SAKARYA UNIVERSITY
INSTITUTE OF SCIENCE AND TECHNOLOGY

SOME Δ-CONVERGENCE AND STRONG CONVERGENCE THEOREMS RELATED TO FIXED POINTS ON CAT(κ) AND HYPERBOLIC SPACES

Ph.D. THESIS

Aynur ŞAHİN

Department	:	**MATHEMATICS**
Field of Science	:	**FUNCTIONS THEORY AND FUNCTIONAL ANALYSIS**
Supervisor	:	**Prof. Dr. Metin BAŞARIR**

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This thesis has been accepted unanimously / ~~with majority of votes~~ by the examination committee on 20.11.2014.

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PREFACE

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TABLE OF CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

$\mathbb{R}$	: The set of real numbers
$\mathbb{C}$	: The set of complex numbers
ℓ_∞	: The space of all bounded sequences
ℓ_p	: The space of all sequences such that $\sum \lvert x_k \rvert^p < \infty$
c	: The space of all convergent sequences
c_0	: The space of all null sequences
$C[a,b]$	: The space of all continuous real functions on a closed interval $[a,b]$
$\mathbb{R}^n$	: The n-dimensional Euclidean space
$F(T)$	: The set of all fixed points of the mapping T
$T^n(x)$	: n^{th} iteration of x under T
$\mathbb{N}$	: The set of natural numbers
$\varnothing$	: The empty set
M_κ^2	: The two-dimensional model space
D_κ	: The diameter of M_κ^2
$\Delta(x,y,z)$	: The geodesic triangle
$\overline{\Delta}(x,y,z)$	: The comparison triangle for $\Delta(x,y,z)$
$r(\{x_n\})$	: The asymptotic radius of $\{x_n\}$
$r_K(\{x_n\})$	: The asymptotic radius of $\{x_n\}$ with respect to K
$A(\{x_n\})$	: The asymptotic center of $\{x_n\}$
$A_K(\{x_n\})$	: The asymptotic center of $\{x_n\}$ with respect to K
$W_\Delta(x_n)$	: The union of asymptotic centers of all subsequences of $\{x_n\}$
F	: The set of common fixed points of mappings

LIST OF FIGURES

SUMMARY

Key Words: CAT(κ) Space, Fixed Point, Iterative Process, Strong Convergence, Δ-Convergence, Hyperbolic Space.

This thesis consists of seven chapters. In the first chapter, some basic definitions and theorems are given. In the second chapter, some fundamental definitions and theorems related to the concepts of CAT(κ) space and hyperbolic space, are given.

In the first part of the third chapter, the strong and Δ-convergence of the SP-iteration process are studied for nonexpansive mappings in a CAT(0) space. In the second part of this chapter, the strong and Δ-convergence of an iteration process for approximating a common fixed point of nonexpansive mappings are proved in a uniformly convex hyperbolic space.

In the first part of the fourth chapter, the strong and Δ-convergence of the S-iteration process are proved for mappings satisfying condition (C) in a CAT(0) space. In the second part of this chapter, the strong and Δ-convergence of the new three-step iteration process are examined for mappings of this type in a CAT(0) space. In the last part of it, some results on the strong and Δ-convergence of the S-iteration and the Noor iteration processes are given for nonself mappings in a CAT(0) space.

In the first part of the fifth chapter, the strong and Δ-convergence of some iteration process are proved for k-strictly pseudo-contractive mappings in a CAT(0) space. In the second part of this chapter, a new class of mappings is introduced and the Δ-convergence of the new multi-step iteration and the S-iteration processes are examined for mappings of this type in a CAT(0) space. Also some results on the strong convergence of these iteration processes are obtained for contractive-like mappings in a CAT(0) space. In the last part of it, the strong convergence of the modified S-iteration process is studied for asymptotically quasi-nonexpansive mappings in a CAT(0) space.

In the first part of the sixth chapter, the strong and Δ-convergence theorems of the modified S-iteration and the modified two-step iteration processes are given for total asymptotically nonexpansive mappings in a CAT(0) space. In the last part of it, some results on the strong and Δ-convergence of the modified SP-iteration process are obtained for total asymptotically nonexpansive mappings in hyperbolic spaces.

In the last section of this thesis, the main results, which were obtained, are summarized.

CAT(κ) VE HİPERBOLİK UZAYLARDA SABİT NOKTALARA İLİŞKİN BAZI Δ-YAKINSAMA VE KUVVETLİ YAKINSAMA TEOREMLERİ

ÖZET

Anahtar Kelimeler: CAT(κ) Uzayı, Sabit Nokta, İterasyon Yöntemi, Kuvvetli Yakınsama, Δ-Yakınsama, Hiperbolik Uzay.

Bu tez çalışması yedi bölümden oluşmaktadır. Birinci bölümde, bazı temel tanım ve teoremler verildi. İkinci bölümde ise, CAT(κ) uzayı ve hiperbolik uzay kavramları ile ilgili bazı temel tanım ve teoremler verildi.

Üçüncü bölümün ilk kısmında, CAT(0) uzayında genişlemeyen dönüşümler için SP-iterasyon yönteminin kuvvetli ve Δ-yakınsaması çalışıldı. Aynı bölümün ikinci kısmında ise, düzgün konveks hiperbolik uzayda bir iterasyon yönteminin genişlemeyen dönüşümlerin ortak sabit noktasına kuvvetli ve Δ-yakınsaması ispatlandı.

Dördüncü bölümün ilk kısmında, CAT(0) uzayında (C) şartını sağlayan dönüşümler için S-iterasyon yönteminin kuvvetli ve Δ-yakınsaması ispatlandı. Aynı bölümün ikinci kısmında, CAT(0) uzayında yine bu dönüşümler için üç adımlı bir iterasyon yönteminin kuvvetli ve Δ-yakınsaması incelendi. Son kısmında ise, yine CAT(0) uzayında kendi üzerine olmayan dönüşümler için S-iterasyon ve Noor iterasyon yönteminin kuvvetli ve Δ-yakınsaması üzerine bazı sonuçlar verildi.

Beşinci bölümün ilk kısmında CAT(0) uzayında *k*-strictly pseudo contractive dönüşümler için bazı iterasyon yöntemlerinin kuvvetli ve Δ-yakınsaması ispatlandı. Aynı bölümün ikinci kısmında, yeni bir dönüşüm sınıfı tanımlandı ve CAT(0) uzayında bu dönüşüm sınıfı için çok adımlı bir iterasyon ve S-iterasyon yönteminin Δ-yakınsaması incelendi. Aynı zamanda CAT(0) uzayında contractive-like dönüşümler için bu iterasyon yöntemlerinin kuvvetli yakınsaması üzerine bazı sonuçlar elde edildi. Son kısmında ise, CAT(0) uzayında asimptotik quasi genişlemeyen dönüşümler için modified S-iterasyon yönteminin kuvvetli yakınsaması çalışıldı.

Altıncı bölümün ilk kısmında, CAT(0) uzayında total asimptotik genişlemeyen dönüşümler için modified S-iterasyon ve modified iki adımlı iterasyon yöntemlerinin kuvvetli ve Δ-yakınsama teoremleri verildi. Son kısmında ise, hiperbolik uzayda total asimptotik genişlemeyen dönüşümler için modified SP-iterasyon yönteminin kuvvetli ve Δ-yakınsaması üzerine bazı sonuçlar elde edildi.

Son bölümde ise elde edilen temel sonuçlar özetlendi.

CHAPTER 1. INTRODUCTION

In this section; review of the literature, some definitions and preliminaries, which are necessary throughout this thesis, are given.

1.1. Basic Facts and Definitions

Definition 1.1.1. [1] A metric space is a pair (X,d), consisting of a nonempty set X and a metric function $d: X\times X \to \mathbb{R}$ such that, for all x,y,z in X, the following conditions hold,

(M1) $d(x,y)=0$ if and only if $x=y$,

(M2) $d(x,y)=d(y,x)$,

(M3) $d(x,z)\leq d(x,y)+d(y,z)$.

Example 1.1.2. [2] Let $X=\mathbb{R}$, the set of all real numbers. For $x,y\in X$, define $d(x,y)=|x-y|$. Then (X,d) is a metric space. This is called the metric space $\mathbb{R}$ with the usual absolute metric.

Example 1.1.3. [3] The metric space $\mathbb{R}^2$, called the Euclidean plane, is obtained if we take the set of ordered pairs of real numbers, written $x=(x_1,x_2)$, $y=(y_1,y_2)$ and the Euclidean metric defined by $d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$.

Definition 1.1.4. [1] Let (X,d) be a metric space. A sequence $x=\{x_n\}$ is called a convergent sequence (with limit l) if for every $\varepsilon>0$, there exists $N=N(\varepsilon)$ such that $d(x_n,l)<\varepsilon$, for all $n\geq N$. We write $x_n \to l$ $(n\to\infty)$ or $\lim_{n\to\infty} x_n = l$.

Definition 1.1.5. [1] Let (X,d) be a metric space. A sequence $x=\{x_n\}$ is called a Cauchy sequence if $d(x_n,x_m)\to 0\,(n,m\to\infty)$, i.e., for all $\varepsilon>0$, there exists $N=N(\varepsilon)$ such that $d(x_n,x_m)<\varepsilon$ for all $n,m>N$.

Remark 1.1.6. [1] A convergent sequence in a metric space has a unique limit. Every convergent sequence is also a Cauchy sequence, but not conversely, in general. If a Cauchy sequence has a convergent subsequence then the whole sequence is convergent.

Definition 1.1.7. [1] A metric space (X,d) is called complete if every Cauchy sequence is convergent (to a point of X). Explicitly, we require that if $d(x_n,x_m)\to 0$ $(n,m\to\infty)$, then there exists $x\in X$ such that $d(x_n,x)\to 0$ $(n\to\infty)$.

Example 1.1.8. [1] The set of real numbers $\mathbb{R}$ with the usual metric forms a complete metric space.

Definition 1.1.9. [1] Let (X,d) and (Y,ρ) be metric spaces. Then $T:X\to Y$ is called continuous at $x_0\in X$ if for every $\varepsilon>0$, there exists $\delta=\delta(\varepsilon,x_0)>0$ such that $d(x,x_0)<\delta$ implies $\rho(T(x),T(x_0))<\varepsilon$. The mapping T is called continuous on X if it is continuous at each point of X.

Definition 1.1.10. [4] Let T be a mapping from a metric space (X,d) into another metric space (Y,ρ). Then T is said to be uniformly continuous on X if for given $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that $\rho(T(x),T(y))<\varepsilon$ whenever $d(x,y)<\delta$ for all $x,y\in X$.

Definition 1.1.11. [1] A linear space over a field $\mathbb{F}$, is a nonempty set X with two operations:

$$\begin{aligned} +:X\times X&\to X & \qquad \cdot:\mathbb{F}\times X&\to X \\ (x,y)&\to x+y & (\lambda,x)&\to \lambda x \end{aligned}$$

such that for all $\lambda, \mu \in \mathbb{F}$ and elements (vectors) $x, y, z \in X$ we have

(i) $x + y = y + x$,

(ii) $x + (y + z) = (x + y) + z$,

(iii) there exists $\theta \in X$ such that $x + \theta = \theta + x = x$,

(iv) there exists $(-x) \in X$ such that $x + (-x) = (-x) + x = \theta$,

(v) $1 \cdot x = x$,

(vi) $\lambda(x + y) = \lambda x + \lambda y$,

(vii) $(\lambda + \mu)x = \lambda x + \mu x$,

(viii) $(\lambda\mu)x = \lambda(\mu x)$.

If $\mathbb{F} = \mathbb{R}$, X is called real linear space and if $\mathbb{F} = \mathbb{C}$, X is called complex linear space.

Definition 1.1.12. [1] Let X be a (real and complex) linear space. The function

$$\begin{aligned} \|.\| : X &\to \mathbb{R} \\ x &\to \|x\| \end{aligned}$$

satisfies the following conditions for all $x, y \in X$ and $\lambda \in \mathbb{F}$,

(i) $\|x\| = 0 \Leftrightarrow x = \theta$,

(ii) $\|\lambda x\| = |\lambda| \|x\|$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

Then, the function $\|.\|$ is called a norm, the pair of $(X, \|.\|)$ is also called a normed linear space.

Example 1.1.13. [1] $C[a,b]$ is a normed space with $\|x\| = \max |x(t)|$ for $t \in [a,b]$.

Definition 1.1.14. [1] A Banach space X is a complete normed linear space. Completeness means that if $\|x_m - x_n\| \to 0$ $(m,n \to \infty)$ where $x_n \in X$, then there exists $x \in X$ such that $\|x_n - x\| \to 0$ $(n \to \infty)$.

Example 1.1.15. [1] $\mathbb{R}, \mathbb{C}, \ell_\infty, \ell_p$ $(p \geq 1)$, c, c_0 and $C[a,b]$ are Banach spaces.

Definition 1.1.16. [4] Let X be a linear space over field $\mathbb{C}$. An inner product on X is a function $\langle .,. \rangle : X \times X \to \mathbb{C}$ with the following three properties:

(i) $\langle x,x \rangle \geq 0$ for all $x \in X$ and $\langle x,x \rangle = 0$ if and only if $x = \theta$;

(ii) $\langle x,y \rangle = \overline{\langle y,x \rangle}$, where the bar denotes complex conjugation;

(iii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x,z \rangle + \beta \langle y,z \rangle$ for all $x,y,z \in X$ and $\alpha, \beta \in \mathbb{C}$.

The ordered pair $(X, \langle .,. \rangle)$ is called an inner product space. Sometimes, it is called a pre-Hilbert space. $\langle x,y \rangle$ is called inner product of two elements $x, y \in X$.

Remark 1.1.17. [1] Each inner product space is a normed linear space under $\|x\| = \sqrt{\langle x,x \rangle}$.

Definition 1.1.18. [1] A Hilbert space H is a complete inner product space, i.e., a Banach space whose norm is generated by an inner product.

Example 1.1.19. [3] The n-dimensional Euclidean space $\mathbb{R}^n$ is a Hilbert space with inner product defined by

$$\langle x,y \rangle = x_1.y_1 + x_2.y_2 + ... + x_n.y_n$$

where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$.

1.2. Some Basic Notations of Fixed Point Theory

Definition 1.2.1. [5] Let X be a nonempty set and $T: X \to X$ be a self mapping. We say that $x \in X$ is a fixed point of T if $T(x) = x$ and the set of all fixed points of T is denoted by $F(T)$.

Example 1.2.2. [5]

(i) If $X = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$, then $F(T) = \{-2\}$;

(ii) If $X = \mathbb{R}$ and $T(x) = x^2 - x$, then $F(T) = \{0, 2\}$;

(iii) If $X = \mathbb{R}$ and $T(x) = x + 2$, then $F(T) = \varnothing$;

(iv) If $X = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$.

Definition 1.2.3. [5] Let X be any nonempty set and $T: X \to X$ be a self mapping. For any given $x \in X$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$; we call $T^n(x)$ the n^{th} iteration of x under T. In order to simplify the notations we will often use Tx instead of $T(x)$.

Definition 1.2.4. [5] The mapping $T^n (n \geq 1)$ is called the n^{th} iteration of T. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0} \subset X$ given by $x_n = Tx_{n-1} = T^n x_0$, $n = 1, 2, \ldots$ is called the sequence of successive approximations with the initial value x_0. It is also known as the Picard iteration starting at x_0.

For a given self mapping the following properties obviously hold:

(i) $F(T) \subset F(T^n)$, for each $n \in \mathbb{N}$;

(ii) $F(T^n) = \{x\}$, for some $n \in \mathbb{N} \;\Rightarrow\; F(T) = \{x\}$.

The inverse of (ii) is not true, in general, as shown by the next example.

Example 1.2.5. [5] Let $T:\{1,2,3\}\to\{1,2,3\}$, $T(1)=3,\ T(2)=2$ and $T(3)=1$. Then $F(T^2)=\{1,2,3\}$ but $F(T)=\{2\}$.

Definition 1.2.6. [5] Let (X,d) be a metric space. A mapping $T:X\to X$ is called

(i) Lipschitzian (or L-Lipschitzian) if there exists a constant $L>0$ such that $d(Tx,Ty)\le Ld(x,y),$ for all $x,y\in X$;

(ii) (strict) contraction (or a-contraction) if T is a-Lipschitzian, with $a\in[0,1)$;

(iii) nonexpansive if T is 1-Lipschitzian;

(iv) contractive if $d(Tx,Ty)<d(x,y),$ for all $x,y\in X,\ x\neq y$.

Remark 1.2.7. The class of contractive mappings includes contraction mappings, whereas the class of nonexpansive mappings is larger than contractive mappings. Moreover, each nonexpansive mapping is a Lipschitzian mapping.

Remark 1.2.8. [4] If T is a Lipschitzian mapping, then T is a uniformly continuous.

Definition 1.2.9. [4] Let (X,d) be a metric space. A mapping $T:X\to X$ is called

(i) quasi-nonexpansive if $F(T)\neq\varnothing$ and $d(Tx,p)\le d(x,p)$ for all $x\in X$ and $p\in F(T)$;

(ii) asymptotically nonexpansive if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty}k_n=1$ such that $d(T^nx,T^ny)\le k_nd(x,y)$ for all $x,y\in X$ and $n\in\mathbb{N}$;

(iii) uniformly L-Lipschitzian if there exists a constant $L>0$ such that $d(T^nx,T^ny)\le Ld(x,y)$ for all $x,y\in K$ and $n\in\mathbb{N}$.

Remark 1.2.10. [4] The class of quasi-nonexpansive mappings and asymptotically nonexpansive mappings includes nonexpansive mappings. Moreover, each asymptotically nonexpansive mapping is a uniformly L-Lipschitzian mapping with $L=\sup_{n\in\mathbb{N}}\{k_n\}$.

Definition 1.2.11. [6] Let (X,d) be a metric space. A mapping $T:X\to X$ is called asymptotically quasi nonexpansive if there exists a sequence $\{u_n\}\in[0,\infty)$ with $\lim_{n\to\infty} u_n=0$ and such that

$$d(T^n x,p)\le(1+u_n)d(x,p)$$

for all $x\in X$ and $p\in F(T)\neq\varnothing$.

Remark 1.2.12. [6] The class of asymptotically quasi-nonexpansive mappings is larger than that of quasi-nonexpansive mappings and asymptotically nonexpansive mappings.

Definition 1.2.13. ([7, Definition 2.1]) Let (X,d) be a metric space. A mapping $T:X\to X$ is called total asymptotically nonexpansive if there exist non-negative real sequences $\{\mu_n\}$, $\{v_n\}$ with $\mu_n\to 0, v_n\to 0\ (n\to\infty)$ and a strictly increasing continous function $\zeta:[0,\infty)\to[0,\infty)$ with $\zeta(0)=0$ such that

$$d(T^n x,T^n y)\le d(x,y)+v_n\zeta(d(x,y))+\mu_n$$

for all $x,y\in X$ and $n\in\mathbb{N}$.

Remark 1.2.14. [7] Each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $v_n=k_n-1$, $\mu_n=0$, $\forall n\in\mathbb{N}$, $\zeta(t)=t$, $\forall t\ge 0$.

Definition 1.2.15. [8] Let (X,d) be a metric space. A mapping $T:X\to X$ is said to satisfy condition (C) if

$$\frac{1}{2}d(x,Tx)\le d(x,y) \text{ implies } d(Tx,Ty)\le d(x,y),$$

for all $x,y\in X$.

Remark 1.2.16. [8]

(i) Every nonexpansive mapping satisfies condition (C).

(ii) Assume that a mapping T satisfies condition (C) and has a fixed point. Then T is a quasi-nonexpansive mapping.

Eaxmple 1.2.17. [8] Define a mapping T on $[0,3]$ by

$$Tx = \begin{cases} 0 & \text{if } x \neq 3 \\ 1 & \text{if } x = 3. \end{cases}$$

Then T satisfies condition (C), but T is not nonexpansive.

Eaxmple 1.2.18. [8] Define a mapping T on $[0,3]$ by

$$Tx = \begin{cases} 0 & \text{if } x \neq 3 \\ 2 & \text{if } x = 3. \end{cases}$$

Then $F(T) \neq \varnothing$ and T is quasi-nonexpansive, but T does not satisfy condition (C).

Definition 1.2.19. [9] Let K be a nonempty subset of a metric space (X,d). A mapping $T: K \to K$ is said to be demi-compact if for any bounded sequence $\{x_n\}$ in K such that $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = p \in K$.

Definition 1.2.20. [10] Let K be a nonempty subset of a metric space (X,d). A mapping $T: K \to K$ is said to satisfy condition (I) if there exists a non-decreasing function $f:[0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$d(x, T(x)) \geq f(d(x, F(T))) \quad \text{for all } x \in K.$$

Remark 1.2.21. It is clear that the condition (I) is weaker than both the compactness of K and the demi-compactness of the nonexpansive mapping T.

Definition 1.2.22. [11] A sequence $\{x_n\}$ in a metric space (X,d) is said to be Fejér monotone with respect to K (a subset of X) if $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in K$ and $n \in \mathbb{N}$.

Lemma 1.2.23. [11] Let K be a nonempty closed subset of a complete metric space (X,d) and let $\{x_n\}$ be Fejér monotone with respect to K. Then $\{x_n\}$ converges strongly to some $p\in K$ if and only if $\lim_{n\to\infty} d(x_n,K)=0$.

1.3. Some Iteration Processes

Definition 1.3.1. [5] Let (X,d) be a metric space, K be a closed subset of X and $T:K\to K$ be a self mapping. For a given $x_0\in X$, the Picard iteration is the sequence $\{x_n\}$ defined by

$$x_n=T(x_{n-1})=T^n(x_0),\ \ n\in\mathbb{N}. \qquad \textbf{(1.3.1)}$$

The sequence defined by (1.3.1) is known as the sequence of successive approximations.

When the contractive conditions are slightly weaker, then the Picard iterations doesn't need to converge to a fixed point of the operator T, and some other iteration procedures must be considered.

Example 1.3.2. [5] Let $K=[0,1]$ and $T:[0,1]\to[0,1]$, $Tx=1-x$ for all $x\in[0,1]$. Then T is nonexpansive, T has a unique fixed point, $F(T)=\left\{\frac{1}{2}\right\}$, but, for any $x_0=a\neq\frac{1}{2}$, the Picard iteration (1.3.1) yields an oscillatory sequence $a,\ 1-a,\ a,\ 1-a,\ \ldots$. Since this sequence is not convergent for $a\neq\frac{1}{2}$, then the Picard iteration (1.3.1) no longer converge to a fixed point of T.

Definition 1.3.3. [12] Let (X,d) be a metric space, K be a nonempty convex subset of X and $T:K\to K$ be a self mapping. Let $\{\alpha_n\}$ be a sequence of real numbers in $[0,1]$. For an arbitrary $x_1\in K$, define a sequence $\{x_n\}$ in K by

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_nTx_n,\ \ \ n\in\mathbb{N}. \qquad \textbf{(1.3.2)}$$

Then $\{x_n\}$ is called the Mann iteration.

Example 1.3.4. [5] Let $K=\left[\frac{1}{2},2\right]$ and $T:K\to K$, $Tx=\frac{1}{x}$, for all $x\in K$. Then the Mann iteration (1.3.2) converges to the unique fixed point of T.

Definition 1.3.5. [13] Let K be a nonempty convex subset of a metric space (X,d) and $T:K\to K$ be a self mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of real numbers in $[0,1]$. For an arbitrary $x_1\in K$, define a sequence $\{x_n\}$ in K by

$$\begin{cases} x_{n+1}=(1-\alpha_n)x_n+\alpha_n Ty_n, \\ y_n=(1-\beta_n)x_n+\beta_n Tx_n,\ n\in\mathbb{N}. \end{cases} \tag{1.3.3}$$

Then $\{x_n\}$ is called the Ishikawa iteration.

Remark 1.3.6. [5] Despite this apparent similarity and the fact that, for $\beta_n=0$, the Ishikawa iteration (1.3.3) is reduced to the Mann iteration, there is not a general dependence between convergence results for the Mann iteration and the Ishikawa iteration.

Definition 1.3.7. [14] Let K be a nonempty convex subset of a metric space (X,d) and $T:K\to K$ be a self mapping. The Noor iteration, starting from $x_1\in K$, is a sequence $\{x_n\}$ in K defined by

$$\begin{cases} x_{n+1}=(1-\alpha_n)x_n+\alpha_n Ty_n, \\ y_n=(1-\beta_n)x_n+\beta_n Tz_n, \\ z_n=(1-\gamma_n)x_n+\gamma_n Tx_n,\ n\in\mathbb{N}, \end{cases} \tag{1.3.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences of real numbers in $[0,1]$.

