Constraint algorithm for k-presymplectic Hamiltonian systems. Application to singular field theories

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9 March 2009

Abstract

The k-symplectic formulation of field theories is especially simple, since only tangent and cotangent bundles are needed in its description. Its defining elements show a close relationship with those in the symplectic formulation of mechanics. It will be shown that this relationship also stands in the presymplectic case. In a natural way, one can mimick the presymplectic constraint algorithm to obtain a constraint algorithm that can be applied to k-presymplectic field theory, and more particularly to the Lagrangian and Hamiltonian formulations of field theories defined by a singular Lagrangian, as well as to the unified Lagrangian-Hamiltonian formalism (Skinner–Rusk formalism) for k-presymplectic field theory. Two examples of application of the algorithm are also analyzed.

Key words: *k-symplectic manifold, k-presymplectic manifold, constraint algorithm, field theories, Lagrangian formalism, Hamiltonian formalism*

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1 Introduction

Systems of singular differential equations have been a matter of increasing interest in theoretical physics and in some technical areas such as engineering of electric networks or control theory. The fundamental characteristic of these kinds of systems is that the existence and uniqueness of solutions are not assured. In particular, this situation arises in mechanics when dynamical systems described by singular Lagrangians are considered, and also when considering systems of PDE's associated with field theories described by singular Lagrangians (such as, for instance, electromagnetism), as well as in some other applications related to optimal control theories. Furthermore, these systems do not have a nice Hamiltonian description, since not all the momenta are available and, in general, the equations have no solution everywhere.

Bergmann and Dirac were pioneer in solving the problem for the Hamiltonian formalism of singular mechanical systems, by developing a constraint algorithm which gives, in the favourable cases, a final constraint submanifold where admissible solutions to the dynamics exist (in the sense that the dynamical evolution remains on this manifold) [9]. Their main aim was to apply this procedure to field theories. Afterwards, a lot of work was done in order to geometrize this algorithm. The first important step was the work by Gotay *et al* [15], and its application to the Lagrangian formalism [13, 14]. Other algorithms were given later, in order to find consistent solutions of the dynamical equations in the Lagrangian formalism of singular systems (including the problem of finding holonomic solutions) [3, 21, 42], and afterwards, new geometric algorithms were developed to be applied both in the Hamiltonian and the Lagrangian formalisms, as well as to other kinds of more general systems og singular differential equations [17, 19, 20, 37, 41, 44].

The Lagrangian and Hamiltonian descriptions of field theories is the natural extension of time-dependent mechanics. Therefore, in order to understand the constraint algorithm for field theories in a covariant formalism, the first step was to develop the algorithmic procedures for time-dependent systems. This work was provided mainly in [5, 6, 16, 27, 30, 31, 35, 36, 38, 50].

There are several alternative models for describing geometrically first-order classical field theories. From a conceptual point of view, the simplest one is the *k-symplectic formalism*, which is the generalization to field theories of the standard symplectic formalism used as the geometric framework for describing autonomous dynamical systems. In this sense, the k-symplectic formalism is used to give a geometric description of certain kinds of field theories: in a local description, those theories whose Lagrangians or Hamiltonians depend on the fields and on the partial derivatives of the fields, or the corresponding moments, but not on the space-time coordinates [40]. The foundations of the k-symplectic formalism are the k-symplectic manifolds [1, 2, 32]. Historically, it is based on the so-called *polysymplectic formalism* developed by Günther [18], who introduced the concept of *polysymplectic manifold*. Then, k-symplectic manifolds are polysymplectic manifolds which have *Darboux-type coordinates* [32]. A natural extension of this formalism is the *k-cosymplectic formalism*, where k-cosymplectic manifolds are used to describe geometrically field theories involving space-time coordinates or analogous ones, on the Lagrangian or the Hamiltonian [33, 34]. This is the generalization to field theories of the cosymplectic formalism geometrically describing non-autonomous mechanical systems. One of the advantages of these formalisms is that one only needs the tangent and cotangent bundle of

a manifold to develop them.

It is worth noting that G. Sardanashvily *et al* [12, 47] developed a polysymplectic formalism for classical field theories which differs from the one proposed by Günther. (See also [22] for more details on the polysymplectic formalism.) In addition, we must remark that the soldering form on the linear frames bundles is a polysymplectic form, and its study and applications to field theory constitute the n-symplectic formalism developped by L. K. Norris [39, 43].

Working within the framework of the k-symplectic description for these theories, we present in this paper a geometric algorithm for finding the maximal submanifold where there are consistent solutions to the field equations of singular theories. This algorithm is a generalization of the *presymplectic constraint algorithm* for presymplectic dynamical systems [15], and gives an intrinsic description of all the constraint submanifolds. The problem is stated in a generic way for k-presymplectic Hamiltonian systems, in order to give a solution to both Lagrangian and Hamiltonian field theories, as well as other possible kinds of systems of partial differential equations. In this framework, the solutions to these equations are given geometrically by integrable k-vector fields in the manifold where the equations are stated. In this way, a constraint algorithm can be developed giving a sequence of submanifolds which, in the best case, ends in some final constraint submanifold where field equations have consistent solutions (k-vector fields), although not necessarily integrable. The general problem of integrability is not addressed in this paper, only discussed in the examples. Finally, Lagrangian and Hamiltonian field theories are particular cases where the above results are applied straightforwardly, although in the Lagrangian case the problem of finding holonomic solutions must be also analized. In addition, the unified Lagrangian-Hamiltonian formalism of Skinner–Rusk [49], which was adapted recently for k-symplectic field theories [45], constitutes a framework where this algorithm is applied in a very natural way. A description of constraint algorithms for other geometrical models of field theories (*multisymplectic*) was made in [10, 11, 28, 29].

The paper is organized as follows. In section 2 k-symplectic structures are reviewed, as well as the corresponding Hamiltonian systems. Section 3 is devoted to k-presymplectic Hamiltonian systems and the presymplectic constraint algorithm. In section 4 the particular case of field theories described by a Lagrangian function is considered, either in Lagrangian or in Hamiltonian formalism. Finally, the application to the Skinner–Rusk formalism and two examples are studied in section 5.

Manifolds and maps are assumed to be smooth. Sum over crossed repeated indices is understood.

2 k-symplectic Hamiltonian systems

2.1 k-symplectic manifolds. The bundle of k^1-covelocities

Definition 1. *A k-symplectic structure on a differentiable manifold M of dimension $N = n + kn$ is a family $(\omega^1, \ldots, \omega^k; V)$, where each ω^A is a closed 2-form, and V is an integrable*

nk-dimensional tangent distribution on M such that

$$(i) \qquad \omega^A|_{V \times V} = 0 \text{ for each } A,$$

$$(ii) \qquad \cap_{A=1}^{k} \text{Ker}\, \omega^A = \{0\}.$$

Then (M, ω^A, V) is called a k-symplectic manifold.

Theorem 1. *[1, 32]. Let (ω^A, V) be a k-symplectic structure on M. For every point of M there exists a neighbourhood U and local coordinates (q^i, p_i^A) $(1 \le i \le n, 1 \le A \le k)$ such that, on U,*

$$\omega^A = dq^i \wedge dp_i^A, \quad V = \left\langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \right\rangle_{i=1,\dots,n}.$$

These are called *Darboux* or *canonical coordinates* of the k-symplectic manifold.

The canonical model of a k-symplectic manifold is $(\mathrm{T}_k^1)^*Q = \mathrm{T}^*Q \oplus \overset{k}{\dots} \oplus \mathrm{T}^*Q$, the *bundle of k^1-covelocities* of an n-dimensional differentiable manifold Q, which has the natural projections

$$\begin{array}{ccccccc} \pi^A \colon (\mathrm{T}_k^1)^*Q & \to & \mathrm{T}^*Q & ; & \pi_Q^1 \colon (\mathrm{T}_k^1)^*Q & \to & Q \\ (q; \alpha_q^1, \dots, \alpha_q^k) & \mapsto & (q; \alpha_q^A) & ; & (q; \alpha_q^1, \dots, \alpha_q^k) & \mapsto & q \end{array}.$$

$(\mathrm{T}_k^1)^*Q$ is endowed with the canonical forms

$$\theta^A = (\pi^A)^*\theta, \quad \omega^A = (\pi^A)^*\omega = -(\pi^A)^*\mathrm{d}\theta = -\mathrm{d}\theta^A,$$

where θ and ω are the Liouville 1-form and the canonical symplectic form on T^*Q.

