Magic coverings and the Kronecker product

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Abstract

In this paper we study a relationship existing among (super) magic coverings and the well known Kronecker product.

Keyworks: magic covering, super magic covering, Kronecker product.

1 Introduction

For the undefined concepts and notation found in this paper we refer the reader to [5], [7] or [12]. Let $G = (V, E)$ and $H = (V', E')$ be simple graphs. We say that G admits an $H-$*covering* if every edge of G belongs to at least one subgraph of G isomorphic to H. In an H-covering of G, we denote by the H-*degree* of a vertex the number of subgraphs isomorphic to H that the vertex belongs to, and by an H-*sequence*, a sequence of all H-degrees of the graph. For $m, n \in \mathbb{Z}, m < n$ we write $[m, n] = \{m, m+1, \ldots, n\}$. Furthermore, G is $H-$*magic* if there is a bijective mapping $f : V \cup E \longrightarrow [1, |V| + |E|]$ such that if $H_1 = (V_1, E_1)$ is a subgraph of G isomorphic to H then $\sum_{u\in V_1} f(u) + \sum_{e\in E_1} f(e)$ does not depend on H_1.

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If $f(V) = [1, |V|]$ then we say that G admits an *H–super magic covering.* When G admits an H-(super) magic covering we simply say that G is *H-(super) magic.* The concept of H–(super) magic covering was first introduced by A. Gutiérrez and A. Lladó [8].

In this paper we introduce the concept of H-(super) magicness for digraphs. We introduce this concept since it will prove to be useful for studying the H-(super) magic properties of several types of graphs. It is worthwhile to point out that we do not pretend, in this paper, to start the study of the H-(super) magic properties of digraphs as a goal. We study them just as a way to obtain results involving graphs.

Let $\vec{G} = (V, E)$ and $\vec{H} = (V', E')$ be two oriented digraphs where und($\vec{G}$)=G and und($\vec{H}$)=H. We say that $\vec{G}$ is $\vec{H}$-(super) magic if G is H-(super) magic.

The main goal of the paper is the use of the Kronecker product in order to obtain results involving labelings. It is not the first time that this idea is used. In [6] Figueroa-Centeno et al. used a similar one in order to study (super) edge magic labelings of graphs. In fact they introduced a more general product, that was used in order to generate labelings. Unfortunately, when working with coverings, we need to restrict ourselves to the Kronecker product, and further generalizations do not seem to work. Also Bača et al., [1], restricted the product defined in [6] in order to find lower bounds for the number of super edge magic labelings of the disjoint union of path–like trees. See either [2] or [3] for the definition and further information regarding path–like trees.

2 The main result

Let us denote by $+_n$ the sum of two elements in the group $\mathbb{Z}_n$. Next we introduce the $\mathbb{Z}_n$–property for a digraph $\vec{G} = (V, E)$ as follows. A digraph $\vec{G} = (V, E)$ has the *$\mathbb{Z}_n$–property* if the vertices of $\vec{G}$ can be labeled with the elements of $\mathbb{Z}_n$ in such a way that if $(u, v) \in E$ and u is labeled with i then v is labeled with $i +_n 1$, and no vertex is labeled with more than one label. Such a labeling will be called a *$\mathbb{Z}_n$–labeling.*

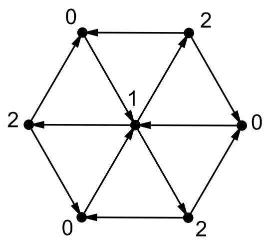

Figure 1: A $\mathbb{Z}_3$-label of the wheel

A *weakly connected component* of a digraph $\vec{G}$ is a connected component of its underlying graph. If $\vec{G}$ has the $\mathbb{Z}_n$–property then every weakly connected component of $\vec{G}$ admits exactly n different $\mathbb{Z}_n$–labelings. Moreover, each one of them is perfectly determined by the label on any vertex of the component.