Remark 1.3.8. If we take $\gamma_n=0$ for all $n\in\mathbb{N}$, (1.3.4) is reduced to the Ishikawa iteration and we take $\beta_n=\gamma_n=0$ for all $n\in\mathbb{N}$, (1.3.4) is reduced to the Mann iteration.

Definition 1.3.9. [15] For a convex subset K of a metric space (X,d) and a self mapping T on K, the iterative sequence $\{x_n\}$ of the S-iteration process is generated from $x_1 \in K$ and is defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n,\ n \in \mathbb{N}, \end{cases} \tag{1.3.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

Remark 1.3.10. [15] The S-iteration process (1.3.5) is independent of the Mann and Ishikawa iteration processes. The rate of convergence of S-iteration process is similar to the Picard iteration process, but faster than the Mann iteration process for contraction mappings.

Definition 1.3.11. [16] Let K be a nonempty convex subset of a metric space (X,d) and $T: K \to K$ be a self mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of real numbers in $[0,1]$. For an arbitrary $x_1 \in K$, define a sequence $\{x_n\}$ in K by

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n,\ n \in \mathbb{N}. \end{cases} \tag{1.3.6}$$

Then $\{x_n\}$ is called the new two-step iteration.

Remark 1.3.12. If we take $\beta_n = 0$ for all $n \in \mathbb{N}$, the new two-step iteration (1.3.6) is reduced to the Mann iteration.

Definition 1.3.13. [17] Let K be a nonempty convex subset of a metric space (X,d) and $T: K \to K$ be a self mapping. Define a sequence $\{x_n\}$ in K by

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n,\ n \in \mathbb{N}, \end{cases} \tag{1.3.7}$$

where $x_1 \in K$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. Then $\{x_n\}$ is called the SP-iteration.

Remark 1.3.14. [17] The Mann, Ishikawa, Noor and SP-iterations are equivalent and the SP-iteration converges better than the others for the class of continuous and non-decreasing functions. Clearly, the new two-step and Mann iterations are special cases of the SP-iteration.

CHAPTER 2. THE CAT(κ) SPACE AND THE HYPERBOLIC SPACE

In this section; some fundamental definitions and lemmas related to the concepts of CAT(κ) space and hyperbolic space, are given.

2.1. The CAT(κ) Space

The terminology “CAT(κ)” was coined by Gromov [18]. The initials are in honor of E. **C**artan, A. D. **A**lexanderov and V. A. **T**oponogov whom considered similar conditions in varying degrees of generality.

Definition 2.1.1. [19] Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0)=x$, $c(l)=y$ and $d(c(t),c(t'))=|t-t'|$ for all $t,t' \in [0,l]$ (in particular, $l=d(x,y)$). The image of c is called a geodesic segment with endpoints x and y. When it is unique, this geodesic is denoted by $[x,y]$.

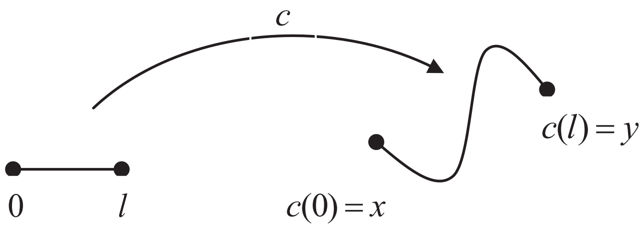

Figure 2.1. The geodesic segment

Definition 2.1.2. [19] The space (X,d) is said to be a geodesic metric space (or, more briefly, a geodesic space) if every two points of X are joined by a geodesic, and X is said to be a uniquely geodesic space if there is exactly one geodesic joining x to y for all $x, y \in X$.

Definition 2.1.3. [19] Given $r > 0$, a metric space (X, d) is said to be r-geodesic if for every pair of points $x, y \in X$ with $d(x, y) < r$, there is a geodesic joining x to y and X is said to be a r-uniquely geodesic if there is a unique geodesic segment joining each such pair of points x and y.

Definition 2.1.4. [19] Let (X, d) be a geodesic space. A subset Y of X is said to be convex if Y includes every geodesic segment joining any two of its points.

Definition 2.1.5. [19] Given a real number κ, let M_κ^2 denote the following metric spaces:

(i) if $\kappa = 0$ then M_κ^2 is the Euclidean plane $\mathbb{R}^2$;

(ii) if $\kappa < 0$ then M_κ^2 is the real hyperbolic space H^2 with the metric scaled by a factor of $1/\sqrt{-\kappa}$;

(iii) if $\kappa > 0$ then M_κ^2 is the 2-dimensional sphere S^2 with the metric scaled by a factor of $1/\sqrt{\kappa}$.

Definition 2.1.6. [19] The diameter of M_κ^2 is denoted by

$$D_\kappa = \begin{cases} \dfrac{\pi}{\sqrt{\kappa}} & \kappa > 0, \\ +\infty & \kappa \le 0. \end{cases}$$

Definition 2.1.7. [19] A geodesic triangle $\Delta(x, y, z)$ in a geodesic metric space (X, d) consists of three points $x, y, z \in X$ and three geodesic segments $[x, y], [y, z], [z, x]$. A comparison triangle of $\Delta(x, y, z)$ is a geodesic triangle $\overline{\Delta}(x, y, z) = \Delta(\overline{x}, \overline{y}, \overline{z})$ in M_κ^2 with vertices $\overline{x}, \overline{y}, \overline{z}$ such that $d(x, y) = d(\overline{x}, \overline{y})$, $d(y, z) = d(\overline{y}, \overline{z})$ and $d(z, x) = d(\overline{z}, \overline{x})$. The point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a comparison point in $\overline{\Delta}$ for $p \in [x, y]$ if $d(x, p) = d(\overline{x}, \overline{p})$. Comprasion points on $[y, z]$ and $[z, x]$ are defined similarly.

Remark 2.1.8. [19] If $\kappa \leq 0$ then such a $\overline{\Delta}$ always exists; if $\kappa > 0$ then it exists provided the perimeter $d(x, y) + d(y, z) + d(z, x)$ of Δ is less than $2D_\kappa$; in both cases it is unique up to isometry of M_κ^2.

Definition 2.1.9. [19] Let X be a geodesic space and let κ be a real number. Let Δ be a geodesic triangle in X with perimeter less than $2D_\kappa$. Let $\overline{\Delta}$ in M_κ^2 be a comparison triangle for Δ. Then X is said to satisfy the CAT(κ) inequality if for all $p, q \in \Delta$ and all comparison points $\overline{p}, \overline{q} \in \overline{\Delta}$,

$$d(p, q) \leq d(\overline{p}, \overline{q}).$$

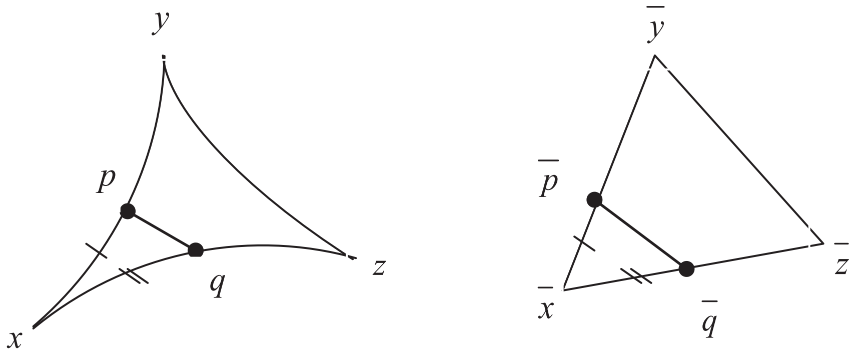

Figure 2.2. The CAT(κ) inequality

Definition 2.1.10. [19]

(iv) If $\kappa \leq 0$, then X is called a CAT(κ) space (more briefly, "X is CAT(κ)") if X is a geodesic space all of whose geodesic triangles satisfy the CAT(κ) inequality.

(v) If $\kappa > 0$, then X is called a CAT(κ) space if X is D_κ-geodesic and all geodesic triangle in X with perimeter less than $2D_\kappa$ satisfy the CAT(κ) inequality.

Example 2.1.11. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [19]), Euclidean buildings (see [20]), $\mathbb{R}$-trees (see [21]), the complex Hilbert ball with a hyperbolic metric (see [22]) and many others.

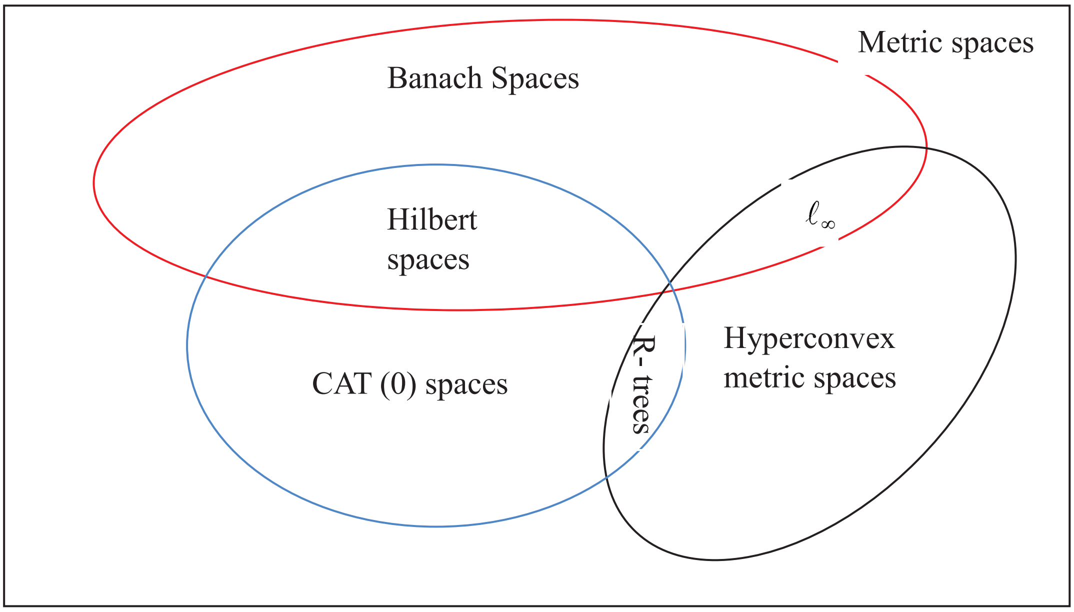

Figure 2.3. The relation between some spaces

Hilbert spaces (in which the CAT(0) inequality is an equality); the only Banach spaces that are CAT(0). $\mathbb{R}$-trees; the only hyperconvex metric spaces that are CAT(0).

Fact 2.1.12. If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the mid-point of the segment $[y_1, y_2]$, then the CAT(0) inequality implies that

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

This is the (CN) inequality of Bruhat and Tits [23]. In fact (see [19, p.163]), a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality.

Remark 2.1.13. ([19, p.165]) It is worth mentioning that the results in a CAT(0) space can be applied to any CAT(κ) space with $\kappa \leq 0$ since any CAT(κ) space is a CAT(κ') space for every $\kappa' \geq \kappa$.

Fact 2.1.14. ([24, Lemma 2.3]) Let X be a CAT(0) space and let $x, y \in X$ such that $x \neq y$. Then

$$[x, y] = \{(1-t)x \oplus ty; t \in [0,1]\}.$$

Lemma 2.1.15. ([24, Lemmas 2.4, 2.5]) Let X be a CAT(0) space. Then the following inequalities hold:

(i) $d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$,

(ii) $d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2$,

for all $t \in [0,1]$ and $x, y, z \in X$.

Lemma 2.1.16. ([25, Lemma 2.7]) Let X be a complete CAT(0) space and let $x \in X$. Suppose that $\{t_n\}$ is a sequence in $[a,b]$ for some $a, b \in (0,1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that

$$\limsup_{n\to\infty} d(x_n, x) \leq r, \ \limsup_{n\to\infty} d(y_n, x) \leq r, \ \lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$$

for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Fixed point theory in a CAT(0) space has been first studied by Kirk (see [26, 27]). He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory in a CAT(0) space has been rapidly developed and many papers have appeared (see [23, 28-30]). It is worth mentioning that fixed point theorems in a CAT(0) space (especially in $\mathbb{R}$-trees) can be applied to graph theory, biology and computer science (see [21, 31-34]).

We now give the definition and collect some basic properties of the Δ-convergence.

Definition 2.1.17. [35] Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\},$$

and the asymptotic radius $r_K(\{x_n\})$ of $\{x_n\}$ with respect to $K \subset X$ is given by

$$r_K(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in K\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\},$$

and the asymptotic center $A_K(\{x_n\})$ of $\{x_n\}$ with respect to $K \subset X$ is the set

$$A_K(\{x_n\}) = \{x \in K : r(x,\{x_n\}) = r_K(\{x_n\})\}.$$

Proposition 2.1.18. ([35, Proposition 3.2]) Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X and let K be a closed convex subset of X, then $A(\{x_n\})$ and $A_K(\{x_n\})$ are singletons.

The notion of Δ-convergence in a general metric space was introduced by Lim [36]. Kirk and Panyanak [37] used the concept of Δ-convergence introduced by Lim [36] to prove on the CAT(0) space analogs of some Banach space results which involve weak convergence. Further, Dhompongsa and Panyanak [24] obtained the Δ-convergence theorems for the Picard, Mann and Ishikawa iterations in a CAT(0) space.

Definition 2.1.19. ([36, 37]) A sequence $\{x_n\}$ in a CAT(0) space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ and x is called the Δ-limit of $\{x_n\}$.

Remark 2.1.20. [37] Every CAT(0) space satisfies the Opial property, i.e., if $\{x_n\}$ is a sequence in K and $\Delta-\lim_{n\to\infty} x_n = x$, then for each $y(\neq x)\in K$,

$$\limsup_{n\to\infty} d(x_n,x) < \limsup_{n\to\infty} d(x_n,y).$$

Lemma 2.1.21. ([37, p.3690]) Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.

Lemma 2.1.22. ([38, Proposition 2.1]) Let K be a nonempty closed convex subset of a complete CAT(0) space and let $\{x_n\}$ be a bounded sequence in K. Then the asymptotic center of $\{x_n\}$ is in K.

Lemma 2.1.23. ([24, Lemma 2.8]) If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\})=\{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\})=\{u\}$ and the sequence $\{d(x_n,u)\}$ is convergent then $x=u$.

Nanjaras and Panyanak [35] gave the concept of "$\rightharpoonup$" convergence and a connection between this convergence and Δ-convergence.

Definition 2.1.24. [35] Let C be a closed convex subset of a CAT(0) space X and $\{x_n\}$ be a bounded sequence in C. Denote the notation

$$\{x_n\}\rightharpoonup w \Leftrightarrow \Phi(w)=\inf_{x\in C}\Phi(x) \tag{2.1.1}$$

where $\Phi(x)=\limsup_{n\to\infty} d(x_n,x)$.

Proposition 2.1.25. ([35, Proposition 3.12]) Let C be a closed convex subset of a CAT(0) space X and $\{x_n\}$ be a bounded sequence in C. Then $\Delta-\lim_{n\to\infty} x_n = w$ implies that $\{x_n\}\rightharpoonup w$.

2.2. The Hyperbolic Space and Relation with the CAT(0) Space

Kohlenbach [39] introduced the hyperbolic spaces, defined below, which play a significant role in many branches of mathematics.

Definition 2.2.1. A hyperbolic space (X,d,W) is a metric space (X,d) together with a mapping $W: X\times X\times[0,1]\to X$ satisfying

(W1) $d(z,W(x,y,\lambda))\leq(1-\lambda)d(z,x)+\lambda d(z,y)$,

(W2) $d(W(x,y,\lambda_1),W(x,y,\lambda_2))=|\lambda_1-\lambda_2|d(x,y)$,

(W3) $W(x,y,\lambda)=W(y,x,1-\lambda)$

(W4) $d(W(x,z,\lambda),W(y,w,\lambda))\leq(1-\lambda)d(x,y)+\lambda d(z,w)$

for all $x,y,z,w\in X$ and $\lambda,\lambda_1,\lambda_2\in[0,1]$.

Definition 2.2.2. A subset K of a hyperbolic space X is convex if $W(x,y,\lambda)\in K$ for all $x,y\in K$ and $\lambda\in[0,1]$.

Remark 2.2.3. If a space satisfies only (W1), it coincides with the convex metric space introduced by Takahashi [40]. The concept of hyperbolic space in [39] is more restrictive than the hyperbolic type introduced by Goebel and Kirk [41] since (W1)-(W3) together are equivalent to (X,d,W) being a space of hyperbolic type in [41]. Also it is slightly more general than the hyperbolic space defined by Reich and Shafrir [42].

Remark 2.2.4. The class of hyperbolic spaces in [39] contains all normed linear spaces and convex subsets thereof, $\mathbb{R}$-trees, the Hilbert ball with the hyperbolic metric (see [22]), Cartesian products of Hilbert balls, Hadamard manifolds and CAT(0) spaces (see [19]), as special cases.

Example 2.2.5. [43] Let B_H be an open unit ball in a complex Hilbert space $(H,\langle . \rangle)$ with respect to the metric (also known as the Kobayashi distance)

$$k_{B_H}(x,y) = \arg\tanh\ (1-\sigma(x,y))^{\frac{1}{2}},$$

where

$$\sigma(x,y) = \frac{(1-\|x\|^2)(1-\|y\|^2)}{\left|1-\langle x,y\rangle\right|^2} \text{ for all } x,y \in B_H.$$

Then (B_H, k_{B_H}, W) is a hyperbolic space where $W(x,y,\lambda)$ defines a unique point $(1-\lambda)x \oplus \lambda y$ in a unique geodesic segment $[x,y]$ for all $x,y \in B_H$.

Definition 2.2.6. A hyperbolic space (X,d,W) is said to be

(i) [40] strictly convex if for any $x,y \in X$ and $\lambda \in [0,1]$, there exists a unique element $z \in X$ such that $d(z,x) = \lambda d(x,y)$ and $d(z,y) = (1-\lambda)d(x,y)$;

(ii) [44] uniformly convex if for all $u,x,y \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that $d\left(W\left(x,y,\frac{1}{2}\right),u\right) \le (1-\delta)r$ whenever $d(x,u) \le r$, $d(y,u) \le r$ and $d(x,y) \ge \varepsilon r$.

Remark 2.2.7. [30] A uniformly convex hyperbolic space is strictly convex.

Definition 2.2.8. [43] A mapping $\eta : (0,\infty)\times(0,2] \to (0,1]$, which provides such a for given $r > 0$ and $\varepsilon \in (0,2]$, is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε), i.e., $\forall \varepsilon > 0$, $\forall r_2 \ge r_1 > 0$, $\eta(r_2,\varepsilon) \le \eta(r_1,\varepsilon)$.

It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that “bounded sequences have unique asymptotic centers with respect to closed convex subsets”. The following lemma is due to Leustean [45] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 2.2.9. ([45, Proposition 3.3]) Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.

Lemma 2 2.10. ([46, Lemma 2.5]) Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that

$$\limsup_{n\to\infty} d\left(x_n, x\right) \leq r,\ \limsup_{n\to\infty} d\left(y_n, x\right) \leq r,\ \lim_{n\to\infty} d\left(W(x_n, y_n, \alpha_n), x\right) = r$$

for some $r \geq 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0.$

Lemma 2.2.11. ([46, Lemma 2.6]) Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and let $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y.$

CHAPTER 3. SOME CONVERGENCE RESULTS FOR NONEXPANSIVE MAPPINGS

In this section, some strong and Δ-convergence theorems for nonexpansive mappings are proved.

3.1. The Strong and Δ-Convergence of SP-Iteration for Nonexpansive Mappings on CAT(0) Spaces

In this subsection, we prove the strong and Δ-convergence theorems of SP-iteration for nonexpansive mappings on a CAT(0) space.

Now, we apply the SP-iteration in a CAT(0) space for nonexpansive mappings as follows.

Definition 3.1.1. Let X be a CAT(0) space, K be a nonempty convex subset of X and $T:K\to K$ be a nonexpansive mapping. The SP-iteration, starting from $x_1\in K$, is the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1}=(1-\alpha_n)y_n\oplus\alpha_nTy_n, \\ y_n=(1-\beta_n)z_n\oplus\beta_nTz_n, \\ z_n=(1-\gamma_n)x_n\oplus\gamma_nTx_n,\ n\in\mathbb{N}, \end{cases} \tag{3.1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

Lemma 3.1.2. ([37, Proposition 3.7]) Let K be a nonempty closed convex subset of a complete CAT(0) space X and $f:K\to X$ be a nonexpansive mapping. Then the conditions, $\{x_n\}$ Δ-converges to x and $d(x_n,f(x_n))\to 0$, imply $x\in K$ and $f(x)=x$.

We give the following lemma which is used later.

Lemma 3.1.3. Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T:K\to K$ be a nonexpansive mapping with $F(T)\neq\varnothing$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$, $\{\gamma_n\}$ be a sequence in $[\varepsilon,1-\varepsilon]$ for some $\varepsilon\in(0,1)$ and $\{x_n\}$ be defined by the iteration process (3.1.1). Then

(vi) $\lim_{n\to\infty} d(x_n,p)$ exists for all $p\in F(T)$,

(vii) $\lim_{n\to\infty} d(x_n,Tx_n)=0$.

Proof. (i) Let $p\in F(T)$. By (3.1.1) and Lemma 2.1.15(i), we have

$$\begin{aligned} d(z_n,p) &= d((1-\gamma_n)x_n\oplus\gamma_n Tx_n,p) \\ &\le (1-\gamma_n)d(x_n,p)+\gamma_n d(Tx_n,p) \\ &\le (1-\gamma_n)d(x_n,p)+\gamma_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3.1.2}$$

Also, we get

$$\begin{aligned} d(y_n,p) &= d((1-\beta_n)z_n\oplus\beta_n Tz_n,p) \\ &\le (1-\beta_n)d(z_n,p)+\beta_n d(Tz_n,p) \\ &\le (1-\beta_n)d(z_n,p)+\beta_n d(z_n,p) \\ &= d(z_n,p). \end{aligned} \tag{3.1.3}$$

Then we obtain

$$d(y_n,p)\le d(x_n,p). \tag{3.1.4}$$

Using (3.1.1) and Lemma 2.1.15(i), we have

$$\begin{aligned} d(x_{n+1},p) &= d((1-\alpha_n)y_n\oplus\alpha_n Ty_n,p) \\ &\le (1-\alpha_n)d(y_n,p)+\alpha_n d(Ty_n,p) \\ &\le (1-\alpha_n)d(y_n,p)+\alpha_n d(y_n,p) \\ &= d(y_n,p). \end{aligned} \tag{3.1.5}$$

Combining (3.1.4) and (3.1.5), we get

$$d(x_{n+1},p)\le d(x_n,p).$$

This implies that the sequence $\{d(x_n, p)\}$ is non-increasing and bounded below, and so $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. This completes the proof of part (i).

(ii) Let

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{3.1.6}$$

Firstly, we will prove that $\lim_{n\to\infty} d(y_n, p) = c.$ By (3.1.5) and (3.1.6), $\liminf_{n\to\infty} d(y_n, p) \geq c.$ Also, from (3.1.4) and (3.1.6), $\limsup_{n\to\infty} d(y_n, p) \leq c.$ Then we obtain

$$\lim_{n\to\infty} d(y_n, p) = c. \tag{3.1.7}$$

Secondly, we will prove that $\lim_{n\to\infty} d(z_n, p) = c.$ From (3.1.2) and (3.1.3), we have

$$d(y_n, p) \leq d(z_n, p) \leq d(x_n, p).$$

This gives

$$\lim_{n\to\infty} d(z_n, p) = c. \tag{3.1.8}$$

Next, by Lemma 2.1.15(ii),

$$\begin{aligned}
d(z_n, p)^2 &= d((1-\gamma_n)x_n \oplus \gamma_n Tx_n, p)^2 \\
&\leq (1-\gamma_n)d(x_n, p)^2 + \gamma_n d(Tx_n, p)^2 - \gamma_n(1-\gamma_n)d(x_n, Tx_n)^2 \\
&\leq (1-\gamma_n)d(x_n, p)^2 + \gamma_n d(x_n, p)^2 - \gamma_n(1-\gamma_n)d(x_n, Tx_n)^2 \\
&= d(x_n, p)^2 - \gamma_n(1-\gamma_n)d(x_n, Tx_n)^2.
\end{aligned}$$

Thus,

$$\gamma_n(1-\gamma_n)d(x_n, Tx_n)^2 \leq d(x_n, p)^2 - d(z_n, p)^2,$$

so that

$$\begin{aligned}
d(x_n, Tx_n)^2 &\leq \frac{1}{\gamma_n(1-\gamma_n)}\left[d(x_n, p)^2 - d(z_n, p)^2\right] \\
&\leq \frac{1}{\varepsilon^2}\left[d(x_n, p)^2 - d(z_n, p)^2\right].
\end{aligned}$$

Now using (3.1.6) and (3.1.8), $\limsup_{n\to\infty} d(x_n, Tx_n) \le 0$ and hence, $\lim_{n\to\infty} d(x_n, Tx_n) = 0.$ This completes the proof of part (ii).

