4 \mid n \text{ and } n/4 \text{ is odd, if } P^2 \equiv 1 \pmod 7, \\ 4 \mid n \text{ and } n/4 \text{ is odd, if } P^2 \equiv 2 \pmod 7, \\ 3 \mid n \text{ and } n/3 \text{ is odd, if } P^2 \equiv 4 \pmod 7. \end{cases}$$

Lemma 3.3.

$$7 \mid U_n \Leftrightarrow \begin{cases} 2 \mid n, \text{ if } 7 \mid P, \\ 8 \mid n, \text{ if } P^2 \equiv 1 \pmod 7, \\ 16 \mid n, \text{ if } P^2 \equiv 2 \pmod 7, \\ 6 \mid n, \text{ if } P^2 \equiv 4 \pmod 7. \end{cases}$$

We state the following three lemmas without proof.

Lemma 3.4. All positive integer solutions of the equation $x^2 - 7y^2 = 2$ are given by $(x, y) = \big(3\big(U_{m+1}(16,-1) - U_m(16,-1)\big),\ 17U_m(16,-1) - U_{m+1}(16,-1)\big)$ with $m \ge 0$.

Lemma 3.5. All positive integer solutions of the equation $x^2 - 7y^2 = -3$ are given by

$$(x, y) = \big(2U_{m+1}(16,-1) + 5U_m(16,-1),\ 2U_{m+1}(16,-1) - 4U_m(16,-1)\big)$$

or

$(x, y) = \big(5U_m(16,-1) + 2U_{m-1}(16,-1),\ 2U_m(16,-1) - U_{m-1}(16,-1)\big)$ with $m \ge 0$.

Lemma 3.6. All positive integer solutions of the equation $x^2-3y^2=1$ are given by $(x,y)=\left(V_m(4,-1)/2,\ U_m(4,-1)\right)$ with $m\geq 1$.

Lemma 3.7. The equation $x^4-7y^2=-3$ has no positive integer solutions.

Proof: Assume that the equation $x^4-7y^2=-3$ for some $x,y>0$. If y is odd, then, it follows that $x^4\equiv 4(\text{mod}\,8)$, which is impossible. Thus, y is even and therefore x is odd. Note that the equation $x^4-7y^2=-3$ implies that

$$(x^2)^2-7y^2=-3.$$

By Lemma 3.5, we get

$$x^2=2U_{m+1}(16,-1)+5U_m(16,-1)$$

or

$$x^2=5U_m(16-1)+2U_{m-1}(16,-1).$$

Assume that $x^2=2U_{m+1}(16,-1)+5U_m(16,-1)$. Since x is odd, it is seen from (2.41) that m is odd. Besides, $x^2\equiv 2U_{m+1}(16,-1)(\text{mod}\,5)$, which implies that $5\,|\,U_{m+1}$. It can be easily shown that $5\,|\,U_m(16,-1)$ iff $3\,|\,m$. Thus, we get $m+1=3k$ for some $k>0$. Since m is odd, k is even and therefore $k=2q$. Hence, we have $m=6q-1$ with $q>0$. And so, by (2.5), we get

$$x^2=2U_{2\cdot 3q}(16,-1)+5U_{2\cdot 3q-1}(16,-1)\equiv 2U_0(16,-1)+5U_{-1}(16,-1)\left(\text{mod}\,U_3(16,-1)\right),$$

implying that $x^2\equiv -5(\text{mod}\,17)$, because $17\,|\,U_3(16,-1)$. But this is impossible since

$$\left(\frac{-5}{17}\right)=\left(\frac{-1}{17}\right)\left(\frac{5}{17}\right)=\left(\frac{2}{5}\right)=-1.$$

The details of the proof of the equality $x^2 = 5U_m(16-1) + 2U_{m-1}(16,-1)$, broadly similar to the above, are omitted.

Lemma 3.8. The equation $9x^4 - 21y^2 = -3$ has no positive integer solutions.

Proof: Dividing both sides of the equation above by 3 gives $7y^2 - 3x^4 = 1$. Now let us consider the equation

$$7u^2 - 3v^2 = 1. \tag{3.1}$$

Since the fundamental solution of (3.1) is $2\sqrt{7} + 3\sqrt{3}$, it follows as a consequence of Theorem 2.2 given in [67] that all positive integer solutions of (3.1) are given by $(u,v) = \left(2(U_{n+1} - U_n),\ 3(U_{n+1} + U_n)\right)$, where $U_n = U_n(150,-1)$. Therefore, we have $x^2 = 3(U_{n+1} + U_n)$. It can be easily shown that

$$U_n \equiv \begin{cases} n(\bmod 8), & \text{if } n \text{ is odd}, \\ -n(\bmod 8), & \text{if } n \text{ is even}. \end{cases} \tag{3.2}$$

Hence, if n is odd, then, by (3.2), we have $x^2 \equiv -3n - 3 + 3n \equiv -3(\bmod 8)$, a contradiction. If n is even, then, by (3.2), we get $x^2 \equiv 3n + 3 - 3n \equiv 3(\bmod 8)$, a contradiction.

In [68], when $Q = 1$, Keskin and Karaatlı solved the equations $U_n = 5\square$ and $U_n = 5U_m\square$ under some assumptions on P. They solved the equations $V_n = 5\square$ with P odd and $Q = 1$. They showed that the equation $V_n = 5V_m\square$ has no solutions. These results were presented in the second subchapter of Chapter 2 of this thesis. Here we will solve the equations $U_n = 7\square$, $V_n = 7\square$, $U_n = 7U_m\square$, and $V_n = 7V_m\square$ under the conditions that P is odd and $Q = 1$.

We begin with the following theorem. This result will be used in the solution of the equation $U_n = 7\square$.

Theorem 3.3. If P is an odd integer, then, there is no integer x such that $V_n = 14x^2$.

Proof: Assume that $V_n = 14x^2$ and P is odd. Since $2 | V_n$, we get $3 | n$ by (2.26). The remainder of the proof is split into two cases.

Case I: Assume that $7 | P$ or $P^2 \equiv 4 (\bmod 7)$. Since $7 | V_n$, it is seen from Lemma 3.2 that $2 \nmid n$. Since $3 | n$, we get $n = 3t$ and therefore $2 \nmid t$. Thus we can write $n = 12q \pm 3$. And so, by (2.2), we obtain $V_n = V_{12q \pm 3} \equiv \pm V_3 (\bmod U_6)$, which implies that $14x^2 \equiv \pm 4P \equiv 4 (\bmod 8)$. This shows that $x^2 \equiv 2 (\bmod 4)$, a contradiction.

Case II: Assume that $P^2 \equiv 1 (\bmod 7)$ or $P^2 \equiv 2 (\bmod 7)$. Since $7 | V_n$, it follows that $n = 4t$ for some odd t by Lemma 3.2. Since $3 | n$, we see that $3 | t$ and therefore $t = 6q + 3$. Thus, we can write $n = 24q + 12$. Let $P^2 \equiv 1 (\bmod 7)$. And so, by (2.4), we get

$$V_n = V_{24q+12} = V_{2 \cdot 2 \cdot 6q+12} \equiv V_{12} \equiv -V_0 \equiv -2 (\bmod V_2),$$

which implies that $14x^2 \equiv -2 (\bmod P^2 + 2)$. Hence, we obtain $1 = \left(\frac{-7}{P^2 + 2} \right)$. But this is impossible since

$$\left(\frac{-7}{P^2 + 2} \right) = (-1)^{\frac{P^2+1}{2}} (-1)^{\frac{P^2+1}{2}} \left(\frac{P^2 + 2}{7} \right) = \left(\frac{3}{7} \right) = \left(\frac{-4}{7} \right) = -1.$$

Now let $P^2 \equiv 2 (\bmod 7)$. Since $n = 24q + 12$, it follows from (2.2) that $V_n = V_{24q+12} = V_{2 \cdot 3(4q+2)} \equiv V_0 \equiv 2 (\bmod U_3)$, which implies that $14x^2 \equiv 2 (\bmod P^2 + 1)$, i.e., $7x^2 \equiv 1 \left(\bmod \frac{P^2 + 1}{2} \right)$. But this is also impossible since

$$1=\left(\frac{7}{(P^2+1)/2}\right)=(-1)^{\frac{P^2-1}{4}}\left(\frac{(P^2+1)/2}{7}\right)=\left(\frac{5}{7}\right)=\left(\frac{-2}{7}\right)=-1.$$

By Theorem 2.2.12, we have the following immediate corollary.