Note that, if $\vec{G}$ has the $\mathbb{Z}_n$–property then its line digraph $L(\vec{G})$ (the digraph with vertex set consisting of the arcs of $\vec{G}$ and $((u,v),(w,t))$ is an arc of $L(\vec{G})$ whenever $v = w$) also has the $\mathbb{Z}_n$–property. For instance, the labeling that assigns to $(u,v) \in E(\vec{G})$ the label of v in a $\mathbb{Z}_n$–labeling of $\vec{G}$ is a $\mathbb{Z}_n$–labeling of $L(\vec{G})$. Next we provide several examples of graphs that admit the $\mathbb{Z}_n$–property.

Exemple 2.1

- *Rooted trees oriented with all arcs coming out from the root and such that it is possible to travel from the root to any leaf following the direction of the arrows have the $\mathbb{Z}_n$–property for all n.*
- *Strong oriented cycles of order n have the $\mathbb{Z}_k$–property when $k \mid n$.*
- *If $\vec{G}$ and $\vec{H}$ have the $\mathbb{Z}_n$–property and $u \in V(G)$, $v \in V(H)$ then the digraph with vertex set $V(G) \cup V(H)$ and arc set $E(\vec{G}) \cup E(\vec{H}) \cup (u,v)$ has the $\mathbb{Z}_n$–property.*

If $\vec{G}$ has the $\mathbb{Z}_n$–property, lab is a $\mathbb{Z}_n$–labeling of $\vec{G}$ and $P : a_1, \ldots, a_l$ is a path in G from a_1 to a_l. The weight of P is said to be $w(P)$=lab(a_l)-lab(a_1).

The following are easy remarks regarding $\mathbb{Z}_n$–labelings.

1. The definition of weight is independent of the $\mathbb{Z}_n$–labeling chosen.
2. The weight of a path depends only on the labels of the leaves.
3. The weight of any closed path is zero.
4. Let Q be a path with initial vertex being the terminal vertex of another path P, then the weight of $P \cup Q$ is given by the sum of the weights of P and Q.

The Kronecker product of two digraphs $\vec{G} = (V,E)$ and $\vec{H} = (V',E')$ is the digraph $\vec{G} \bigotimes \vec{H}$ with vertex set $V \times V'$ and with an arc $((a,b),(c,d))$ whenever $(a,c) \in E$ and $(b,d) \in E'$. Let $\vec{C_n}$ be an strong oriented cycle of length n. Assume that the vertex set of $\vec{C_n}$ is $[0, n-1]$ and that i is adjacent to $i +_n 1$.

With this information developed so far in hand, we are now ready to state and proof our first lemma.

Lemma 2.1 *If $\vec{G}$ has the $\mathbb{Z}_n$–property then each weakly connected component of $\vec{G} \bigotimes \vec{C}_n$ also has de $\mathbb{Z}_n$–property. Furthermore, for each $x \in V$, if $i \neq j$ in $\mathbb{Z}_n$ then (x,i) and (x,j) belong to different weakly connected components of $\vec{G} \bigotimes \vec{C}_n$.*

Proof.
Let $x, y \in V$. If (x,i), (y,j) belong to the same weakly connected component of $\vec{G} \bigotimes \vec{C}_n$ then there is a path, in und$(\vec{G} \bigotimes \vec{C}_n)$ from (x,i) to (y,j) and therefore a path $\hat{P}_y$ in G from x to y with weight $w(\hat{P}_y) = j - i$. Hence, if there is a path in und$(\vec{G} \bigotimes \vec{C}_n)$ joining (x,i) and (x,j) then its related path in G has weight zero and therefore, $i = j$. Moreover, the labeling defined by lab(y,j)= $i +_n w(\hat{P}_y)$ is a $\mathbb{Z}_n$–labeling of $\vec{G} \bigotimes \vec{C}_n$. □

Theorem 2.2 *A digraph $\vec{G}$ has the $\mathbb{Z}_n$–property if and only if $\vec{G} \bigotimes \vec{C}_n$ consist of n disjoint copies of $\vec{G}$.*

Proof.
First of all, let us assume that $\vec{G}$ is weakly connected.