If (q^i) $(1 \le i \le n)$ are local coordinates on $U \subset Q$, the induced coordinates (q^i, p_i^A) $(1 \le A \le k)$ on $(\pi_Q^1)^{-1}(U)$ are given by

$$q^i(q; \alpha_q^1, \dots, \alpha_q^k) = q^i(q), \quad p_i^A(q; \alpha_q^1, \dots, \alpha_q^k) = \alpha_q^A\left(\frac{\partial}{\partial q^i}\Big|_q\right).$$

Then we have

$$\theta^A = p_i^A \mathrm{d}q^i, \quad \omega^A = \mathrm{d}q^i \wedge \mathrm{d}p_i^A.$$

Thus, the triple $((\mathrm{T}_k^1)^*Q, \omega^A, V)$, where $V = \text{Ker}\, \mathrm{T}\pi_Q^1$, is a k-symplectic manifold, and the natural coordinates in $(\mathrm{T}_k^1)^*Q$ are Darboux coordinates.

2.2   k-vector fields and integral sections

Let $\mathrm{T}_k^1 M = \mathrm{T}M \oplus \overset{k}{\dots} \oplus \mathrm{T}M$ be the *bundle of k^1-velocities* of a differentiable manifold M. It is endowed with the natural projections

$$\begin{array}{ccccccc} \tau^A \colon \mathrm{T}_k^1 M & \to & \mathrm{T}M & ; & \tau_M^1 \colon \mathrm{T}_k^1 M & \to & M \\ (q, v_{1q}, \dots, v_{kq}) & \mapsto & (q; v_{Aq}) & ; & (q, v_{1q}, \dots, v_{kq}) & \mapsto & q \end{array} \tag{1}$$

Definition 2. *A k-vector field on a manifold M is a section $\mathbf{X} \colon M \longrightarrow \mathrm{T}_k^1 M$ of the projection τ_M^1.*

Therefore, giving a k-vector field $\mathbf{X}$ is the same as giving k vector fields $X_1, \dots, X_k$ on M, obtained as $X_A = \tau_A \circ \mathbf{X}$. We denote $\mathbf{X} = (X_1, \dots, X_k)$.

Remark   The term k-vector field on M is more often applied to the sections of the bundle $\Lambda^k \mathrm{T}M \to M$, that is, contravariant skew-symmetric tensor fields of order k. The k-vector fields $\mathbf{X} = (X_1, \ldots, X_k)$ used here lead to a particular class of such tensor fields, the *decomposable* ones, $X_1 \wedge \ldots \wedge X_k$, which can be associated with distributions on M.

Definition 3. *An* integral section *of the k-vector field $\mathbf{X} = (X_1, \ldots, X_k)$ is a map $\phi \colon J \to M$, defined on an open set $J \subset \mathbb{R}^k$, such that*

$$\mathrm{T}\phi \circ \frac{\partial}{\partial t^A} = X_A \circ \phi,$$

where $t = (t^1, \ldots, t^k)$ denote the canonical coordinates of $\mathbb{R}^k$.
Equivalently, an integral section satisfies the equation

$$\phi^{(1)} = \mathbf{X} \circ \phi,$$

where $\phi^{(1)} \colon J \to \mathrm{T}^1_k M$ is the first prolongation *of ϕ to $\mathrm{T}^1_k M$ defined by*

$$\phi^{(1)}(t) = \left(\phi(t), \mathrm{T}\phi\left(\left.\frac{\partial}{\partial t^1}\right|_t\right), \ldots, \mathrm{T}\phi\left(\left.\frac{\partial}{\partial t^k}\right|_t\right) \right) .$$

A k-vector field $\mathbf{X}$ is integrable *if every point of M belongs to the image of an integral section of $\mathbf{X}$.*

In coordinates, write $X_A = X_A^i \dfrac{\partial}{\partial x^i}$. The ϕ is an integral section of $\mathbf{X}$ if, and only if, the following system of partial differential equations holds:

$$\frac{\partial \phi^i}{\partial t^A} = X_A^i(\phi) .$$

Proposition 1. *A k-vector field $\mathbf{X} = (X_1, \ldots, X_k)$ is integrable if, and only if, $[X_A, X_B] = 0$ for each A, B.*

This is the geometric expression of the integrability condition of the preceding differential equation (see, for instance, [26] or [8]).

2.3   Hamiltonian systems

Definition 4. *Let (M, ω^A, V) be a k-symplectic manifold, and $\alpha \in \Omega^1(M)$ a closed form. (M, ω^A, V, α) is said to be a k-symplectic Hamiltonian system.*

As α is closed, for every point of M there exists a neighbourhood $U \subset M$ and a function $H \in \mathrm{C}^\infty(U)$ such that $\alpha = \mathrm{d}H$ on U. This function is called a *local Hamiltonian function*. If α is exact, then $H \in \mathrm{C}^\infty(M)$ is called a *(global) Hamiltonian function*. These functions are unique up to a constant on each connected component of M. From now on, we will write $\alpha = \mathrm{d}H$.

The *Hamilton–de Donder–Weyl (HDW) equation* for a map $\psi \colon J \to M$ $(J \subset \mathbb{R}^k)$ is

$$i(\psi_A^{(1)})\omega^A = \mathrm{d}H \circ \psi . \tag{2}$$

In canonical coordinates this reads

$$\frac{\partial \psi^i}{\partial t^A} = \frac{\partial H}{\partial p_i^A}, \qquad \frac{\partial \psi_i^A}{\partial t^A} = -\frac{\partial H}{\partial q^i}.$$

where $\psi = (\psi^i, \psi_i^A)$. Recall that, according to our conventions, a sum $\sum_A$ is understood whenever the index A appears twice in upper and lower position.

In order to give an alternative geometrical interpretation of these equations, we introduce the set $\mathfrak{X}_H^k(M)$ of those k-vector fields $\mathbf{X} = (X_1, \dots, X_k)$ on M which are solutions of the *geometric field equation*

$$i(X_A)\,\omega^A = \mathrm{d}H. \tag{3}$$

For k-symplectic Hamiltonian systems, solutions of equation (3) always exist (this is a consequence of the lemma and the theorem in the next section). They are neither unique, nor necessarily integrable.

In canonical coordinates of M, writing $X_A = (X_A)^i \dfrac{\partial}{\partial q^i} + (X_A)_i^B \dfrac{\partial}{\partial p_i^B}$, equation (3) reads

$$\frac{\partial H}{\partial q^i} = -(X_A)_i^A, \quad \frac{\partial H}{\partial p_i^A} = (X_A)^i.$$

This geometric field equation for $\mathbf{X}$ is an alternative formulation of the HDW equation in the following sense:

Proposition 2. *Let $\mathbf{X} = (X_1, \dots, X_k)$ be an integrable k-vector field in M. Every integral section $\psi \colon J \to M$ of $\mathbf{X}$ satisfies the HDW equation (2) if, and only if, $\mathbf{X} \in \mathfrak{X}_H^k(M)$.*

Note however that equations (2) and (3) cannot, in general, be considered as fully equivalent: a solution to the HDW equations may not be an integral section of some integrable k-vector field on M. Solutions ψ that are integral sections of some $\mathbf{X} \in \mathfrak{X}_H^k(M)$ will be called *admissible*, and we will restrict our attention to them.

3 k-presymplectic system and constraint algorithms

3.1 k-presymplectic Hamiltonian systems

To consider singular field theories we have to drop some assumptions in the definition of a k-symplectic structure. So, a family $(\omega^1, \dots, \omega^k)$ ok k closed 2-forms on a smooth manifold M will be called a *k-presymplectic structure*; accordingly, (M, ω^A) will be called a *k-presymplectic manifold*.

The simplest example of a k-presymplectic manifold is provided by any submanifold of a k-symplectic manifold: the pull-back of the k 2-forms by the inclusion map yields k 2-forms on the submanifold.

In some particular k-presymplectic manifolds one can find Poisson-like coordinates, but it is an open question to characterize the necessary and sufficient conditions for these coordinates to exist.

Given a closed 1-form $\alpha \in \Omega^1(M)$, (M, ω^A, α) is said to be a *k-presymplectic Hamiltonian system*. As above, we will write $\alpha = \mathrm{d}H$ (locally or globally).

Then we can also consider the Hamilton–de Donder–Weyl equation $i(\psi_A^{(1)})\omega^A = \mathrm{d}H \circ \psi$, and also the set $\mathfrak{X}_H^k(M)$ of k-vector fields $\mathbf{X}$ that satisfy the geometric field equation $i(X_A)\omega^A = \mathrm{d}H$.

For k-presymplectic systems the existence of solutions of this equation is not assured everywhere on M. We will analyze the existence of solutions on a certain submanifold of M.

3.2 Statement of the problem

The problem we wish to solve arises from the Lagrangian and Hamiltonian k-presymplectic formalisms in field theories, although other kinds of systems could also be stated in this way.

Statement. *Let $(M, \omega^A, \mathrm{d}H)$ be a k-presymplectic Hamiltonian system. We want to find a submanifold C of M and integrable k-vector fields $\mathbf{X} = (X_1, \ldots, X_k) \in \mathfrak{X}^k(M)$ such that*

$$i(X_A)\omega^A \underset{C}{\approx} \mathrm{d}H \tag{4}$$

(this means equality on the points of C) and

$$\mathbf{X} \text{ is tangent to } C$$

(this means that $X_1, \ldots, X_k$ are tangent to C).