Corollary 3.1. The equations $196x^4-(P^2+4)y^2=\pm 4$ have no positive integer solutions.

Theorem 3.4. Let P be odd. If $7\mid P$, then, $V_n=7x^2$ is possible if and only if $(P,n)=(7\square,1)$. If $7\nmid P$, then, $V_n=7x^2$ is impossible, except for the case $(P,n)=(1,4)$.

Proof: Assume that $V_n=7x^2$, $7\mid P$ and P is odd. Then, by Theorem 2.2.7, we get $n=1$ or $n=3$. If $n=1$, then, $V_1=P=7x^2$. Therefore $n=1$ is a solution. If $n=3$, then, $V_3=P(P^2+3)=7x^2$. Since $7\mid P$, it follows that $(P/7)(P^2+3)=x^2$. Clearly, $d=(P/7,P^2+3)=1$ or 3. Let $d=1$. Then, $P=7a^2$ and $P^2+3=b^2$ for some $a,b>0$. This implies that $b^2\equiv 3(\mathrm{mod}\,7)$, which is impossible since $\left(\frac{3}{7}\right)=-1$. Let $d=3$. Then, we have

$$P=21a^2 \text{ and } P^2+3=3b^2. \tag{3.3}$$

It is seen from (3.3) that $3\mid P$ and therefore

$$P=3c \tag{3.4}$$

for some $c>0$. Substituting (3.4) into (3.3), we immediately have the Pell equation $b^2-3c^2=1$. By Lemma 3.6, we have $(b,c)=\left(V_m(4,-1)/2,\ U_m(4,-1)\right)$ for some $m\geq 1$. On the other hand, since $3c=21a^2$, we get $c=7a^2$. So, we are interested in the solutions $U_m(4,-1)=7\square$. Since $7\mid U_4(4,-1)$, it can be easily shown that

$7 \mid U_m(4,-1)$ if and only if $m=4k$ for some $k \geq 1$. Then, by (2.11), it follows that $7\square = U_{4k}\left(4,-1\right) = U_{2k}\left(4,-1\right)V_{2k}\left(4,-1\right)$. From (2.29) and (2.41), it is seen that $\left(U_{2k}\left(4,-1\right), V_{2k}\left(4,-1\right)\right) = 2$. Then, either

$$U_{2k}\left(4,-1\right) = 2u^2 \text{ and } V_{2k}\left(4,-1\right) = 14v^2 \tag{3.5}$$

or

$$U_{2k}\left(4,-1\right) = 14u^2 \text{ and } V_{2k}\left(4,-1\right) = 2v^2 \tag{3.6}$$

for some positive integers u and v. From now on and until the end of this paragraph, instead of $U_n(4,-1)$ and $V_n(4,-1)$, we will write U_n and V_n, respectively. Suppose (3.5) is satisfied. Clearly, $7 \mid V_{2k}$. Since $7 \mid V_2$, it can be easily shown that k is odd. Let $k = 4q \pm 1$. By (2.5), we get

$$2u^2 = U_{2(4q\pm1)} \equiv U_{\pm2}(\bmod U_4).$$

Since $8 \mid U_4$, the previous congruence becomes $2u^2 \equiv \pm 4(\bmod 8)$, which is impossible. Suppose (3.6) is satisfied. We show that if $V_n = 2v^2$, then, $3 \mid n$. Let $n = 6q + r$, $0 \leq r \leq 5$. Then, by (2.6), it follows that $V_n \equiv V_r(\bmod U_3)$, implying that $2v^2 \equiv V_r(\bmod 5)$ since $5 \mid U_3$. From this, it follows that $r = 0$ or 3. This shows that $3 \mid n$. Returning to the equation $V_{2k} = 2v^2$, we have $k = 3r$. Thus $V_{6r} = V_{3\cdot 2r} = V_{2r}(V_{2r}^2 - 3) = 2v^2$ by (2.15). This implies that $v^2 = \frac{V_{2r}}{2}(V_{2r}^2 - 3)$. On the other hand, since $V_n^2 - 12U_n^2 = 4$ by (2.13), we see that $3 \nmid V_n$. Thus, $\left(\frac{V_{2r}}{2},\ V_{2r}^2 - 3\right) = 1$. Then, we have $V_{2r}^2 - 3 = a^2$, which is impossible.

Now we consider the case $P^2 \equiv 1(\bmod 7)$. Since $7 \mid V_n$, it follows from Lemma 3.2 that $n = 4t$ for some odd integer t. Let $t > 1$. We can write $t = 4q \pm 1$ with $q \geq 1$ and therefore $n = 4t = 2 \cdot 2^r a \pm 4,$ with a odd and $r \geq 3.$ Thus by (2.4), we get $V_n \equiv -V_4 (\operatorname{mod} V_{2^r}).$ If $r = 3,$ then,

$$7x^2 \equiv -V_4 (\operatorname{mod} V_8) \tag{3.7}$$

and if $r > 3,$ then,

$$7x^2 \equiv -V_4 (\operatorname{mod} V_{2^r}). \tag{3.8}$$

Since $V_8 = V_4^2 - 2$ by (2.12), it follows that $V_8 \equiv -2 (\operatorname{mod} V_4)$ and therefore $V_{2^r} \equiv 2 (\operatorname{mod} V_4).$ Note that $V_4 = P^4 + 4P^2 + 2.$ Since $P^2 \equiv 1 (\operatorname{mod} 7),$ we see that $7 \mid V_4$ and therefore by (2.12), we have $V_8 \equiv -2 (\operatorname{mod} 7)$ and $V_{2^r} \equiv 2 (\operatorname{mod} 7).$ Besides, since P is odd, it follows that $V_4 \equiv 7 (\operatorname{mod} 8)$ and $V_8 \equiv 7 (\operatorname{mod} 8).$ Also,

$$\left(\frac{-1}{V_{2^r}} \right) = -1$$

by (2.9). Assume that $r = 3,$ so that, by (3.7), we have

$$\left(\frac{7}{V_8} \right) = \left(\frac{-1}{V_8} \right) \left(\frac{V_4}{V_8} \right).$$

But this is impossible since

$$\left(\frac{7}{V_8} \right) = (-1) \left(\frac{V_8}{7} \right) = (-1) (\frac{-2}{7}) = 1, \ \left(\frac{-1}{V_8} \right) = -1$$

and

$$\left(\frac{V_4}{V_8} \right) = (-1) \left(\frac{V_8}{V_4} \right) = (-1) \left(\frac{-2}{V_4} \right) = 1.$$

Now assume that $r > 3,$ so that (3.8) is satisfied. Then, it follows that

$$\left(\frac{7}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{V_4}{V_{2^r}}\right).$$

But this is also impossible since

$$\left(\frac{7}{V_{2^r}}\right)=(-1)\left(\frac{V_{2^r}}{7}\right)=(-1)\left(\frac{2}{7}\right)=-1,\ \left(\frac{-1}{V_{2^r}}\right)=-1$$

and

$$\left(\frac{V_4}{V_{2^r}}\right)=(-1)\left(\frac{V_{2^r}}{V_4}\right)=(-1)\left(\frac{2}{V_4}\right)=-1.$$

Hence, we conclude that $t=1.$ Then, $n=4$ and therefore $V_4=(P^2+2)^2-2=V_n=7x^2.$ Now, we consider the equation $u^2-7v^2=2$ with $u=P^2+2.$ By Lemma 3.4, we get

$$P^2+2=3\big(U_{m+1}(16,-1)-U_m(16,-1)\big).$$

From now on and until the end of the case $P^2\equiv 1(\operatorname{mod} 7),$ instead of $U_m(16,-1),$ we will write $U_m.$ Let $m=4q+r,\ 0\le r\le 3.$ Then, by (2.5), it follows that

$$U_{4q+r}\equiv U_r(\operatorname{mod} U_2),$$

leading to

$$P^2+2\equiv 3(U_{r+1}-U_r)(\operatorname{mod} 16)$$

since $16\,|\,U_2.$ A simple calculation shows that $r=0$ and therefore $4\,|\,m.$ So, we can write $m=12q,\ 12q+4$ or $12q+8.$ If $m=12q+4,$ then, we obtain

$$P^2+2\equiv 3(U_{12q+5}-U_{12q+4})\equiv 3(U_5-U_4)(\bmod U_3)$$

by (2.5). Since $5\,|\,U_3$, we immediately have $P^2+2\equiv 0(\bmod 5)$, which is impossible. The remainder of the proof is split into two cases.