Suppose that $\vec{G}$ has the $\mathbb{Z}_n$–property and let $x_s \in V(G)$. The weakly connected component that contains (x_s, i) is isomorphic to $\vec{G}$ since the function defined by the rule $x \longmapsto (x, l)$ where $l - i$ is the weight of any path in und$(\vec{G} \bigotimes \vec{C}_n)$ from (x_s, i) to (x, l), is an injective morphism of digraphs. Hence each weakly connected component of $\vec{G} \bigotimes \vec{C}_n$ is a copy of $\vec{G}$.

Conversely, let $u, v \in V(G)$ and label u with j where $0 \leq j \leq n-1$. There exists $k \in [0, 1]$ such that (v, k) and (u, j) belong to the same weakly connected component of the product $\vec{G} \bigotimes \vec{C}_n$. If we assign k to vertex v, then we obtain a $\mathbb{Z}_n$–labeling of $\vec{G}$ since:

- If $(v, w) \in E(\vec{G})$ then $((v, k), (w, k +_n 1)) \in E(\vec{G} \bigotimes \vec{C}_n)$ and w is labeled with $k +_n 1$.
- Assume to the contrary that this construction produces a vertex w with two different labels, namely k and l. Then (w, k) and (w, l) belong to the same weakly connected component in $\vec{G} \bigotimes \vec{C}_n$, and such a connected component cannot be a copy of $\vec{G}$.

Finally, if $\vec{G} = \vec{G}_1 \cup \ldots \cup \vec{G}_k$ is a decomposition of $\vec{G}$ into weakly connected components, we apply the result to each of the components, since $\vec{G} \bigotimes \vec{C}_n = (\vec{G}_1 \bigotimes \vec{C}_n) \cup \ldots \cup (\vec{G}_k \bigotimes \vec{C}_n)$. □

Assume that we have a labeling on the vertices of $\vec{G}$, that assigns different labels to different vertices, and that we identify each vertex with its label. If $\vec{G}$ has the $\mathbb{Z}_n$–property then given a labeling on the vertices of $\vec{G}$ and a permutation, σ of $[0, n-1]$ we consider a labeling on $\vec{G} \bigotimes \vec{C}_n$ as follows:

$$\text{lab}_\sigma(u, j) = n(u-1) + \sigma(j) + 1 \quad \text{where} \quad u \in V(G),\ j \in [0, n-1]. \tag{1}$$

This labeling has the following properties.

Lemma 2.2 *Let σ and τ be two permutations of $[0, n-1]$.*

1. *The labels of $\vec{G} \bigotimes \vec{C}_n$ defined by equation (1) are different. Moreover, if the labels of G are all labels in $[1, k]$, then the labels of $\vec{G} \bigotimes \vec{C}_n$ defined by lab$_\sigma$ are in $[1, nk]$.*
2. *If there exists $k \in [0, n-1]$ such that $\sigma(j) = \tau(j +_n k), \forall j$, then the labels of the component of (u, j) labeled by lab$_\sigma$ coincide with the labels of the component of $(u, j +_n k)$ labeled by lab$_\tau$.*
3. *If lab$_\sigma$ =lab$_\tau$ then σ and τ are the same cyclic permutation.*

Proof.
Points 1. and 2. are clearly true. If lab$_\sigma$ =lab$_\tau$ then the labels of lab$_\sigma$ on the component that contains (u, k) must coincide with the labels assigned by lab$_\tau$ on the component that contains

the label $(u, k +_n l)$ for some l. Therefore, if the weight of any path from u to v is j then the labels $\mathrm{lab}_\sigma(v, k +_n j)$ and $\mathrm{lab}_\tau(v, k +_n j +_n l)$ must coincide. Thus, $\sigma(s) = \tau(s +_n l)$ for all s.□