As stated in the introduction, we will focus on the consistency of the equation and will not address the integrability condition in generality.

Given a submanifold C of M, with natural embedding $\jmath_C \colon C \hookrightarrow M$, let $\mathrm{T}_1^k \jmath_C \colon \mathrm{T}_k^1 C \to \mathrm{T}_k^1 M$ be the natural extension of $\jmath_C$ to the k-tangent bundles, and denote its image as $\underline{\mathrm{T}_k^1 C} = \mathrm{T}_1^k \jmath_C(\mathrm{T}_k^1 C)$.

We can define the map

$$
\begin{array}{cccc}
\flat_k \colon & \mathrm{T}_k^1 M & \to & \mathrm{T}^* M \\
& (p; v_{p_1}, \ldots, v_{p_k}) & \mapsto & (p, i(v_{A_p})\omega_p^A)
\end{array}
\tag{5}
$$

and denote by $(\mathrm{T}C)_{\flat_k}^\perp$ the annihilator of the image of $\underline{\mathrm{T}_k^1 C}$ by $\flat_k$; that is,

$$(\mathrm{T}C)_{\flat_k}^\perp = [\flat_k(\underline{\mathrm{T}_k^1 C})]^0 = \{u_p \in \mathrm{T}M \mid \forall (v_{p1}, \ldots, v_{pk}) \in \underline{\mathrm{T}_k^1 C}, \ \langle \sum i(v_{pA})\omega_p^A, u_p \rangle = 0\}.$$

We call $(\mathrm{T}C)_{\flat_k}^\perp$ the *k-presymplectic orthogonal complement* of $\underline{\mathrm{T}_k^1 C}$ in $\mathrm{T}_k^1 M$.

In particular, for $C = M$ we have:

Lemma 1. $(\mathrm{T}M)_{\flat_k}^\perp = \{(p, u_p) \in \mathrm{T}M \mid u_p \in \bigcap_{A=1}^k \mathrm{Ker}\,\omega_p^A\}$

(Proof) For every $p \in M$ and $(v_{p1}, \ldots, v_{pk}) \in (\mathrm{T}_k^1)_p M$, if $u_p \in \bigcap_{A=1}^k \mathrm{Ker}\,\omega_p^A$, then we have $(i(v_{Ap})\omega_p^A)(u_p) = -i(v_{Ap})\,i(u_p)\omega_p^A = 0$, and therefore $u_p \in (\mathrm{T}_p M)_{\flat_k}^\perp$.

Conversely, if $u_p \in (\mathrm{T}_p M)^{\perp}_{\flat_k}$, then $(i(v_{Ap})\omega^A_p)(u_p) = 0$ for every $(v_{p1}, \ldots, v_{pk}) \in (\mathrm{T}^1_k)_p M$. Then taking any $(v_{p1}, 0, \ldots, 0)$ with $v_{p1} \neq 0$ we conclude that $u_p \in \operatorname{Ker} \omega^1_p$; and analogously for the others. ∎

The main result is the following:

Theorem 2. *Let C be a submanifold of M. The following conditions are equivalent:*

- *there exists a k-vector field $\mathbf{X} = (X_1, \ldots, X_k) \in \mathfrak{X}^k(M)$, tangent to C, such that equation (4) holds*

-
$$i(Y_p)(\mathrm{d}H)_p = 0 \quad \text{for every } p \in C, \ Y_p \in (\mathrm{T}_p C)^{\perp}_{\flat_k}. \tag{6}$$

(*Proof*) ($\Longrightarrow$) If there exists a k-vector field $\mathbf{X} = (X_1, \ldots, X_k) \in \mathfrak{X}^k(M)$, tangent to C such that equation (4) holds, then, for every $p \in C$ and $Y_p \in (\mathrm{T}_p C)^{\perp}_{\flat_k}$,

$$0 = [i(X_{Ap})\omega^A_p](Y_p) = i(Y_p)\, i(X_{Ap})\omega^A_p = i(Y_p)(\mathrm{d}H)_p\,.$$

($\Longleftarrow$) If (6) holds, then

$$(\mathrm{d}H)_p \in [(\mathrm{T}C)^{\perp}_{\flat_k}]^0 = [[\flat_k(\underline{\mathrm{T}^1_k C})]^0]^0 = \flat_k(\underline{\mathrm{T}^1_k C}),$$

and hence there exists $(X_{p1}, \ldots, X_{pk}) \in \underline{(\mathrm{T}^1_k)_p}C$ such that (4) holds. ∎

3.3 k-presymplectic constraint algorithm

The application of the above result leads to an algorithmic procedure which gives a sequence of subsets $\ldots \subset C_j \subset \ldots C_2 \subset C_1 \subset M$. We will assume that:

Assumption. *Every subset C_j of this sequence is a regular submanifold of M.*

These submanifolds are sequentially obtained from the analysis of the consistency of a linear equation, namely eq. (4) at each point:

$$i(X_{Ap})\omega^A_p = (\mathrm{d}H)_p.$$

First, $C_1 \hookrightarrow M$ is the submanifold of M where this equation is consistent:

$$C_1 = \{p \in M \mid \exists \mathbf{X}_p \text{ such that } i(X_{Ap})\omega^A_p = (\mathrm{d}H)_p\}.$$

So, there exist k-vector fields $\mathbf{X}$ on M which satisfy equation (4) on the sumbanifold C_1. However, in general these $\mathbf{X}$ may not be tangent to C_1. Therefore, we consider the submanifold

$$C_2 = \{p \in C_1 \mid \exists \mathbf{X}_p \in \mathrm{T}^1_k(C_1) \text{ such that } i(X_{Ap})\omega^A_p = (\mathrm{d}H)_p\}\,,$$

and so on. Following this process, we obtain a sequence of constraint submanifolds

$$\ldots \hookrightarrow C_j \hookrightarrow \ldots C_2 \hookrightarrow C_1 \hookrightarrow M$$

where, taking into account Theorem 2, each submanifold C_j is geometrically defined by

$$C_j = \{p \in C_{j-1} \mid i(Y_p)(\mathrm{d}H)_p = 0 \text{ for every } Y_p \in (\mathrm{T}_p C_{j-1})_{\flat_k}^{\perp}\}.$$

For every $j \geq 1$, C_j is called the *jth constraint submanifold*.

If we denote by $\mathfrak{X}(C_j)_{\flat_k}^{\perp}$ the set of vector fields Y in M such that $Y_p \in (\mathrm{T}_p C_j)_{\flat_k}^{\perp}$, then one can obtain constraint functions $\{\xi_\mu\}$ defining each C_j from a local basis $\{Z_1, \ldots, Z_r\}$ of vector fields of $\mathfrak{X}(C_{j-1})_{\flat_k}^{\perp}$ by setting $\xi_\mu = i(Z_\mu)\mathrm{d}H$.

The technical procedure to obtain these constraints is the following:

- To obtain a local basis $\{Z_1, \ldots, Z_r\}$ of vector fields of $\bigcap_{A=1}^{k} \mathrm{Ker}\, \omega^A$.

- To apply Theorem 2 to obtain a set of independent constraint functions $\xi_\mu = i(Z_\mu)\mathrm{d}H$, defining $C_1 \hookrightarrow M$.

- To calculate $\mathbf{X} = (X_1, \ldots, X_k)$, solutions to (4) on C_1.

- To impose the tangency condition of $X_1, \ldots, X_k$ on the constraints ξ_μ.

- To iterate the last item until no new constraints appear.

This is the *k-presymplectic constraint algorithm*. We have two possibilities:

- There exists an integer $j > 0$ such that $C_{j+1} = C_j \equiv C_f$. In this case, C_f is called the *final constraint submanifold*, and there exist a family of k-vector fields $\mathbf{X}^f = (X_1^f, \ldots, X_k^f)$ in M, tangent to C_f, such that (3) holds on C_f, that is,

$$[i(X_A^f)\omega^A - \mathrm{d}H]|_{C_f} = 0. \tag{7}$$

  This is the situation which is interesting to us.

- There exists an integer $j > 0$ such that $C_j = \emptyset$. This means that the equations have no solution on a submanifold of M.

4 k-symplectic field theory

4.1 The bundle of k^1-velocities

The Lagrangian formalism of k-symplectic field theories uses the bundle of k^1-velocities of a manifold as phase space. First we introduce the canonical structures which this manifold is endowed with.

Let $\mathrm{T}_k^1 Q = \mathrm{T}Q \oplus \overset{k}{\cdots} \oplus \mathrm{T}Q$ be the *bundle of k^1-velocities* of Q, with natural projections $\tau^A \colon \mathrm{T}_k^1 Q \to \mathrm{T}Q$ and $\tau_Q^1 \colon \mathrm{T}_k^1 Q \to Q$, given in (1).