Case I: Let $m=12q$ with $q\geq 0$. If $q>0$, then, we can write $m=12q=2\cdot 2^r\cdot 3a$, with a odd and $r\geq 1$. Thus by (2.7), we get

$$P^2+2=3(U_{2\cdot 2^r\cdot 3a+1}-U_{2\cdot 2^r\cdot 3a})\equiv -3(\bmod V_{2^r}),$$

leading to

$$P^2\equiv -5(\bmod V_{2^r}/2). \tag{3.9}$$

If $r\geq 2$, then, a simple calculation shows that $V_{2^r}\equiv 2(\bmod 8)$ and $V_{2^r}\equiv -1(\bmod 5)$. Thus, $V_{2^r}/2\equiv 1(\bmod 4)$ and $V_{2^r}/2\equiv 2(\bmod 5)$. From (3.9), it is seen that

$$1=\left(\frac{-1}{V_{2^r}/2}\right)\left(\frac{5}{V_{2^r}/2}\right).$$

But this is impossible since

$$\left(\frac{-1}{V_{2^r}/2}\right)=1$$

and

$$\left(\frac{5}{V_{2^r}/2}\right)=\left(\frac{V_{2^r}/2}{5}\right)=\left(\frac{2}{5}\right)=-1.$$

Hence, we get $r=1$. By (2.7), it follows that $P^2+2=3(U_{2\cdot 6a+1}-U_{2\cdot 6a})\equiv -3(\bmod V_6)$, i.e., $P^2\equiv -5(\bmod V_6/2)$. This implies that

$$1 = \left(\frac{-1}{V_6/2}\right)\left(\frac{5}{V_6/2}\right).$$

Using the fact that $V_6/2 \equiv 1 \pmod 5$ and $V_6/2 \equiv 3 \pmod 4$, we readily obtain

$$1 = \left(\frac{-1}{V_6/2}\right)\left(\frac{5}{V_6/2}\right) = (-1)\left(\frac{V_6/2}{5}\right) = (-1)\left(\frac{1}{5}\right) = -1,$$

a contradiction. Thus, we get $q = 0$. Then, $m = 0$ and therefore $P^2 + 2 = 3$. This gives that $P = 1$.

Case II: Let $m = 12q + 8$ with $q \geq 0$. This implies that $m = 12u - 4$ for some $u > 0$. Then, by (2.7), we get

$$P^2 + 2 = 3(U_{2\cdot 3\cdot 2q-3} - U_{2\cdot 3\cdot 2q-4}) \equiv 3(U_{-3} - U_{-4}) \equiv 3(U_4 - U_3) \pmod{V_3}.$$

A simple calculation shows that $11 \mid V_3$, $U_4 \equiv 5 \pmod{11}$, and $U_3 \equiv 2 \pmod{11}$. Thus, it is seen that $P^2 \equiv 7 \pmod{11}$, which is impossible since $\left(\frac{7}{11}\right) = \left(\frac{-4}{11}\right) = -1.$

Assume that $P^2 \equiv 2 \pmod 7$. Since $7 \mid V_n$, it follows from Lemma 3.2 that $n = 4t$ for some odd integer t. Similar arguments used for the case when $P^2 \equiv 1 \pmod 7$ show that $P = 1$. But this is impossible since $P^2 \equiv 2 \pmod 7$.

Assume that $P^2 \equiv 4 \pmod 7$. Since $7 \mid V_n$, it follows that $n = 3t$ for some odd integer t by Lemma 3.2. Hence, $V_n = V_{3t} = V_t(V_t^2 + 3)$ from (2.15). Clearly, $(V_t, V_t^2 + 3) = 1$ or 3. Let $(V_t, V_t^2 + 3) = 1$. Then, either

$$V_t = a^2,\ V_t^2 + 3 = 7b^2 \tag{3.10}$$

or

$$V_t = 7a^2,\ V_t^2 + 3 = b^2 \tag{3.11}$$

for some positive integer a and b. But the two relations (3.11) lead to $b^2 \equiv 3 (\bmod 7)$, which is impossible, hence (3.10) is satisfied. Solving the systems of equations $V_t = a^2,\ V_t^2 + 3 = 7b^2$ gives $a^4 - 7b^2 = -3$, which has no positive integer solutions by Lemma 3.7. It is obvious that (3.11) is not satisfied. Because, we get $b^2 \equiv 3 (\bmod 7)$ in this case. Let $(V_t,\ V_t^2 + 3) = 3$. This implies that either

$$V_t = 3a^2,\ V_t^2 + 3 = 21b^2 \tag{3.12}$$

or

$$V_t = 21a^2,\ V_t^2 + 3 = 3b^2 \tag{3.13}$$

for some $a, b > 0$. Assume that (3.12) is satisfied. Then, we get $9a^4 - 21b^2 = -3$, which has no positive integer solutions by Lemma 3.8. Now assume that (3.13) is satisfied. Since $3 \mid V_t$ and t is odd, it follows from Lemma 3.1 that $3 \mid P$. On the other hand, it is seen that $V_t^2 = V_{2t} - 2$ by (2.12). Combining the equation $V_t^2 = V_{2t} - 2$ with $V_t^2 + 3 = 3b^2$ gives $V_{2t} = 3b^2 - 1$. Let $t > 1$. Then, we can write $t = 4q \pm 1 = 2^r z \pm 1$ with z odd and $r \geq 2$. And so, by (2.4), we get $V_{2t} = V_{2.2^r z \pm 2} \equiv -V_2 (\bmod V_{2^r})$, implying that

$$3b^2 \equiv -(P^2 + 2 - 1) \equiv -U_3 (\bmod V_{2^r}).$$

This means that $\left(\dfrac{-3U_3}{V_{2^r}}\right) = 1$. We have $\left(\dfrac{-1}{V_{2^r}}\right) = -1$ and $\left(\dfrac{U_3}{V_{2^r}}\right) = 1$ by (2.9) and (2.31), respectively. On the other hand, $V_{2^r} \equiv 2 (\bmod 3)$ by (2.34), leading to

$$\left(\frac{3}{V_{2^r}}\right) = (-1)^{\frac{V_{2^r} - 1}{2}} \left(\frac{V_{2^r}}{3}\right) = (-1)\left(\frac{2}{3}\right) = 1.$$

Therefore,

$$\left(\frac{-3U_3}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{U_3}{V_{2^r}}\right)=(-1)(1)(1)=-1,$$

which contradicts the displayed relation a few lines above. Hence, $t=1$ and therefore $V_1=P=21\square$. But this contradicts the fact that $P^2\equiv 4(\bmod 7)$.

By using Theorem 2.2.12, we have the following corollary.

Corollary 3.2. The equations $49x^4-(P^2+4)y^2=\pm 4$ have positive integer solutions only when $P=1$ or $P=5a^2$ with a odd.

Theorem 3.5. If P is odd, then, a relation of the form $V_n=7V_mx^2$, with $V_m\neq 1$, is possible if and only if $P^2=-3+7\square$, (hence P is given by Lemma 3.5) and $(n,m)=(3,1)$.

Proof: The strategy of the proof is as follows. We will prove that, when $V_m\neq 1$ and either P is divisible by 7, or $P^2\equiv 1,2(\bmod 7)$, then, the equation $V_n=7V_mx^2$ is impossible. And then, we will prove that, if $P^2\equiv 4(\bmod 7)$, then, $(n,m)=(3,1)$. Note that, in this last case, the relation $V_3=7V_1x^2$ is equivalent to $P^2-7x^2=-3$, hence P is obtained by applying Lemma 3.5.