Theorem 2.3 *Let $\vec{G}$ be a $\vec{C}_n$-magic digraph such that $\vec{G}$ has the $\mathbb{Z}_n$–property. Let $a_0, \dots, a_{n-1}$, $b_0, \dots, b_{n-1}$ be the labels of the vertices and arcs respectively of a subdigraph $\vec{H}$ of $\vec{G}$ isomorphic to $\vec{C}_n$. Then, for each pair of circular permutations σ, τ of $[0, n-1]$ the sum of labels of lab_σ and lab_τ, defined in (1), of the vertices and arcs in each copy of H in $und(\vec{G} \bigotimes \vec{C}_n)$ is respectively,*

$$n(\sum_{i=0}^{n-1} a_i - \frac{n-1}{2}) \quad \text{and} \quad n(\sum_{i=0}^{n-1} b_i - \frac{n-1}{2}). \tag{2}$$

Proof.
On H and $L(H)$, the sum of the labels of the vertices is $\sum_{i=0}^{n-1} a_i$ and $\sum_{i=0}^{n-1} b_i$ respectively. Therefore the sum of the labels of the vertices defined in (1) over each copy of H in $und(\vec{G} \bigotimes \vec{C}_n)$ and each copy of $L(H)$ in $und(L(\vec{G}) \bigotimes \vec{C}_n)$ is given respectively by

$$\sum_{i=0}^{n-1}(n(a_i - 1) + \sigma(i) + 1) \quad \text{and} \quad \sum_{i=0}^{n-1}(n(b_i - 1) + \tau(i) + 1).$$

This completes the proof. □

Corollary 2.1 *Let $\vec{G}$ be a $\vec{C}_n$-(super)magic digraph. If $\vec{G}$ has the Z_n-property, then $\vec{G} \bigotimes \vec{C}_n$ is $\vec{C}_n$-(super)magic. Furthermore, for each $\vec{C}_n$-(super)magic labeling of $\vec{G}$ there are $((n-1)!)^2$ non-isomorphic $\vec{C}_n$-super-magic labelings of $\vec{G} \bigotimes \vec{C}_n$.*

Proof.
The first part of the corollary is clearly true. For the second part, switching σ and τ in equation (1), we find $(n-1)!$ non-isomorphic labelings for the vertices and $(n-1)!$ non-isomorphic labelings for the edges. Combining all these labelings, we obtain the desired result. □

3 S–magic partitions

In this section we introduce a new related problem, that the theory developed so far helps us to partially solve. Let us define S–magic partitions next.

We say that a graph G is $\mathbb{Z}_n$–orientable if there is an oriented digraph $\vec{G}$ with the $\mathbb{Z}_n$–property and its underlying graph is G.

Let $S = \{s_1 \leq s_2 \leq \ldots \leq s_k\} \subset \mathbb{N}$ be a multiset of k elements. We say that $[1, nk]$ admits an S–magic partition into sets $B_1, B_2, \ldots, B_n$, all of them of cardinality k if $B_1, B_2, \ldots, B_n$ is a partition of $[1, nk]$ and there exist bijections: $\{f_i : B_i \longrightarrow S\}_{i=[1,n]}$, such that

$$\sum_{x \in B_i} x f_i(x) = \sum_{x \in B_j} x f_j(x), \text{ for all } i, j \in [1, n].$$

With this definition in hand we are ready to state and prove the following theorem.

Theorem 3.1 *Let $S = \{s_1 \leq s_2 \leq \ldots \leq s_k\}$ be a multiset of natural numbers and assume that there is a graph G with a C_n-covering and C_n-sequence $s_1 \leq s_2 \leq \ldots \leq s_k$. If G is $\mathbb{Z}_n$–orientable then $[1, nk]$ admits an S–magic partition into n sets of cardinality k.*