If (q^i) are local coordinates on $U \subset Q$, the induced coordinates (q^i, v_A^i) on $(\tau_Q^1)^{-1}(U)$ are

$$q^i(v_{1q}, \ldots, v_{kq}) = q^i(q) \quad , \quad v_A^i(v_{1q}, \ldots, v_{kq}) = v_{Aq}(q^i).$$

For $Z_q \in T_qQ$, its *vertical A-lift* at $(v_{1q}, \ldots, v_{kq}) \in T_k^1Q$ is the vector $(Z_q)^{V_A}$, tangent to the fiber $(\tau_Q^1)^{-1}(q) \subset T_k^1Q$, given by

$$(Z_q)^{V_A}(v_{1q}, \ldots, v_{kq}) = \frac{d}{ds}\Big|_{s=0}(v_{1q}, \ldots, v_{(A-1)q}, v_{Aq} + sZ_q, v_{(A+1)q}, \ldots, v_{kq})$$

If $Z_q = a^i \frac{\partial}{\partial q^i}\big|_q$, then $(Z_q)^{V_A}(v_{1q}, \ldots, v_{kq}) = a^i \frac{\partial}{\partial v_A^i}\big|_{(v_{1q}, \ldots, v_{kq})}$.

The *canonical k-tangent structure* on T_k^1Q is the set $(S^1, \ldots, S^k)$ of tensor fields of type $(1,1)$ defined by

$$S^A(w_q)(Z_{w_q}) = (T_{w_q}(\tau_Q^1)(Z_{w_q}))^{V_A}(w_q) \ ,$$

for $w_q = (v_{1q}, \ldots, v_{kq}) \in T_k^1Q, \ Z_{w_q} \in T_{w_q}(T_k^1Q)$.

In coordinates we have $S^A = \frac{\partial}{\partial v_A^i} \otimes dq^i$.

The *Liouville vector field* $\Delta \in \mathfrak{X}(T_k^1Q)$ is the infinitesimal generator of the flow $\psi \colon \mathbb{R} \times T_k^1Q \longrightarrow T_k^1Q$

$$\psi(s; v_{1q}, \ldots, v_{kq}) = (e^s v_{1q}, \ldots, e^s v_{kq}).$$

Observe that $\Delta = \Delta_1 + \ldots + \Delta_k$, where each $\Delta_A \in \mathfrak{X}(T_K^1Q)$ is the infinitesimal generator of the flow $\psi^A \colon \mathbb{R} \times T_k^1Q \longrightarrow T_k^1Q$

$$\psi^A(s; v_{1q}, \ldots, v_{kq}) = (v_{1q}, \ldots, v_{(A-1)q}, e^s v_{Aq}, v_{(A+1)q}, \ldots, v_{kq})$$

In local coordinates we have $\ \Delta = \sum_{A=1}^k \Delta_A = v_A^i \frac{\partial}{\partial v_A^i}$.

Now we want to characterize the integrable k-vector fields on T_k^1Q such that their integral sections are first prolongations $\phi^{(1)}$ of maps $\phi \colon \mathbb{R}^k \to Q$. Remember that a k-vector field in T_k^1Q is a section $\mathbf{\Gamma} \colon T_k^1Q \longrightarrow T_k^1(T_k^1Q)$ of the canonical projection $\tau_{T_k^1Q} \colon T_k^1(T_k^1Q) \to T_k^1Q$. Then:

Definition 5. *A second order partial differential equation (SOPDE) is a k-vector field $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_k)$ on T_k^1Q which is also a section of the projection $T_k^1\tau \colon T_k^1(T_k^1Q) \to T_k^1Q$; that is,*

$$T_k^1\tau \circ \mathbf{\Gamma} = \mathrm{Id}_{T_k^1Q} \ ,$$

or, what is equivalent, $T_{w_q}\tau \cdot \Gamma_A(w_q) = v_{Aq}$, *for* $w_q = (v_{1q}, \ldots, v_{kq}) \in T_k^1Q$.

If the local expression of the k-vector field $\mathbf{\Gamma} = (\Gamma_A)$ on T_k^1Q is $\Gamma_A = (\Gamma_A)^i \frac{\partial}{\partial q^i} + (\Gamma_A)_B^i \frac{\partial}{\partial v_B^i}$, then $\mathbf{\Gamma}$ is a SOPDE iff $(\Gamma_A)^i = v_A^i$:

$$\Gamma_A(q^i, v_A^i) = v_A^i \frac{\partial}{\partial q^i} + (\Gamma_A)_B^i \frac{\partial}{\partial v_B^i} \ ,$$

where $(\Gamma_A)_B^i$ are functions locally defined in T_k^1Q.

If $\psi \colon \mathbb{R}^k \to T_k^1Q$ is an integral section of $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_k)$, locally given by $\psi(t) = (\psi^i(t), \psi_B^i(t))$, then from the last expression and Definition 3 we deduce

$$\frac{\partial \psi^i}{\partial t^A}\Big|_t = \psi_A^i(t) \, , \quad \frac{\partial \psi_B^i}{\partial t^A}\Big|_t = (\Gamma_A)_B^i(\psi(t)) \, .$$

Proposition 3. *Let $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_k)$ be an integrable* SOPDE. *If ψ is an integral section of $\mathbf{\Gamma}$ then $\psi = \phi^{(1)}$, where $\phi^{(1)}$ is the first prolongation of the map $\phi = \tau \circ \psi \colon \mathbb{R}^k \xrightarrow{\psi} T_k^1 Q \xrightarrow{\tau} Q$, and ϕ is a solution of the system of second order partial differential equations*

$$\frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(t) = (\Gamma_A)_B^i \left(\phi^i(t), \frac{\partial \phi^i}{\partial t^C}(t) \right) . \tag{8}$$

Conversely, if $\phi \colon \mathbb{R}^k \to Q$ is any map satisfying (8), then $\phi^{(1)}$ is an integral section of $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_k)$.

From (8) we deduce that if $\mathbf{\Gamma}$ is an integrable SOPDE then $(\Gamma_A)_B^i = (\Gamma_B)_A^i$.

Finally, using the canonical k-tangent structure of $T_k^1 Q$, we have that a k-vector field $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_k)$ on $T_k^1 Q$ is a SOPDE if, and only if, $S^A(\Gamma_A) = \Delta_A$ (A fixed).

4.2 k-symplectic Lagrangian field theory

Let $L \in C^\infty(T_k^1 Q)$ be a *Lagrangian function*. We define the *Lagrangian forms*

$$\theta_L^A = {}^t(S^A) \circ dL \in \Omega^1(T_k^1 Q) , \quad \omega_L^A = -d\theta_L^A \in \Omega^2(T_k^1 Q) .$$

and the *Lagrangian energy function*

$$E_L = \Delta(L) - L \in C^\infty(T_k^1 Q) .$$

They have local expressions $\theta_L^A = \dfrac{\partial L}{\partial v_A^i} \, dq^i$, $\omega_L^A = dq^i \wedge d\left(\dfrac{\partial L}{\partial v_A^i} \right)$, $E_L = v_A^i \dfrac{\partial L}{\partial v_A^i} - L$.

We introduce the *Legendre map* of L, which is its fibre derivative $FL \colon T_k^1 Q \longrightarrow (T_k^1)^* Q$. It can be defined as follows: for $q \in Q$, $u_q \in T_q Q$, $(v_{1q}, \ldots, v_{kq}) \in (T_k^1)_q Q$,

$$[FL(v_{1q}, \ldots, v_{kq})]^A(u_q) = \frac{d}{ds} L(v_{1q}, \ldots, v_{Aq} + s u_q, \ldots, v_{kq})|_{s=0}.$$

Locally, $FL(q^i, v_A^i) = \left(q^i, \dfrac{\partial L}{\partial v_A^i} \right)$. Furthermore, we have that $\theta_L^A = FL^*(\theta^A)$, $\omega_L^A = FL^*(\omega^A)$.

Definition 6. *The Lagrangian L is* regular *if the following equivalent conditions hold:*

1. $\left(\dfrac{\partial^2 L}{\partial v_A^i \partial v_B^j} \right)$ *is everywhere nonsingular.*

2. *The second fibre derivative $FL \colon T_k^1 Q \longrightarrow (T_k^1)^* Q \otimes (T_k^1)^* Q$ is everywhere nonsingular.*

3. *FL is a local diffeomorphism.*

4. *$(T_k^1 Q, \omega_L^A, V = \operatorname{Ker} T\tau_Q^1)$ is a k-symplectic manifold.*

The Lagrangian L is called hyperregular *if FL is a global diffeomorphism.*

We must point out that, in field theories, the notion of regularity is not uniquely defined (for other approaches see, for instance, [4, 7, 23, 24, 25]).