Case I: Assume that $7\mid P$ and $V_n=7V_mx^2$. Since $7\mid V_n$, it follows from Lemma 3.2 that $n\geq 3$ is odd. Furthermore, since $V_m\mid V_n$, there exists an odd integer t such that $n=mt$ by (2.27). Thus, m is odd. Therefore, we have $V_n\equiv nP(\bmod P^2)$ and $V_m\equiv mP(\bmod P^2)$ by Lemma 2.1.1. This shows that $nP\equiv 7mPx^2(\bmod P^2)$, i.e., $n\equiv 7mx^2(\bmod P)$. Since $7\mid P$, it is obvious that $7\mid n$. Since $7\mid n$ and $n=mt$, it is seen that $7\mid mt$. Assume that $7\mid t$. Then, $t=7s$ for some odd positive integer s and therefore $n=mt=7ms$. By (2.22), we immediately have

$$7V_m x^2 = V_n = V_{7ms} = V_{ms}\left(V_{2ms}^3 + V_{2ms}^2 - 2V_{2ms} - 1\right) = V_{ms}\left(V_{ms}^6 + 7V_{ms}^4 + 14V_{ms}{}^2 + 7\right),$$

by (2.24). This implies that 7 divides the parenthesis, i.e.,

$$7 \mid \left(V_{2ms}^3 + V_{2ms}^2 - 2V_{2ms} - 1\right).$$

Hence, we get

$$x^2 = \frac{V_{ms}}{V_m}\left(\frac{V_{2ms}^3 + V_{2ms}^2 - 2V_{2ms} - 1}{7}\right).$$

We have

$$\left(\frac{V_{ms}}{V_m}, \frac{V_{2ms}^3 + V_{2ms}^2 - 2V_{2ms} - 1}{7}\right) = \left(\frac{V_{ms}}{V_m}, \frac{V_{ms}^6 + 7V_{ms}^4 + 14V_{ms}{}^2 + 7}{7}\right) = 1.$$

Then,

$$V_{ms} = V_m a^2 \text{ and } V_{2ms}^3 + V_{2ms}^2 - 2V_{2ms} - 1 = 7b^2$$

for some $a, b > 0.$ By Theorem 2.2.5, we have $ms = 3,$ $m = 1,$ and $P = 1$ or $ms = m.$ If $m = 1$ and $P = 1,$ then, we see that $V_m = V_1 = P = 1,$ which is impossible since $V_m \neq 1.$ If $ms = m,$ then, $s = 1.$ Since $n = 7ms,$ we get $n = 7m.$ By (2.4), it follows that $V_n = V_{7m} = V_{8m-m} = V_{2\cdot 4m-m} \equiv -V_{-m} (\operatorname{mod} V_4),$ implying that $7V_m x^2 \equiv V_m (\operatorname{mod} V_4).$ Since V_4 is odd, it follows by (2.29) that $(U_{2m}, V_4) = 1.$ But $U_{2m} = U_m V_m,$ by (2.11), hence $(V_m, V_4) = 1.$ Therefore, the congruence becomes $7x^2 \equiv 1 (\operatorname{mod} V_4).$ Using the fact that $7 \mid P,$ we have

$$1 = \left(\frac{7}{V_4}\right) = (-1)^{\frac{V_4 - 1}{2}}\left(\frac{V_4}{7}\right) = (-1)\left(\frac{2}{7}\right) = -1,$$

a contradiction. Now assume that $7 \nmid t$, so that $7 \mid m$. So, we can write $m = 7^r a$ with $7 \nmid a$ and $r \geq 1$. By (2.25), we get $V_m = V_{7^r a} = 7V_{7^{r-1}a}(7a_1 + 1)$ for some positive integer a_1. Thus, we conclude that $V_m = V_{7^r a} = 7^r V_a (7a_1 + 1)(7a_2 + 1)\cdots(7a_r + 1)$ for some $a_i > 0$ with $1 \leq i \leq r$. Let $A = (7a_1 + 1)(7a_2 + 1)\cdots(7a_r + 1)$. As a consequence, $V_m = 7^r V_a A$. It is clear that $7 \nmid A$. In a similar way, we see that $V_n = V_{7^r at} = 7^r V_{at}(7b_1 + 1)(7b_2 + 1)\cdots(7b_r + 1)$ for some $b_j > 0$ with $1 \leq j \leq r$. Thus, we have $V_n = 7^r V_{at} B$, where $B = (7b_1 + 1)(7b_2 + 1)\cdots(7b_r + 1)$. Clearly, $7 \nmid B$. Upon substituting the values of V_n and V_m into $V_n = 7V_m x^2$, we obtain $7^r V_{at} B = 7 \cdot 7^r V_a A x^2$, implying that $V_{at} B = 7V_a A x^2$. By Lemma 2.1.1, it is seen that $atPB \equiv 7aPAx^2 (\mathrm{mod}\, P^2)$, which gives that $atB \equiv 7aAx^2 (\mathrm{mod}\, P)$. Since $7 \mid P$, it follows that $7 \mid atB$. But this is impossible since $7 \nmid a$, $7 \nmid t$, and $7 \nmid B$.

Case II: Assume that $P^2 \equiv 1 (\mathrm{mod}\, 7)$. From Lemma 3.2, it is seen that $n = 4t$ for some odd positive integer t. Therefore, we can write $n = 12q$ for some odd q or $n = 12q \pm 4$ for some even q. Firstly, let $n = 12q$. And so, by (2.2), we get $V_n = V_{12q} = V_{2.6q} \equiv V_0 (\mathrm{mod}\, U_6)$. Since $U_6 = P^5 + 4P^3 + 3P$ and P is odd, it is easily seen that $8 \mid U_6$. Hence, we have $V_n \equiv 2 (\mathrm{mod}\, 8)$. Secondly, let $n = 12q \pm 4$. Then, we immediately have from (2.2) that $V_n = V_{12q \pm 4} = V_{2.6q \pm 4} \equiv V_{\pm 4} (\mathrm{mod}\, U_6)$, implying that $V_n \equiv V_4 (\mathrm{mod}\, 8)$. Using the fact that $V_4 = P^4 + 4P^2 + 2$ and P is odd, we obtain $V_4 \equiv 7 (\mathrm{mod}\, 8)$ in this case. Hence, we conclude that $V_n \equiv 2, 7 (\mathrm{mod}\, 8)$. On the other hand, since $V_m \mid V_n$, we get $n = ms$ for some odd s by (2.27). It is known that $4 \mid n$ and s is odd. Hence, we see that $4 \mid m$ and therefore $m = 4u$ for some odd u. And so, with arguments similar to those a few lines above, we have $V_m \equiv 2, 7 (\mathrm{mod}\, 8)$. Thus, $7V_m \equiv 14, 49 \equiv 6, 1 (\mathrm{mod}\, 8)$. As a consequence,

$$V_n = 7V_m x^2 \equiv 1,6\begin{Bmatrix}0\\1\\4\end{Bmatrix} \equiv 0,1,4,6 \pmod 8.$$

But this contradicts the fact that $V_n \equiv 2,7 \pmod 8$.

Case III: Assume that $P^2 \equiv 2 \pmod 7$. Since $7 \mid V_n$, it follows from Lemma 3.2 that $n = 4t$ for some odd t. Furthermore, since $V_m \mid V_n$, there exists an odd integer $s(>1)$ such that $n = ms$ by (2.27). Thus, we can write $s = 4q \pm 1$ for some $q \geq 1$. Since $4 \mid n$ and s is odd, it is seen that m is even and also $4 \mid m$. Upon substituting $n = ms$ and $s = 4q \pm 1$ into V_n, we get $V_n = V_{ms} = V_{m(4q\pm1)} = V_{2\cdot 2mq \pm m} \equiv V_m \pmod{U_{2m}}$ by (2.2). This implies that $7V_m x^2 \equiv V_m \pmod{U_m V_m}$ by (2.11). Dividing both sides of the congruence by V_m gives $7x^2 \equiv 1 \pmod{U_m}$. Since $4 \mid m$, it is clear from (2.28) that $U_4 \mid U_m$. Since $U_4 = U_2 V_2$ by (2.11), the preceding congruence becomes $7x^2 \equiv 1 \pmod{V_2}$, i.e.,

$$7x^2 \equiv 1 \pmod{P^2 + 2}. \tag{3.14}$$

This means that $\left(\frac{7}{P^2+2}\right) = 1$. Using $P^2 \equiv 2 \pmod 7$, we get

$$1 = \left(\frac{7}{P^2+2}\right) = (-1)^{\frac{P^2+1}{2}} \left(\frac{P^2+2}{7}\right) = (-1)\left(\frac{4}{7}\right) = -1,$$

a contradiction.