Proof.
Let $\vec{G}$ be a $\mathbb{Z}_n$-orientation of G. We label univocally the elements of $V(G)$ and the elements of $V(\vec{G} \bigotimes \vec{C}_n)$ with all elements of $[1, k]$ and the elements of $[1, nk]$ respectively, where the elements of $V(\vec{G} \bigotimes \vec{C}_n)$ are labeled according with formula (1). By Theorem 2.2, we know that $\vec{G} \bigotimes \vec{C}_n$ is isomorphic to the disjoint union of n disjoint components, all of them isomorphic to $\vec{G}$. We denote by B_i the set of labels of component i. By Lemma 2.2, $B_1, \ldots, B_n$ is a partition of $[1, nk]$. For $x \in B_i$, we let $f_i(x)$ to be the C_n-degree of the vertex labeled x. Let $C^1, \ldots, C^l$ be the C_n-covering of $\vec{G}$ and $C_i^1, \ldots, C_i^l$ be the induced C_n-covering of the i^{th} component of the product $\vec{G} \bigotimes \vec{C}_n$. We denote by lab (H) the sum of the labels of H. By Theorem 2.3 $\text{lab}(C_i^k) = \text{lab}(C_j^k)$, for each pair $i, j \in [1, n]$ and $k \in [1, l]$. Therefore,

$$\sum_{x \in B_i} x \cdot f_i(x) = \sum_{k \in [1,l]} \text{lab}(C_i^k) = \sum_{k \in [1,l]} \text{lab}(C_j^k) = \sum_{x \in B_j} x \cdot f_j(x), \ \forall i, j \in [1, n]$$

□

Exemple 3.2 *Let us see that $[1, 18]$ admits and S-magic partition into 3 sets of cardinality 6, for $S = \{1, 1, 1, 3, 3, 3\}$. Let $\vec{G}$ be the digraph that in Figure 2 is multiplied by $\vec{C}_3$. The product $\vec{G} \bigotimes \vec{C}_3$ results into three copies of $\vec{G}$.*

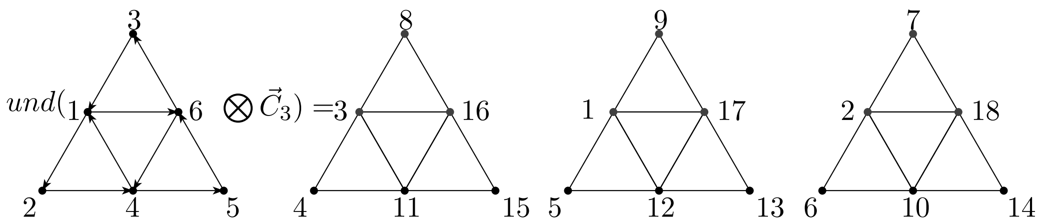

Figure 2: An S-magic partition of $[1, 18]$

The labels of each copy are the elements of each part of an S-magic partition:

$$\begin{cases} 1 \cdot 4 + 1 \cdot 8 + 1 \cdot 15 + 3 \cdot 3 + 3 \cdot 11 + 3 \cdot 16 &= 117 \\ 1 \cdot 5 + 1 \cdot 9 + 1 \cdot 13 + 3 \cdot 1 + 3 \cdot 12 + 3 \cdot 17 &= 117 \\ 1 \cdot 6 + 1 \cdot 7 + 1 \cdot 14 + 3 \cdot 2 + 3 \cdot 10 + 3 \cdot 18 &= 117 \end{cases}$$

4 $\mathbb{Z}_n$–orientable graphs

In this section we show that to decide whether a graph G is $\mathbb{Z}_n$–orientable or not is an NP–complete problem. We end the section giving families of C_n–magic graphs that are $\mathbb{Z}_n$–orientable.

Proposition 4.1 *Let G be a graph. The following statements are equivalent:*

(i) G is $\mathbb{Z}_3$–orientable.

(ii) G is tripartite.

(iii) G is 3–colorable.