Now, we give the Δ-convergence theorem of SP-iteration in a CAT(0) space.

Theorem 3.1.4. Let $X, K, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}$ satisfy the hypotheses of Lemma 3.1.3. Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. By Lemma 3.1.3, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Also, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Thus $\{x_n\}$ is bounded. Let $W_\Delta(x_n) = \cup A(\{u_n\}),$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_\Delta(x_n) \subseteq F(T).$ Let $u \in W_\Delta(x_n).$ Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}.$ By Lemmas 2.1.21 and 2.1.22, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in K.$ By Lemma 3.1.2, $v \in F(T).$ By Lemma 3.1.3(i), $\lim_{n\to\infty} d(x_n, v)$ exists. Now, we claim that $u = v.$ On the contrary, assume that $u \ne v.$ Then, by the uniqueness of asymptotic centers, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) \\
&\le \limsup_{n\to\infty} d(u_n, u) \\
&< \limsup_{n\to\infty} d(u_n, v) \\
&= \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v).
\end{aligned}$$

That is a contradiction. Thus $u = v \in F(T)$ and $W_\Delta(x_n) \subseteq F(T).$ To show that the sequence $\{x_n\}$ Δ-converges to a fixed point of T, we will show that $W_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}.$ We have already seen that $u = v$ and $v \in F(T).$ Finally, we claim that $x = v.$ If not, then the existence of $\lim_{n\to\infty} d(x_n, v)$ and the uniqueness of asymptotic centers imply that

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x) \\ &\le \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, v) \\ &= \limsup_{n\to\infty} d(v_n, v),\end{aligned}$$

a contradiction and hence $x = v \in F(T)$. Therefore, $W_\Delta(x_n) = \{x\}$. As a result, the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

We give the strong convergence theorems on CAT(0) space as follows.

Theorem 3.1.5. Let $X, K, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}$ satisfy the hypotheses of Lemma 3.1.3. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Proof. If $\{x_n\}$ converges to $p \in F(T)$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \le d(x_n, F(T)) \le d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Now, $d(x_{n+1}, p) \le d(x_n, p)$ gives

$$\inf_{p\in F(T)} d(x_{n+1}, p) \le \inf_{p\in F(T)} d(x_n, p),$$

which means that $d(x_{n+1}, F(T)) \le d(x_n, F(T))$ and so $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus, by hypothesis, $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Next, we will show that $\{x_n\}$ is a Cauchy sequence in K. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a constant n_0 such that

$$d(x_n, F(T)) < \frac{\varepsilon}{4},$$

for all $n \ge n_0$. In particular, $\inf\{d(x_{n_0}, p) : p \in F(T)\} < \frac{\varepsilon}{4}$. Thus there exists $p^* \in F(T)$ such that

$$d(x_{n_0}, p^*) < \frac{\varepsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p^*) + d(x_n, p^*) \leq 2d(x_{n_0}, p^*) < 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a complete CAT(0) space X, it must be convergent to a point in K. Let $\lim_{n\to\infty} x_n = p \in K$. Now, $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(p, F(T)) = 0$ and the closedness of $F(T)$ forces p to be in $F(T)$. Therefore, the sequence $\{x_n\}$ converges strongly to a fixed point p of T.

Theorem 3.1.6. Let $X, K, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}$ satisfy the hypotheses of Lemma 3.1.3 and $T : K \to K$ be a nonexpansive mapping satisfying condition (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 3.1.3(i), $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Let this limit be c, where $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose that $c > 0$. As proved in Theorem 3.1.5, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also, by Lemma 3.1.3(ii), we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from condition (I) that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is,

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0.$$

Since $f : [0, \infty) \to [0, \infty)$ is a non-decreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, therefore we obtain

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

The conclusion now follows from Theorem 3.1.5.

Since SP-iteration is reduced to the new two-step iteration when $\alpha_n = 0$ for all $n \in \mathbb{N}$ and to the Mann iteration when $\alpha_n = \beta_n = 0$ for all $n \in \mathbb{N}$, we obtain the following corollaries.

Corollary 3.1.7. Let $X, K, T, \{\gamma_n\}$ satisfy the hypotheses of Lemma 3.1.3 and $\{x_n\}$ be defined by the iteration process (1.3.6). Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T. Further, if $\{x_n\}$ is defined by the iteration process (1.3.2), the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Corollary 3.1.8. Let $X, K, \{\gamma_n\}$ satisfy the hypotheses of Lemma 3.1.3, $T: K \to K$ be a nonexpansive mapping satisfying condition (I) and $\{x_n\}$ be defined by the iteration process (1.3.6). Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T. Also, if $\{x_n\}$ is defined by the iteration process (1.3.2), the sequence $\{x_n\}$ converges strongly to a fixed point of T.

3.2. The Strong and Δ-Convergence of an Iteration Process for Nonexpansive Mappings in Uniformly Convex Hyperbolic Spaces

In this subsection, we establish some strong and Δ-convergence theorems of an iteration process for approximating a common fixed point of three nonexpansive mappings in a uniformly convex hyperbolic space.

Khan, Cho and Abbas [47] introduced a new iteration process in Banach spaces. We now modify this iteration in hyperbolic spaces as follows.

Definition 3.2.1. Let K be a nonempty convex subset of a hyperbolic space X and T, S, Q be three nonexpansive self mappings on K. The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = W(Tx_n, Sy_n, \alpha_n), \\ y_{n+1} = W(x_n, Qx_n, \beta_n), \ n \in \mathbb{N}, \end{cases} \tag{3.2.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$.

Remark 3.2.2. It is worth mentioning that the iteration process (3.2.1) coincides with the iteration process of Khan, Cho and Abbas [47] when $W(x,y,\alpha)=(1-\alpha)x+\alpha y$ and X is a uniformly convex Banach space. Moreover, this iteration is reduced to the S-iteration process of Khan and Abbas [48] in a CAT(0) space if $W(x,y,\alpha)=(1-\alpha)x\oplus\alpha y$ and $T=S=Q.$ It is also reduced to Ishikawa iteration when $T=I,\ S=Q$, Mann iteration when $T=Q=I$ and Picard iteration when $T=S,\ Q=I.$

We give the following key lemmas.

Lemma 3.2.3. Let K be a nonempty, closed and convex subset of a hyperbolic space X and T,S,Q be three nonexpansive self mappings on K with $F\neq\emptyset$. Then the sequence $\{x_n\}$ defined by (3.2.1) is Fejér monotone with respect to $F.$

Proof. Let $p\in F.$ Using (3.2.1), we have

$$\begin{aligned} d(y_n,p)&=d(W(x_n,Qx_n,\beta_n),p)\\ &\leq(1-\beta_n)d(x_n,p)+\beta_n d(Qx_n,p)\\ &\leq(1-\beta_n)d(x_n,p)+\beta_n d(x_n,p)\\ &=d(x_n,p). \end{aligned} \tag{3.2.2}$$

Thus from (3.2.2), we get

$$\begin{aligned} d(x_{n+1},p)&=d(W(Tx_n,Sy_n,\alpha_n),p)\\ &\leq(1-\alpha_n)d(Tx_n,p)+\alpha_n d(Sy_n,p)\\ &\leq(1-\alpha_n)d(x_n,p)+\alpha_n d(y_n,p)\\ &\leq(1-\alpha_n)d(x_n,p)+\alpha_n d(x_n,p)\\ &=d(x_n,p). \end{aligned}$$

Hence $\{x_n\}$ is Fejér monotone with respect to $F.$

Lemma 3.2.4. Let K be a nonempty, closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and T,S,Q be three nonexpansive self mappings on K such that $d(x_n,Sx_n)\leq d(Tx_n,Sx_n)$ and $F\neq\varnothing$. Let the sequence $\{x_n\}$ be as defined in (3.2.1). Then

$$\lim_{n\to\infty} d(x_n,Tx_n)=\lim_{n\to\infty} d(x_n,Sx_n)=\lim_{n\to\infty} d(x_n,Qx_n)=0.$$

Proof. Let $p\in F$. By Lemma 3.2.3, it follows that $\lim_{n\to\infty} d(x_n,p)$ exists. We may assume that $\lim_{n\to\infty} d(x_n,p)=r$. The case $r=0$ is trivial. Next, we deal with the case $r>0$. By (3.2.2), we obtain

$$\begin{aligned}\limsup_{n\to\infty} d(Sy_n,p)&\leq\limsup_{n\to\infty} d(y_n,p)\\&\leq\lim_{n\to\infty} d(x_n,p)=r.\end{aligned}$$

Moreover, we have $\limsup\limits_{n\to\infty} d(Tx_n,p)\leq r$. Since

$$\lim_{n\to\infty} d(x_{n+1},p)=\lim_{n\to\infty} d(W(Tx_n,Sy_n,\alpha_n),p)=r,$$

Lemma 2.2.10 gives

$$\lim_{n\to\infty} d(Tx_n,Sy_n)=0. \qquad \textbf{(3.2.3)}$$

Next

$$\begin{aligned}d(x_{n+1},p)&\leq(1-\alpha_n)d(Tx_n,p)+\alpha_n d(Sy_n,p)\\&\leq(1-\alpha_n)d(Tx_n,Sy_n)+(1-\alpha_n)d(Sy_n,p)+\alpha_n d(Sy_n,p)\\&\leq d(y_n,p)+(1-\alpha_n)d(Tx_n,Sy_n)\end{aligned}$$

yields that $\liminf_{n\to\infty} d(y_n,p)\geq r$. But by (3.2.2), $\limsup_{n\to\infty} d(y_n,p)\leq r$. Hence

$$\lim_{n\to\infty} d(y_n,p)=\lim_{n\to\infty} d(W(x_n,Qx_n,\beta_n),p)=r.$$

Since $\limsup_{n\to\infty} d(Qx_n,p)\leq r$ and $\lim_{n\to\infty} d(x_n,p)=r$, Lemma 2.2.10 guarantees

$$\lim_{n\to\infty} d(x_n,Qx_n)=0. \qquad \textbf{(3.2.4)}$$

By virtue of (3.2.4), we get

$$\begin{aligned} d(Sx_n, Sy_n) &\le d(x_n, y_n) \\ &= d(x_n, W(x_n, Qx_n, \beta_n)) \\ &\le \beta_n d(x_n, Qx_n) \to 0 \text{ as } n \to \infty. \end{aligned} \qquad \textbf{(3.2.5)}$$

From the hypothesis $d(x_n, Sx_n) \le d(Tx_n, Sx_n)$, we have

$$\begin{aligned} d(x_n, Sx_n) &\le d(Tx_n, Sx_n) \\ &\le d(Tx_n, Sy_n) + d(Sy_n, Sx_n). \end{aligned}$$

It follows from (3.2.3) and (3.2.5) that $\lim_{n\to\infty} d(x_n, Sx_n) = 0.$ Since

$$d(x_n, Tx_n) \le d(x_n, Sx_n) + d(Sx_n, Sy_n) + d(Sy_n, Tx_n),$$

we conclude that $\lim_{n\to\infty} d(x_n, Tx_n) = 0.$ The proof is completed.

Now we prove the Δ-convergence of the iteration process defined by (3.2.1) in a hyperbolic space.

Theorem 3.2.5. Let K, X, T, S, Q and $\{x_n\}$ be the same as in Lemma 3.2.4. Then the sequence $\{x_n\}$ Δ-converges to some $p \in F.$

Proof. It follows from Lemma 3.2.3 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F.$ This implies that the sequence $\{x_n\}$ is bounded. Hence $\{x_n\}$ has a Δ-convergent subsequence. We now prove that every Δ-convergent subsequence of $\{x_n\}$ has a unique Δ-limit in $F.$ Let u and v be the Δ-limits of the subsequences $\{u_n\}$ and $\{v_n\}$ of $\{x_n\}$, respectively. Then, $A(\{u_n\}) = \{u\}$ and $A(\{v_n\}) = \{v\}$. By Lemma 3.2.4, we have

$$\lim_{n\to\infty} d(u_n, Tu_n) = \lim_{n\to\infty} d(u_n, Su_n) = \lim_{n\to\infty} d(u_n, Qu_n) = 0. \qquad \textbf{(3.2.6)}$$

We claim that $u \in F.$ So, we calculate

$$\begin{aligned} d(Tu, u_n) &\le d(Tu, Tu_n) + d(Tu_n, u_n) \\ &\le d(u, u_n) + d(Tu_n, u_n). \end{aligned}$$

Taking limsup on both sides of the above inequality and using (3.2.6), we have

$$r(Tu,\{u_n\}) = \limsup_{n\to\infty} d(Tu,u_n) \le \limsup_{n\to\infty} d(u,u_n) = r(u,\{u_n\}).$$

By the uniqueness of asymptotic centers implies that $Tu = u.$ A similar argument shows that $Su = u$ and $Qu = u$. This means that $u \in F$. By the same method, we can also prove that $v \in F$. Finally, we claim that $u = v.$ If not, then by the uniqueness of asymptotic centers, we have

$$\begin{aligned}\limsup_{n\to\infty} d(x_n,u) &= \limsup_{n\to\infty} d(u_n,u)\\ &< \limsup_{n\to\infty} d(u_n,v)\\ &= \limsup_{n\to\infty} d(x_n,v)\\ &= \limsup_{n\to\infty} d(v_n,v)\\ &< \limsup_{n\to\infty} d(v_n,u)\\ &= \limsup_{n\to\infty} d(x_n,u).\end{aligned}$$

a contradiction and hence $u = v \in F.$ Consequently, $\{x_n\}$ Δ-converges to a point in F.

Next we discuss the strong convergence of the iteration process defined by (3.2.1) in a hyperbolic space.

Theorem 3.2.6. Let K,X,T,S,Q and $\{x_n\}$ be the same as in Lemma 3.2.4. Then $\{x_n\}$ converges strongly to some $p \in F$ if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F) = 0$. It follows from Lemma 3.2.3 that $\lim_{n\to\infty} d(x_n,F)$ exists. Thus by hypothesis, $\lim_{n\to\infty} d(x_n,F) = 0.$ Again by Lemma 3.2.3, $\{x_n\}$ is Fejér monotone with respect to $F.$ Thus Lemma 1.2.23 implies that $\{x_n\}$ converges strongly to a point p in $F.$

Remark 3.2.7. In Theorem 3.2.6, the condition $\liminf_{n\to\infty} d(x_n,F) = 0$ may be replaced with $\limsup_{n\to\infty} d(x_n,F) = 0$.

Example 3.2.8. Let $\mathbb{R}$ be the real line with the usual absolute metric and $T,S,Q:\mathbb{R}\to\mathbb{R}$ be three mappings defined by $T(x)=1-x$, $S(x)=\frac{2x+1}{4}$ and $Q(x)=\frac{1}{2}$. It is noticed in [47, p.10] that T and S satisfy the condition $d(x_n,Sx_n)\le d(Tx_n,Sx_n)$. Additionally T,S and Q are nonexpansive mappings. Clearly, $F=\left\{\frac{1}{2}\right\}$. Set $\alpha_n=\frac{n}{2n+1}$ and $\beta_n=\frac{2n}{3n+1}$ for all $n\in\mathbb{N}$. Thus, the conditions of Lemma 3.2.4 are fulfilled. Therefore the results of Theorem 3.2.5 and Theorem 3.2.6 can be easily seen.

Khan and Fukhar-ud-din [49] introduced the so-called condition (A') for two mappings and gave an improved version of it in [50] as follows.

Definition 3.2.9. Two mappings $T,S:K\to K$ with $F\neq\emptyset$ are said to satisfy the condition (A') if there exists a non-decreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0, f(r)>0$ for all $r\in(0,\infty)$ such that either $d(x,Tx)\ge f(d(x,F))$ or $d(x,Sx)\ge f(d(x,F))$ for all $x\in K$.

This condition becomes condition (I) whenever $S=T$. We can modify this definition for three mappings as follows.

Definition 3.2.10. Let T,S and Q be three nonexpansive self mappings on K with $F\neq\emptyset$. These mappings are said to satisfy condition (B) if there exists a non-decreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0, f(r)>0$ for all $r\in(0,\infty)$ such that $d(x,Tx)\ge f(d(x,F))$ or $d(x,Sx)\ge f(d(x,F))$ or $d(x,Qx)\ge f(d(x,F))$ for all $x\in K$.

The condition (B) is reduced to the condition (A') when $Q=T$. We use the condition (B) to study strong convergence of $\{x_n\}$ defined in (3.2.1).

Theorem 3.2.11. Let K,X,T,S,Q and $\{x_n\}$ be the same as in Lemma 3.2.4. If T,S,Q satisfy the condition (B), then $\{x_n\}$ converges strongly to a point in F.

Proof. By Lemma 3.2.3, $\lim_{n\to\infty} d(x_n,F)$ exists. Also, by Lemma 3.2.4,

$$\lim_{n\to\infty} d(x_n,Tx_n)=\lim_{n\to\infty} d(x_n,Sx_n)=\lim_{n\to\infty} d(x_n,Qx_n)=0.$$

By using the condition (B), we get $\lim_{n\to\infty} f(d(x_n,F))=0$. Since f is a non-decreasing function with $f(0)=0$, it follows that $\lim_{n\to\infty} d(x_n,F)=0$. Therefore Theorem 3.2.6 implies that $\{x_n\}$ converges strongly to a point in F.

Now, we obtain the following strong convergence theorem.

Theorem 3.2.12. Under the assumptions of Lemma 3.2.4, if one of the mappings T,S and Q is demi-compact or K is compact, then $\{x_n\}$ converges strongly to a point in F.

Proof. It is clear that the condition (B) is weaker than both the compactness of K and the demi-compactness of one of the nonexpansive mappings T,S and Q. Therefore we have the result by Theorem 3.2.11.

Remark 3.2.13.

(i) Theorems 3.2.5, 3.2.6, 3.2.11 extend the corresponding results of Khan and Abbas [48] from CAT(0) space to the general setup of uniformly convex hyperbolic spaces.

(ii) Theorems 3.2.5, 3.2.6, 3.2.11, 3.2.12 contain the corresponding theorems proved for the Ishikawa iteration when $T=I,S=Q$, for the Mann iteration when $T=Q=I$ and for the Picard iteration when $T=S,Q=I$. Then these theorems improve and generalize some results of Dhompongsa and Panyanak [24].

If we take $Q = T$ in Theorems 3.2.5, 3.2.6, 3.2.11, 3.2.12 we get the following corollary, yet it is new in the literature.

Corollary 3.2.14. Let K be a nonempty, closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and T, S be two nonexpansive self mappings on K such that $F \neq \varnothing$. Let the sequence $\{x_n\}$ be defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = W(Tx_n, Sy_n, \alpha_n), \\ y_{n+1} = W(x_n, Tx_n, \beta_n), \ n \in \mathbb{N}. \end{cases} \tag{3.2.7}$$

(i) Then the sequence $\{x_n\}$ Δ-converges to some $p \in F$.

(ii) Then $\{x_n\}$ converges strongly to some $p \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$.

(iii) If T and S satisfy the condition (A'), then $\{x_n\}$ converges strongly to a point in F.

(iv) If one of the mappings T and S is demi-compact or K is compact, then $\{x_n\}$ converges strongly to a point in F.

Remark 3.2.15. Note that the iteration process (3.2.7) has two nonexpansive mappings T, S and the condition $d(x_n, Sx_n) \leq d(Tx_n, Sx_n)$ doesn't need to get convergence of this iteration.

CHAPTER 4. SOME CONVERGENCE RESULTS FOR MAPPINGS SATISFYING CONDITION (C)

In this section, some strong and Δ-convergence theorems for mappings satisfying condition (C) are proved in CAT(0) spaces.

4.1. The Strong and Δ-Convergence of S-Iteration in CAT(0) Spaces

In this subsection, we prove the strong and Δ-convergence theorems of S-iteration process for mappings satisfying condition (C) in a CAT(0) space.

Khan and Abbas [48] modified the S-iteration process in CAT(0) spaces for nonexpansive mappings as follows.

Definition 4.1.1. Let K be a nonempty, closed, convex subset of a CAT(0) space X and $T:K\to K$ be a nonexpansive mapping. The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n \oplus \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n \oplus \beta_n Tx_n,\ n\in\mathbb{N}, \end{cases} \tag{4.1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences such that $a\le \alpha_n, \beta_n \le b$ for all $n\in\mathbb{N}$ and for some $a,b\in(0,1)$.

Lemma 4.1.2. [51] Let K be a closed convex subset of a complete CAT(0) space X and $T:K\to K$ be a mapping satisfying condition (C). Then,

$$d(x,Ty)\le 3d(x,Tx)+d(x,y) \text{ for all } x,y\in K.$$

Before proving strong and Δ-convergence theorems, we need the following lemmas.

Lemma 4.1.3. Let K be a nonempty, closed, convex subset of a complete CAT(0) space X, $T:K\to K$ be a mapping satisfying condition (C) and $\{x_n\}$ be a sequence defined by the iteration process (4.1.1). If $F(T)\neq\varnothing$, then $\lim_{n\to\infty} d(x_n,p)$ exists for all $p\in F(T)$.

Proof. Set $y_n=(1-\beta_n)x_n\oplus\beta_n Tx_n$, $n\in\mathbb{N}$. Since T is a mapping satisfying condition (C) and $p\in F(T)$, we have $d(Ty_n,p)\le d(y_n,p)$ and $d(Tx_n,p)\le d(x_n,p)$ for all $n\in\mathbb{N}$. By combining these inequalities and Lemma 2.1.15(i), we get

$$\begin{aligned} d(x_{n+1},p) &= d((1-\alpha_n)Tx_n\oplus\alpha_n Ty_n,p)\\ &\le (1-\alpha_n)d(Tx_n,p)+\alpha_n d(Ty_n,p)\\ &\le (1-\alpha_n)d(x_n,p)+\alpha_n d(y_n,p). \end{aligned} \tag{4.1.2}$$

Also,

$$\begin{aligned} d(y_n,p) &= d((1-\beta_n)x_n\oplus\beta_n Tx_n,p)\\ &\le (1-\beta_n)d(x_n,p)+\beta_n d(Tx_n,p)\\ &\le (1-\beta_n)d(x_n,p)+\beta_n d(x_n,p)\\ &= d(x_n,p). \end{aligned} \tag{4.1.3}$$

Using (4.1.2) and (4.1.3), we have

$$d(x_{n+1},p)\le d(x_n,p).$$

This implies $d(x_n,p)$ is non-increasing and bounded below, and so $\lim_{n\to\infty} d(x_n,p)$ exists for all $p\in F$. This completes the proof.

Lemma 4.1.4. Let $X,K,T,\{x_n\}$ satisfy the hypotheses of Lemma 4.1.3. Then, $F(T)$ is nonempty if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n,Tx_n)=0$.