Case IV: Assume that $P^2 \equiv 4 \pmod 7$. Since $7 \mid V_n$, it follows from Lemma 3.2 that $n = 3t$ for some odd positive integer t. Moreover, since $V_m \mid V_n$, it is obvious that $n = ms$ for some odd $s > 1$ by (2.27). And so, we can write $s = 4q \pm 1$ with $q \geq 1$. Thus, we get $n = ms = 4qm \pm m$. From now on, we divide the proof into two subcases.

Subcase (i): Let $3 \mid m$. Then, by (2.28), it is clear that $U_3 \mid U_m$. Substituting $n = 4qm \pm m$ into V_n and using (2.2) and (2.11), we obtain $V_n = V_{4qm \pm m} = V_{2 \cdot 2mq \pm m} \equiv V_{\pm m} (\bmod U_m V_m)$, i.e., $7x^2 \equiv \pm 1 (\bmod U_m)$. Since $U_3 \mid U_m$ and $U_3 = P^2 + 1$, we conclude that

$$7x^2 \equiv \pm 1 (\bmod P^2 + 1). \tag{3.15}$$

It is clear from (3.15) that

$$7x^2 \equiv \pm 1 \left(\bmod \frac{P^2 + 1}{2} \right).$$

Let $7x^2 \equiv 1 \left(\bmod \frac{P^2 + 1}{2} \right)$. This shows that

$$\left(\frac{7}{(P^2 + 1)/2} \right) = 1.$$

Since $P^2 \equiv 4 (\bmod 7)$, it follows that $\frac{P^2 + 1}{2} \equiv -1 (\bmod 7)$. Hence, we get

$$1 = \left(\frac{7}{\left(P^2 + 1\right)/2} \right) = \left(-1\right)^{\frac{P^2 - 1}{4}} \left(\frac{\left(P^2 + 1\right)/2}{7} \right) = \left(\frac{-1}{7} \right) = -1,$$

a contradiction. Similarly $7x^2 \equiv -1 \left(\bmod \frac{P^2 + 1}{2} \right)$ leads to a contradiction.

Subcase (ii): Let $3 \nmid m$. Since $3 \mid n$ and $n = ms$, it follows that $3 \mid s$ and therefore $s = 3k$ for some odd k. Thus, we get $n = ms = 3mk$. Substituting this into V_n gives

$V_n = V_{3mk} = V_{mk}\left(V_{mk}^2+3\right)$ by (2.15). This implies that $7V_m x^2 = V_{mk}\left(V_{mk}^2+3\right)$, i.e., $7x^2 = \frac{V_{mk}}{V_m}\left(V_{mk}^2+3\right)$. Clearly, $d = \left(\frac{V_{mk}}{V_m}, V_{mk}^2+3\right) = 1$ or 3. Let $d=1$. Then, either

$$V_{mk} = V_m a^2,\ V_{mk}^2+3 = 7b^2 \tag{3.16}$$

or

$$V_{mk} = 7V_m a^2,\ V_{mk}^2+3 = b^2 \tag{3.17}$$

for some $a,b>0$. We immediately see that (3.17) is not satisfied. Because the only possible value of V_{mk} for which $V_{mk}^2+3=b^2$ is $V_{mk}=1$, which is impossible. Assume that (3.16) is satisfied. Then, by Theorem 2.2.5, we obtain $mk=3$, $m=1$, and $P=1$ or $mk=m.$ If $mk=3$ and $P=1$, then $V_{mk}{}^2+3 = V_3{}^2+3 = (P^3+3P)^2+3 = 19 = 7b^2$, which is impossible. If $mk=m$, then, $k=1$. So, it is sufficent to consider the equation $V_m^2+3=7b^2$. From (2.12), it follows that $V_{2m}+1=7b^2$. Assume that $m>1$. Since m is odd, we can write $2m = 2(4q\pm 1) = 2\left(2^r z\right)\pm 2$ with z odd and $r\geq 2$. Then, by (2.4), we get

$$V_{2m} = V_{2\cdot 2^r z\pm 2} \equiv -V_2 \equiv -(P^2+2)(\bmod V_{2^r}),$$

implying that

$$7b^2 \equiv -(P^2+2-1) \equiv -U_3(\bmod V_{2^r}).$$

This means that $1 = \left(\frac{-7U_3}{V_{2^r}}\right)$. We have $\left(\frac{-1}{V_{2^r}}\right) = -1$ and $\left(\frac{U_3}{V_{2^r}}\right) = 1$ by (2.31) and (2.9), respectively. On the oher hand, it is easy to see that $V_{2^r} \equiv 6(\bmod 7)$ when $P^2 \equiv 4(\bmod 7)$. Thus, we get

$$\left(\frac{7}{V_{2^r}}\right) = (-1)^{\frac{V_{2^r}-1}{2}}\left(\frac{V_{2^r}}{7}\right) = (-1)\left(\frac{6}{7}\right) = -1\left(\frac{-1}{7}\right) = 1.$$

Combining the above, we see that

$$1=\left(\frac{-7U_3}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{7}{V_{2^r}}\right)\left(\frac{U_3}{V_{2^r}}\right)=(-1)(1)(1)=-1,$$

a contradiction. Hence, we get $m=1$ and therefore $n=3m=3$. Substituting $m=1$ into $V_{2m}+1=7b^2$ gives $P^2-7b^2=-3$ which has solutions by Lemma 3.5. Thus, $(m,n)=(1,3)$ is a solution. Let $d=3$. Then, we obtain

$$V_{mk}=3V_m a^2,\ V_{mk}^2+3=21b^2 \tag{3.18}$$

or

$$V_{mk}=21V_m a^2,\ V_{mk}^2+3=3b^2 \tag{3.19}$$

for some $a,b>0$. Assume that (3.18) is satisfied. Then, by Theorem 3.2, the only possible values of mk and m for which $V_{mk}=3V_m a^2$ are $mk=3$ and $m=1$. This implies that $V_3^2+3=21b^2$. Thus, we get $V_3^2\equiv 4(\text{mod}\,7)$. This is impossible since $7\,|\,V_3$. Now assume that (3.19) is satisfied. Since $3\,|\,V_{mk}$ and mk is odd, it is seen that $3\,|\,P$ by Lemma 3.1. On the other hand, $V_{mk}^2=V_{2mk}-2$ by (2.12). Combining the equations $V_{mk}^2=V_{2mk}-2$ and $V_{mk}^2+3=3b^2$, we get $V_{2mk}=3b^2-1$. Let $mk=4q\pm1=2^r z\pm1$ with z odd and $r\geq 2$. And so, by (2.4), we obtain

$$V_{2mk}=V_{2\cdot 2^r z\pm 2}\equiv -V_2(\text{mod}\,V_{2^r}),$$

implying that

$$3b^2\equiv -(P^2+2-1)\equiv -U_3(\text{mod}\,V_{2^r}).$$

This shows that

$$1=\left(\frac{-3U_3}{V_{2^r}}\right).$$

We have $\left(\frac{-1}{V_{2^r}}\right)=-1$ by (2.9), $\left(\frac{U_3}{V_{2^r}}\right)=1$ by (2.31), and $V_{2^r}\equiv 2(\bmod 3)$ by (2.34).

Thus,

$$\left(\frac{3}{V_{2^r}}\right)=(-1)^{\frac{V_{2^r}-1}{2}}\left(\frac{V_{2^r}}{3}\right)=(-1)\left(\frac{2}{3}\right)=1.$$

Combining the above, we see that

$$1=\left(\frac{-3U_3}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{U_3}{V_{2^r}}\right)=(-1)(1)(1)=-1,$$

a contradiction.

Theorem 3.6. If P is odd, then, $U_n=7x^2$ is possible if and only if $P=7\square$ and $n=2$.