Our purpose, however, is the study of singular Lagrangians, *i.e.*, those which are not regular. Following [13], we will deal with singular Lagrangians satisfying some regularity conditions:

Definition 7. *A singular Lagrangian L is* almost-regular *if*

1. $\mathcal{P} := FL(T_k^1 Q)$ *is a closed submanifold of* $(T_k^1)^* Q$.

2. FL *is a submersion onto its image.*

3. *The fibres* $FL^{-1}(p)$, *for every* $p \in \mathcal{P}$, *are connected submanifolds of* $T_k^1 Q$.

If L is regular, $(T_k^1 Q, \omega_L^A, E_L)$ is a k-symplectic Lagrangian system, otherwise it is a k-presymplectic Lagrangian system. Therefore, $(T_k^1 Q, \omega_L^A, dE_L)$ is a k-symplectic or a k-presymplectic Hamiltonian system, depending on the regularity of L.

In a natural chart of $T_k^1 Q$ we have the *Euler–Lagrange (EL) equations* for L, which are

$$\frac{\partial}{\partial t^A} \left(\frac{\partial L}{\partial v_A^i} \Big|_{\varphi(t)} \right) = \frac{\partial L}{\partial q^i} \Big|_{\varphi(t)} , \quad v_A^i(\varphi(t)) = \frac{\partial \varphi^i}{\partial t^A} , \tag{9}$$

whose solutions are maps $\varphi \colon \mathbb{R}^k \to T_k^1 Q$ that are first prolongations to $T_k^1 Q$ of maps $\phi = \tau_Q^1 \circ \varphi \colon \mathbb{R}^k \to Q$; that is, φ are *holonomic*. We will show that these equations can be given a geometric interpretation using the k-presymplectic structure.

Indeed, consider a map $\varphi \colon \mathbb{R}^k \to T_k^1 Q$ which is holonomic. Then the Euler–Lagrange equations for φ can be also written as

$$i(\varphi_A^{(1)})\omega_L^A = dE_L . \tag{10}$$

As in our general discussion on k-presymplectic Hamiltonian systems, a convenient way to represent the solutions of these equations can be set in terms of k-vector fields. Let us introduce the set $\mathfrak{X}_L^k(T_k^1 Q)$ of k-vector fields $\boldsymbol{\Gamma} = (\Gamma_1, \ldots, \Gamma_k)$ in $T_k^1 Q$ which are solutions of

$$i(\Gamma_A)\omega_L^A = dE_L . \tag{11}$$

If $\Gamma_A = (\Gamma_A)^i \dfrac{\partial}{\partial q^i} + (\Gamma_A)_B^i \dfrac{\partial}{\partial v_B^i}$ locally, then (11) is equivalent to

$$\left(\frac{\partial^2 L}{\partial q^i \partial v_A^j} - \frac{\partial^2 L}{\partial q^j \partial v_A^i} \right)(\Gamma_A)^j - \frac{\partial^2 L}{\partial v_A^i \partial v_B^j}(\Gamma_A)_B^j = v_A^j \frac{\partial^2 L}{\partial q^i \partial v_A^j} - \frac{\partial L}{\partial q^i}$$

$$\frac{\partial^2 L}{\partial v_B^j \partial v_A^i}(\Gamma_A)^i = \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} v_A^i . \tag{12}$$

If, in addition, $\boldsymbol{\Gamma}$ is required to be a SOPDE, *i.e.* $(\Gamma_A)^i = v_A^i$, then the above equations are equivalent to

$$\frac{\partial^2 L}{\partial q^j \partial v_A^i} v_A^j + \frac{\partial^2 L}{\partial v_A^i \partial v_B^j}(\Gamma_A)_B^j = \frac{\partial L}{\partial q^i} .$$

These equations imply that, if $\mathbf{\Gamma}$ is an integrable SOPDE, its integral sections are holonomic and they are solutions to the EL-equations.

If L is regular, solutions to (11) always exist, although they are neither unique, nor necessarily integrable. However, if $\mathbf{\Gamma}$ is integrable, then the second group of equations (12) imply that its integral sections are holonomic and they are solutions to the EL-equations. Hence $\mathbf{\Gamma}$ is a SOPDE.

If L is not regular then, in general, equations (11) have no solutions everywhere in $\mathrm{T}_k^1 Q$ but, in the most favourable situations, they do in a submanifold of $\mathrm{T}_k^1 Q$ which is obtained by applying the k-presymplectic constraint algorithm developed in Section 3.3. Nevertheless, solutions to equations (11) are not necessarily SOPDE's (unless it is required as an additional condition). In addition, if they are integrable, their integral sections are not necessarily holonomic, and thus they are not solutions to the EL-equations (9). The geometric analysis of this problem must be done in a separate way. (For the multisymplectic formalism of field theories, a study of this problem can be found in [29]).

4.3 k-symplectic Hamiltonian field theory

The Hamiltonian formalism of k-symplectic regular field theories uses the bundle of k^1-covelocities of a manifold as phase space.

So, consider the k-symplectic manifold $((\mathrm{T}_k^1)^* Q, \omega^A, V)$, and let $H \in \mathrm{C}^\infty((\mathrm{T}_k^1)^* Q)$ be a *Hamiltonian function*. Then $((\mathrm{T}_k^1)^* Q, \omega^A, \mathrm{d}H)$ is a k-symplectic Hamiltonian system.

In particular, if $(\mathrm{T}_k^1 Q, \omega_L^A, \mathrm{d}E_L)$ is a *Lagrangian system*, then:

- If L is hyperregular, we may define the Hamiltonian $H = E_L \circ FL^{-1}$, and $((\mathrm{T}_k^1)^* Q, \omega^A, \mathrm{d}H)$ is the *k-symplectic Hamiltonian system associated with L*.

- If L is almost-regular, let $\mathcal{P}$ be the image of the Legendre map, and $\jmath_0 \colon \mathcal{P} \hookrightarrow (\mathrm{T}_k^1)^* Q$ the corresponding embedding, and denote by $FL_0 \colon \mathrm{T}_k^1 Q \to \mathcal{P}$ the restriction of the Legendre map defined by $\jmath_0 \circ FL_0 = FL$. Then, the condition of almost-regularity implies that there exists $H_0 \in \mathrm{C}^\infty(\mathcal{P})$ such that $(FL_0)^*(H_0) = E_L$. Furthermore, we can define $\omega_0^A = \jmath_0^*(\omega^A)$. With these definitions, the triple $(\mathcal{P}, \omega_0^A, \mathrm{d}H_0)$ is the *k-presymplectic Hamiltonian system associated with L*, and the corresponding Hamiltonian field equation (3) is

$$i(X_A^0)\omega_0^A = dH_0$$

where $\mathbf{X}^0 = (X_1^0, \dots, X_k^0)$ (if it exists) is a k-vector field on $\mathcal{P}$. Once again, in general, this equation has no solutions everywhere in $\mathcal{P}$ but, in the most favourable situations, they do in a submanifold of $\mathcal{P}$ which is obtained applying the k-presymplectic constraint algorithm developed in Section 3.3.

5   Applications and examples

5.1   The Skinner–Rusk unified formalism for k-symplectic field theory

The so-called *Skinner–Rusk formalism* [48, 49] was developed in order to give a geometrical unified formalism for describing mechanical systems. It incorporates all the characteristics of Lagrangian and Hamiltonian descriptions of these systems. This formalism has been generalized to the k-symplectic description of first-order field theories in [45]. Next we outline the main features of this formalism.

Let us consider the direct sum $T_k^1 Q \oplus (T_k^1)^* Q$ (of vector bundles over Q), with coordinates (q^i, v_A^i, p_i^A), and denote by $pr_1 \colon T_k^1 Q \oplus (T_k^1)^* Q \to T_k^1 Q$ and $pr_2 \colon T_k^1 Q \oplus (T_k^1)^* Q \to (T_k^1)^* Q$ the canonical projections. In this manifold, we have some canonical structures.

First, if $((\omega_0)_1, \ldots, (\omega_0)_k)$ is the canonical k-symplectic structure on $(T_k^1)^* Q$, its pull-back through pr_2 yields a k-presymplectic structure $(\Omega_1, \ldots, \Omega_k)$ on $T_k^1 Q \oplus (T_k^1)^* Q$: the 2-forms are defined by $\Omega_A = (pr_2)^*(\omega_0)_A$.