Proof. Suppose that $F(T)$ is nonempty and $p\in F(T)$. Then, by Lemma 4.1.3, $\lim_{n\to\infty} d(x_n,p)$ exists and $\{x_n\}$ is bounded. Set

$$\lim_{n\to\infty} d(x_n,p)=c \tag{4.1.4}$$

and $y_n = (1-\beta_n)x_n \oplus \beta_n Tx_n$, for $n \in \mathbb{N}$. We first prove that $\lim_{n\to\infty} d(y_n, p) = c$. By (4.1.2), we have

$$d(x_{n+1}, p) \leq (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p).$$

This gives that

$$\alpha_n d(x_n, p) \leq d(x_n, p) + \alpha_n d(y_n, p) - d(x_{n+1}, p)$$

or

$$\begin{aligned} d(x_n, p) &\leq d(y_n, p) + \frac{1}{\alpha_n}\left[d(x_n, p) - d(x_{n+1}, p)\right] \\ &\leq d(y_n, p) + \frac{1}{a}\left[d(x_n, p) - d(x_{n+1}, p)\right]. \end{aligned}$$

This implies that

$$c \leq \liminf_{n\to\infty} d(y_n, p). \tag{4.1.5}$$

By (4.1.3) and (4.1.4), $\limsup_{n\to\infty} d(y_n, p) \leq c$. By combining this inequality and (4.1.5), we get

$$\lim_{n\to\infty} d(y_n, p) = c. \tag{4.1.6}$$

Next, by Lemma 2.1.15(ii),

$$\begin{aligned} d(y_n, p)^2 &= d((1-\beta_n)x_n \oplus \beta_n Tx_n, p)^2 \\ &\leq (1-\beta_n)d(x_n, p)^2 + \beta_n d(Tx_n, p)^2 - \beta_n(1-\beta_n)d(x_n, Tx_n)^2 \\ &\leq d(x_n, p)^2 - \beta_n(1-\beta_n)d(x_n, Tx_n)^2. \end{aligned}$$

Thus

$$\beta_n(1-\beta_n)d(x_n, Tx_n)^2 \leq d(x_n, p)^2 - d(y_n, p)^2$$

so that

$$\begin{aligned} d(x_n, Tx_n)^2 &\leq \frac{1}{\beta_n(1-\beta_n)}\left[d(x_n, p)^2 - d(y_n, p)^2\right] \\ &\leq \frac{1}{a(1-b)}\left[d(x_n, p)^2 - d(y_n, p)^2\right]. \end{aligned}$$

Using (4.1.4) and (4.1.6), we get $\limsup_{n\to\infty} d(x_n, Tx_n) \leq 0.$ Hence,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0.$ Let $A(\{x_n\}) = \{x\}.$ Then $x \in K$, by Lemma 2.1.22. Since T is a mapping satisfying condition (C), we have, by Lemma 4.1.2,

$$d(x_n, Tx) \leq 3d(x_n, Tx_n) + d(x_n, x),$$

which implies

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, Tx) &\leq \limsup_{n\to\infty} \left[3d(x_n, Tx_n) + d(x_n, x)\right] \\ &= \limsup_{n\to\infty} d(x_n, x).\end{aligned}$$

By the uniqueness of asymptotic centers, we get $Tx = x.$ Therefore, x is a fixed point of $T.$ This completes the proof.

Now, we prove the Δ-convergence theorem of S-iteration process in CAT(0) space.

Theorem 4.1.5. Let $X, K, T, \{x_n\}$ satisfy the hypotheses of Lemma 4.1.3 with $F(T) \neq \varnothing.$ Then $\{x_n\}$ Δ-converges to a fixed point of $T.$

Proof. Lemma 4.1.4 guarantees that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. We claim that $W_\Delta(x_n) \subseteq F(T).$ Let $u \in W_\Delta(x_n).$ Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}.$ By Lemmas 2.1.21 and 2.1.22, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in K.$ Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$ and T is a mapping satisfying condition (C), then, by Lemma 4.1.2,

$$d(v_n, Tv) \leq 3d(v_n, Tv_n) + d(v_n, v).$$

By taking limsup and using the Opial property, we obtain $v \in F(T).$ By Lemma 4.1.3, $\{d(x_n, v)\}$ converges. Then, by using Lemma 2.1.23, we have $u = v \in F(T).$

This shows that $W_\Delta(x_n) \subseteq F(T)$. Next, we show that $W_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Since $u \in W_\Delta(x_n) \subseteq F(T)$, $\{d(x_n, u)\}$ converges. Again, by using Lemma 2.1.23, $x = u \in F(T)$. Therefore, $W_\Delta(x_n) = \{x\}$. As a result, the iteration sequence $\{x_n\}$ Δ-converges to a fixed point of T.

We briefly discuss the strong convergence of S-iteration process in a CAT(0) space setting in Theorems 4.1.6 and 4.1.7.

Theorem 4.1.6. ([52, Theorem 3.4]) Let $X, K, T, \{x_n\}$ satisfy the hypotheses of Lemma 4.1.3 and $T: K \to K$ be a mapping satisfying condition (I) with $F(T) \neq \varnothing$. Then, $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 4.1.7. Let $X, K, T, \{x_n\}$ satisfy the hypotheses of Lemma 4.1.3 with $F(T) \neq \varnothing$ and K be compact subset of X. Then, $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Lemma 4.1.4 guarentees that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z \in K$. By Lemma 4.1.2, we have

$$d(x_{n_k}, Tz) \leq 3d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, z) \text{ for all } k \in \mathbb{N}.$$

Letting $k \to \infty$, we have $\{x_{n_k}\}$ converges to Tz. This implies $Tz = z$, that is $z \in F(T)$. By Lemma 4.1.3, we have $\lim_{n\to\infty} d(x_n, z)$ exists, thus z is the strong limit of the sequence $\{x_n\}$. As a result, the iteration sequence $\{x_n\}$ converges strongly to a fixed point of T.

4.2. The Strong and Δ-Convergence of New Three-Step Iteration in CAT(0) Spaces

In this subsection, we apply a new three-step iteration process into a CAT(0) space and present some results on the strong and Δ-convergence of the new three-step iteration for mappings satisfying condition (C) in a CAT(0) space.

Karakaya et. al. [53] established a new three-step iteration method in a Banach space. We modified this iteration process into a CAT(0) space as follows.

Definition 4.2.1. Let K be a nonempty, closed, convex subset of a CAT(0) space X and $T: K \to K$ be a mapping satisfying condition (C). The new three-step iteration sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n - \beta_n) y_n \oplus \alpha_n T y_n \oplus \beta_n T z_n, \\ y_n = (1-a_n - b_n) z_n \oplus a_n T z_n \oplus b_n T x_n, \\ z_n = (1-c_n) x_n \oplus c_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{4.2.1}$$

where $\{a_n + b_n\}_{n=1}^{\infty}, \{\alpha_n + \beta_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty} \subset [0,1]$.

Remark 4.2.2. [53] Some special cases of the new three-step iteration process given by (4.2.1), as follows.

(i) If $c_n = 1$ and $\alpha_n = \beta_n = a_n = b_n = 0$ for all $n \in \mathbb{N}$, then (4.2.1) is reduced to the Picard iteration.

(ii) If $\alpha_n = \beta_n = a_n = b_n = 0$ for all $n \in \mathbb{N}$, then (4.2.1) is reduced to the Mann iteration.

(iii) If $c_n = b_n = 0$ and $\alpha_n + \beta_n = 1$ for all $n \in \mathbb{N}$, then (4.2.1) is reduced to the S-iteration.

(iv) If $\alpha_n = a_n = b_n = 0$ for all $n \in \mathbb{N}$, then (4.2.1) is reduced to the new two-step iteration.

(v) If $\beta_n = b_n = 0$ for all $n \in \mathbb{N}$, then (4.2.1) is reduced to the SP-iteration.

Lemma 4.2.3. ([54, Lemma 2.5 (1)]) Let X be a CAT(0) space. Then

$$d(z_\alpha, z) \leq \sum_{i=1}^{n} \alpha_i d(z_i, z),$$

where $\alpha_1, \alpha_2, ..., \alpha_n \in [0,1]$ with $\sum_{i=1}^{n} \alpha_i = 1$, $z, z_i \in X$, $1 \leq i \leq n$ and $z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus ... \oplus \alpha_n z_n$.

We give a basic property of the new three-step iterative sequence $\{x_n\}$ defined by (4.2.1) for mappings satisfying condition (C).

Lemma 4.2.4. Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T : K \to K$ be a mapping satisfying condition (C) with $F(T) \neq \varnothing$. Assume that $\{x_n\}$ be a sequence defined by (4.2.1) such that $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty} \subset [0,1]$. Then $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$.

Proof. Since T is a mapping satisfying condition (C) and $p \in F(T)$, we have

$$\frac{1}{2} d(p,Tp) = 0 \leq d(p,z) \text{ for all } z \in K.$$

It implies that $d(Tp,Tz) \leq d(p,z)$ for all $z \in K$. Then, by Lemma 2.1.15(i), we get

$$\begin{aligned} d(z_n, p) &= d((1-c_n)x_n \oplus c_n T x_n, p) \\ &\leq (1-c_n) d(x_n, p) + c_n d(Tx_n, p) \\ &\leq (1-c_n) d(x_n, p) + c_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \qquad \textbf{(4.2.2)}$$

Also, by Lemma 4.2.3, we obtain

$$\begin{aligned}
d(y_n,p) &= d((1-a_n-b_n)z_n \oplus a_nTz_n \oplus b_nTx_n,p)\\
&\le (1-a_n-b_n)d(z_n,p)+a_nd(Tz_n,p)+b_nd(Tx_n,p)\\
&\le (1-a_n-b_n)d(z_n,p)+a_nd(z_n,p)+b_nd(x_n,p)\\
&\le (1-b_n)d(x_n,p)+b_nd(x_n,p)\\
&= d(x_n,p). \qquad \textbf{(4.2.3)}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
d(x_{n+1},p) &= d((1-\alpha_n-\beta_n)y_n \oplus \alpha_nTy_n \oplus \beta_nTz_n,p)\\
&\le (1-\beta_n)d(y_n,p)+\beta_nd(z_n,p). \qquad \textbf{(4.2.4)}
\end{aligned}$$

By combining (4.2.2), (4.2.3) and (4.2.4), we obtain

$$d(x_{n+1},p)\le d(x_n,p). \qquad \textbf{(4.2.5)}$$

This implies that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in F(T)$.

By using Lemmas 2.1.16 and 4.2.4, we prove the other property of the iterative sequence (4.2.1) for mappings satisfying condition (C).

Lemma 4.2.5. Let K be a nonempty closed convex subset of a complete CAT(0) space X, $T:K\to K$ be a mapping satisfying condition (C) with $F(T)\neq\varnothing$ and $\{x_n\}$ be a sequence defined by (4.2.1). If $\{a_n\}_{n=1}^{\infty},\{b_n\}_{n=1}^{\infty},\{\alpha_n\}_{n=1}^{\infty}\subset[0,1]$ and $\{\beta_n\}_{n=1}^{\infty},\{c_n\}_{n=1}^{\infty}\subset[a,b]$ for some $a,b\in(0,1)$, then $\lim_{n\to\infty} d(x_n,Tx_n)=0$.

Proof. Let $p\in F(T)$. By Lemma 4.2.4, there exists at least one $c\in[0,\infty)$ such that

$$\lim_{n\to\infty} d(x_n,p)=c. \qquad \textbf{(4.2.6)}$$

The case $c=0$ is trivial. Next, we deal with the case $c>0$. First, we prove that $\lim_{n\to\infty} d(z_n,p)=c.$ By (4.2.3) and (4.2.4), we have

$$\begin{aligned}
d(x_{n+1},p) &\le (1-\beta_n)d(y_n,p)+\beta_nd(z_n,p)\\
&\le (1-\beta_n)d(x_n,p)+\beta_nd(z_n,p).
\end{aligned}$$

It follows that

$$\beta_nd(x_n,p)\le d(x_n,p)+\beta_nd(z_n,p)-d(x_{n+1},p)$$

or

$$\begin{aligned} d(x_n,p) &\le d(z_n,p)+\frac{1}{\beta_n}\left[d(x_n,p)-d(x_{n+1},p)\right] \\ &\le d(z_n,p)+\frac{1}{a}\left[d(x_n,p)-d(x_{n+1},p)\right]. \end{aligned}$$

This gives $c \le \liminf_{n\to\infty} d(z_n,p)$. Also, from (4.2.2) and (4.2.6), we obtain $\limsup_{n\to\infty} d(z_n,p) \le c$. Then we get $\lim_{n\to\infty} d(z_n,p)=c$. Since $d(Tx_n,p)\le d(x_n,p)$ for all $n\in\mathbb{N}$, by (4.2.6), we obtain

$$\limsup_{n\to\infty} d(Tx_n,p)\le c.$$

Also, we have

$$c=\lim_{n\to\infty} d(z_n,p)=\lim_{n\to\infty} d((1-c_n)x_n \oplus c_nTx_n,p).$$

By Lemma 2.1.16, we can conclude that $\lim_{n\to\infty} d(x_n,Tx_n)=0$.

Theorem 4.2.6. ([55, Theorem 8]) Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T:K\to K$ be a mapping satisfying condition (C) with $F(T)\ne\varnothing$. Assume that $\{x_n\}$ be a sequence defined by (4.2.1) such that $\{a_n\}_{n=1}^{\infty},\{b_n\}_{n=1}^{\infty},\{\alpha_n\}_{n=1}^{\infty}\subset[0,1]$ and $\{\beta_n\}_{n=1}^{\infty},\{c_n\}_{n=1}^{\infty}\subset[a,b]$ for some $a,b\in(0,1)$. Then $\{x_n\}$ Δ-converges to an element of $F(T)$.

We now discuss the strong convergence of the new three-step iteration for mappings satisfying condition (C) in a CAT(0) space setting.

Theorem 4.2.7. Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T:K\to K$ be a mapping satisfying condition (C) with $F(T)\ne\varnothing$. Let $\{x_n\}$ be a sequence defined by (4.2.1) such that $\{a_n\}_{n=1}^{\infty},\{b_n\}_{n=1}^{\infty},\{\alpha_n\}_{n=1}^{\infty},\{\beta_n\}_{n=1}^{\infty},\{c_n\}_{n=1}^{\infty}\subset[0,1]$. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n,F(T))=0$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. It follows from Lemma 4.2.4 that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by hypothesis, $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Again by Lemma 4.2.4, $\{x_n\}$ is is Fejér monotone with respect to $F(T)$. Since T is quasi-nonexpansive, it is known by [29, Lemma 1.1] that $F(T)$ is always closed. Thus Lemma 1.2.23 implies that $\{x_n\}$ converges strongly to a point p in $F(T)$.

Remark 4.2.8.

(i) In Theorem 4.2.7, the condition $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ may be replaced with $\limsup_{n\to\infty} d(x_n, F(T)) = 0$.

(ii) Theorem 4.2.7 generalizes Theorem 2 of [56] since every nonexpansive mapping satisfies condition (C) and (4.2.1) is reduced to the SP-iteration when $\beta_n = b_n = 0$ for all $n \in \mathbb{N}$.

(iii) Theorem 4.2.7. is an extension of Theorem 2 of [48] since every nonexpansive mapping satisfies condition (C) and (4.2.1) is reduced to the S-iteration when $c_n = b_n = 0$ and $\alpha_n + \beta_n = 1$ for all $n \in \mathbb{N}$.

Theorem 4.2.9. Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T : K \to K$ be a mapping satisfying condition (C) with $F(T) \neq \varnothing$. For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined by (4.2.1) such that $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty} \subset [0,1]$ and $\{\beta_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty} \subset [a,b]$ for some $a, b \in (0,1)$.

(i) If T is demi-compact, then $\{x_n\}$ converges strongly to a fixed point of T.

(ii) If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. (i) It follows from Lemma 4.2.4 that $\{x_n\}$ is a bounded sequence. Also, by Lemma 4.2.5, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, by the demi-compactness of T, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z \in K$ as $k \to \infty$. The rest of the proof closely follows the proof of Theorem 4.1.7.

(ii) It follows from Lemma 4.2.4 that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Further, by the condition (I) and Lemma 4.2.5, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. The conclusion now follows from Theorem 4.2.7.

Remark 4.2.11. If we take $\alpha_n = a_n = b_n = 0$ for all $n \in \mathbb{N}$, the new three-step iteration is reduced to the new two-step iteration. Then theorems in this subsection contain the corresponding theorems proved for the new two-step iteration.

4.3. The Strong and Δ-Convergence Theorems for Nonself Mappings on CAT(0) Spaces

In this subsection, we study the S-iteration and the Noor iteration processes for nonself mappings satisfying condition (E) on a CAT(0) space.

In 2011, Falset et. al. [57] introduced condition (E) as follows.

Definition 4.3.1. ([57, Definition 2]) Let K be a bounded closed convex subset of a complete CAT(0) space X. A mapping $T: K \to X$ is called to satisfy condition (E_μ) on K, if there exists $\mu \geq 1$ such that

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y),$$

for all $x, y \in K$. The mapping T is said to satisfy condition (E) on K whenever T satisfies condition (E_μ) for some $\mu \geq 1$.

Remark 4.3.2. [57] Every mapping satisfying condition(C) satisfies condition (E).

Now, we apply the S-iteration and the Noor iteration processes in a CAT(0) space for a nonself mapping satisfying condition (E) as follows.

Definition 4.3.3. Let K be a nonempty closed convex subset of a complete CAT(0) space X with the nearest point projection P from X into K. Let $T:K\to X$ be a nonself mapping satisfying condition (E) with $F(T)\neq\emptyset$. Then, we give the following iteration processes;

$$\begin{cases} x_1\in K, \\ x_{n+1}=P((1-\alpha_n)Tx_n\oplus\alpha_n Ty_n) \\ y_n=P((1-\beta_n)x_n\oplus\beta_n Tx_n),\ n\in\mathbb{N}, \end{cases} \tag{4.3.1}$$

and

$$\begin{cases} x_1\in K, \\ x_{n+1}=P((1-\alpha_n)x_n\oplus\alpha_n Ty_n), \\ y_n=P((1-\beta_n)x_n\oplus\beta_n Tz_n), \\ z_n=P((1-\gamma_n)x_n\oplus\gamma_n Tx_n),\ n\in\mathbb{N}, \end{cases} \tag{4.3.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[\varepsilon,1-\varepsilon]$ for some $\varepsilon\in(0,1)$.

Lemma 4.3.4. ([19, Proposition 2.4]) Let K be a convex subset of X which is complete in the induced metric. Then for every $x\in X$, there exists a unique point $P(x)\in K$ such that $d(x,P(x))=\inf\{d(x,y):y\in K\}$. Moreover, the mapping $x\to P(x)$ is nonexpansive retract from X onto K.

Lemma 4.3.5. Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T:K\to X$ be a nonself mapping satisfying condition (E) with $F(T)\neq\emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\varepsilon,1-\varepsilon]$ for some $\varepsilon\in(0,1)$ and $\{x_n\}$ be defined by the iteration process (4.3.1). Then,

(i) $\lim_{n\to\infty} d(x_n,x^*)$ exists for all $x^*\in F(T)$.

(ii) $\lim_{n\to\infty} d(x_n,Tx_n)=0$.

Proof. (i) Suppose that $F(T)$ is nonempty and $x^*\in F(T)$. Since the nearest point projection $P:X\to K$ is nonexpansive by Lemma 4.3.4 and T is a mapping satisfying condition (E), we have

$$\begin{aligned} d(y_n,x^*) &= d(P((1-\beta_n)x_n \oplus \beta_n Tx_n), Px^*) \\ &\le d((1-\beta_n)x_n \oplus \beta_n Tx_n), x^*) \\ &\le (1-\beta_n)d(x_n,x^*) + \beta_n d(Tx_n,x^*) \\ &\le (1-\beta_n)d(x_n,x^*) + \beta_n(\mu d(Tx^*,x^*) + d(x_n,x^*)) \\ &= (1-\beta_n)d(x_n,x^*) + \beta_n d(x_n,x^*) \\ &= d(x_n,x^*). \end{aligned} \quad \textbf{(4.3.3)}$$

Also, we have

$$\begin{aligned} d(x_{n+1},x^*) &= d(P((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n), Px^*) \\ &\le d((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n, x^*) \\ &\le (1-\alpha_n)d(Tx_n,x^*) + \alpha_n d(Ty_n,x^*) \\ &\le (1-\alpha_n)(\mu d(Tx^*,x^*) + d(x_n,x^*)) + \alpha_n(\mu d(Tx^*,x^*) + d(y_n,x^*)) \\ &= (1-\alpha_n)d(x_n,x^*) + \alpha_n d(y_n,x^*). \end{aligned} \quad \textbf{(4.3.4)}$$

Using (4.3.3) and (4.3.4), we obtain

$$d(x_{n+1},x^*) \le d(x_n,x^*).$$

This implies that $\lim_{n\to\infty} d(x_n,x^*)$ exists for all $x^* \in F(T)$. This completes the proof of part (i).

(ii) Let

$$\lim_{n\to\infty} d(x_n,x^*) = c. \quad \textbf{(4.3.5)}$$

If $c=0$, by the condition (E) for some $\mu \ge 1$, we obtain

$$\begin{aligned} d(x_n,Tx_n) &\le d(x_n,x^*) + d(x^*,Tx_n) \\ &\le d(x_n,x^*) + \mu d(x^*,Tx^*) + d(x_n,x^*). \end{aligned}$$

Therefore, $\lim_{n\to\infty} d(x_n,Tx_n) = 0$. Let $c > 0$. Firstly, we will prove that $\lim_{n\to\infty} d(y_n,x^*) = c$. From (4.3.4), we have

$$d(x_{n+1},x^*) \le (1-\alpha_n)d(x_n,x^*) + \alpha_n d(y_n,x^*).$$

This gives that

$$\alpha_n d(x_n,x^*) \le d(x_n,x^*) + \alpha_n d(y_n,x^*) - d(x_{n+1},x^*)$$

or

$$d(x_n,x^*)\le d(y_n,x^*)+\frac{1}{\alpha_n}\left[d(x_n,x^*)-d(x_{n+1},x^*)\right]$$
$$\le d(y_n,x^*)+\frac{1}{\varepsilon}\left[d(x_n,x^*)-d(x_{n+1},x^*)\right].$$

This shows

$$c=\liminf_{n\to\infty} d(x_n,x^*)\le \liminf_{n\to\infty} d(y_n,x^*)+\lim_{n\to\infty}\frac{1}{\varepsilon}\left[d(x_n,x^*)-d(x_{n+1},x^*)\right]$$

so that

$$c\le \liminf_{n\to\infty} d(y_n,x^*). \qquad \textbf{(4.3.6)}$$

By (4.3.3) and (4.3.5), we get $\limsup_{n\to\infty} d(y_n,x^*)\le c.$ By combining this inequality and (4.3.6), we obtain

$$\lim_{n\to\infty} d(y_n,x^*)=c. \qquad \textbf{(4.3.7)}$$

By Lemma 2.1.15(ii), we get

$$\begin{aligned}
d(y_n,x^*)^2 &= d(P((1-\beta_n)x_n\oplus\beta_nTx_n),Px^*)^2\\
&\le d((1-\beta_n)x_n\oplus\beta_nTx_n,x^*)^2\\
&\le (1-\beta_n)d(x_n,x^*)^2+\beta_nd(Tx_n,x^*)^2-\beta_n(1-\beta_n)d(x_n,Tx_n)^2\\
&\le (1-\beta_n)d(x_n,x^*)^2+\beta_n(\mu d(Tx^*,x^*)+d(x_n,x^*))^2-\beta_n(1-\beta_n)d(x_n,Tx_n)^2\\
&= (1-\beta_n)d(x_n,x^*)^2+\beta_nd(x_n,x^*)^2-\beta_n(1-\beta_n)d(x_n,Tx_n)^2\\
&= d(x_n,x^*)^2-\beta_n(1-\beta_n)d(x_n,Tx_n)^2.
\end{aligned}$$

Therefore,

$$\beta_n(1-\beta_n)d(x_n,Tx_n)^2\le d(x_n,x^*)^2-d(y_n,x^*)^2$$

so that

$$d(x_n,Tx_n)^2\le \frac{1}{\beta_n(1-\beta_n)}\left[d(x_n,x^*)^2-d(y_n,x^*)^2\right]$$
$$\le \frac{1}{\varepsilon^2}\left[d(x_n,x^*)^2-d(y_n,x^*)^2\right].$$

Using (4.3.5) and (4.3.7), we get $\limsup_{n\to\infty} d(x_n,Tx_n)\le 0.$ Hence, $\lim_{n\to\infty} d(x_n,Tx_n)=0.$ This completes the proof of part (ii).

Now, we give the Δ-convergence theorem of the S-iteration process in a CAT(0) space.

Theorem 4.3.6. Let $X,K,T,\{\alpha_n\},\{\beta_n\},\{x_n\}$ satisfy the hypotheses of Lemma 4.3.5. Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. By Lemma 4.3.5, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n,Tx_n)=0.$ We claim that $W_\Delta(x_n)\subseteq F(T).$ Let $u\in W_\Delta(x_n).$ Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\})=\{u\}.$ By Lemmas 2.1.21 and 2.1.22, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta-\lim_{n\to\infty} v_n = v\in K.$ Since T is a mapping satisfying condition (E) and we get

$$d(v_n,Tv)\le \mu d(v_n,Tv_n)+d(v_n,v)$$

for some $\mu\ge 1.$ Also $\lim_{n\to\infty} d(v_n,Tv_n)=0,$ we get

$$\begin{aligned}\limsup_{n\to\infty} d(v_n,Tv) &\le \limsup_{n\to\infty}(\mu d(v_n,Tv_n)+d(v_n,v))\\ &= \limsup_{n\to\infty} d(v_n,v).\end{aligned}$$

By using the uniqueness of asymptotic center, we obtain $v\in F(T).$ The rest of the proof closely follows the proof of Theorem 4.1.5.

We briefly discuss the strong convergence of the S-iteration process in a CAT(0) space.