Proof: Assume that $U_n=7x^2$ for some $x>0$. Since $7\,|\,U_n$, it follows that $n=2t$ for some positive integer t by Lemma 3.3. And so, by (2.11), we get $U_n=U_{2t}=U_tV_t=7x^2$. Clearly, $(U_t,V_t)=1$ or 2 by (2.29). Let $(U_t,V_t)=1$. Then, either

$$U_t=a^2,\ V_t=7b^2 \tag{3.20}$$

or

$$U_t=7a^2,\ V_t=b^2 \tag{3.21}$$

for some $a,b>0$. Assume that (3.20) is satisfied. Then, by Theorem 3.4, the possible values of t for which $V_t=7x^2$ are $t=1$ when $7\,|\,P$ and $t=4, P=1$ when $P^2\equiv 1(\bmod 7)$. If $t=1$, then, $n=2$ and therefore $P=7\square$ is a solution. If $t=4$ and

$P=1$, then, $U_4=P^3+2P=3=a^2$, which is impossible in integers. Now assume that (3.21) is satisfied. Since $7\,|\,U_t$, it is seen from Lemma 3.3 that t is even. Let $t=2m.$ Then, by (2.12), we see that $b^2=V_t=V_{2m}=V_m^2\pm 2,$ which is impossible. Let $\left(U_t,V_t\right)=2.$ Then, either

$$U_t=2a^2,\ V_t=14b^2 \tag{3.22}$$

or

$$U_t=14a^2,\ V_t=2b^2 \tag{3.23}$$

for some $a,b>0.$ According to Theorem 3.3, (3.22) cannot hold. Assume that (3.23) is satisfied. Then, by Theorem 2.2.3, we have $t=6$ and $P=1,5.$ But this is also impossible. For, otherwise we would have $14a^2=U_6=U_3V_3=(P^2+1)(P^3+3P),$ which is impossible for $P=1,5.$

By Theorems 2.2.12 and 2.2.14, we give the following corollary.

Corollary 3.3. The equations $x^2-49(P^2+4)y^4=\pm 4$ and $49x^4-7Px^2y-y^2=\pm 1$ have positive integer solutions only when $P=7a^2$ with a odd.

Theorem 3.7. Let P be odd, $m>1$ and $U_m\neq 1.$ The equation $U_n=7U_mx^2$ has solution only when $P^2\equiv 1 (\mathrm{mod}\, 7),$ in which case, the only solution is given by $(n,m,P,x)=(8,4,1,1).$

Proof: Assume that $U_n=7U_mx^2$ with $m>1.$ Since $U_m\,|\,U_n,$ it follows from (2.28) that $m\,|\,n.$ Thus, $n=mt$ for some positive integer $t.$ It is easy to see that $n\neq m.$ Then, we have $t>1.$ On the other hand, since $7\,|\,U_n,$ it is seen that n is even by Lemma 3.3. Since n is even and $n=mt,$ either m or t is even.

Case I: t is even. Then, $t = 2s$ for some $s > 0$. By (2.11), we have $U_n = U_{2ms} = U_{ms}V_{ms} = 7U_m x^2$. This yields that $(U_{ms}/U_m)V_{ms} = 7x^2$. Clearly, $(U_{ms}/U_m, V_{ms}) = 1$ or 2 by (2.29). If $(U_{ms}/U_m, V_{ms}) = 1$, then, either

$$U_{ms} = U_m a^2,\ V_{ms} = 7b^2 \tag{3.24}$$

or

$$U_{ms} = 7U_m a^2,\ V_{ms} = b^2 \tag{3.25}$$

for some positive integers a and b. By Theorem 3.4, the identity (3.24) is impossible when $P^2 \equiv 2 \pmod 7$ or $P^2 \equiv 4 \pmod 7$. If $7 \mid P$, then, by Theorem 3.4, we have $ms = 1$. But this contradicts the fact that $m > 1$. If $P^2 \equiv 1 \pmod 7$, then, by Theorem 3.4, it follows that $ms = 4$ and $P = 1$. Since $m > 1$, we get $m = 4$, $s = 1$ or $m = 2$, $s = 2$. Let $m = 4$, $s = 1$. Since $t = 2s$ and $n = mt$, we get $n = 8$. Hence, $U_8 = 7U_4 x^2$, implying by (2.11) that $V_4 = 7x^2$. Since $P = 1$, we obtain $x = 1$. So, $(n, m, P, x) = (8, 4, 1, 1)$ is a solution. Now, let $m = 2$, $s = 2$. Then, we readily obtain $n = 8$ and therefore $U_8 = 7U_2 x^2$. By (2.11), it follows that $V_2 V_4 = 7x^2$. Since $7 \mid V_4$, we get $V_2 \frac{V_4}{7} = x^2$. Clearly, $\left(V_2, \frac{V_4}{7}\right) = 1$ by (2.29) and (2.26). Then, $V_2 = a^2$, $V_4 = 7b^2$ for some $a, b > 0$. Since $P = 1$, it follows that $V_2 = P^2 + 2 = 3 = a^2$, which is impossible. If (3.25) is satisfied, then, by Theorem 2.2.2, we have $ms = 3$ and $P = 1$ or 3. Since $m > 1$ and $ms = 3$, it follows that $m = 3$. This implies that $U_3 = 7U_3 x^2$, which is impossible. If $(U_{ms}/U_m,\ V_{ms}) = 2$, then, either

$$U_{ms} = 2U_m a^2,\ V_{ms} = 14b^2 \tag{3.26}$$

or

$$U_{ms} = 14U_m a^2,\ V_{ms} = 2b^2 \tag{3.27}$$

for some positive integers a and b. Clearly, (3.26) is excluded by Theorem 3.3. Suppose (3.27) is satisfied. Then, by Theorem 2.2.3, we have $ms = 6$ and $P = 1, 5$.

Since $m>1$, it follows that $m=2,3$ or 6. If $m=2$, then, $U_6=14U_2a^2$, implying that $(P^2+1)(P^2+3)=14a^2$ which is impossible in integers for the case when $P=1,5$. If $m=3$, then, $U_6=14U_3a^2$, implying that $(P^3+3P)=14a^2$, which is impossible. Lastly, if $m=6$, then, $U_6=14U_6a^2$, implying that $1=14a^2$, which is also impossible.

Case II: t is odd. Since $n=mt$ and n is even, it follows that m is even. Let $m=2s$. Then, it follows that $n=2st$ and so, by (2.11), we get $U_n=U_{2st}=U_{st}V_{st}=7U_{2st}x^2=7U_{st}V_{st}x^2$. This implies that $\frac{U_{st}}{U_s}\frac{V_{st}}{V_s}=7x^2$. Clearly, $d=\left(\frac{U_{st}}{U_s},\frac{V_{st}}{V_s}\right)=1$ or 2. Let $d=1$. Then, either

$$U_{st}=U_sa^2,\ V_{st}=7V_sb^2 \tag{3.28}$$

or

$$U_{st}=7U_sa^2,\ V_{st}=V_sb^2 \tag{3.29}$$

for some $a,b>0$. Suppose (3.28) is satisfied. Then, by Theorem 3.5, we get $s=1$ and $st=3$. This implies that $U_3=U_1a^2$, that is, $P^2+1=a^2$, which is impossible. Suppose (3.29) is satisfied. Then, by Theorem 2.2.5, we obtain $st=3$, $s=1$, and $P=1$ or $st=s$. If $st=3$, $s=1$, and $P=1$, then from $U_{st}=7U_sa^2$, we have $U_3=7U_1a^2$, leading to $2=7a^2$, which is impossible. If $st=s$, then again from $U_{st}=7U_sa^2$, we have $1=7a^2$, which is impossible. Let $d=2$. Then, either

$$U_{st}=2U_sa^2,\ V_{st}=14V_sb^2 \tag{3.30}$$

or

$$U_{st}=14U_sa^2,\ V_{st}=2V_sb^2 \tag{3.31}$$

for some positive integers a and b. Assume that (3.30) is satisfied. Then, by Theorem 2.2.4, the possible values of st, s, and P for which $U_{st} = 2U_s a^2$ are $st = 3$, $s = 2$, $P = 1$; $st = 6$, $s = 2$, $P = 1$; $st = 12$, $s = 3$, $P = 1$; $st = 12$, $s = 6$, $P = 1$; or $st = 12$, $s = 6$, $P = 5$. A simple computation shows that $V_{st} = 14V_s b^2$ is impossible under all the conditions that when $P = 1$. If $P = 5$, then, this is impossible for the case when $7 \mid P$ or $P^2 \equiv 1, 2 \pmod 7$. On the other hand, since $7 \mid V_{st}$, it follows from Lemma 3.2 that $st = 3r$ with r odd for the case when $P^2 \equiv 4 \pmod 7$. This means that st is odd. But this contradicts the fact that $st = 12$ is even. Assume that (3.31) is satisfied. Then, by Theorem 2.2.6, we get $s = 1$ and $P = 1$. Since $m = 2s$, it follows that $m = 2$. Substituting this value of m into $U_n = 7U_m x^2$ gives $U_n = 7U_2 x^2 = 7x^2$. By Theorem 3.6, the equation $U_n = 7x^2$ is possible if and only if $n = 2$. As a consequence, we have $m = 2$ and $n = 2$. But this is impossible since $n \neq m$.

