Combinatorial vs. Algebraic Characterizations of Pseudo-Distance-Regularity Around a Set *

M. Cámara, J. Fàbrega, M.A. Fiol, and E. Garriga

Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya
Jordi Girona 1-3, Mòdul C3, Campus Nord
08034 Barcelona, Catalonia (Spain)

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Abstract

Given a simple connected graph Γ and a subset of its vertices C, the pseudo-distance-regularity around C generalizes, for not necessarily regular graphs, the notion of completely regular code. Up to know, most of the characterizations of pseudo-distance-regularity has been derived from a combinatorial definition. In this paper we propose an algebraic (Terwilliger-like) approach to this notion, showing its equivalence with the combinatorial one. This allows us to give new proofs of known results, and also to obtain new characterizations which do not depend on the so-called C-spectrum of Γ, but only on the positive eigenvector of its adjacency matrix. In the way, we also obtain some results relating the local spectra of a vertex set and its antipodal. As a consequence of our study, we obtain a new characterization of a completely regular code C, in terms of the number of walks in Γ with an endvertex in C.

Keywords: Pseudo-Distance-regular graph; Adjacency matrix; Local spectrum; Orthogonal predistance polynomials; Terwilliger algebras; Completely regular code.

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1 Preliminaries

Pseudo-distance-regularity is a natural generalization of distance-regularity which extends this notion to not necessary regular graphs. The key point of this generalization relays

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Emails: mcamara@ma4.upc.edu, jfabrega@ma4.upc.edu, fiol@ma4.upc.edu, egarriga@ma4.upc.edu.

on defining an adequate weight for each vertex in such a way that we obtain a "regularized" graph. Since its introduction in [7], the study of pseudo-distance-regularity produced several interesting results, specially in the area of quasi-spectral characterizations of distance-regularity [4, 7] and completely regular codes [5, 6]. This study was based on the combinatorial definition of pseudo-distance-regularity around a vertex, which comes up naturally from the notion of distance-regularity around a vertex. Among the variety of techniques used in these works, two concepts stand out: the local spectrum (of a single vertex or a subset of vertices) and certain families of orthogonal polynomials.

Our work in this paper is motivated by the connection existing between pseudo-distance-regularity and the study developed by Terwilliger [11] in the context of association schemes. In his work, he introduced the subconstituent algebra (also known as Terwilliger algebra) with respect to a vertex of a graph and defined the notion of thin module in this algebra. As commented by the third and fourth authors in [3, 5], the study of pseudo-distance-regularity around a vertex i is equivalent to the thin character of the minimum module containing its characteristic vector $\boldsymbol{e}_i$. The aim of this paper is to extend this parallelism from a single vertex to a set of vertices.

The plan of the paper is as follows. In the rest of this section we first give some notation on graphs and their spectra. In Section we introduce the local spectrum of a vertex set, discussing some of its properties. Special attention is paid to the relation between the local spectra of two antipodal subsets of vertices. Section 3 is devoted to explain the concept of pseudo-distance-regularity around a vertex set, in combinatorial sense, and to review some its known quasi-spectral characterizations. In the case of regular graphs, this concept coincides with that of a completely regular code. Our main results are in Section 4, where we extend the (algebraic) definition of Terwilliger to a set of vertices in any graph, and prove its equivalence with the combinatorial approach. This allows us to give new proofs of known results, and also to obtain new characterizations which do not depend on the so-called C-local spectrum, but only on the positive eigenvector of the adjacency matrix. As a consequence, we obtain a new characterization of a completely regular code C, in terms of the number of walks having an endvertex in C.

Throughout this paper $\Gamma = (V, E)$ stands for a simple connected graph with vertex set $V = \{1, 2, \ldots, n\}$ and $\mathcal{V}$ denotes the space of the formal linear combinations of its vertices. The adjacencies in Γ, $\{i, j\} \in E$, are denoted by $i \sim j$ and $\Gamma_k(i) = \{j | \partial(i, j) = k\}$ represents the set of vertices at distance k from i, where $\partial(\cdot, \cdot)$ is the distance function in Γ. For simplicity we will write $\Gamma(i)$ instead of $\Gamma_1(i)$. Every vertex i is associated to the i-th unitary (or characteristic) vector $\boldsymbol{e}_i \in \mathbb{R}^n$, and, consequently, $\mathcal{V}$ is identified with $\mathbb{R}^n$. With this identification in mind, the adjacency matrix of Γ, $\boldsymbol{A}$, can be seen as the matrix of an endomorphism in $\mathcal{V}$ with respect to the basis $\{\boldsymbol{e}_i\}_{i \in V}$.