Theorem 4.3.7. ([58, Theorem 3.4]) Let $X,K,T,\{\alpha_n\},\{\beta_n\},\{x_n\}$ satisfy the hypotheses of Lemma 4.3.5 and T be a mapping satisfying condition (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 4.3.8. Let $X, K, T, \{\alpha_n\}, \{\beta_n\}, \{x_n\}$ satisfy the hypotheses of Lemma 4.3.5 and K be a compact subset of X. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 4.3.5(ii), we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to z \in K$. Since T is a nonself mapping satisfying condition (E), we have

$$d(x_{n_k}, Tz) \leq \mu d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, z)$$

for some $\mu \geq 1$, for all $k \in \mathbb{N}$. Letting $k \to \infty$, we have $\{x_{n_k}\}$ converges to Tz. This implies $Tz = z$, that is $z \in F(T)$. By Lemma 4.3.5(i), we have $\lim_{n\to\infty} d(x_n, z)$ exists, thus z is the strong limit of the sequence $\{x_n\}$. Therefore the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Lemma 4.3.9. Let K be a nonempty closed convex subset of a complete CAT(0) space X and $T: K \to X$ be a nonself mapping satisfying condition (E) with $F(T) \neq \varnothing$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$ and $\{x_n\}$ be defined by the iteration process (4.3.2). Then,

(1) $\lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F(T)$.

(2) $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. (i) Suppose that $F(T)$ is nonempty and $x^* \in F(T)$. By Lemma 4.3.4, the nearest point projection $P: X \to K$ is nonexpansive mapping. By the condition (E) for some $\mu \geq 1$, we obtain

$$\begin{aligned}
d(z_n, x^*) &= d(P((1-\gamma_n)x_n \oplus \gamma_n T x_n), Px^*) \\
&\le d((1-\gamma_n)x_n \oplus \gamma_n T x_n), x^*) \\
&\le (1-\gamma_n) d(x_n, x^*) + \gamma_n d(Tx_n, x^*) \\
&\le (1-\gamma_n) d(x_n, x^*) + \gamma_n (\mu d(Tx^*, x^*) + d(x_n, x^*)) \\
&= (1-\gamma_n) d(x_n, x^*) + \gamma_n d(x_n, x^*) \\
&= d(x_n, x^*). \qquad \textbf{(4.3.9)}
\end{aligned}$$

Similarly, using by (4.3.9), we have

$$d(y_n, x^*) \le d(x_n, x^*). \qquad \textbf{(4.3.10)}$$

Also, we get

$$\begin{aligned}
d(x_{n+1}, x^*) &= d(P((1-\alpha_n)x_n \oplus \alpha_n T y_n), Px^*) \\
&\le d((1-\alpha_n)x_n \oplus \alpha_n T y_n, x^*) \\
&\le (1-\alpha_n) d(x_n, x^*) + \alpha_n d(Ty_n, x^*) \\
&\le (1-\alpha_n) d(x_n, x^*) + \alpha_n (\mu d(Tx^*, x^*) + d(y_n, x^*)) \\
&= (1-\alpha_n) d(x_n, x^*) + \alpha_n d(y_n, x^*) \qquad \textbf{(4.3.11)}
\end{aligned}$$

Using (4.3.10) and (4.3.11), we obtain $d(x_{n+1}, x^*) \le d(x_n, x^*)$. This implies that $\lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F(T)$. This completes the proof of part (i).

(ii) Let

$$\lim_{n\to\infty} d(x_n, x^*) = c. \qquad \textbf{(4.3.12)}$$

If $c = 0$ then we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Let $c > 0$. By the same method in the proof of Lemma 4.3.5(ii), we obtain

$$\lim_{n\to\infty} d(y_n, x^*) = c. \qquad \textbf{(4.3.13)}$$

Similarly, using by (4.3.13), we have

$$\lim_{n\to\infty} d(z_n, x^*) = c. \qquad \textbf{(4.3.14)}$$

By Lemma 2.1.15(ii), we get

$$
\begin{aligned}
d(z_n,x^*)^2 &= d(P((1-\gamma_n)x_n \oplus \gamma_n Tx_n), Px^*)^2 \\
&\le d((1-\gamma_n)x_n \oplus \gamma_n Tx_n), x^*)^2 \\
&\le (1-\gamma_n)d(x_n,x^*)^2 + \gamma_n d(Tx_n,x^*)^2 - \gamma_n(1-\gamma_n)d(x_n,Tx_n)^2 \\
&\le (1-\gamma_n)d(x_n,x^*)^2 + \gamma_n(\mu d(Tx^*,x^*) + d(x_n,x^*))^2 - \gamma_n(1-\gamma_n)d(x_n,Tx_n)^2 \\
&= (1-\gamma_n)d(x_n,x^*)^2 + \gamma_n d(x_n,x^*)^2 - \gamma_n(1-\gamma_n)d(x_n,Tx_n)^2 \\
&= d(x_n,x^*)^2 - \gamma_n(1-\gamma_n)d(x_n,Tx_n)^2.
\end{aligned}
$$

This gives that

$$\gamma_n(1-\gamma_n)d(x_n,Tx_n)^2 \le d(x_n,x^*)^2 - d(z_n,x^*)^2$$

or

$$
\begin{aligned}
d(x_n,Tx_n)^2 &\le \frac{1}{\gamma_n(1-\gamma_n)}\left[d(x_n,x^*)^2 - d(z_n,x^*)^2\right] \\
&\le \frac{1}{\varepsilon^2}\left[d(x_n,x^*)^2 - d(z_n,x^*)^2\right].
\end{aligned}
$$

Using (4.3.12) and (4.3.14), we obtain $\lim_{n\to\infty} d(x_n,Tx_n)=0$. This completes the proof of part (ii).

We give following theorems related to the strong and Δ-convergence of the Noor iteration process which their proofs are similar arguments of Theorem 4.3.6, Theorem 4.3.7 and Theorem 4.3.8, respectively.

Theorem 4.3.10. Let $X, K, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}$ satisfy the hypotheses of Lemma 4.3.9. Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Theorem 4.3.11. Let $X, K, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}$ satisfy the hypotheses of Lemma 4.3.9. If T satisfies condition (I) then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 4.3.12. Let $X, K, T, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{x_n\}$ satisfy the hypotheses of Lemma 4.3.9. If K is a compact subset of X then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Remark 4.3.13. It should be noted that Theorems 4.3.6-4.3.12 contain the corresponding theorems proved for a mapping satisfying condition (C).

CHAPTER 5. THE CONVERGENCE RESULTS FOR SOME ITERATIVE PROCESSES IN CAT(0) SPACE

In this section, the strong and Δ-convergence theorems of some iteration processes are proved in a CAT(0) space.

5.1. The Strong and Δ-Convergence of Some Iterative Algorithms for k-Strictly Pseudo-Contractive Mappings

In this subsection, we prove the Δ-convergence theorems of the cyclic algorithm and the new multi-step iteration for k-strictly pseudo-contractive mappings and give also the strong convergence theorem of the modified Halpern's iteration for these mappings in a CAT(0) space.

Definition 5.1.1. [59] Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping $T:C\to H$ is said to be k-strictly pseudo-contractive if there exists a constant $k\in[0,1)$ such that

$$\|Tx-Ty\|^2 \le \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2,\ \forall x,y\in C.$$

Remark 5.1.2. [59] The class of k-strictly pseudo-contractive includes the class of nonexpansive mappings T on C as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive.

Definition 5.1.3. [59] The mapping T is said to be pseudo-contractive if $k=1$ and T is said to be strongly pseudo-contractive if there exists a constant $\lambda\in(0,1)$ such that $T-\lambda I$ is pseudo-contractive.

Remark 5.1.4. [59] The class of k-strictly pseudo-contractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings.

Remark 5.1.5. [60-62] The class of strongly pseudo-contractive mappings is independent from the class of k-strictly pseudo-contractive mappings.

Many authors have been devoted the studies on the problems of finding fixed points for k-strictly pseudo-contractive mappings (see, [59, 63-65]). Motivated by these results, we define the concept of k-strictly pseudo-contractive mapping in a CAT(0) space as follows.

Definition 5.1.6. Let C be a nonempty closed convex subset of a CAT(0) space X. A mapping $T: C \to C$ is said to be k-strictly pseudo-contractive if there exists a constant $k \in [0,1)$ such that

$$d(Tx,Ty)^2 \le d(x,y)^2 + k(d(x,Tx)+d(y,Ty))^2, \ \forall x,y \in C. \tag{5.1.1}$$

Gürsoy, Karakaya and Rhoades [66] introduced a new multi-step iteration in a Banach space. We modified this iteration in a CAT(0) space as follows.

Definition 5.1.7. Let C be a closed convex subset of a CAT(0) space X. For an arbitrary fixed order $k \ge 2$,

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)y_n^1 \oplus \alpha_n T y_n^1, \\ y_n^1 = (1-\beta_n^1)y_n^2 \oplus \beta_n^1 T y_n^2, \\ y_n^2 = (1-\beta_n^2)y_n^3 \oplus \beta_n^2 T y_n^3, \\ \vdots \\ y_n^{k-2} = (1-\beta_n^{k-2})y_n^{k-1} \oplus \beta_n^{k-2} T y_n^{k-1}, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1} T x_n, \ n \ge 0, \end{cases}$$

or, in short,

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)y_n^1 \oplus \alpha_n T y_n^1, \\ y_n^i = (1-\beta_n^i)y_n^{i+1} \oplus \beta_n^i T y_n^{i+1}, i = 1, 2, ..., k-2, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1} T x_n, \ n \geq 0. \end{cases} \quad \textbf{(5.1.2)}$$

By taking $k=3$ and $k=2$ in (5.1.2), we obtain the SP-iteration and the two-step iteration, respectively.

Acedo and Xu [67] introduced a cyclic algorithm in a Hilbert space. We modify this algorithm in a CAT(0) space as follows.

Definition 5.1.8. Let $x_0 \in C$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. The cyclic algorithm generates a sequence $\{x_n\}$ in the following way:

$$\begin{cases} x_1 = \alpha_0 x_0 \oplus (1-\alpha_0) T_0 x_0, \\ x_2 = \alpha_1 x_1 \oplus (1-\alpha_1) T_1 x_1, \\ \vdots \\ x_N = \alpha_{N-1} x_{N-1} \oplus (1-\alpha_{N-1}) T_{N-1} x_{N-1}, \\ x_{N+1} = \alpha_N x_N \oplus (1-\alpha_N) T_0 x_N, \\ \vdots \end{cases}$$

or, in short,

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T_{[n]} x_n, \ n \geq 0, \quad \textbf{(5.1.3)}$$

where $T_{[n]} = T_i$, with $i = n(\mathrm{mod}\, N), 0 \leq i \leq N-1$. By taking $T_{[n]} = T$ for all n in (5.1.3), we obtain the Mann iteration.

By using the convergence defined in (2.1.1), we obtain the demiclosedness principle for k-strictly pseudo-contractive mappings in a CAT(0) space.

Theorem 5.1.9. Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T: C \to C$ be a k-strictly pseudo-contractive mapping such that

$k\in\left[0,\frac{1}{2}\right)$ and $F(T)\neq\varnothing$. Let $\{x_n\}$ be a bounded sequence in C such that $\Delta-\lim_{n\to\infty}x_n=w$ and $\lim_{n\to\infty}d(x_n,Tx_n)=0$. Then $Tw=w$.

Proof. By the hypothesis, $\Delta-\lim_{n\to\infty}x_n=w$. From Proposition 2.1.25, we get $\{x_n\}\rightharpoonup w$. Then we obtain $A(\{x_n\})=\{w\}$ by Lemma 2.1.22 (see [35]). Since $\lim_{n\to\infty}d(x_n,Tx_n)=0$, then we get

$$\Phi(x)=\limsup_{n\to\infty}d(x_n,x)=\limsup_{n\to\infty}d(Tx_n,x) \tag{5.1.4}$$

for all $x\in C$. In (5.1.4) by taking $x=Tw$, we have

$$\begin{aligned}\Phi(Tw)^2&=\limsup_{n\to\infty}d(Tx_n,Tw)^2\\&\le\limsup_{n\to\infty}\{d(x_n,w)^2+k(d(x_n,Tx_n)+d(w,Tw))^2\}\\&\le\limsup_{n\to\infty}d(x_n,w)^2+k\limsup_{n\to\infty}(d(x_n,Tx_n)+d(w,Tw))^2\\&=\Phi(w)^2+kd(w,Tw)^2.\end{aligned} \tag{5.1.5}$$

The (CN) inequality implies that

$$d\left(x_n,\frac{w\oplus Tw}{2}\right)^2\le\frac{1}{2}d(x_n,w)^2+\frac{1}{2}d(x_n,Tw)^2-\frac{1}{4}d(w,Tw)^2.$$

Letting $n\to\infty$ and taking limsup on the both sides of the above inequality, we get

$$\Phi\left(\frac{w\oplus Tw}{2}\right)^2\le\frac{1}{2}\Phi(w)^2+\frac{1}{2}\Phi(Tw)^2-\frac{1}{4}d(w,Tw)^2.$$

Since $A(\{x_n\})=\{w\}$, we have

$$\Phi(w)^2\le\Phi\left(\frac{w\oplus Tw}{2}\right)^2\le\frac{1}{2}\Phi(w)^2+\frac{1}{2}\Phi(Tw)^2-\frac{1}{4}d(w,Tw)^2$$

which implies that

$$d(w,Tw)^2\le 2\Phi(Tw)^2-2\Phi(w)^2. \tag{5.1.6}$$

By (5.1.5) and (5.1.6), we get $(1-2k)d(w,Tw)^2 \leq 0$. Since $k \in \left[0, \frac{1}{2}\right)$, then we have $Tw = w$ as desired.

Now, we prove the Δ-convergence of the new multi-step iteration for k-strictly pseudo-contractive mappings in a CAT(0) space.

Theorem 5.1.10. Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T: C \to C$ be a k-strictly pseudo-contractive mapping such that $k \in \left[0, \frac{1}{2}\right)$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n^i\}$, $i = 1, 2, \ldots, k-2$ be sequences in $[a, b]$ for some $a, b \in (0,1)$ and $k < 1-b$. Let $\{x_n\}$ be a sequence defined by (5.1.2). Then the sequence $\{x_n\}$ is Δ-convergent to a fixed point of T.

Proof. Let $p \in F(T)$. From (5.1.1), (5.1.2) and Lemma 2.1.15(ii), we have

$$\begin{aligned}
d(x_{n+1}, p)^2 &= d((1-\alpha_n)y_n^1 \oplus \alpha_n Ty_n^1, p)^2 \\
&\leq (1-\alpha_n)d(y_n^1, p)^2 + \alpha_n d(Ty_n^1, p)^2 - \alpha_n(1-\alpha_n)d(y_n^1, Ty_n^1)^2 \\
&\leq (1-\alpha_n)d(y_n^1, p)^2 + \alpha_n\{d(y_n^1, p)^2 + kd(y_n^1, Ty_n^1)^2\} - \alpha_n(1-\alpha_n)d(y_n^1, Ty_n^1)^2 \\
&= d(y_n^1, p)^2 - \alpha_n((1-\alpha_n) - k)d(y_n^1, Ty_n^1)^2 \\
&\leq d(y_n^1, p)^2.
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
d(y_n^1, p)^2 &= d((1-\beta_n^1)y_n^2 \oplus \beta_n^1 Ty_n^2, p)^2 \\
&\leq (1-\beta_n^1)d(y_n^2, p)^2 + \beta_n^1 d(Ty_n^2, p)^2 - \beta_n^1(1-\beta_n^1)d(y_n^2, Ty_n^2)^2 \\
&\leq (1-\beta_n^1)d(y_n^2, p)^2 + \beta_n^1\{d(y_n^2, p)^2 + kd(y_n^2, Ty_n^2)^2\} - \beta_n^1(1-\beta_n^1)d(y_n^2, Ty_n^2)^2 \\
&= d(y_n^2, p)^2 - \beta_n^1((1-\beta_n^1) - k)d(y_n^2, Ty_n^2)^2 \\
&\leq d(y_n^2, p)^2.
\end{aligned}$$

Continuing the above process we have

$$d(x_{n+1}, p) \leq d(y_n^2, p) \leq \ldots \leq d(y_n^{k-1}, p) \leq d(x_n, p). \quad \textbf{(5.1.7)}$$

This inequality guarentees that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Let $\lim_{n\to\infty} d(x_n, p) = r$. By using (5.1.7), we get

$$\lim_{n\to\infty} d(y_n^{k-1}, p) = r.$$

By Lemma 2.1.15(ii), we also have

$$\begin{aligned} d(y_n^{k-1}, p)^2 &= d((1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1}Tx_n, p)^2 \\ &\le (1-\beta_n^{k-1})d(x_n, p)^2 + \beta_n^{k-1}d(Tx_n, p)^2 - \beta_n^{k-1}(1-\beta_n^{k-1})d(x_n, Tx_n)^2 \\ &\le (1-\beta_n^{k-1})d(x_n, p)^2 + \beta_n^{k-1}\{d(x_n, p)^2 + kd(x_n, Tx_n)^2\} \\ &\quad - \beta_n^{k-1}(1-\beta_n^{k-1})d(x_n, Tx_n)^2 \\ &= d(x_n, p)^2 - \beta_n^{k-1}((1-\beta_n^{k-1}) - k)d(x_n, Tx_n)^2, \end{aligned}$$

which implies that

$$d(x_n, Tx_n)^2 \le \frac{1}{a((1-b)-k)}[d(x_n, p)^2 - d(y_n^{k-1}, p)^2].$$

Thus $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. To show that the sequence $\{x_n\}$ is Δ-convergent to a fixed point of T, we prove that $W_\Delta(x_n) \subseteq F(T)$ and $W_\Delta(x_n)$ consists of exactly one point. Let $u \in W_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.1.21 and 2.1.22, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta-\lim_{n\to\infty} v_n = v \in K$. By Theorem 5.1.9, we have $v \in F(T)$ and by Lemma 2.1.23, we have $u = v \in F(T)$. This shows that $W_\Delta(x_n) \subseteq F(T)$. Now we prove that $W_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Finally, since $\{d(x_n, v)\}$ is convergent, we have $x = v \in F(T)$ by Lemma 2.1.23. This shows $W_\Delta(x_n) = \{x\}$. This completes the proof.

Also, we prove the Δ-convergence of the cyclic algorithm for k-strictly pseudo-contractive mappings in a CAT(0) space.

Theorem 5.1.11. Let C be a nonempty closed convex subset of a complete CAT(0) space X and $N \ge 1$ be an integer. Let, for each $0 \le i \le N-1$, $T_i : C \to C$ be k_i-

strictly pseudo-contractive mappings for some $0 \le k_i < \frac{1}{2}$. Let $k = \max\{k_i; 0 \le i \le N-1\}$, $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$ and $k < a$. Let $F = \cap_{i=0}^{N-1} F(T_i) \neq \varnothing$. For $x_0 \in C$, let $\{x_n\}$ be a sequence defined by (5.1.3). Then the sequence $\{x_n\}$ is Δ-convergent to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.

Proof. Let $p \in F$. Using (5.1.1), (5.1.3) and Lemma 2.1.15(ii), we have

$$\begin{aligned}
d(x_{n+1}, p)^2 &= d(\alpha_n x_n \oplus (1-\alpha_n) T_{[n]} x_n, p)^2 \\
&\le \alpha_n d(x_n, p)^2 + (1-\alpha_n) d(T_{[n]} x_n, p)^2 - \alpha_n (1-\alpha_n) d(x_n, T_{[n]} x_n)^2 \\
&\le \alpha_n d(x_n, p)^2 + (1-\alpha_n)\{d(x_n, p)^2 + k d(x_n, T_{[n]} x_n)^2\} - \alpha_n (1-\alpha_n) d(x_n, T_{[n]} x_n)^2 \\
&= d(x_n, p)^2 - (1-\alpha_n)(\alpha_n - k) d(x_n, T_{[n]} x_n)^2 \qquad \textbf{(5.1.8)} \\
&\le d(x_n, p)^2.
\end{aligned}$$

This inequality guarentees that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$. By (5.1.8), we also have

$$\begin{aligned}
d(x_n, T_{[n]} x_n)^2 &\le \frac{1}{(1-\alpha_n)(\alpha_n - k)} \left[d(x_n, p)^2 - d(x_{n+1}, p)^2 \right] \\
&\le \frac{1}{(1-b)(a-k)} \left[d(x_n, p)^2 - d(x_{n+1}, p)^2 \right].
\end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, p)$ exists, we obtain $\lim_{n\to\infty} d(x_n, T_{[n]} x_n) = 0$. The rest of the proof closely follows the proof of Theorem 5.1.10.

Hu [68] introduced a modified Halpern's iteration. We modify this iteration in a CAT(0) space as follows.

Definition 5.1.12. [69] For an arbitrary initial value $x_0 \in C$ and a fixed anchor $u \in C$, the iterative sequence $\{x_n\}$ is defined by

$$\begin{cases} x_{n+1} = \alpha_n u \oplus (1-\alpha_n) y_n, \\ y_n = \dfrac{\beta_n}{1-\alpha_n} x_n \oplus \dfrac{\gamma_n}{1-\alpha_n} T x_n, \ n \ge 0, \end{cases} \qquad \textbf{(5.1.9)}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are three real sequences in $(0,1)$ satisfying $\alpha_n+\beta_n+\gamma_n=1$.

Remark 5.1.13. [69] Clearly, the iterative sequence (5.1.9) is a natural generalization of the well known iterations.

(i) If we take $\beta_n=0$ for all n in (5.1.9), then the sequence (5.1.9) is reduced to the Halpern's iteration in [70].

(ii) If we take $\alpha_n=0$ for all n in (5.1.9), then the sequence (5.1.9) is reduced to the Mann iteration.

Definition 5.1.14. [71] A continous linear functional μ on ℓ_∞, the Banach space of bounded real sequences, is called a Banach limit if $\|\mu\|=\mu(1,1,...)=1$ and $\mu(a_n)=\mu(a_{n+1})$ for all $\{a_n\}_{n=1}^{\infty}\subset\ell_\infty$.

Lemma 5.1.15. ([71, Proposition 2]) Let $\{a_1,a_2,...\}\in\ell_\infty$ be such that $\mu(a_n)\le 0$ for all Banach limits μ and $\limsup_{n\to\infty}(a_{n+1}-a_n)\le 0$. Then, $\limsup_{n\to\infty}a_n\le 0$.

Lemma 5.1.16. Let C be a nonempty closed convex subset of a complete CAT(0) space X, $T:C\to C$ be a k-strictly pseudo-contractive mapping with $k\in[0,1)$ and $S:C\to C$ be a mapping defined by $Sz=kz\oplus(1-k)Tz$, for $z\in C$. Let $u\in C$ be fixed. For each $t\in[0,1]$, the mapping $S_t:C\to C$ defined by

$$S_tz=tu\oplus(1-t)Sz=tu\oplus(1-t)(kz\oplus(1-t)Tz),\ \text{for}\ z\in C,$$

has a unique fixed point $z_t\in C$, that is,

$$z_t=S_t(z_t)=tu\oplus(1-t)S(z_t). \qquad \textbf{(5.1.10)}$$

Proof. As it has been proven in [72], if T is a k-strictly pseudo-contractive mapping with $k\in[0,1)$, S is a nonexpansive mapping such that $F(S)=F(T)$. Then, from Lemma 2.1 in [73], the mapping S_t has a unique fixed point $z_t\in C$.

Lemma 5.1.17. Let X, C, T and S be as in Lemma 5.1.16. Then, $F(T) \neq \emptyset$ if and only if $\{z_t\}$ given by (5.1.10) remains bounded as $t \to 0$. In this case, the following statements hold:

(1) $\{z_t\}$ converges to the unique fixed point z of T which is nearest to u,

(2) $d(u,z)^2 \leq \mu d(u,x_n)^2$ for all Banach limits μ and all bounded sequences $\{x_n\}$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0.$

Proof. If $F(T) \neq \emptyset$, then we have $F(S) = F(T) \neq \emptyset$. Also, if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we obtain that

$$d(x_n, Sx_n) = d(x_n, kx_n \oplus (1-k)Tx_n) \leq (1-k)d(x_n, Tx_n) \to 0 \text{ as } n \to \infty.$$

Thus, from Lemma 2.2 in [73], the rest of the proof of this lemma can be seen.