CHAPTER 4. CONCLUSIONS AND SUGGESTIONS

In this thesis we dealt with the generalized Fibonacci numbers $U_n(P,Q)$ and Lucas numbers $V_n(P,Q)$ of the form kx^2 with the special consideration that P is odd and $Q=\pm1$. The cases $k=5$ and $k=7$ are the ones of interest to our thesis. The main tools that we employed are the Jacobi symbol that we made extensive use of it, divisibility properties, and congruences concerning generalized Fibonacci and Lucas numbers. In the second subchapter of Chapter 2 of this thesis, we, assuming $Q=1$, considered the equations $U_n(P,1)=5x^2$ and $U_n(P,1)=5U_m(P,1)x^2$ under some assumptions on P. Besides, we considered the equation $V_n(P,1)=5x^2$ for the case when P is odd. We also considered the equation $V_n(P,1)=5V_m(P,1)x^2$ and proved that this equation has no solutions. Applying the results of findings, we solved some Diophantine equations. This work has been published in International Journal of Number Theory [68]. In the third subchapter of Chapter 2 we considered the similar problem for the case when $Q=-1$. Finally, in Chapter 3, for all odd values of P, we solved the equations $U_n(P,1)=7x^2$, $U_n(P,1)=7U_m(P,1)x^2$, $V_n(P,1)=7x^2$, and $V_n(P,1)=7V_m(P,1)x^2$. And again applying these results, we solved some Diophantine equations. Chapter 3 and the third subchapter of Chapter 2 are still under consideration in some journals.

Except the works mentioned above, there are various works that can be made. For instance, the equations $U_n(P,-1)=7x^2$, $U_n(P,-1)=7U_m(P,-1)x^2$, $V_n(P,-1)=7x^2$, and $V_n(P,-1)=7V_m(P,-1)x^2$ can be first solved. It is also possible to consider the equations $U_n(P,\pm1)=kx^2$ and $V_n(P,\pm1)=kx^2$ for another special values of prime k such that $k=11,\ 13,\ 17,\ldots,$ and in general for any prime k. Considering the equations $V_n(P,\pm 1)=5x^2$ and $V_n(P,\pm 1)=7x^2$ when P is even is an open problem, yet.

The equations $V_n(P,1)=kx^2$ and $V_n(P,-1)=kx^2$ were solved when P is odd and $k\mid P$ in [58] and [66], respectively. Similarly, it can be investigated the solutions of the equations $V_n(P,\pm 1)=5kx^2$ and $V_n(P,\pm 1)=7kx^2$ under the conditions that P is odd, $k\mid P$ and $k>1$.

In [69], Alexseyev and Tengely showed the finiteness of the terms of the form $am^2+b,$ for fixed integers $a\neq 0$ and $b,$ in a Lucas sequence $U_n(P,Q)$ or $V_n(P,Q)$ with $Q=\pm 1,$ unless this sequence is $V_n(P,Q)$ and $b=\pm 2.$ In [66], Keskin solved the equations $V_n(P,-1)=kx^2\mp 1,$ $V_n(P,-1)=2kx^2\mp 1,$ and $U_n(P,-1)=kx^2\mp 1$ when P is odd, $k\mid P$ and $k>1.$ Moreover, the author solved the equations $V_n(P,-1)=wx^2\mp 1$ for $w\in\{2,3,6\}.$ So, the same problems can be considered for $Q=1.$ Furthermore, the equations $V_n(P,\pm 1)=5kx^2\pm 1$ and $V_n(P,\pm 1)=7kx^2\pm 1$ can be solved when P is odd, $k\mid P$ and $k>1.$ Also, it is possible to consider the equations $V_n(P,\pm 1)=5x^2\pm 1$ and $V_n(P,\pm 1)=7x^2\pm 1.$

REFERENCES

[1] KOSHY, T., Fibonacci and Lucas Numbers with Applications, ser. Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs, and Tracts. New York: Wiley, 2001.

[2] POSAMENTIER, A. S., LEHMANN, I., The fabulous Fibonacci numbers, Amherst, NY: Prometheus Books, 2007.

[3] KESKİN, R., DEMİRTÜRK, B., Fibonacci and Lucas congruences and their applications, Acta. Math. Sin. (Engl. Ser.), 27(4): 725 – 736, 2011.

[4] CARLITZ, L., A note on Fibonacci numbers, Fibonacci Quart. 1(1):15 – 28, 1964.

[5] LUCAS, E., Théorie des fonctions numériques simplement periodiques, Amer. J. Math. 1(2):184 – 240, 1878.

[6] HORADAM, A. F., Basic properties of a certain generalized sequence of numbers, Fibonacci Quart. 3:161 – 176, 1965.

[7] HORADAM, A. F., Generating functions for powers of a certain generalized sequence of numbers, Duke Math. J. 32: 437 – 446, 1965.

[8] HARDY, G. H., WRIGHT, E. M., An introduction to the theory of numbers, 4th ed. London, Oxford University Press, 1962.

[9] HORADAM, A. F., MAHON, J. M., Pell and Pell-Lucas Polynomials, Fibonacci Quartert. 23(1): 7 – 20, 1985.

[10] RIBENBOIM, P., My numbers, my friends, Popular lectures on number theory, New York: Springer, 2000.

[11] KALMAN, D., MENA, R., The Fibonacci numbers-exposed, Math. Mag. 76:167 – 181, 2003.

[12] MUSKAT, J. B., Generalized Fibonacci and Lucas sequences and rootfinding methods, Math. Comp. $61:365-372$, 1993.

[13] RABINOWITZ, S., Algorithmic manipulation of Fibonacci identities, in Applications of Fibonacci numbers Vol. 6 (Kluwere Academic Publishers, 1996), pp. $389-408$.

[14] LUCAS, E., Sur la theorie des nombres premiers, Atti R. Acad. Sc. Torino Math. $11:928-937$, $1875-1876$.

[15] CARMICHAEL, R. D., On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$, Annals of Math. $15:30-70$, 1913.

[16] MCDANIEL, W. L., The g. c. d. in Lucas sequences and Lehmer number sequences, Fibonacci Quart. $29:24-29$, 1991.

[17] RIBENBOIM, P., MCDANIEL, W. L., The square terms in Lucas sequences, J. Number Theory $58:104-123$, 1996.

[18] RIBENBOIM, P., MCDANIEL, W. L., Squares in Lucas sequences having an even first parameter, Colloq. Math. $78:29-34$, 1998.

[19] RIBENBOIM, P., MCDANIEL, W. L., On Lucas sequences terms of the form kx^2, in: Number Theory (Turku, 1999), de Gruyter, Berlin, 2001, $293-303$.

[20] HILTON, P., PEDERSEN, J., SOMER, L., On Lucasian Numbers, Fibonacci Quart. $35(1):43-47$, 1997.

[21] NAGELL, T., Introduction to Number Theory, Stockholm-New York: Almqvist & Wiksell; John Wiley & Sons, Inc., 1951.

[22] M. J. JACOBSON, M. J., WILLIAMS, H. C., Solving the Pell equation, ser. CMS Books in Mathematics, New York: Springer, 2009.

[23] LENSTRA, H. W., Solving the Pell equation, Notices Am. Math. Soc. $49(2):182-192$, 2002.

[24] ROBERTSON, J. P., Solving the generalized Pell equation $x^2 - Dy^2 = N$, 2003. Online Available: http://hometown.aol.com/jpr2718/pell.pdf

[25] MOLLIN, R. A., Quadratic Diophantine equations $x^2 - Dy^2 = c^n$, Ir. Math. Soc. Bull. $58:55-68,\ 2006$.

[26] MORDELL, L. J., Diophantine equations, ser. Pure and Applied Mathematics. London-New Yor: Academic Press, 1969, vol. 30.