We can also define the so-called coupling function $\mathcal{C} \colon T_k^1 Q \oplus (T_k^1)^* Q \longrightarrow \mathbb{R}$ by

$$\mathcal{C}(v_{1q}, \ldots, v_{kq}, \alpha_q^1, \ldots, \alpha_q^k) := \langle \alpha_q^A, v_{Aq} \rangle.$$

Now, consider a Lagrangian $L \in \mathrm{C}^\infty(T_k^1 Q)$. We can define a Hamiltonian function $\mathcal{H} \in \mathrm{C}^\infty(T_k^1 Q \oplus (T_k^1)^* Q)$ as $\mathcal{H} = \mathcal{C} - pr_1^*(L)$:

$$\mathcal{H}(v_{1q}, \ldots, v_{kq}, \alpha_q^1, \ldots, \alpha_q^k) = \mathcal{C}(v_{1q}, \ldots, v_{kq}, \alpha_q^1, \ldots, \alpha_q^k) - L(v_{1q}, \ldots, v_{kq}),$$

which in local coordinates reads $\mathcal{H} = \alpha_i^A v_A^i - L(q^i, v_A^i)$.

Then $\left(T_k^1 Q \oplus (T_k^1)^* Q, \Omega_A, \mathcal{H} \right)$ is a k-presymplectic Hamiltonian system; where $\cap_{A=1}^k \Omega_A$ is locally generated by the vector fields $\left\{ \dfrac{\partial}{\partial v_A^i} \right\}$. We look for the solutions of its HDW equation which are integral sections $\psi \colon \mathbb{R}^k \to T_k^1 Q \oplus (T_k^1)^* Q$ of some integrable k-vector field $\mathbf{Z} = (Z_1, \ldots, Z_k)$ on $T_k^1 Q \oplus (T_k^1)^* Q$, satisfying

$$\imath_{Z_A} \Omega_A = d\mathcal{H}. \tag{13}$$

This equation gives various kinds of information. In fact, writing locally each Z_A as

$$Z_A = (Z_A)^i \frac{\partial}{\partial q^i} + (Z_A)_B^i \frac{\partial}{\partial v_B^i} + (Z_A)_i^B \frac{\partial}{\partial p_i^B} \ ,$$

equation (13) amounts to the following conditions:

$$p_i^A = \frac{\partial L}{\partial v_A^i} \circ pr_1 \,, \quad (Z_A)^i = v_A^i \,, \quad (Z_B)_i^B = \frac{\partial L}{\partial q^i} \circ pr_1 \,.$$

The first group of equations are algebraic rather than differential, and they define a submanifold M_L of $T_k^1 Q \oplus_Q (T_k^1)^* Q$ where the equation (13) has solution. These constraints can also be obtained by computing $i \left(\dfrac{\partial}{\partial v_A^i} \right) d\mathcal{H}$, as noted in the discussion of the k-presymplectic

constraint algorithm. Observe that the submanifold M_L is just the graph of the Legendre map FL defined by the Lagrangian L, and hence it is diffeomorphic to $T_k^1 Q$. We denote by $\jmath \colon M_L \to T_k^1 Q \oplus_Q (T_k^1)^* Q$ the natural embedding.

The second group of equations are a holonomy condition which means that the k-velocity part of the integral sections of the k-vector field $\mathbf{Z}$ is the lift of a section $\phi \colon \mathbb{R}^k \to Q$.

The third group of equations establishes some relations among some of the coefficients $(Z_A)_i^B$ of the vector fields Z_A.

Given a solution $\mathbf{Z} = (Z_1, \ldots, Z_k)$ of equation (13), the vector fields Z_A are tangent to the submanifold M_L if, and only if, the functions $\mathcal{L}_{Z_A} \left(p_j^B - \dfrac{\partial L}{\partial v_B^j} \circ pr_1 \right)$ vanish at the points of M_L, for every A, B, j (the symbol $\mathcal{L}$ denotes the Lie derivative). Taking into account the above results, this is equivalent to

$$(Z_A)_j^B = \frac{\partial^2 L}{\partial v_B^j \partial q^i} v_A^i + \frac{\partial^2 L}{\partial v_B^j \partial v_C^i} (Z_A)_C^i \,. \tag{14}$$

In general, equations (13) have not a unique solution. If L is regular, taking into account the above results, one can define local k-vector fields $(Z_1, \ldots, Z_k)$ on a neighborhood of each point in M_L which are solutions to (13). The vector field Z_A may be locally given by

$$(Z_A)^i = v_A^i \,, \quad (Z_A)_i^B = \frac{1}{k} \frac{\partial L}{\partial q^i} \delta_A^B \,,$$

with $(Z_A)_B^i$ given by equation (14). Then, using a partition of the unity, one can construct global k-vector fields which are solutions to (13). When the Lagrangian L is singular one cannot assure the existence of consistent solutions for equation (13). Then, in the best cases, the constraint algorithm will provide a constraint submanifold $\mathcal{P}_f$ where these solutions exist.

If $\mathbf{Z}$ is an integrable k-vector field solution to (13), then every integral section of $\mathbf{Z}$ is of the form $\psi = (\psi_L, \psi_H)$, with $\psi_L = pr_1 \circ \psi \colon \mathbb{R}^k \to T_k^1 Q$, and as ψ takes values in M_L then $\psi_H = FL \circ \psi_L$; in fact,

$$\psi_H(t) = (pr_2 \circ \psi)(t) = (\psi^i(t), \psi_i^A(t)) = \left(\psi^i(t), \frac{\partial L}{\partial v_A^i} \bigg|_{\psi_L(t)} \right) = (FL \circ \psi_L)(t) \,.$$

Furthermore, it can be proved (see [45]) that ψ_L is the canonical lift $\phi^{(1)}$ of the projected section $\phi = \tau_Q \circ pr_1 \circ \psi \colon \mathbb{R}^k \to Q$, which is a solution to the Euler-Lagrange field equations, and that, if L is regular, then $\psi_H = FL \circ \psi_L$ is a solution to the Hamilton-De Donder-Weyl field equations, where the Hamiltonian H is locally given by $H \circ FL = E_L$. In the almost-regular case, this last result also holds, but the sections ψ, ψ_L and ψ_H take values not on M_L, $T_k^1 Q$ and $(T_k^1)^* Q$, but in the final constraint submanifold $\mathcal{P}_f$ and on the projection submanifolds $pr_1(\mathcal{P}_f) \hookrightarrow T_k^1 Q$ and $pr_2(\mathcal{P}_f) \hookrightarrow (T_k^1)^* Q$, respectively.

In this way, every constraint, differential equation, etc. in the unified formalism can be translated to the Lagrangian or the Hamiltonian formalisms by restriction to the first or the second factors of the product bundle. In particular, the constraint conditions $p_i^A - \dfrac{\partial L}{\partial v_A^i} \circ pr_1 = 0$ generate, by pr_2-projection, the primary constraints of the Hamiltonian formalism for singular

Lagrangians (i.e., the image of the Legendre transformation, $FL(T_k^1 Q) \subset (T_k^1)^* Q$), and they are the primary Hamiltonian constraints.

5.2 Example 1

Let us study a simple example, the electromagnetic field in 2 dimensions. The base manifold is $Q = \mathbb{R}^2$, with local coordinates (q^1, q^2), and $k = 2$. The induced coordinates on $T_2^1 \mathbb{R}^2$ are $(q^1, q^2, v_1^1, v_2^1, v_1^2, v_2^2)$. The electromagnetic field Lagrangian is $L = \frac{1}{2}(v_2^1 + v_1^2)^2$ (see [25]).

The canonical 2-tangent structure on $T_2^1 \mathbb{R}^2$, (S^1, S^2), is $S^1 = \frac{\partial}{\partial v_1^1} \otimes dq^1 + \frac{\partial}{\partial v_1^2} \otimes dq^2$ and $S^2 = \frac{\partial}{\partial v_2^1} \otimes dq^1 + \frac{\partial}{\partial v_2^2} \otimes dq^2$, and the Liouville vector field reads as $\Delta = \Delta_1 + \Delta_2 = v_1^1 \frac{\partial}{\partial v_1^1} + v_1^2 \frac{\partial}{\partial v_1^2} + v_2^1 \frac{\partial}{\partial v_2^1} + v_2^2 \frac{\partial}{\partial v_2^2}$.

The Lagrangian forms are

$$\theta_L^1 = dL \circ S^1 = (v_2^1 + v_1^2)dq^2 \quad , \quad \theta_L^2 = dL \circ S^2 = (v_2^1 + v_1^2)dq^1,$$

$$\omega_L^1 = -d\theta_L^1 = dq^2 \wedge dv_2^1 + dq^2 \wedge dv_1^2 \quad , \quad \omega_L^2 = -d\theta_L^2 = dq^1 \wedge dv_2^1 + dq^1 \wedge dv_1^2,$$

and the Lagrangian energy function is

$$E_L = \Delta(L) - L = (v_2^1 + v_1^2)^2 - \frac{1}{2}(v_2^1 + v_1^2)^2 = \frac{1}{2}(v_2^1 + v_1^2)^2.$$

Since $\operatorname{Ker}\omega_L^1 \cap \operatorname{Ker}\omega_L^2 = \left\langle \frac{\partial}{\partial v_1^1}, \frac{\partial}{\partial v_2^2} \right\rangle$, L is not regular and $(T_2^1 \mathbb{R}^2, (\omega_L^1, \omega_L^2), dE_L)$ is a 2-presymplectic Hamiltonian system.