Lemma 5.1.18. ([74, Lemma 2.1]) Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the condition

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\sigma_n, \ \forall n \geq 0,$$

where $\{\gamma_n\}$ and $\{\sigma_n\}$ are sequences of real numbers such that

(1) $\{\gamma_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} \gamma_n = \infty,$

(2) either $\limsup_{n\to\infty} \sigma_n \leq 0$ or $\sum_{n=1}^{\infty} |\gamma_n \sigma_n| < \infty.$

Then, $\lim_{n\to\infty} a_n = 0.$

Theorem 5.1.19. Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T: C \to C$ be a k-strictly pseudo-contractive mapping such that $0 \leq k < \dfrac{\beta_n}{1-\alpha_n} < 1$ and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (5.1.9). Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0,$

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $\lim_{n\to\infty} \beta_n \neq k$ and $\lim_{n\to\infty} \gamma_n \neq 0$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. We divide the proof into three steps. In the first step we show that $\{x_n\}$, $\{y_n\}$ and $\{Tx_n\}$ are bounded sequences. In the second step we show that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ Finally, we show that $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest to u.

First step: Take any $p \in F(T)$, then, from Lemma 2.1.15(ii) and (5.1.9), we have

$$\begin{aligned}
d(y_n, p)^2 &\le \frac{\beta_n}{1-\alpha_n} d(x_n, p)^2 + \frac{\gamma_n}{1-\alpha_n} d(Tx_n, p)^2 - \frac{\beta_n \gamma_n}{(1-\alpha_n)^2} d(x_n, Tx_n)^2 \\
&\le \frac{\beta_n}{1-\alpha_n} d(x_n, p)^2 + \frac{\gamma_n}{1-\alpha_n} (d(x_n, p)^2 + k d(x_n, Tx_n)^2) - \frac{\beta_n \gamma_n}{(1-\alpha_n)^2} d(x_n, Tx_n)^2 \\
&= d(x_n, p)^2 - \frac{\gamma_n}{1-\alpha_n} \left(\frac{\beta_n}{1-\alpha_n} - k \right) d(x_n, Tx_n)^2 \\
&\le d(x_n, p)^2.
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
d(x_{n+1}, p)^2 &\le \alpha_n d(u, p)^2 + (1-\alpha_n) d(y_n, p)^2 - \alpha_n (1-\alpha_n) d(u, y_n)^2 \\
&\le \alpha_n d(u, p)^2 + (1-\alpha_n) \left\{ d(x_n, p)^2 - \frac{\gamma_n}{1-\alpha_n} \left(\frac{\beta_n}{1-\alpha_n} - k \right) d(x_n, Tx_n)^2 \right\} \\
&\quad - \alpha_n (1-\alpha_n) d(u, y_n)^2 \\
&= \alpha_n d(u, p)^2 + (1-\alpha_n) d(x_n, p)^2 - \gamma_n \left(\frac{\beta_n}{1-\alpha_n} - k \right) d(x_n, Tx_n)^2 \\
&\quad - \alpha_n (1-\alpha_n) d(u, y_n)^2 \qquad \textbf{(5.1.11)} \\
&\le \alpha_n d(u, p)^2 + (1-\alpha_n) d(x_n, p)^2 \\
&\le \max\{d(u, p)^2, d(x_n, p)^2\}.
\end{aligned}$$

By induction,

$$d(x_{n+1}, p)^2 \le \max\{d(u, p)^2, d(x_0, p)^2\}.$$

This proves the boundedness of the sequence $\{x_n\}$, which leads to the boundedness of $\{Tx_n\}$ and $\{y_n\}$.

Second step: In fact, we have from (5.1.11) (for some appropriate constant $M>0$) that

$$\begin{aligned} d(x_{n+1},p)^2 &\le \alpha_n d(u,p)^2 + (1-\alpha_n)d(x_n,p)^2 - \gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2 \\ &= \alpha_n(d(u,p)^2 - d(x_n,p)^2) + d(x_n,p)^2 - \gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2 \\ &\le \alpha_n M + d(x_n,p)^2 - \gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2, \end{aligned}$$

which implies that

$$\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2 - \alpha_n M \le d(x_n,p)^2 - d(x_{n+1},p)^2. \qquad \textbf{(5.1.12)}$$

If $\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2 - \alpha_n M \le 0$, then

$$d(x_n,Tx_n)^2 \le \frac{\alpha_n}{\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)}M,$$

and hence the desired result is obtained by the conditions (C1) and (C3).

If $\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2 - \alpha_n M > 0$, then following (5.1.12), we have

$$\sum_{n=0}^{m}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2 - \alpha_n M\right] \le d(x_0,p)^2 - d(x_{m+1},p)^2 \le d(x_0,p)^2.$$

That is

$$\sum_{n=0}^{\infty}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2 - \alpha_n M\right] < \infty.$$

Thus

$$\lim_{n\to\infty}\left[\gamma_n\left(\frac{\beta_n}{1-\alpha_n}-k\right)d(x_n,Tx_n)^2-\alpha_n M\right]=0.$$

Then we get

$$\lim_{n\to\infty} d(x_n,Tx_n)=0. \quad \textbf{(5.1.13)}$$

Third step: Using the condition (C1) and (5.1.13), we obtain

$$\begin{aligned}d(x_{n+1},x_n)&\le d(x_{n+1},Tx_n)+d(Tx_n,x_n)\\&\le \alpha_n d(u,Tx_n)+(1-\alpha_n)d(y_n,Tx_n)+d(Tx_n,x_n)\\&\le \alpha_n d(u,Tx_n)+(1-\alpha_n)\left(\frac{\beta_n}{1-\alpha_n}d(x_n,Tx_n)\right)+d(Tx_n,x_n)\\&=\alpha_n d(u,Tx_n)+(\beta_n+1)d(x_n,Tx_n)\to 0 \text{ as } n\to\infty.\end{aligned}$$

Also, from (5.1.13), we have

$$d(x_n,y_n)\le\frac{\gamma_n}{1-\alpha_n}d(x_n,Tx_n)\to 0 \text{ as } n\to\infty. \quad \textbf{(5.1.14)}$$

Let $z=\lim_{t\to 0} z_t$, where z_t is given by (5.1.10) in Lemma 5.1.16. Then, z is the point of $F(T)$ which is nearest to u. By Lemma 5.1.17(2), we have $\mu(d(u,z)^2-d(u,x_n)^2)\le 0$ for all Banach limits μ. Let $a_n=d(u,z)^2-d(u,x_n)^2$. Moreover, since $\lim_{n\to\infty} d(x_{n+1},x_n)=0$, we get

$$\limsup_{n\to\infty}(a_{n+1}-a_n)=0.$$

By Lemma 5.1.15, we obtain

$$\limsup_{n\to\infty}(d(u,z)^2-d(u,x_n)^2)\le 0. \quad \textbf{(5.1.15)}$$

It follows from the condition (C1) and (5.1.14) that

$$\limsup_{n\to\infty}(d(u,z)^2-(1-\alpha_n)d(u,y_n)^2)=\limsup_{n\to\infty}(d(u,z)^2-d(u,x_n)^2) \quad \textbf{(5.1.16)}$$

By (5.1.15) and (5.1.16), we have

$$\limsup_{n\to\infty}(d(u,z)^2-(1-\alpha_n)d(u,y_n)^2)\le 0. \quad \textbf{(5.1.17)}$$

We observe that

$$\begin{aligned} d(x_{n+1},z)^2 &\le \alpha_n d(u,z)^2 + (1-\alpha_n)d(y_n,z)^2 - \alpha_n(1-\alpha_n)d(u,y_n)^2 \\ &\le \alpha_n d(u,z)^2 + (1-\alpha_n)d(x_n,z)^2 - \alpha_n(1-\alpha_n)d(u,y_n)^2 \\ &= (1-\alpha_n)d(x_n,z)^2 + \alpha_n[d(u,z)^2 - (1-\alpha_n)d(u,y_n)^2]. \end{aligned}$$

It follows from the condition (C2) and (5.1.17), using Lemma 5.1.18, that $\lim_{n\to\infty} d(x_n,z) = 0$. This completes the proof of Theorem 5.1.19.

We obtain the following corollary as a direct consequence of Theorem 5.1.19.

Corollary 5.1.20. Let X, C and T be as Theorem 5.1.19. Let $\{\alpha_n\}$ be a real sequence in $(0,1)$ satisfying the conditions (C1) and (C2) of Theorem 5.1.19. For a constant $\delta \in (k,1)$, an arbitrary initial value $x_0 \in C$ and a fixed anchor $u \in C$, let the sequence $\{x_n\}$ be defined by

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)(\delta x_n \oplus (1-\delta)Tx_n),\ n \ge 0. \tag{5.1.18}$$

Then the sequence $\{x_n\}$ is strongly convergent to a fixed point of T.

Proof. If, in proof of Theorem 5.1.19, we take $\beta_n = (1-\alpha_n)\delta$ and $\gamma_n = (1-\alpha_n)(1-\delta)$, then we get the desired conclusion.

Remark 5.1.21. Theorem 5.1.19 contains the strong convergence theorems of the iterative sequences (5.1.9) and (5.1.18) for nonexpansive mappings in a CAT(0) space. Also, this theorem contains the corresponding theorems proved for these iterative sequences in a Hilbert space.

5.2. The Strong and Δ-Convergence of New Multi-Step and S-Iteration Processes

In this subsection, we introduce a new class of mappings and prove the demiclosedness principle for mappings of this type in a CAT(0) space. Also, we obtain the strong and Δ-convergence theorems of new multi-step and S-iteration processes in a CAT(0) space.

Definition 5.2.1. [75] Let T be a self mapping on a metric space X. The mapping T is called a contractive-like mapping if there exist a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\varphi:[0,\infty) \to [0,\infty)$ with $\varphi(0)=0$ such that, for all $x, y \in X$,

$$d(Tx,Ty) \le \delta d(x,y) + \varphi(d(x,Tx)). \tag{5.2.1}$$

Remark 5.2.2. [75] This mapping is more general than those considered by Berinde [76, 77], Harder and Hicks [78], Zamfirescu [79], Osilike and Udomene [80].

By taking $\delta = 1$ in (5.2.1), we define a new class of mappings as follows.

Definition 5.2.3. [81] The mapping T is called a generalized nonexpansive mapping if there exists a non-decreasing and continuous function $\varphi:[0,\infty) \to [0,\infty)$ with $\varphi(0)=0$ such that, for all $x, y \in X$,

$$d(Tx,Ty) \le d(x,y) + \varphi(d(x,Tx)). \tag{5.2.2}$$

Remark 5.2.4. For $x \in F(T)$ in (5.2.2), we have

$$d(x,Ty) = d(Tx,Ty) \le d(x,y) + \varphi(d(x,Tx)) = d(x,y).$$

Fact 5.2.5. [81] If X is an interval of $\mathbb{R}$, then $F(T)$ is convex. The same is also true in each space with unique geodesic for each pair of points (*e.g.* metric trees or CAT(0) spaces).

Remark 5.2.6. In the case $\varphi(t)=0$ for all $t \in [0,\infty)$, it is easy to show every nonexpansive mapping satisfies (5.2.2), but the inverse is not necessarily true.

Example 5.2.7. [81] Let $X=[0,2]$, $d(x,y)=|x-y|$, $\varphi(t)=t$ and define T by

$$T(x) = \begin{cases} 0 & \text{if } x \ne 2, \\ 1 & \text{if } x = 2. \end{cases}$$

By taking $x=2$ and $y=1.5$, we have

$$d(T(2),T(1.5))=1<1.5=d(2,1.5)+\varphi(d(2,T(2)))$$

but

$$d(T(2),T(1.5))=1>0.5=d(2,1.5).$$

Therefore T is a generalized nonexpansive mapping, but T is not nonexpansive mapping.

Remark 5.2.8. Both a contractive-like mapping and a generalized nonexpansive mapping doesn't need to have a fixed point, even if X is a complete.

Example 5.2.9. [81] Let $X=[0,\infty)$, $d(x,y)=|x-y|$ and define T by

$$T(x)=\begin{cases}1 & \text{if } 0\le x\le 0.8,\\ 0.6 & \text{if } 0.8<x<+\infty.\end{cases}$$

It is proved in [66] that T is a contractive-like mapping. Similarly, one can prove that T is a generalized nonexpansive mapping. But the mapping T has no fixed point.

Remark 5.2.10. By using (5.2.1), it is obvious that if a contractive-like mapping has a fixed point then it is unique. However, if a generalized nonexpansive mapping has a fixed point then it doesn't need to have unique.

Example 5.2.11. [81] Let $\mathbb{R}$ be the real line with the usual absolute metric and let $K=[-1,1]$. Define a mapping $T:K\to K$ by

$$T(x)=\begin{cases}x, & \text{if } x\in[0,1],\\ -x, & \text{if } x\in[-1,0).\end{cases}$$

Now, we show that T is a nonexpansive mapping. In fact, if $x,y\in[0,1]$ or $x,y\in[-1,0)$, then we have

$$|Tx-Ty|=|x-y|.$$

If $x \in [0,1]$ and $y \in [-1,0)$ or $x \in [-1,0)$ and $y \in [0,1]$, then we have

$$|Tx - Ty| = |x + y| \le |x - y|.$$

This implies that T is a nonexpansive mapping and so T is a generalized nonexpansive mapping with $\varphi(t) = 0$ for all $t \in [0,\infty)$. But $F(T) = \{x \in K; 0 \le x \le 1\}$.

By using the convergence defined in (2.1.1), we obtain the demiclosedness principle for the new class of mappings in a CAT(0) space.

Theorem 5.2.12. Let K be a nonempty closed convex subset of a complete CAT(0) space X and let $T : K \to K$ be a generalized nonexpansive mapping with $F(T) \ne \varnothing$. Let $\{x_n\}$ be a bounded sequence in K such that $\Delta - \lim_{n\to\infty} x_n = w$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $Tw = w$.

Proof. By the hypothesis, $\Delta - \lim_{n\to\infty} x_n = w$. From Proposition 2.1.25, we get $\{x_n\} \rightharpoonup w$. Then we obtain $A(\{x_n\}) = \{w\}$ by Lemma 2.1.22. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ then we have

$$\Phi(x) = \limsup_{n\to\infty} d(x_n, x) = \limsup_{n\to\infty} d(Tx_n, x) \tag{5.2.3}$$

for all $x \in K$. By taking $x = Tw$ in (5.2.3), we have

$$\begin{aligned} \Phi(Tw) &= \limsup_{n\to\infty} d(Tx_n, Tw) \\ &\le \limsup_{n\to\infty} \{d(x_n, w) + \varphi(d(x_n, Tx_n))\} \\ &\le \limsup_{n\to\infty} d(x_n, w) + \varphi\left(\limsup_{n\to\infty} d(x_n, Tx_n)\right) \\ &= \limsup_{n\to\infty} d(x_n, w) \\ &= \Phi(w). \end{aligned}$$

The rest of the proof closely follows the pattern of Proposition 3.14 in Nanjaras and Panyanak [35]. Hence $Tw = w$ as desired.

Now, we prove the Δ-convergence of the new multi-step iteration process for the new class of mappings in a CAT(0) space.

Theorem 5.2.13. Let K be a nonempty closed convex subset of a complete CAT(0) space X, $T:K\to K$ be a generalized nonexpansive mapping with $F(T)\neq\varnothing$ and let $\{x_n\}$ be a sequence defined by (5.1.2) such that $\{\alpha_n\}$, $\{\beta_n^i\}\subset[0,1]$, $i=1,2,...,k-2$ and $\{\beta_n^{k-1}\}\subset[a,b]$ for some $a,b\in(0,1)$. Then the sequence $\{x_n\}$ Δ-converges to the fixed point of T.

Proof. Let $p\in F(T)$. From (5.1.2), (5.2.2) and Lemma 2.1.15(i), we have

$$\begin{aligned} d(x_{n+1},p) &= d((1-\alpha_n)y_n^1\oplus\alpha_n Ty_n^1,p)\\ &\leq(1-\alpha_n)d(y_n^1,p)+\alpha_n d(Ty_n^1,p)\\ &\leq(1-\alpha_n)d(y_n^1,p)+\alpha_n\{d(y_n^1,p)+\varphi(d(p,Tp))\}\\ &= d(y_n^1,p). \end{aligned}$$

Also, we obtain

$$\begin{aligned} d(y_n^1,p) &= d((1-\beta_n^1)y_n^2\oplus\beta_n^1 Ty_n^2,p)\\ &\leq(1-\beta_n^1)d(y_n^2,p)+\beta_n^1 d(Ty_n^2,p)\\ &\leq(1-\beta_n^1)d(y_n^2,p)+\beta_n^1\{d(y_n^2,p)+\varphi(d(p,Tp))\}\\ &= d(y_n^2,p). \end{aligned}$$

Continuing the above process, we have

$$d(x_{n+1},p)\leq d(y_n^1,p)\leq d(y_n^2,p)\leq...\leq d(y_n^{k-1},p)\leq d(x_n,p). \tag{5.2.4}$$

This implies that $\lim_{n\to\infty} d(x_n,p)$ exists for all $p\in F(T)$. Let $\lim_{n\to\infty} d(x_n,p)=r$. By using (5.2.4), we get

$$\lim_{n\to\infty} d(y_n^{k-1},p)=r.$$

By Lemma 2.1.15(ii), we also have

$$
\begin{aligned}
d(y_n^{k-1},p)^2 &= d((1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1}Tx_n, p)^2 \\
&\le (1-\beta_n^{k-1})d(x_n,p)^2 + \beta_n^{k-1}d(Tx_n,p)^2 - \beta_n^{k-1}(1-\beta_n^{k-1})d(x_n,Tx_n)^2 \\
&\le (1-\beta_n^{k-1})d(x_n,p)^2 + \beta_n^{k-1}\{d(x_n,p)+\varphi(d(p,Tp))\}^2 \\
&\quad -\beta_n^{k-1}(1-\beta_n^{k-1})d(x_n,Tx_n)^2 \\
&= d(x_n,p)^2 - \beta_n^{k-1}(1-\beta_n^{k-1})d(x_n,Tx_n)^2,
\end{aligned}
$$

which implies that

$$
d(x_n,Tx_n)^2 \le \frac{1}{a(1-b)}[d(x_n,p)^2 - d(y_n^{k-1},p)^2].
$$

Thus $\lim_{n\to\infty} d(x_n,Tx_n)=0$. To show that the sequence $\{x_n\}$ Δ-converges to a fixed point of T, we prove that $W_\Delta(x_n)\subseteq F(T)$ and $W_\Delta(x_n)$ consists of exactly one point. Let $u\in W_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\})=\{u\}$. By Lemmas 2.1.21 and 2.1.22, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta-\lim_{n\to\infty} v_n = v\in K$. By Theorem 5.2.12, $v\in F(T)$. By Lemma 2.1.23, we have $u=v\in F(T)$. This shows that $W_\Delta(x_n)\subseteq F(T)$. Now, we prove that $W_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\})=\{u\}$ and let $A(\{x_n\})=\{x\}$. We have already seen that $u=v$ and $v\in F(T)$. Finally, since $\{d(x_n,v)\}$ converges, by Lemma 2.1.23, $x=v\in F(T)$. This shows that $W_\Delta(x_n)=\{x\}$. This completes the proof.

We give following theorem related to the Δ-convergence of the S-iteration process for the new class of mappings in a CAT(0) space.

Theorem 5.2.14. Let K be a nonempty closed convex subset of a complete CAT(0) space X, $T:K\to K$ be a generalized nonexpansive mapping with $F(T)\neq\emptyset$ and let $\{x_n\}$ be a sequence defined by (4.1.1) such that $\{\alpha_n\},\{\beta_n\}\subset[a,b]$ for some $a,b\in(0,1)$. Then the sequence $\{x_n\}$ Δ-converges to the fixed point of T.

Proof. Let $p\in F(T)$. Using (4.1.1), (5.2.2) and Lemma 2.1.15(i), we have

$$\begin{aligned}d(x_{n+1},p) &= d((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n, p)\\ &\le (1-\alpha_n)d(Tx_n,p)+\alpha_n d(Ty_n,p)\\ &\le (1-\alpha_n)\{d(x_n,p)+\varphi(d(p,Tp))\}+\alpha_n\{d(y_n,p)+\varphi(d(p,Tp))\}\\ &= (1-\alpha_n)d(x_n,p)+\alpha_n d(y_n,p). \end{aligned} \tag{5.2.5}$$

Also, we obtain

$$\begin{aligned}d(y_n,p) &= d((1-\beta_n)x_n \oplus \beta_n Tx_n, p)\\ &\le (1-\beta_n)d(x_n,p)+\beta_n d(Tx_n,p)\\ &\le (1-\beta_n)d(x_n,p)+\beta_n\{d(x_n,p)+\varphi(d(p,Tp))\}\\ &= d(x_n,p). \end{aligned} \tag{5.2.6}$$

From (5.2.5) and (5.2.6), we have $d(x_{n+1},p)\le d(x_n,p)$. This implies that $\lim_{n\to\infty} d(x_n,p)$ exists for all $p\in F(T)$. Let

$$\lim_{n\to\infty} d(x_n,p)=r. \tag{5.2.7}$$

Now, we prove that $\lim_{n\to\infty} d(y_n,p)=r$. By (5.2.5), we have

$$d(x_{n+1},p)\le(1-\alpha_n)d(x_n,p)+\alpha_n d(y_n,p).$$

This gives that

$$\alpha_n d(x_n,p)\le d(x_n,p)+\alpha_n d(y_n,p)-d(x_{n+1},p)$$

or

$$\begin{aligned}d(x_n,p) &\le d(y_n,p)+\frac{1}{\alpha_n}[d(x_n,p)-d(x_{n+1},p)]\\ &\le d(y_n,p)+\frac{1}{a}[d(x_n,p)-d(x_{n+1},p)].\end{aligned}$$

This gives $r\le\liminf_{n\to\infty} d(y_n,p)$. By (5.2.6) and (5.2.7), we obtain $\limsup_{n\to\infty} d(y_n,p)\le r$. Then, we get

$$\lim_{n\to\infty} d(y_n,p)=r.$$

By Lemma 2.1.15(ii), we also have

$$\begin{aligned} d(y_n, p)^2 &= d((1-\beta_n)x_n \oplus \beta_n Tx_n, p)^2 \\ &\le (1-\beta_n)d(x_n, p)^2 + \beta_n d(Tx_n, p)^2 - \beta_n(1-\beta_n)d(x_n, Tx_n)^2 \\ &\le (1-\beta_n)d(x_n, p)^2 + \beta_n\{d(x_n, p) + \varphi(d(p, Tp))\}^2 - \beta_n(1-\beta_n)d(x_n, Tx_n)^2 \\ &= d(x_n, p)^2 - \beta_n(1-\beta_n)d(x_n, Tx_n)^2, \end{aligned}$$

which implies that

$$d(x_n, Tx_n)^2 \le \frac{1}{a(1-b)}[d(x_n, p)^2 - d(y_n, p)^2].$$

Thus $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. The rest of the proof follows the pattern of the above theorem.

Now, we prove the strong convergence theorem of the new multi-step iteration process for a contractive-like mapping in a CAT(0) space.