Proof.
Clearly, the graph is tripartite if and only if it is 3–colorable. Let us see that (i) and (ii) are equivalent. If the graph is $\mathbb{Z}_3$–orientable then the vertices of G can be labeled with the elements of $\mathbb{Z}_3$, in such a way that vertices of the same stable set receive the same label. Thus, (i) implies (ii). On the other hand, suppose that (ii) holds and let V_0, V_1, V_2 be the three stable sets. The digraph obtained by orienting the edges from V_i to V_j, for $i = 0, 1, 2$ and $j = i +_3 1$ has the Z_3–property. Thus (i) holds. □

It is well known that the problem of deciding wether a given graph is, or is not 3-colorable (usually known as the 3-COL problem) is an NP–complete problem (see for instance [11]). Therefore we have the following immediate corollary.

Corollary 4.1 *To decide whether a graph G is Z_3–orientable or not is an NP–complete problem.*

4.1 C_n–magic graphs that are $\mathbb{Z}_n$–orientable

We give some families of cycle–magic graphs that are $\mathbb{Z}_n$ orientable. We also show that these families do not include all the known cycle-magic graphs.

- The wheel W_n, $n = 4, 6$ is C_3–magic [9] and is $\mathbb{Z}_3$–orientable. Whereas, for $n \geq 5$ odd W_n is C_3–supermagic [9] but is not $\mathbb{Z}_3$–orientable.

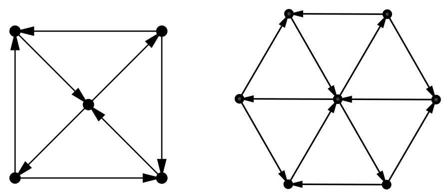

Figure 3: W_4 and W_6.

- For n odd, the books $K_{1,n} \times K_2$ are C_4–supermagic [9] and $\mathbb{Z}_4$–orientable.

- For any $n \geq 2$, the books $K_{1,n} \times K_2$ are C_4–supermagic [10] and $\mathbb{Z}_4$–orientable.
- The windmill $W(r,k)$ (k–cycles of length r with exactly one common vertex), $\forall k \geq 2$, and $r \geq 3$ are C_r–supermagic [9] and $\mathbb{Z}_r$–orientable.
- The graph $\Theta_n(p)$ (n internally disjoint paths of size p, joined by two end vertices) is C_{2p}–supermagic [9] and $\mathbb{Z}_{2p}$–orientable.

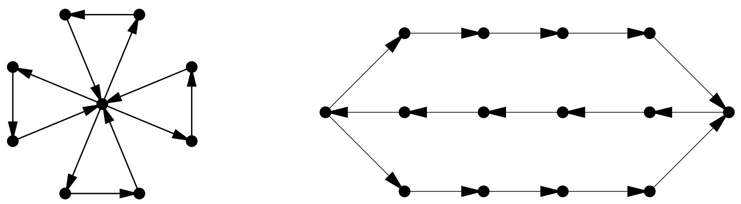

Figure 4: The windmill $W(3,4)$ and $\Theta_3(5)$.

- For any integer $n \geq 2$, the fan $F_n (\cong P_n + K_1)$ is C_3–supermagic [10] and $\mathbb{Z}_3$–orientable.

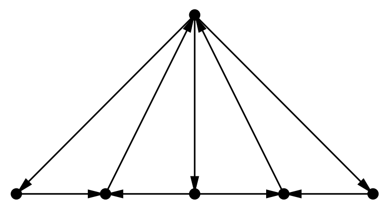

Figure 5: The fan F_5.

- For any integer $n \geq 2$, the ladder $L_n \cong P_n \times K_2$ is C_4–supermagic [10] and $\mathbb{Z}_4$–orientable.
- For any $n \geq 2$, the triangle ladder TL_n of order $2n$ and size $4n-3$ is C_3–supermagic [10] and $\mathbb{Z}_3$–orientable.

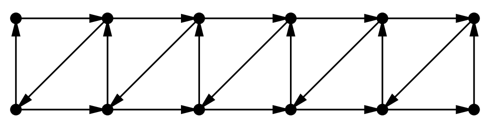

Figure 6: The triangle ladder TL_6.