The field equation is

$$i(X_1)\omega_L^1 + i(X_2)\omega_L^2 = dE_L$$

for a 2-vector field $\mathbf{X} = (X_1, X_2)$ on $T_2^1 \mathbb{R}^2$.

If we write in coordinates

$$X_1 = (X_1)^1 \frac{\partial}{\partial q^1} + (X_1)^2 \frac{\partial}{\partial q^2} + (X_1)_1^1 \frac{\partial}{\partial v_1^1} + (X_1)_2^1 \frac{\partial}{\partial v_2^1} + (X_1)_1^2 \frac{\partial}{\partial v_1^2} + (X_1)_2^2 \frac{\partial}{\partial v_2^2},$$

$$X_2 = (X_2)^1 \frac{\partial}{\partial q^1} + (X_2)^2 \frac{\partial}{\partial q^2} + (X_2)_1^1 \frac{\partial}{\partial v_1^1} + (X_2)_2^1 \frac{\partial}{\partial v_2^1} + (X_2)_1^2 \frac{\partial}{\partial v_1^2} + (X_2)_2^2 \frac{\partial}{\partial v_2^2},$$

then the field equation reads as

$$(X_1)^2(dv_2^1 + dv_1^2) - ((X_1)_2^1 + (X_1)_1^2)dq^2 + (X_2)^1(dv_2^1 + dv_1^2) - ((X_2)_2^1 + (X_2)_1^2)dq^1 = (v_2^1 + v_1^2)(dv_2^1 + dv_1^2).$$

We obtain that

$$(X_1)^2 + (X_2)^1 = v_2^1 + v_1^2, \quad (X_1)_2^1 + (X_1)_1^2 = 0, \quad (X_2)_2^1 + (X_2)_1^2 = 0.$$

Since $dE_L \in (\operatorname{Ker}\omega_L^1 \cap \operatorname{Ker}\omega_L^2)^0$, there are no constraints and the equation has solutions at the whole manifold $T_2^1 \mathbb{R}^2$. The general solution has the form

$$X_1 = (X_1)^1 \frac{\partial}{\partial q^1} + (v_1^2 + A)\frac{\partial}{\partial q^2} + B\frac{\partial}{\partial v_1^1} + C\frac{\partial}{\partial v_2^1} - C\frac{\partial}{\partial v_1^2} + D\frac{\partial}{\partial v_2^2},$$

$$X_2 = (v_2^1 - A)\frac{\partial}{\partial q^1} + (X_2)^2\frac{\partial}{\partial q^2} + E\frac{\partial}{\partial v_1^1} + F\frac{\partial}{\partial v_2^1} - F\frac{\partial}{\partial v_1^2} + G\frac{\partial}{\partial v_2^2},$$

where $(X_1)^1, (X_2)^2, A, B, C, D, E, F$ and G are arbitrary functions.

In order that the integral sections of solutions be holonomic,

$$(X_1)^1 = v_1^1, \quad (X_2)^2 = v_2^2, \quad A = 0, \quad C = E \quad \text{and} \quad D = -F,$$

and, furthermore, we must also demand that $[X_1, X_2] = 0$.

Now we will study the Hamiltonian formalism. Let $(q^1, q^2, p_1^1, p_1^2, p_2^1, p_2^2)$ be the induced coordinates on $(\mathrm{T}_2^1)^*\mathbb{R}^2$. The Legendre map $FL\colon \mathrm{T}_2^1\mathbb{R}^2 \longrightarrow (\mathrm{T}_2^1)^*\mathbb{R}^2$ locally reads

$$FL(q^1, q^2, v_1^1, v_2^1, v_1^2, v_2^2) = (q^1, q^2, p_1^1 = 0, p_1^2 = v_2^1 + v_1^2, p_2^1 = v_2^1 + v_1^2, p_2^2 = 0).$$

The image of FL, $\mathcal{P} := FL(\mathrm{T}_k^1 Q) = \{p_1^1 = 0, p_2^2 = 0, p_1^2 = p_2^1\}$, is a submanifold of $(\mathrm{T}_2^1)^*\mathbb{R}^2$. Let $\jmath_0(q^1, q^2, p) = (q^1, q^2, 0, p, p, 0)$ be the natural embedding and $FL_0\colon \mathrm{T}_k^1 Q \to \mathcal{P}$ the restriction of the Legendre map. We have the Hamiltonian function $H_0 = \frac{1}{2}p^2$ (which is such that $(FL_0)^*H_0 = E_L$), and the 2-forms

$$\omega_0^1 = \jmath_0^*\omega^1 = \jmath_0^*(\mathrm{d}q^1 \wedge \mathrm{d}p_1^1 + \mathrm{d}q^2 \wedge \mathrm{d}p_2^1) = \mathrm{d}q^2 \wedge \mathrm{d}p,$$

$$\omega_0^2 = \jmath_0^*\omega^2 = \jmath_0^*(\mathrm{d}q^1 \wedge \mathrm{d}p_1^2 + \mathrm{d}q^2 \wedge \mathrm{d}p_2^2) = \mathrm{d}q^1 \wedge \mathrm{d}p.$$

With these definitions, $(\mathcal{P}, \omega_0^1, \omega_0^2 \, \mathrm{d}H_0)$ is the 2-presymplectic Hamiltonian system associated with L. The corresponding Hamiltonian field equation is

$$i(Y_1)\omega_0^1 + i(Y_2)\omega_0^2 = \mathrm{d}H_0,$$

where $\mathbf{Y} = (Y_1, Y_2)$ is a 2-vector field on $\mathcal{P}$.

If, in coordinates,

$$Y_1 = (Y_1)^1\frac{\partial}{\partial q^1} + (Y_1)^2\frac{\partial}{\partial q^2} + (Y_1)^0\frac{\partial}{\partial p},$$

$$Y_2 = (Y_2)^1\frac{\partial}{\partial q^1} + (Y_2)^2\frac{\partial}{\partial q^2} + (Y_2)^0\frac{\partial}{\partial p},$$

the equation is

$$(Y_1)^2\mathrm{d}p - (Y_1)^0\mathrm{d}q^2 + (Y_2)^1\mathrm{d}p - (Y_2)^0\mathrm{d}q^1 = p\,\mathrm{d}p$$

and we obtain

$$(Y_1)^2 + (Y_2)^1 = p, \quad (Y_1)^0 = 0, \quad (Y_2)^0 = 0.$$

Since $\operatorname{Ker}\omega_0^1 \cap \operatorname{Ker}\omega_0^2 = \{0\}$, there are no constraints and the equation has solutions at the whole manifold $\mathcal{P}$. The general solution has the form

$$Y_1 = (Y_1)^1\frac{\partial}{\partial q^1} + \left(\frac{1}{2}p + A\right)\frac{\partial}{\partial q^2},$$

$$Y_2 = \left(\frac{1}{2}p - A\right)\frac{\partial}{\partial q^1} + (Y_2)^2\frac{\partial}{\partial q^2},$$

where $(Y_1)^1, (Y_2)^2$ and A are arbitrary functions on $\mathcal{P}$.

5.3   Example 2

In this example we consider two independent variables $(t, s) \in \mathbb{R}^2$, thus $k = 2$. The field components (dependent variables) are $(q, e) \in Q = \mathbb{R}^d \times \mathbb{R}^+$. The corresponding natural coordinates of $\oplus^2 \mathrm{T}Q$ are written $(q, e; q_t, q_s, e_t, e_s)$, and those of $\oplus^2 \mathrm{T}^*Q$ are $(q, e; p^t, p^s, \pi^t, \pi^s)$.

We consider as Lagrangian function

$$L = \frac{1}{2e}(q_t)^2 + \frac{1}{2}m^2 e - \frac{\tau}{2}(q_s)^2,$$

with m, τ parameters, and for instance $(q_t)^2$ is the square of q_t with respect to the Euclidean inner product of $\mathbb{R}^d$. From L we compute the Lagrangian energy

$$E_L = \frac{1}{2e}(q_t)^2 - \frac{\tau}{2}(q_s)^2 - \frac{1}{2}m^2 e$$

and the Legendre map $FL \colon \oplus^2 \mathrm{T}Q \to \oplus^2 \mathrm{T}^*Q$:

$$FL(q, e, q_t, q_s, e_t, e_s) = \left(q, e, \frac{1}{e}q_t, -\tau q_s, 0, 0 \right).$$

It is clear that the primary Hamiltonian constraint submanifold $\mathcal{P}_0 \subset \oplus^2 \mathrm{T}^*Q$ is described by the primary hamiltonian constraints

$$\pi^t \underset{\mathcal{P}_0}{\approx} 0, \quad \pi^s \underset{\mathcal{P}_0}{\approx} 0.$$

This also shows that the Lagrangian L is almost-regular.