Theorem 5.2.15. Let K be a nonempty closed convex subset of a complete CAT(0) space X, $T: K \to K$ be a contractive-like mapping with $F(T) \neq \varnothing$ and let $\{x_n\}$ be a sequence defined by (5.1.2) such that $\{\alpha_n\} \subset [0,1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{\beta_n^i\} \subset [0,1)$, $i = 1, 2, ..., k-1$. Then the sequence $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. Let p be the unique fixed point of T. From (5.1.2), (5.2.1) and Lemma 2.1.15(i), we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)y_n^1 \oplus \alpha_n Ty_n^1, p) \\ &\le (1-\alpha_n)d(y_n^1, p) + \alpha_n d(Ty_n^1, p) \\ &\le (1-\alpha_n)d(y_n^1, p) + \alpha_n\{\delta d(y_n^1, p) + \varphi(d(p, Tp))\} \\ &= (1-\alpha_n)d(y_n^1, p) + \alpha_n \delta d(y_n^1, p) \\ &= [1-\alpha_n(1-\delta)]d(y_n^1, p). \end{aligned}$$

Also, we obtain

$$\begin{aligned} d(y_n^1, p) &= d((1-\beta_n^1)y_n^2 \oplus \beta_n^1 T y_n^2, p) \\ &\le (1-\beta_n^1)d(y_n^2, p) + \beta_n^1 d(Ty_n^2, p) \\ &\le (1-\beta_n^1)d(y_n^2, p) + \beta_n^1\{\delta d(y_n^2, p) + \varphi(d(p, Tp))\} \\ &= (1-\beta_n^1)d(y_n^2, p) + \beta_n^1 \delta d(y_n^2, p) \\ &= [1-\beta_n^1(1-\delta)]d(y_n^2, p). \end{aligned}$$

In a similar fashion, we can get

$$d(y_n^2, p) \le 1-\beta_n^2(1-\delta)]d(y_n^3, p).$$

Continuing the above process we have

$$d(x_{n+1}, p) \le [1-\alpha_n(1-\delta)][1-\beta_n^1(1-\delta)][1-\beta_n^2(1-\delta)]...[1-\beta_n^{k-2}(1-\delta)]d(y_n^{k-1}, p). \tag{5.2.8}$$

In addition, we obtain

$$\begin{aligned} d(y_n^{k-1}, p) &= d((1-\beta_n^{k-1})x_n \oplus \beta_n^{k-1} T x_n, p) \\ &\le (1-\beta_n^{k-1})d(x_n, p) + \beta_n^{k-1} d(Tx_n, p) \\ &\le (1-\beta_n^{k-1})d(x_n, p) + \beta_n^{k-1}\{\delta d(x_n, p) + \varphi(d(p, Tp))\} \\ &= (1-\beta_n^{k-1})d(x_n, p) + \beta_n^{k-1} \delta d(x_n, p) \\ &= [1-\beta_n^{k-1}(1-\delta)]d(x_n, p). \end{aligned} \tag{5.2.9}$$

From (5.2.8) and (5.2.9), we have

$$\begin{aligned} d(x_{n+1}, p) &\le [1-\alpha_n(1-\delta)][1-\beta_n^1(1-\delta)][1-\beta_n^2(1-\delta)]... \\ &\qquad [1-\beta_n^{k-2}(1-\delta)][1-\beta_n^{k-1}(1-\delta)]d(x_n, p) \\ &\le [1-\alpha_n(1-\delta)]d(x_n, p) \\ &\le \prod_{j=0}^{n}[1-\alpha_j(1-\delta)]d(x_0, p) \\ &\le e^{-(1-\delta)\sum_{j=0}^{n}\alpha_j} d(x_0, p). \end{aligned} \tag{5.2.10}$$

Using the fact that $0 \le \delta < 1$, $\alpha_j \in [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, we get that

$$\lim_{n\to\infty} e^{-(1-\delta)\sum_{j=0}^{n}\alpha_j} = 0.$$

This together with (5.2.10) implies that $\lim_{n\to\infty} d(x_{n+1}, p) = 0$. Consequently $x_n \to p \in F(T)$ and this completes the proof.

Remark 5.2.16. In Theorem 5.2.15, the condition $\sum_{n=0}^{\infty}\alpha_n = \infty$ may be replaced with $\sum_{n=0}^{\infty}\beta_n^i = \infty$ for a fixed $i = 1, 2, ..., k-1.$.

We give the strong convergence theorem of the S-iteration process for a contractive-like mapping on a CAT(0) space as follows.

Theorem 5.2.17. Let K be a nonempty closed convex subset of a complete CAT(0) space X, $T: K \to K$ be a contractive-like mapping with $F(T) \neq \varnothing$ and let $\{x_n\}$ be a sequence defined by (4.1.1) such that $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. Then the sequence $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. Let p be the unique fixed point of T. From (5.1.2), (5.2.1) and Lemma 2.1.15(i), we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n, p) \\ &\leq (1-\alpha_n)d(Tx_n, p) + \alpha_n d(Ty_n, p) \\ &\leq (1-\alpha_n)\{\delta d(x_n, p) + \varphi(d(p,Tp))\} + \alpha_n\{\delta d(y_n, p) + \varphi(d(p,Tp))\} \\ &= (1-\alpha_n)\delta d(x_n, p) + \alpha_n \delta d(y_n, p). \end{aligned} \qquad \textbf{(5.2.11)}$$

Similarly, we obtain

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)x_n \oplus \beta_n Tx_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(Tx_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n\{\delta d(x_n, p) + \varphi(d(p,Tp))\} \\ &= (1-\beta_n)d(x_n, p) + \beta_n \delta d(x_n, p) \\ &= (1-\beta_n(1-\delta))d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \qquad \textbf{(5.2.12)}$$

Then from (5.2.11) and (5.2.12), we get that

$$\begin{aligned} d(x_{n+1},p) &\le (1-\alpha_n)\delta d(x_n,p)+\alpha_n\delta d(y_n,p) \\ &\le (1-\alpha_n)\delta d(x_n,p)+\alpha_n\delta d(x_n,p) \\ &\le \delta d(x_n,p) \\ &\vdots \\ &\le \delta^{n+1} d(x_0,p). \end{aligned}$$

If $\delta\in(0,1)$, we obtain $\lim_{n\to\infty} d(x_{n+1},p)=0.$ Thus we have $x_n \to p\in F(T)$. If $\delta=0,$ the result is clear. This completes the proof.

5.3. The Strong Convergence of Modified S-Iteration Process for Asymptotically Quasi-Nonexpansive Mappings

In this subsection, we prove the strong convergence theorems of the modified S-iteration process for asymptotically quasi-nonexpansive mappings on a CAT(0) space.

Agarwal, O'Regan and Sahu [15] introduced the modified S-iteration process which is independent of those of the modified Mann iteration [82] and the modified Ishikawa iteration [83]. We give this iteration process in a CAT(0) space as follows.

Definition 5.3.1. [84] Let K be a nonempty closed convex subset of a complete CAT(0) space X, $T:K\to K$ be an asymptotically quasi-nonexpansive mapping with $F(T)\neq\varnothing$. The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1\in K, \\ x_{n+1}=(1-a_n)T^n x_n \oplus a_n T^n y_n, \\ y_n=(1-b_n)x_n \oplus b_n T^n x_n,\ n\in\mathbb{N}. \end{cases} \tag{5.3.1}$$

By taking $T^n=T$ for all $n\in\mathbb{N}$ in (5.3.1), we obtain the S-iteration process.

Lemma 5.3.2. [85] Let $\{a_n\}$ and $\{u_n\}$ be two sequences of positive real numbers satisfying

$$a_{n+1}\le(1+u_n)a_n$$

for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} u_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Theorem 5.3.3. Let K be a nonempty closed convex subset of a complete CAT(0) space X, $T: K \to K$ be an asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and $\{u_n\}$ be a non-negative real sequence with $\sum_{n=1}^{\infty} u_n < \infty$. Suppose that $\{x_n\}$ is defined by the iteration process (5.3.1). If

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0 \text{ or } \limsup_{n\to\infty} d(x_n, F(T)) = 0,$$

then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $p \in F(T)$. Since T is an asymptotically quasi-nonexpansive mapping, there exists a sequence $\{u_n\} \in [0,\infty)$ with $\lim_{n\to\infty} u_n = 0$ and such that

$$d(T^n x, p) \leq (1+u_n) d(x, p)$$

for all $x \in K$ and $p \in F(T)$. By combining this inequality and Lemma 2.1.15(i), we get

$$\begin{aligned} d(y_n, p) &= d((1-b_n)x_n \oplus b_n T^n x_n, p) \\ &\leq (1-b_n) d(x_n, p) + b_n d(T^n x_n, p) \\ &\leq (1-b_n) d(x_n, p) + b_n (1+u_n) d(x_n, p) \\ &= (1+b_n u_n) d(x_n, p). \end{aligned} \tag{5.3.2}$$

Also,

$$\begin{aligned} d(x_{n+1}, p) &= d((1-a_n)T^n x_n \oplus a_n T^n y_n, p) \\ &\leq (1-a_n) d(T^n x_n, p) + a_n d(T^n y_n, p) \\ &\leq (1-a_n)(1+u_n) d(x_n, p) + a_n (1+u_n) d(y_n, p) \\ &\leq (1-a_n)(1+u_n) d(x_n, p) + a_n (1+u_n)(1+b_n u_n) d(x_n, p) \\ &\leq (1-a_n)(1+u_n) d(x_n, p) + a_n (1+u_n)^2 d(x_n, p) \\ &= (1+u_n)(1-a_n+a_n+a_n u_n) d(x_n, p) \\ &\leq (1+u_n)(1+u_n) d(x_n, p) \\ &= (1+u_n)^2 d(x_n, p). \end{aligned} \tag{5.3.3}$$

When $x \geq 0$ and $1 + x \leq e^x$, we have $(1+x)^2 \leq e^{2x}$. Thus,

$$\begin{aligned} d(x_{n+m}, p) &\leq (1+u_{n+m-1})^2 d(x_{n+m-1}, p) \\ &\leq e^{2u_{n+m-1}} d(x_{n+m-1}, p) \\ &\leq \ldots \\ &\leq e^{2\sum_{k=n}^{n+m-1} u_k} d(x_n, p). \end{aligned}$$

Let $e^{2\sum_{k=n}^{n+m-1} u_k} = M$. Thus, there exits a constant $M > 0$ such that

$$d(x_{n+m}, p) \leq M d(x_n, p)$$

for all $n, m \in \mathbb{N}$ and $p \in F(T)$. By (5.3.3),

$$d(x_{n+1}, p) \leq (1+u_n)^2 d(x_n, p).$$

This gives

$$d(x_{n+1}, F(T)) \leq (1+u_n)^2 d(x_n, F(T)) = (1 + 2u_n + u_n^2) d(x_n, F(T)).$$

Since $\sum_{n=1}^{\infty} u_n < \infty$, we have $\sum_{n=1}^{\infty} (2u_n + u_n^2) < \infty$. Lemma 5.3.2 and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(T)) = 0$ gives that

$$\lim_{n\to\infty} d(x_n, F(T)) = 0. \qquad \textbf{(5.3.4)}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in K. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for each $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, F(T)) < \frac{\varepsilon}{M+1}$$

for all $n > n_1$. Thus, there exists $p_1 \in F(T)$ such that

$$d(x_n, p_1) < \frac{\varepsilon}{M+1} \quad \text{for all } n > n_1$$

and we obtain that

$$\begin{aligned} d(x_{n+m},x_n) &\le d(x_{n+m},p_1)+d(p_1,x_n) \\ &\le Md(x_n,p_1)+d(x_n,p_1) \\ &= (M+1)d(x_n,p_1) \\ &< (M+1)\frac{\varepsilon}{M+1}=\varepsilon \end{aligned}$$

for all $m,n>n_1$. Therefore, $\{x_n\}$ is a Cauchy sequence in K. Since the set K is complete, the sequence $\{x_n\}$ must be convergence to a point in K. Let $\lim_{n\to\infty} x_n = p\in K$. Here after, we show that p is a fixed point. By $\lim_{n\to\infty} x_n = p$, for all $\varepsilon_1>0$, there exists $n_2\in\mathbb{N}$ such that

$$d(x_n,p)<\frac{\varepsilon_1}{2(2+u_1)} \quad \textbf{(5.3.5)}$$

for all $n>n_2$. From (5.3.4), for each $\varepsilon_1>0$, there exists $n_3\in\mathbb{N}$ such that

$$d(x_n,F(T))<\frac{\varepsilon_1}{2(4+3u_1)}$$

for all $n>n_3$. In particular, $\inf\{d(x_{n_3},p):p\in F(T)\}<\dfrac{\varepsilon_1}{2(4+3u_1)}$. Thus, there must exist $p^*\in F(T)$ such that

$$d(x_{n_3},p^*)<\frac{\varepsilon_1}{2(4+3u_1)} \quad \text{for all } n>n_3. \quad \textbf{(5.3.6)}$$

From (5.3.5) and (5.3.6),

$$\begin{aligned} d(Tp,p) &\le d(Tp,p^*)+d(p^*,Tx_{n_3})+d(Tx_{n_3},p^*)+d(p^*,x_{n_3})+d(x_{n_3},p) \\ &\le d(Tp,p^*)+2d(Tx_{n_3},p^*)+d(x_{n_3},p^*)+d(x_{n_3},p) \\ &\le (1+u_1)d(p,p^*)+2(1+u_1)d(x_{n_3},p^*)+d(x_{n_3},p^*)+d(x_{n_3},p) \\ &\le (1+u_1)d(p,x_{n_3})+(1+u_1)d(x_{n_3},p^*)+2(1+u_1)d(x_{n_3},p^*) \\ &\quad +d(x_{n_3},p^*)+d(x_{n_3},p) \\ &= (2+u_1)d(x_{n_3},p)+(4+3u_1)d(x_{n_3},p^*) \\ &< (2+u_1)\frac{\varepsilon_1}{2(2+u_1)}+(4+3u_1)\frac{\varepsilon_1}{2(4+3u_1)}=\varepsilon_1. \end{aligned}$$

Since ε_1 is arbitrary, so $d(Tp, p) = 0$, i.e., $Tp = p$. Therefore, $p \in F(T)$. This completes the proof.

Remark 5.3.4. Let the hypotheses of Theorem 5.3.3 be satisfied and $T : K \to K$ be an asymptotically nonexpansive or quasi-nonexpansive mapping. Since the class of asymptotically quasi-nonexpansive mappings includes quasi-nonexpansive mappings and asymptotically nonexpansive mappings, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Now, we give the following corollaries which have been proved by Theorem 5.3.3.

Corollary 5.3.5. Under the hypotheses of Theorem 5.3.3, T satisfies the following conditions:

(1) $\lim_{n\to\infty} d(x_n, Tx_n) = 0.$

(2) If the sequence $\{z_n\}$ in K satisfies $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, then
$\liminf_{n\to\infty} d(z_n, F(T)) = 0$ or $\limsup_{n\to\infty} d(z_n, F(T)) = 0.$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. It follows from the hypotheses that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. From (2),

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0 \text{ or } \limsup_{n\to\infty} d(x_n, F(T)) = 0,$$

Therefore, the sequence $\{x_n\}$ must converge to a fixed point of T by Theorem 5.3.3.

Corollary 5.3.6. Under the hypothesis of Theorem 5.3.3, T satisfies the following conditions:

(1) $\lim_{n\to\infty} d(x_n, Tx_n) = 0.$

(2) There exists a function $f : [0,\infty) \to [0,\infty)$ which is right-continuous at 0, $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that

$$d(x, Tx) \geq f(d(x, F(T))) \quad \text{for all } x \in K.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. It follows from the hypotheses that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \le \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is, $\lim_{n\to\infty} f(d(x_n, F(T))) = 0.$ Since $f:[0,\infty)\to[0,\infty)$ is right-continuous at 0 and $f(0)=0$, therefore we have $\lim_{n\to\infty} d(x_n, F(T)) = 0.$ Thus, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ and $\limsup_{n\to\infty} d(x_n, F(T)) = 0.$ By Theorem 5.3.3, the sequence $\{x_n\}$ converges strongly to q, a fixed point of T. This completes the proof.

Now, we give the following theorem which has a different hypothesis from Theorem 5.3.3.

Theorem 5.3.7. Let K be a nonempty closed convex subset of a complete CAT(0) space X, $T: K \to K$ be an asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and $\{u_n\}$ be a non-negative real sequence with $\sum_{n=1}^{\infty} u_n < \infty$. Suppose that $\{x_n\}$ is defined by the iteration process (5.3.1). If T is demi-compact and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. From the hypothesis, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Also, since T is demi-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$. Hence,

$$d(p, Tp) = \lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = 0.$$

Thus, $p \in F(T)$. By (5.3.3),

$$d(x_{n+1}, p) \le (1+u_n)^2 d(x_n, p) = (1 + 2u_n + u_n^2) d(x_n, p).$$

By Lemma 5.3.2, $\lim_{n\to\infty} d(x_n, p)$ exists and $x_{n_k} \to p \in F(T)$ gives that $x_n \to p \in F(T)$. This completes the proof.

CHAPTER 6. SOME CONVERGENCE RESULTS FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

In this section, the strong and Δ-convergence theorems for total asymptotically nonexpansive mappings are proved.

6.1. The Strong and Δ-Convergence of Some Iterative Algorithms in CAT(0) Spaces

In this subsection, we get some results which are related to the strong and Δ-convergence of the modified S-iteration and the modified two-step iteration for total asymptotically nonexpansive mappings on a CAT(0) space.

Thianwan [16] introduce the two-step iteration process in a Banach space. We give the modified two-step iteration process in a CAT(0) space as follows.

Definition 6.1.1. Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $T:K\to K$ be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping. The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-a_n)y_n \oplus a_n T^n y_n, \\ y_n = (1-b_n)x_n \oplus b_n T^n x_n, \ n\in\mathbb{N}. \end{cases} \tag{6.1.1}$$

If $b_n = 0$ for each $n\in\mathbb{N}$, then (6.1.1) is reduced to the modified Mann iteration process. By taking $T^n = T$ for all $n\in\mathbb{N}$ in (6.1.1), we obtain the two-step iteration process.

Chang et al. [7] proved demiclosedness principle for total asymptotically nonexpansive mappings in CAT(0) spaces as follows.

Lemma 6.1.2. ([7, Theorem 2.8]) Let K be a closed convex subset of a complete CAT(0) space X and let $T:K\to K$ be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(x_n,Tx_n)=0$ and $\Delta-\lim_{n\to\infty} x_n = w$. Then $Tw=w$.

The following lemma is crucial in the study of iteration processes in metric spaces.

Lemma 6.1.3. ([86, Lemma 2]) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of non-negative real numbers satisfying the inequality

$$a_{n+1}\le(1+\delta_n)a_n+b_n.$$

If $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}b_n<\infty$, then $\lim_{n\to\infty} a_n$ exists.

We prove the Δ-convergence theorem of the modified S-iteration process in a CAT(0) space.

Theorem 6.1.4. Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $T:K\to K$ be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping with $F(T)\ne\varnothing$ and let $\{x_n\}$ be a sequence defined by (5.3.1). If the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty}v_n<\infty,\ \sum_{n=1}^{\infty}\mu_n<\infty,\ \sum_{n=1}^{\infty}a_n<\infty;$

(ii) there exists a constant $M^*>0$ such that $\zeta(r)\le M^*r, r\ge 0;$

(iii) $\{b_n\}$ is the sequence in $[0,1];$

(iv) $\sum_{n=1}^{\infty}\sup\{d(z,T^nz):z\in B\}<\infty$ for each bounded subset B of K;

(v) there exist constants $b,c\in(0,1)$ with $0<b(1-c)\le\frac{1}{2}$ such that $\{a_n\}\subset[b,c]$.

Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. We divide the proof of Theorem 6.1.4 into three steps.

Step I. First we prove that $\lim_{n\to\infty} d(x_n,p)$ exists for each $p\in F(T)$. In fact for each $p\in F(T)$, by Lemma 2.1.15(i), we have

$$\begin{aligned}
d(y_n,p) &= d((1-b_n)x_n\oplus b_nT^nx_n,p)\\
&\le(1-b_n)d(x_n,p)+b_nd(T^nx_n,p)\\
&\le(1-b_n)d(x_n,p)+b_n\{d(x_n,p)+v_n\zeta(d(x_n,p))+\mu_n\}\\
&\le(1+b_nv_nM^*)d(x_n,p)+b_n\mu_n\\
&\le(1+v_nM^*)d(x_n,p)+\mu_n.
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
d(x_{n+1},p) &= d((1-a_n)T^nx_n\oplus a_nT^ny_n,p)\\
&\le(1-a_n)d(T^nx_n,p)+a_nd(T^ny_n,p)\\
&\le(1-a_n)\{d(x_n,p)+v_n\zeta(d(x_n,p))+\mu_n\}+a_nLd(y_n,p)\\
&\le(1-a_n)\{(1+v_nM^*)d(x_n,p)+\mu_n\}+a_nL\{(1+v_nM^*)d(x_n,p)+\mu_n\}\\
&=\{(1-a_n)(1+v_nM^*)+a_nL(1+v_nM^*)\}d(x_n,p)+(1+a_n(L-1))\mu_n\\
&=\{1+a_n(L-1)+v_nM^*(1+a_n(L-1))\}d(x_n,p)+(1+a_n(L-1))\mu_n.
\end{aligned}$$

It follows from condition (i) and Lemma 6.1.3 that $\lim_{n\to\infty} d(x_n,p)$ exists.

Step II. Next we prove that

$$\lim_{n\to\infty} d(x_n,Tx_n)=0. \tag{6.1.2}$$

In fact, it follows from Step I that for all $p\in F(T)$, $\lim_{n\to\infty} d(x_n,p)$ exists, so we can assume that $\lim_{n\to\infty} d(x_n,p)=r.$ Since

$$
\begin{aligned}
d(T^n y_n, p) &= d(T^n y_n, T^n p) \\
&\le d(y_n, p) + v_n \zeta(d(y_n, p)) + \mu_n \\
&\le (1 + v_n M^*) d(y_n, p) + \mu_n \\
&\le (1 + v_n M^*)\{(1 + v_n M^*) d(x_n, p) + \mu_n\} + \mu_n \\
&= (1 + v_n M^*)(1 + v_n M^*) d(x_n, p) + (2 + v_n M^*)\mu_n,
\end{aligned}
$$

then we have $\limsup_{n\to\infty} d(T^n y_n, p) \le r$. Similarly, we obtain $\limsup_{n\to\infty} d(T^n x_n, p) \le r$. On the other hand, since

$$\lim_{n\to\infty} d((1-a_n)T^n x_n \oplus a_n T^n y_n, p) = \lim_{n\to\infty} d(x_{n+1}, p) = r,$$

by Lemma 2.1.16, we have

$$\lim_{n\to\infty} d(T^n x_n, T^n y_n) = 0. \qquad \textbf{(6.1.3)}$$

Since

$$d(x_{n+1}, T^n x_n) \le d((1-a_n)T^n x_n \oplus a_n T^n y_n, T^n x_n) \le a_n d(T^n y_n, T^n x_n)$$

from (6.1.3), we obtain

$$\lim_{n\to\infty} d(x_{n+1}, T^n x_n) = 0. \qquad \textbf{(6.1.4)}$$

From condition (iv), we have

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0. \qquad \textbf{(6.1.5)}$$

Hence from (6.1.4) and (6.1.5), we get

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \qquad \textbf{(6.1.6)}$$

Since T is a uniformly L-Lipschitzian mapping, from (6.1.5) and (6.1.6) we have that

$$
\begin{aligned}
d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + d(T^{n+1} x_{n+1}, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\
&\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + L d(x_{n+1}, x_n) + L d(T^n x_n, x_n) \\
&= (1+L) d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + L d(T^n x_n, x_n) \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

The equation (6.1.2) is proved.

Step III. To show that the sequence $\{x_n\}$ Δ-converges to a fixed point of T, we prove that $W_\Delta(x_n) \subseteq F(T)$ and $W_\Delta(x_n)$ consists of exactly one point. Let $u \in W_\Delta(x_n)$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.1.21 and 2.1.22, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in K$. By Lemma 6.1.2, $v \in F(T)$. Since $\{d(u_n, v)\}$ converges, by Lemma 2.1.23, $u = v \in F(T)$. This shows that $W_\Delta(x_n) \subseteq F(T)$. Now we prove that $W_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F(T)$. Finally, since $\{d(x_n, v)\}$ converges, by Lemma 2.1.23, $x = v \in F(T)$. This shows that $W_\Delta(x_n) = \{x\}$. This completes the proof.

Now we give an example of such mappings which are total asymptotically nonexpansive and uniformly L-Lipschitzian as in Theorem 6.1.4.

Example 6.1.5. Let $\mathbb{R}$ be the real line with the usual absolute metric and let $K = [-1,1]$. Define two mappings $T, S : K \to K$ by

$$T(x) = \begin{cases} -2\sin\dfrac{x}{2}, & \text{if } x \in [0,1] \\ 2\sin\dfrac{x}{2}, & \text{if } x \in [-1,0) \end{cases} \quad \text{and} \quad S(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

It is proved in [87, Example 3.1] that both T and S are asymptotically nonexpansive mappings. Therefore they are total asymptotically nonexpansive and uniformly L-Lipschitzian mappings. Additionally, $F(T) = \{0\}$ and $F(S) = \{x \in K; 0 \le x \le 1\}$.

We give the characterization of strong convergence for the modified S-iteration process in a CAT(0) space as follows.

Theorem 6.1.6. Let $X, K, T, \{a_n\}, \{b_n\}, \{x_n\}$ satisfy the hypotheses of Theorem 6.1.4. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Teorem 6.1.4 (Step I), for all $p \in F(T)$,

$$d(x_{n+1}, p) \le \{1 + a_n(L-1) + v_n M^*(1 + a_n(L-1))\} d(x_n, p) + (1 + a_n(L-1))\mu_n.$$

This implies that

$$d(x_{n+1}, F(T)) \le \{1 + a_n(L-1) + v_n M^*(1 + a_n(L-1))\} d(x_n, F(T)) + (1 + a_n(L-1))\mu_n.$$

By Lemma 6.1.3, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(T)) = 0$. The conclusion now follows from Theorem 3.1.5.

Theorem 6.1.7. ([88, Theorem 3]) Let $X, K, T, \{a_n\}, \{b_n\}, \{x_n\}$ satisfy the hypotheses of Theorem 6.1.4 and let T be a mapping satisfying condition (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Remark 6.1.8. Theorems 6.1.4, 6.1.6, 6.1.7 contain the some results of Khan and Abbas [48, Theorems 1-3] since each nonexpansive mapping is a total asymptotically nonexpansive mapping.