[27] SHOREY, T. N., STEWART, C. L., Pure powers in recurrence sequences and some related Diophantine equations, J. Number Theory, $27:324-325$, 1987.

[28] STEVENHAGEN, P., A density conjecture for the negative Pell equation, in computational algebra and number theory, based on a meeting on computational algebra and number theory, held at Sydney University, Sydney, Australia, November of 1992, ser. Math. Appl., Dordr. Dordrecht: Kluwer Academic Publishers, $325:187-200,\ 1995$.

[29] NIVEN, I., ZUCKERMAN, H. S., MONTGOMARY, H. L., An Introduction to the Theory of Numbers, John Wiley & Sons, Inc., Toronto, 1991.

[30] VAJDA, S., Fibonacci & Lucas numbers, and the golden section, Theory and applications, ser. Ellis Horwood Books in Mathematics and its Applications, Chichester-New York: Ellis Horwood Ltd.; Halsted Press, 1989.

[31] VOROBIEV, N. N., Fibonacci Numbers, Birkhäuser; 2002 edition (October 4, 2013).

[32] JONES, J. P., Representation of solutions of Pell equations using Lucas sequences, Acta Academia Pead. Agr. Sectio Mathematicae, $30:75-86$, 2003.

[33] KESKİN, R., Solutions of some quadratic Diophantine equations, Computer and Mathematics With Applications, $60(8):2225-2230,\ 2010$.

[34] MCDANIEL, W. L., Diophantine representation of Lucas sequences, Fibonacci Quart. $33(1):59-63,\ 1995$.

[35] ZHIWEI, S., Singlefold, Diophantine representation of the sequence $u_0 = 0,\ u_1 = 1,$ and $u_{n+2} = mu_{n+1} + u_n$, Pure and Applied Logic, pp. $97-101$, Beijing Univ. Press, Beijing, 1992.

[36] LJUNGGREN, W., Zur Theorie der Gleichung $x^2+1=Dy^4$, Avh. Norsk. Vid. Akad. Oslo, $5:1-27$, 1942.

[37] COHN, J. H. E., On square Fibonacci numbers, J. Lond. Math. Soc. $39:537-540$, 1964.

[38] ALFRED, U., On square Lucas numbers, Fibonacci Quart. $2:11-12$, 1964.

[39] BURR, S. A., On the occurrence of squares in Lucas sequences, Amer. Math. Soc. Notices (Abstract 63T-302) 10 (1963) p. 367.

[40] WYLER, O., Solution of problem 5080, Amer. Math. Monthly, $71:220-222$, 1964.

[41] COHN, J. H. E., Eight Diophantine equations, Proc. London Math. Soc. $3(16):153-166$, 1966.

[42] COHN, J. H. E., Lucas and Fibonacci numbers and some Diophantine equations, Proc. Glasgow Math. Assoc. $7:24-28$, 1965.

[43] COHN, J. H. E., Square Fibonacci numbers, etc., Fibonacci Quart. $2:109-113$, 1964.

[44] ROBBINS, N., On Fibonacci numbers of the form px^2, where p is prime, Fibonacci Quart. $21(3):266-271$, 1983.

[45] ROBBINS, N., Lucas numbers of the form px^2, where p is prime, Internat. J. Math. & Math. Sci. $14(4):697-704$, 1991.

[46] PETHŐ, A., The Pell sequence contains only trivial perfect Powers, In: Sets, Graphs and Numbers, Colloq. Math. Soc. János Bolyai 60, North-Holland, Amsterdam-New York, pp. $561-568$, 1991.

[47] COHN, J. H. E., Perfect Pell powers, Glasgow Math. J. $38:19-20$, 1996.

[48] COHN, J. H. E., Five Diophantine equations, Math. Scand. $21: 61-70$, 1967.

[49] BREMNER, A., TZANAKIS, N., Lucas sequences whose 12th or 9th term is a square, J. Number Theory, $107: 215-227$, 2004.

[50] BREMNER, A., TZANAKIS, N., On squares in Lucas sequences, J. Number Theory, $124: 511-520$, 2007.

[51] BREMNER, A., TZANAKIS, N., Lucas sequences whose nth term is a square or an almost square, Acta Arith. $126: 261-280$, 2007.

[52] KAGAWA, T., TERAI, N., Squares in Lucas sequences and some Diophantine equations, Manuscripta Math. $96: 195-202$, 1998.

[53] MIGNOTTE, M., PETHŐ, A., Sur les carrés dans certaines suites de Lucas, Sém. Théories Nombr. Bordeaux, $5: 333-341$, 1993.

[54] NAKAMULA, K., PETHŐ, A., Squares in binary recurrence sequences, in: Number Theory, K. Györy et al. (eds.), de Gruyter, Berlin, 1998, $409-421$.

[55] COHN, J. H. E., Squares in some recurrent sequences, Pacific J. Math. $41: 631-646$, 1972.

[56] KESKİN, R., YOSMA, Z., On Fibonacci and Lucas numbers of the form cx^2, J. Integer Seq. 14 (2011), Article ID 11.9.3, 12 pp.

[57] KESKİN, R., ŞİAR, Z., Positive integer solutions of the Diophantine equations $x^2 - L_n xy + (-1)^n y^2 = \pm 5^r$, Proc. Indian Acad. Sci. (Math. Sci.), $124(3): 301-313$, 2014.

[58] ŞİAR, Z., KESKİN, R., The square terms in generalized Lucas sequences, Mathematika, $60: 85-100$, 2014.

[59] SHOREY, T. N., STEWART, C. L., On the Diophantine equation $ax^{2t} + bx^t y + cy^2 = 1$ and pure powers in recurrence sequences, Math. Scand. $52: 24-36$, 1983.

[60] ŞİAR, Z., KESKİN, R., Some new identities concerning generalized Fibonacci and Lucas numbers, Hacettepe Journal of Mathematics and Statistics, 42(3): $211-222$, 2013.

[61] RIBENBOIM, P., The Book of Prime Number Records, Springer-Verlag, New York, 1989.

[62] BURTON, D. M., Elementary Number Theory, Mc Graw--Hill Group, Inc., New York, 1998.

[63] DEMİRTÜRK, B., KESKİN, R., Integer solutions of some Diophantine equations via Fibonacci and Lucas numbers, J. Integer Seq. 12 (2009), Article ID 09.8.7, 10 pp.

[64] ROBBINS, N., Fibonacci numbers of the form cx^2, where $1 \leq c \leq 1000$, Fibonacci Quart. 28: $306-315$, 1990.

[65] BOSMA, W., CANNON, J., PLAYOUST, C., The Magma algebra system. I: The user language, J. Symbolic Comp., 24 $(3-4)$: $235-265$, 1997.

[66] KESKİN, R., Generalized Fibonacci and Lucas numbers of the form wx^2 and $wx^2 \pm 1$, Bulletin of the Korean Mathematical Society, 51(4): $1041-1054$, 2014.

[67] KESKİN, R., ŞİAR, Z., Positive integer solutions of some Diophantine equations in terms of integer sequences, submitted.

[68] KESKİN, R., KARAATLI, O., Generalized Fibonacci and Lucas numbers of the form $5x^2$, International Journal of Number Theory, 11(3): $931-944$, 2015.

[69] ALEXSEYEV, M. A., TENGELY, S., On Integral Points on Biquadratic Curves and Near-Multiples of Squares in Lucas Sequences, J. Integer Seq. 17 (2014), Article ID 14.6.6, 15 pp.

RESUME

Olcay Karaatlı was born in Ankara on June 30, 1985. He received his education, from the first grade through the starting university, in Ankara. The year after graduating the high school, he joined to Mathematics Department of the Faculty of Arts and Science of Uludağ University, Bursa, in 2004, and he graduated from the university in 2008. He completed the last part of his university education at Wroclaw University of Technology, Department of Mathematics and Computer Sciences, Poland. In September 2008, he started his Master of Science at Sakarya University and he completed it under the supervision of Prof. Dr. Refik KESKİN with the dissertation title Triangular Numbers. Between July 2009 and December 2010, he worked as a research assistant at Ağrı İbrahim Çeçen University, Department of Mathematics. He became a member of Mathematics Deparment of the Faculty of Arts and Science of Sakarya University on December 2010. He still works as a research assistant at the same department.

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