Hamiltonian formalism

Using (q, e, p^t, p^s) as coordinates on the submanifold $\mathcal{P}_0$, its 2-presymplectic structure —the pull-back ot the canonical 2-symplectic structure of $\oplus^2 \mathrm{T}^*Q$— is given by $\omega_0^t = \mathrm{d}q \wedge \mathrm{d}p_q^t$ and $\omega_0^s = \mathrm{d}q \wedge \mathrm{d}p_q^s$ —in these expressions a summation over the invisible vector indices of q and the momenta is implicit. Then

$$\mathrm{Ker}\,\omega_0^t \cap \mathrm{Ker}\,\omega_0^s = \left\langle \frac{\partial}{\partial e} \right\rangle.$$

The Hamiltonian function on $\mathcal{P}_0$ is

$$H_0 = \frac{e}{2}(p^t)^2 - \frac{1}{2}m^2 e - \frac{1}{2\tau}(p^s)^2.$$

Consider $\mathbf{X} = (X_t, X_s)$, a 2-vector field on $\mathcal{P}_0$:

$$
\begin{aligned}
X_t &= F_t \frac{\partial}{\partial q} + f_t \frac{\partial}{\partial e} + F_t^t \frac{\partial}{\partial p^t} + F_t^s \frac{\partial}{\partial p^s} \\
X_s &= F_s \frac{\partial}{\partial q} + f_s \frac{\partial}{\partial e} + F_s^t \frac{\partial}{\partial p^t} + F_s^s \frac{\partial}{\partial p^s}
\end{aligned}
$$

(where the capital F's are also vector functions). The Hamiltonian field equation for it is $i(X_t)\omega_0^t + i(X_s)\omega_0^s = \mathrm{d}H_0$:

$$F_t \mathrm{d}p^t + F_s \mathrm{d}p^s - (F_t^t + F_s^s)\mathrm{d}q = e\,p^t \mathrm{d}p^t - \frac{p^s}{\tau}\mathrm{d}p^s + \frac{1}{2}\left((p^t)^2 - m^2\right)\mathrm{d}e,$$

which partly determines the coefficients of $\mathbf{X}$:

$$F_t = e\,p^t \,, \ \ F_s = -\frac{1}{\tau}p^s \,, \ \ F_t^t + F_s^s = 0\,,$$

and imposes as a consistency condition the secondary hamiltonian constraint

$$\frac{1}{2}\left((p^t)^2 - m^2\right) \underset{\mathcal{P}_1}{\approx} 0;$$

this can also be obtained as $\xi = i\left(\dfrac{\partial}{\partial e}\right)\mathrm{d}H = \tfrac{1}{2}((p^t)^2 - m^2) \approx 0$.

Imposing the tangency of $\mathbf{X}$ to $\mathcal{P}_1$ yields no more constraints and determines partly some coefficients of $\mathbf{X}$:

$$\mathcal{L}_{X_t}\xi = p^t F_t^t \approx 0, \quad \mathcal{L}_{X_s}\xi = p^t F_s^t \approx 0,$$

from which the final dynamics on $\mathcal{P}_1$ is given by

$$\begin{aligned}
X_t &= e\,p^t \frac{\partial}{\partial q} + f_t \frac{\partial}{\partial e} + F_t^t \frac{\partial}{\partial p^t} + F_t^s \frac{\partial}{\partial p^s} \\
X_s &= -\frac{1}{\tau}p^s \frac{\partial}{\partial q} + f_s \frac{\partial}{\partial e} + F_s^t \frac{\partial}{\partial p^t} - F_t^t \frac{\partial}{\partial p^s},
\end{aligned}$$

with f_t, f_s, F_t^s arbitrary functions, and F_t^t, F_s^t arbitrary but orthogonal to p^t.

Consider the particular case of $d = 1$ —the q variable is just a scalar. The submanifold $\mathcal{P}_1$ is given by the constraint $p^t = m$ (or $p^t = -m$). Then, in coordinates (q, e, p^s), the dynamics reads

$$\begin{aligned}
X_t &= me\frac{\partial}{\partial q} + f_t \frac{\partial}{\partial e} + F_t^s \frac{\partial}{\partial p^s} \\
X_s &= -\frac{1}{\tau}p^s \frac{\partial}{\partial q} + f_s \frac{\partial}{\partial e}.
\end{aligned}$$

The analysis of the integrability of the 2-vector field $\mathbf{X} = (X_t, X_s)$ relies on the computation of

$$[X_t, X_s] = -\left(\frac{1}{\tau}F_t^s + mf_s\right)\frac{\partial}{\partial q} + \left(\mathcal{L}_{X_t}f_s - \mathcal{L}_{X_s}f_t\right)\frac{\partial}{\partial e} - \left(\mathcal{L}_{X_s}F_t^s\right)\frac{\partial}{\partial p^s}\,.$$

Setting it to zero determines $F_t^s = -\tau m f_s$ and a set of two nonlinear PDEs for f_s, f_t:

$$-\frac{1}{\tau}p^s \frac{\partial f_s}{\partial q} + f_s \frac{\partial f_s}{\partial e} = 0\,,$$

$$me\frac{\partial f_s}{\partial q} + f_t \frac{\partial f_s}{\partial e} - \tau m f_s \frac{\partial f_s}{\partial p^s} + \frac{1}{\tau}p^s \frac{\partial f_t}{\partial q} - f_s \frac{\partial f_t}{\partial e} = 0\,.$$

Certainly there are solutions to this equations, as for instance the one given by $f_t = f_s = 0$. However, it does not seem easy to give an explicit description of the whole set of these solutions.

Finally, once one has an integrable 2-vector field $\mathbf{X}$, a map $\psi\colon \mathbb{R}^2 \to \mathcal{P}_1$, $(t, s) \mapsto (q, e, p^s)$, is an integral section iff it satisfies

$$\frac{\partial q}{\partial t} = me, \quad \frac{\partial q}{\partial s} = -\frac{p^s}{\tau}, \quad \frac{\partial e}{\partial t} = f_t, \quad \frac{\partial e}{\partial s} = f_s, \quad \frac{\partial p^s}{\partial t} = F_t^s, \quad \frac{\partial p^s}{\partial s} = 0.$$

Lagrangian formalism

The Lagrangian analysis can be performed in a similar way. Let us describe it more briefly. Using natural coordinates $(q, e; q_t, e_t, q_s, e_s)$ on $\oplus^2 \mathrm{T}Q$, the 2-presymplectic structure induced by L is described by $\omega^t = \dfrac{1}{e}\, \mathrm{d}q \wedge \mathrm{d}q_t - \dfrac{1}{e^2} q_t\, \mathrm{d}q \wedge \mathrm{d}e$ and $\omega^s = -\tau\, \mathrm{d}q \wedge \mathrm{d}q_s$. Then

$$\operatorname{Ker}\omega^t \cap \operatorname{Ker}\omega^s = \left\langle \frac{\partial}{\partial e_t}, \frac{\partial}{\partial e_t}, e\frac{\partial}{\partial e} + q_t\frac{\partial}{\partial q_t} \right\rangle.$$

An arbitrary 2-vector field $\mathbf{X} = (X_t, X_s)$ on $\oplus^2 \mathrm{T}Q$ reads

$$
\begin{aligned}
X_t &= F_t\frac{\partial}{\partial q} + f_t\frac{\partial}{\partial e} + F_{tt}\frac{\partial}{\partial q_t} + F_{ts}\frac{\partial}{\partial q_s} f_{tt}\frac{\partial}{\partial e_t} + f_{ts}\frac{\partial}{\partial e_s}, \\
X_s &= F_s\frac{\partial}{\partial q} + f_s\frac{\partial}{\partial e} + F_{st}\frac{\partial}{\partial q_t} + F_{ss}\frac{\partial}{\partial q_s} f_{st}\frac{\partial}{\partial e_t} + f_{ss}\frac{\partial}{\partial e_s}.
\end{aligned}
$$

If it has to satisfy the second-order condition, one has moreover

$$F_t = q_t, \quad F_s = q_s \quad f_t = e_t, \quad f_s = e_s.$$

The field equation for it is $i(X_t)\omega^t + i(X_s)\omega^s = \mathrm{d}E_L$ (maybe on a certain submanifold). This determines some of the coefficients and defines just one primary Lagrangian constraint, $\chi = \dfrac{1}{2}\left(\dfrac{(q_t)^2}{e^2} - m^2\right)$. The tangency to this submanifold does not yield new constraints, and some functions in $\mathbf{X}$ remain arbitrary. This happens regardless of whether we impose the second-order condition or not, the only difference being in the number of remaining arbitrary functions.

Acknowledgments

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