Now, we give the Δ-convergence theorem of the modified two-step iteration process in a CAT(0) space.

Theorem 6.1.9. Let $X, K, T, \{b_n\}$ satisfy the hypotheses of Theorem 6.1.4, $\{a_n\}$ be a sequence in $[0,1]$ and let $\{x_n\}$ be a sequence defined by (6.1.1). If the conditions (i)-(iv) in Theorem 6.1.4 are satisfied, then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. First we will prove that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. As proved in Theorem 6.1.4, we have

$$d(y_n, p) \le (1 + v_n M^*) d(x_n, p) + \mu_n. \qquad \textbf{(6.1.7)}$$

Since T is a uniformly L-Lipschitzian mapping, from (6.1.7) we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-a_n)y_n \oplus a_n T^n y_n, p) \\ &\le (1-a_n)d(y_n, p) + a_n d(T^n y_n, p) \\ &\le (1-a_n)d(y_n, p) + a_n L d(y_n, p) \\ &= (1+a_n(L-1))d(y_n, p) \\ &\le (1+a_n(L-1))\{(1+v_n M^*)d(x_n, p) + \mu_n\} \\ &= \{1+a_n(L-1) + v_n M^*(1+a_n(L-1))\}d(x_n, p) + (1+a_n(L-1))\mu_n. \end{aligned}$$

It follows from Lemma 6.1.3 that $\lim_{n\to\infty} d(x_n, p)$ exists. Next we prove that $\lim_{n\to\infty} d(x_n, Tx_n) = 0.$ From condition (iv), we have

$$\lim_{n\to\infty} d(x_n, T^n x_n) = \lim_{n\to\infty} d(y_n, T^n y_n) = 0. \tag{6.1.8}$$

By the above equality, we get

$$d(T^n x_n, T^n y_n) \le L d(x_n, y_n) \le L b_n d(x_n, T^n x_n) \le L d(x_n, T^n x_n) \to 0 \text{ as } n \to \infty. \tag{6.1.9}$$

Since

$$d(x_{n+1}, T^n y_n) \le d((1-a_n)y_n \oplus a_n T^n y_n, T^n y_n) \le (1-a_n)d(y_n, T^n y_n)$$

from (6.1.8), we obtain

$$\lim_{n\to\infty} d(x_{n+1}, T^n y_n) = 0. \tag{6.1.10}$$

From (6.1.8), (6.1.9) and (6.1.10) we have that

$$d(x_n, x_{n+1}) \le d(x_n, T^n x_n) + d(T^n x_n, T^n y_n) + d(T^n y_n, x_{n+1}) \to 0 \text{ as } n \to \infty.$$

The rest of the proof follows the pattern of the Theorem 6.1.4.

Remark 6.1.10. Theorem 6.1.9 contains the main result of Chang et. al. [7, Theorem 3.5] since the modified two-step iteration is reduced to the modified Mann iteration. Also, Theorem 6.1.9 contains the main result of Nanjaras and Panyanak [35, Theorem 5.7] since each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping.

We give following theorems related to the strong convergence of the modified two-step iteration process which their proofs are similar arguments of Theorem 6.1.6 and Theorem 6.1.7, respectively.

Theorem 6.1.11. Let $X, K, T, \{a_n\}, \{b_n\}, \{x_n\}$ satisfy the hypotheses of Theorem 6.1.9. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0.$

Theorem 6.1.12. Let $X, K, T, \{a_n\}, \{b_n\}, \{x_n\}$ satisfy the hypotheses of Theorem 6.1.9 and let T be a mapping satisfying condition (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

6.2. The Strong and Δ-Convergence of Modified SP-Iteration Scheme in Hyperbolic Spaces

In this subsection, we prove some strong and Δ-convergence theorems of the modified SP-iteration process for approximating a fixed point of total asymptotically nonexpansive mappings in hyperbolic spaces.

The following iteration process is a translation of the SP-iteration from Banach spaces to hyperbolic spaces.

Definition 6.2.1. [89] Let K be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : K \to K$ be a uniformly L-Lipschitzian and total asymptotically nonexpansive mapping. The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = W(y_n, T^n y_n, \alpha_n), \\ y_n = W(z_n, T^n z_n, \beta_n), \\ z_n = W(x_n, T^n x_n, \gamma_n), \quad n \in \mathbb{N}. \end{cases} \tag{6.2.1}$$

Remark 6.2.2. ([90, Theorem 3.1]) Every total asymptotically nonexpansive mapping defined on a nonempty bounded closed convex subset of a complete uniformly convex hyperbolic space always has a fixed point.

We give Δ-convergence of the modified SP-iterative sequence $\{x_n\}$ defined by (6.2.1) for total asymptotically nonexpansive mappings in hyperbolic spaces.

Theorem 6.2.3. Let K be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T: K \to K$ be a uniformly L-Lipschitzian and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. If the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty} v_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$;

(ii) there exist constants $a, b \in (0,1)$ such that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a,b]$;

(iii) there exists a constant $M > 0$ such that $\zeta(r) \leq Mr,\ \forall r \geq 0$,

Then the sequence $\{x_n\}$ defined by (6.2.1), Δ-converges to a fixed point of T.

Proof. We divide our proof into three steps.

Step 1. First we prove that the following limits exist:

$$\lim_{n\to\infty} d(x_n, p) \text{ for each } p \in F(T) \text{ and } \lim_{n\to\infty} d(x_n, F(T)). \qquad \textbf{(6.2.2)}$$

Since T is a total asymptotically nonexpansive mapping, by the condition (iii), we get

$$\begin{aligned} d(z_n, p) &= d(W(x_n, T^n x_n, \gamma_n), p) \\ &\leq (1-\gamma_n) d(x_n, p) + \gamma_n d(T^n x_n, p) \\ &\leq (1-\gamma_n) d(x_n, p) + \gamma_n \{d(x_n, p) + v_n \zeta(d(x_n, p)) + \mu_n\} \\ &= d(x_n, p) + \gamma_n v_n \zeta(d(x_n, p)) + \gamma_n \mu_n \\ &\leq (1 + \gamma_n v_n M) d(x_n, p) + \gamma_n \mu_n \qquad \textbf{(6.2.3)} \end{aligned}$$

and

$$\begin{aligned} d(y_n, p) &= d(W(z_n, T^n z_n, \beta_n), p) \\ &\le (1-\beta_n) d(z_n, p) + \beta_n d(T^n z_n, p) \\ &\le (1-\beta_n) d(z_n, p) + \beta_n \{d(z_n, p) + v_n \zeta(d(z_n, p)) + \mu_n\} \\ &\le (1+\beta_n v_n M) d(z_n, p) + \beta_n \mu_n. \end{aligned} \quad \textbf{(6.2.4)}$$

Substituting (6.2.3) into (6.2.4) and simplifying it, we have

$$\begin{aligned} d(y_n, p) &\le (1+\beta_n v_n M)\{(1+\gamma_n v_n M) d(x_n, p) + \gamma_n \mu_n\} + \beta_n \mu_n \\ &\le (1 + v_n M(\beta_n + \gamma_n + \beta_n \gamma_n v_n M)) d(x_n, p) + \mu_n(\beta_n + \gamma_n + \beta_n \gamma_n v_n M). \end{aligned} \quad \textbf{(6.2.5)}$$

Similarly, we obtain

$$d(x_{n+1}, p) \le (1+\alpha_n v_n M) d(y_n, p) + \alpha_n \mu_n. \quad \textbf{(6.2.6)}$$

Combining (6.2.5) and (6.2.6), we have

$$d(x_{n+1}, p) \le (1+\sigma_n) d(x_n, p) + \xi_n, \ \forall n \ge 1 \text{ and } p \in F(T), \quad \textbf{(6.2.7)}$$

and so

$$d(x_{n+1}, F(T)) \le (1+\sigma_n) d(x_n, F(T)) + \xi_n, \ \forall n \ge 1,$$

where $\sigma_n = v_n M(\alpha_n + \beta_n + \gamma_n + v_n M(\alpha_n \beta_n + \beta_n \gamma_n + \alpha_n \gamma_n + \alpha_n \beta_n \gamma_n v_n M))$ and $\xi_n = \alpha_n + \beta_n + \gamma_n + v_n M(\alpha_n \beta_n + \beta_n \gamma_n + \alpha_n \gamma_n + \alpha_n \beta_n \gamma_n v_n M)$. By virtue of the condition (i),

$$\sum_{n=1}^{\infty} \sigma_n < \infty \text{ and } \sum_{n=1}^{\infty} \xi_n < \infty.$$

By Lemma 6.1.3, $\lim_{n\to\infty} d(x_n, F(T))$ and $\lim_{n\to\infty} d(x_n, p)$ exist for each $p \in F(T)$.

Step 2. Next we prove that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. In fact, it follows from (6.2.2) that $\lim_{n\to\infty} d(x_n, p)$ exists for each given $p \in F(T)$. We may assume that $\lim_{n\to\infty} d(x_n, p) = r$. The case $r = 0$ is trivial. Next, we deal with the case $r > 0$. Taking limsup on both sides in the inequality (6.2.5), we have

$$\limsup_{n\to\infty} d(y_n, p) \le r. \quad \textbf{(6.2.8)}$$

Since

$$\begin{aligned} d(T^n y_n, p) &\le d(y_n, p) + v_n\zeta(d(y_n, p)) + \mu_n \\ &\le (1+v_n M)d(y_n, p) + \mu_n, \ \forall n \ge 1, \end{aligned}$$

we have

$$\limsup_{n\to\infty} d(T^n y_n, p) \le r. \tag{6.2.9}$$

In addition,

$$\lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d(W(y_n, T^n y_n, \alpha_n), p) = r. \tag{6.2.10}$$

With the help of (6.2.8)-(6.2.10) and Lemma 2.2.10, we have

$$\lim_{n\to\infty} d(y_n, T^n y_n) = 0. \tag{6.2.11}$$

On the other hand, since

$$\begin{aligned} d(x_{n+1}, p) &\le d(x_{n+1}, T^n y_n) + d(T^n y_n, p) \\ &\le (1-\alpha_n)d(y_n, T^n y_n) + (1+v_n M)d(y_n, p) + \mu_n \end{aligned}$$

we have $\liminf_{n\to\infty} d(y_n, p) \ge r$. Combined with (6.2.8), it yields that $\lim_{n\to\infty} d(y_n, p) = r$. This implies that

$$\lim_{n\to\infty} d(W(z_n, T^n z_n, \beta_n), p) = r. \tag{6.2.12}$$

Taking limsup on both sides in the inequality (6.2.3), we have

$$\limsup_{n\to\infty} d(z_n, p) \le r. \tag{6.2.13}$$

Since

$$\begin{aligned} d(T^n z_n, p) &\le d(z_n, p) + v_n\zeta(d(z_n, p)) + \mu_n \\ &\le (1+v_n M)d(z_n, p) + \mu_n, \ \forall n \ge 1, \end{aligned}$$

we have

$$\limsup_{n\to\infty} d(T^n z_n, p) \le r. \tag{6.2.14}$$

With the help of (6.2.12)-(6.2.14) and Lemma 2.2.10, we have

$$\lim_{n\to\infty} d(z_n, T^n z_n) = 0. \quad \textbf{(6.2.15)}$$

By the same method, we can also prove that

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0. \quad \textbf{(6.2.16)}$$

By (6.2.11), we get

$$d(x_{n+1}, y_n) \le d(W(y_n, T^n y_n, \alpha_n), y_n) \le \alpha_n d(y_n, T^n y_n) \to 0 \text{ as } n \to \infty.$$

In a similar way, we have

$$d(y_n, z_n) \le \beta_n d(z_n, T^n z_n) \to 0 \text{ as } n \to \infty$$

and

$$d(z_n, x_n) \le \alpha_n d(x_n, T^n x_n) \to 0 \text{ as } n \to \infty.$$

It follows that

$$d(x_{n+1}, x_n) \le d(x_{n+1}, y_n) + d(y_n, z_n) + d(z_n, x_n) \to 0 \text{ as } n \to \infty. \quad \textbf{(6.2.17)}$$

Since T is uniformly L-Lipschitzian, therefore we obtain

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} x_{n+1}) + d(T^{n+1} x_{n+1}, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\ &\le (1+L) d(x_{n+1}, x_n) + d(x_{n+1}, T^{n+1} x_{n+1}) + L d(T^n x_n, x_n). \end{aligned}$$

Hence, (6.2.16) and (6.2.17) imply that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \quad \textbf{(6.2.18)}$$

Step 3. Now we prove the sequence $\{x_n\}$ Δ-converges to a fixed point of T. In fact, for each $p \in F(T)$, $\lim_{n\to\infty} d(x_n, p)$ exists. This implies that the sequence $\{x_n\}$ is bounded. Hence by virtue of Lemma 2.2.9, $\{x_n\}$ has a unique asymptotic center $A_K(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A_K(\{u_n\}) = \{u\}$. Then, by (6.2.18), we have

$$\lim_{n\to\infty} d(u_n, Tu_n) = 0. \quad \textbf{(6.2.19)}$$

We claim that $u \in F(T)$. In fact, we define a sequence $\{z_m\}$ in K by $z_m = T^m u$. So, we calculate

$$\begin{aligned} d(z_m,u_n) &\le d(T^m u,T^m u_n) + d(T^m u_n,T^{m-1}u_n) + \ldots + d(Tu_n,u_n) \\ &\le d(u,u_n) + v_n\zeta(d(u,u_n)) + \mu_n + \sum_{i=1}^{m} d(T^i u_n,T^{i-1}u_n) \\ &\le (1+v_n M)d(u,u_n) + \mu_n + \sum_{i=1}^{m} d(T^i u_n,T^{i-1}u_n). \end{aligned} \tag{6.2.20}$$

Since T is uniformly L-Lipschitzian, from (6.2.20), we have

$$d(z_m,u_n) \le (1+v_n M)d(u,u_n) + \mu_n + mLd(Tu_n,u_n).$$

Taking limsup on both sides of the above estimate and using (6.2.19), we have

$$r(z_m,\{u_n\}) = \limsup_{n\to\infty} d(z_m,u_n) \le \limsup_{n\to\infty} d(u,u_n) = r(u,\{u_n\}).$$

This implies that $|r(z_m,\{u_n\}) - r(u,\{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 2.2.11 that $\lim_{m\to\infty} T^m u = u$. Utilizing the uniform continuity of T, we have that

$$Tu = T(\lim_{m\to\infty} T^m u) = \lim_{m\to\infty} T^{m+1} u = u.$$

Hence $u \in F(T)$. Moreover, $\lim_{n\to\infty} d(x_n,u)$ exists by (6.2.2). Suppose that $x \ne u$. By the uniqueness of asymptotic centers, we have

$$\begin{aligned} \limsup_{n\to\infty} d(u_n,u) &< \limsup_{n\to\infty} d(u_n,x) \\ &\le \limsup_{n\to\infty} d(x_n,x) \\ &< \limsup_{n\to\infty} d(x_n,u) \\ &= \limsup_{n\to\infty} d(u_n,u) \end{aligned}$$

a contradiction. Hence $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$, that is, $\{x_n\}$ Δ-converges to $x \in F(T)$. The proof is completed.

We now discuss the strong convergence of the modified SP-iteration for total asymptotically nonexpansive mappings in hyperbolic spaces.

Theorem 6.2.4. Let K, X, T and $\{x_n\}$ be the same as in Theorem 6.2.3. Suppose that the conditions (i)-(iii) in Theorem 6.2.3 are satisfied. Then $\{x_n\}$ converges strongly to some $p \in F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0.$

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0.$ It follows from (6.2.2) that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(T)) = 0.$ Next, we show that $\{x_n\}$ is a Cauchy sequence. In fact, it follows from (6.2.7) that for any $p \in F(T)$

$$d(x_{n+1}, p) \le (1+\sigma_n) d(x_n, p) + \xi_n, \ \forall n \ge 1,$$

where $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\sum_{n=1}^{\infty} \xi_n < \infty.$ Hence for any posititve integers n, m, we have

$$d(x_{n+m}, x_n) \le d(x_{n+m}, p) + d(p, x_n) \le (1+\sigma_{n+m-1}) d(x_{n+m-1}, p) + \xi_{n+m-1} + d(x_n, p).$$

Since for each $x \ge 0,\ 1 + x \le e^x,$ we have

$$\begin{aligned} d(x_{n+m}, x_n) &\le e^{\sigma_{n+m-1}} d(x_{n+m-1}, p) + \xi_{n+m-1} + d(x_n, p) \\ &\le e^{\sigma_{n+m-1}+\sigma_{n+m-2}} d(x_{n+m-2}, p) + e^{\sigma_{n+m-1}} \xi_{n+m-2} + \xi_{n+m-1} + d(x_n, p) \\ &\le \ldots \\ &\le e^{\sum_{i=n}^{n+m-1} \sigma_i} d(x_n, p) + e^{\sum_{i=n+1}^{n+m-1} \sigma_i} \xi_n + e^{\sum_{i=n+2}^{n+m-2} \sigma_i} \xi_{n+1} + \ldots \\ &\quad + e^{\sigma_{n+m-1}} \xi_{n+m-2} + \xi_{n+m-1} + d(x_n, p) \\ &\le (1+N) d(x_n, p) + N \sum_{i=n}^{n+m-1} \xi_i, \end{aligned}$$

where $N = e^{\sum_{i=1}^{\infty} \sigma_i} < \infty.$ Therefore we have

$$d(x_{n+m}, x_n) \le (1+N) d(x_n, F(T)) + N \sum_{i=n}^{n+m-1} \xi_i \to 0 \ \text{ as } n, m \to \infty.$$

This shows that $\{x_n\}$ is a Cauchy sequence in K. Since K is a closed subset in a complete hyperbolic space X, it is complete. We can assume that $\{x_n\}$ converges strongly to some point $p^* \in K$. It is easy to prove that $F(T)$ is closed subset in K, so is $F(T)$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, we obtain $p^* \in F(T)$. This completes the proof.

Remark 6.2.5. In Theorem 6.2.4, the condition $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ may be replaced with $\limsup_{n\to\infty} d(x_n, F(T)) = 0$.

Example 6.2.6. Let $\mathbb{R}$ be the real line with the usual absolute metric and let $K = [-1,1]$. Define two mappings $T_1, T_2 : K \to K$ by

$$T_1(x) = \begin{cases} -2\sin\dfrac{x}{2}, & \text{if } x \in [0,1] \\ 2\sin\dfrac{x}{2}, & \text{if } x \in [-1,0) \end{cases} \quad \text{and} \quad T_2(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

It is proved in [87, Example 3.1] that both T_1 and T_2 are asymptotically nonexpansive mappings with $k_n = 1$, $\forall n \geq 1$. Therefore they are total asymptotically nonexpansive mappings with $v_n = \mu_n = 0$, $\forall n \geq 1$, $\zeta(t) = t$, $\forall t \geq 0$. Additionally, they are uniformly L-Lipschitzian mappings with $L = 1$. Clearly, $F(T_1) = \{0\}$ and $F(T_2) = \{x \in K; 0 \leq x \leq 1\}$. Set

$$\alpha_n = \frac{n}{2n+1}, \beta_n = \frac{n}{3n+1} \text{ and } \gamma_n = \frac{n}{4n+1} \text{ for all } n \geq 1. \qquad \textbf{(6.2.21)}$$

Thus, the conditions of Theorem 6.2.3 are fulfilled. Therefore the results of Theorem 6.2.3 and Theorem 6.2.4 can be easily seen.

Example 6.2.7. Let $\mathbb{R}$ be the real line with the usual absolute metric and let $K = [0,\infty)$. Define two mappings $S_1, S_2 : K \to K$ by $S_1(x) = \sin x$ and $S_2(x) = x$. It is proved in [91, Example 1] that both S_1 and S_2 are total asymptotically nonexpansive mappings with $v_n = \frac{1}{n^2}$, $\mu_n = \frac{1}{n^3}$, $\forall n \geq 1$. Additionally, they are uniformly L-Lipschitzian mappings with $L = 1$. Clearly, $F(S_1) = \{0\}$ and $F(S_2) = \{x \in K; 0 \leq x < \infty\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be the same as in (6.2.21).

Similarly, the conditions of Theorem 6.2.3 are satisfied. So, the results of Theorem 6.2.3 and Theorem 6.2.4 also can be received.

Theorem 6.2.8. Under the assumptions of Theorem 6.2.3, if T is demi-compact, then $\{x_n\}$ converges strongly to a fixed point of T

Proof. It follows from (6.2.2) that $\{x_n\}$ is a bounded sequence. Also, by (6.2.18), we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, by demi-compactness of T, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some point $p \in K$. Moreover, by the uniform continuity of T, we have

$$d(p,Tp) = \lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = 0.$$

This implies that $p \in F(T)$. Again, by (6.2.2), $\lim_{n\to\infty} d(x_n, p)$ exists. Hence p is the strong limit of the sequence $\{x_n\}$. As a result, $\{x_n\}$ converges strongly to a fixed point p of T.

Theorem 6.2.9. Under the assumptions of Theorem 6.2.3, if T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By virtue of (6.2.2), $\lim_{n\to\infty} d(x_n, F(T))$ exists. Further, by the condition (I) and (6.2.18), we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Now Theorem 6.2.4 implies that $\{x_n\}$ converges strongly to a point p in $F(T)$.

Remark 6.2.10. Theorems 6.2.3-6.2.9 contain the corresponding theorems proved for asypmtotically nonexpansive mappings when $v_n = k_n - 1$, $\mu_n = 0$, $\forall n \geq 1$, $\zeta(t) = t$, $\forall t \geq 0$.

CHAPTER 7. RESULTS AND SUGGESTIONS

In this section, the results obtained from the previous sections of thesis will be summarized. The third, fourth, fifth and sixth sections of this thesis equipped with original works.

In the first part of third section, we prove the strong and Δ-convergence theorems of SP-iteration for nonexpansive mappings on CAT(0) spaces. Since SP-iteration is reduced to the new multi-step and the Mann iterations, then these results extend and generalize some works in the literature. In the second part of third section, we get some results on the strong and Δ-convergence of the iteration process of Khan et. al. [47] for nonexpansive mappings in uniformly convex hyperbolic spaces. Moreover, we give an example to support our reults. These results generalize some results which is given in [24, 47, 48].

There are three parts in the fourth chapter. In the first part of it, we study the S-iteration process for mappings satisfying condition (C) which are weaker than nonexpansive mappings in CAT(0) spaces and generalize some results of Khan and Abbas [48]. In the second part of this chapter, we present the strong and Δ-convergence theorems of the new three-step iteration for mappings satisfying condition (C) in CAT(0) spaces. Since every nonexpansive mapping satisfies condition (C) and the new three-step iteration is reduced to the new two-step iteration, S-iteration and SP-iteration processes, then these results extend and improve some results in the literature. In the last part of it, we study the S-iteration and the Noor iteration processes for nonself mappings satisfying condition (E) in CAT(0) spaces. These results generalize some results of Khan and Abbas [48], Razani and Salahifard [92] and Razani and Shabani [93].

In the first part of fifth section, we prove the Δ-convergence theorems of the cyclic algorithm and the new multi-step iteration for k-strictly pseudo-contractive mappings and give also the strong convergence theorem of the modified Halpern's iteration for these mappings in a CAT(0) space. In the second part of it, we introduce a new class of mappings and examine the properties of these mappings with some examples. We prove the Δ-convergence theorems of the new multi-step iteration and S-iteration processes for mappings of this type in a CAT(0) space. Also, we present the strong convergence theorems of these iteration processes for contractive-like mappings in a CAT(0) space. In the last part of it, we give the strong convergence theorems of the modified S-iteration process for asymptotically quasi-nonexpansive mappings in a CAT(0) space. These results presented in this section extend and improve some works for a CAT(0) space in the literature.

In the first part of chapter 6, we get some results which are related to the strong and Δ-convergence theorems of the modified S-iteration and the modified two-step iteration processes for total asymptotically nonexpansive mappings in a CAT(0) space. Also, an example which satisfies our main result, have been given. These results extend and improve the corresponding ones announced by Chang et. al. [7], Nanjaras and Panyanak [35] and Khan and Abbas [48] and many others. In the second part of this chapter, we prove some strong and Δ-convergence theorems of the modified SP-iteration process for total asymptotically nonexpansive mappings in hyperbolic spaces by employing recent technical results of Khan et. al. [46]. Moreover, we give some examples to support our results. These results generalize some recent results given in [17, 56].

These results related to CAT(0) space can be generalized to CAT(κ) spaces and hyperbolic spaces. They can be researched in the future.

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RESUME

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