A *chain graph* G was defined by Barrientos [4] as a graph with blocks $B_1, B_2, \ldots, B_k$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block-cut-vertex graph is a path. (Where the block-cut-vertex of G, is a bipartite graph H with stables sets the set of blocks and the set of the cut–vertices of G, and with an edge in H between a cut-vertex and a block, if the cut–vertex belongs to the block). A kC_n–path is a chain graph where each block is identical and isomorphic to the cycle C_n.

- For any $k \geq 2$, $n \geq 3$, the kC_n–path is C_n–supermagic [10] and $\mathbb{Z}_n$–orientable.

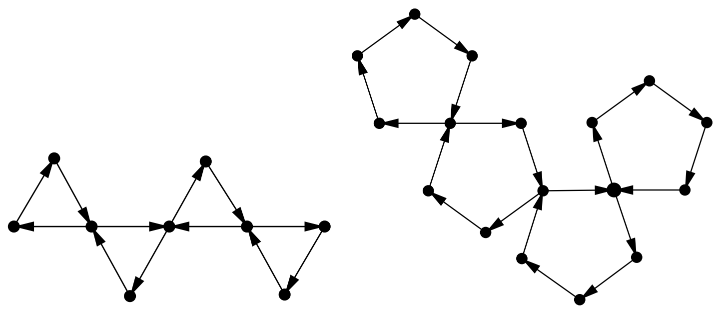

Figure 7: A $4C_3$–path and a $4C_5$–path.

- For any positive integer n, the join of a star of size n with one isolated vertex, $K_{1,n} + K_1$, is C_3–supermagic [10] and $\mathbb{Z}_3$–orientable.

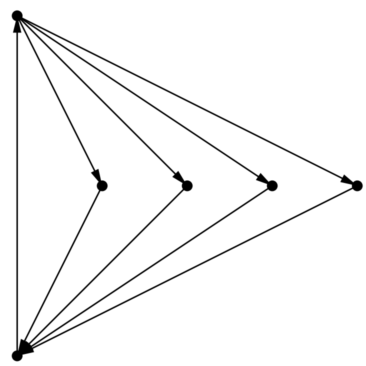

Figure 8: The join of a star with an isolated vertex.

5 The Kronecker product of path–magic graphs

Theorem 5.1 *Let $\vec{G}$ be a $\vec{P}_n$-magic digraph such that $\vec{G}$ has the $\mathbb{Z}_n$–property. If $a_0, \ldots, a_{n-1}$, $b_0, \ldots, b_{n-2}$ are the labels of the vertices and arcs respectively of a subdigraph $\vec{H}$ of $\vec{G}$ isomorphic to $\vec{P}_n$, then for any pair of circular permutations σ and τ the sum of labels defined in (1) of the vertices and arcs in the n–copies of H in $\vec{G} \bigotimes \vec{C}_n$ is respectively,*

$$n(\sum_{i=0}^{n-1} a_i - \frac{n-1}{2}) \quad \text{and} \quad n(\sum_{i=0}^{n-2} b_i - \frac{n-1}{2}) - k, \quad k = 1, 2, \ldots, n. \tag{3}$$

Proof.
In each copy of H in $\vec{G}$, the sum of the labels of the vertices is $\sum_{i=0}^{n-1} a_i$ and the sum of the labels of the arcs is $\sum_{i=0}^{n-2} b_i$. Therefore the sum of the labels of the vertices defined in (1) over each copy of H in $\vec{G} \bigotimes \vec{C}_n$, and the corresponding $L(\vec{G}) \bigotimes \vec{C}_n$ is given respectively by

$$\sum_{i=0}^{n-1}(n(a_i - 1) + \sigma(i) + 1) \quad \text{and} \quad \sum_{i=0}^{n-2}(n(b_i - 1) + \tau(i) + 1),$$

where $\sum_{i=0}^{n-2}(\tau(i) + 1) = \frac{n(n-1)}{2} - k$ for some $k \in \{1, \ldots, n\}$. This completes the proof. □

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