The set of different eigenvalues of $\boldsymbol{A}$ is denoted by $\operatorname{ev}\Gamma := \{\lambda_0, \lambda_1, \ldots, \lambda_d\}$, where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the spectrum of Γ is defined by

$$\operatorname{sp}\Gamma := \operatorname{sp}\boldsymbol{A} = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \cdots, \lambda_d^{m(\lambda_0)}\},$$

where $m(\lambda_l)$ stands for the multiplicity of the eigenvalue λ_l. From the Perron-Frobenius

Theorem for nonnegative matrices, we have that $\lambda_0 \geq |\lambda_d|$ and equality is attained if and only if Γ is a bipartite graph; see e.g. [1]. Moreover, $m(\lambda_0) = 1$ and every non-null vector of $\mathrm{Ker}(\boldsymbol{A} - \lambda_0 \boldsymbol{I})$ has all its components either positive or negative. We denote by $\boldsymbol{\nu} \in \mathrm{Ker}(\boldsymbol{A} - \lambda_0 \boldsymbol{I})$ the unique positive eigenvector with minimum component equal to one. Let us remark that in the case of δ-regular graphs we have that $\lambda_0 = \delta$ and the vector $\boldsymbol{\nu}$ turns out to be the all-1 vector $\boldsymbol{j}$.

Note that $\mathcal{V}$ is a module over the quotient ring $\mathbb{R}[x]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the polynomial $Z = \prod_{l=0}^{d}(x - \lambda_l)$, which vanishes in $\boldsymbol{A}$, with product defined by

$$p\boldsymbol{u} := p(\boldsymbol{A})\boldsymbol{u} \qquad \text{for every } p \in \mathbb{R}[x]/\mathcal{I} \text{ and } \boldsymbol{u} \in \mathcal{V}.$$

Remark that the orthogonal projection of $\mathcal{V}$ onto the eigenspace $\mathcal{E}_l = \mathrm{Ker}(\boldsymbol{A} - \lambda_l \boldsymbol{I})$, for some $0 \leq l \leq d$, can be written as

$$\boldsymbol{E}_l \boldsymbol{u} = Z_l \boldsymbol{u}, \quad \boldsymbol{u} \in \mathcal{V},$$

where $Z_l = \frac{(-1)^l}{\pi_l} \prod_{0 \leq h \leq d (h \neq l)} (x - \lambda_l)$ and $\pi_l := \prod_{0 \leq h \leq d (h \neq l)} |\lambda_h - \lambda_l|$.

2 The local spectrum of a vertex set and its antipodal

Given a nonempty set C of vertices of Γ, we consider the map $\boldsymbol{\rho} : \mathcal{P}(V) \to \mathcal{V}$ defined by $\boldsymbol{\rho}\emptyset = \boldsymbol{0}$ and $\boldsymbol{\rho} C = \sum_{i \in C} \nu_i \boldsymbol{e}_i$ for $C \neq \emptyset$ and denote by $\boldsymbol{e}_C$ the normalized vector $\boldsymbol{\rho} C / \|\boldsymbol{\rho} C\|$. If $\boldsymbol{e}_C = \boldsymbol{z}_C(\lambda_0) + \boldsymbol{z}_C(\lambda_1) + \cdots + \boldsymbol{z}_C(\lambda_d)$ is the spectral decomposition of $\boldsymbol{e}_C$; that is $\boldsymbol{z}_C(\lambda_l) = \boldsymbol{E}_l \boldsymbol{e}_C \in \mathcal{E}_l$, $l = 0, 1, \ldots, d$, the *C-multiplicity* (or *C-local multiplicity*) of the eigenvalue λ_l is defined by $m_C(\lambda_l) = \|\boldsymbol{z}_C(\lambda_l)\|^2$. Note that, since

$$\boldsymbol{E}_0 \boldsymbol{e}_C = \frac{1}{\|\boldsymbol{\rho} C\|} \frac{\langle \boldsymbol{\rho} C, \boldsymbol{\nu} \rangle}{\|\boldsymbol{\nu}\|^2} \boldsymbol{\nu} = \frac{1}{\|\boldsymbol{\rho} C\|} \sum_{i \in C} \nu_i \frac{\nu_i}{\|\boldsymbol{\nu}\|^2} \boldsymbol{\nu} = \frac{\|\boldsymbol{\rho} C\|}{\|\boldsymbol{\nu}\|^2} \boldsymbol{\nu},$$

we get $m_C(\lambda_0) = \frac{\|\boldsymbol{\rho} C\|^2}{\|\boldsymbol{\nu}\|^2}$. Then, if $\mu_0 (= \lambda_0), \mu_1, \ldots, \mu_{d_C}$ are the eigenvalues with non-zero C-multiplicity, the *C-spectrum* (or *C-local spectrum*) is defined by

$$\mathrm{sp}_C \Gamma := \{\mu_0^{m_C(\mu_0)}, \mu_1^{m_C(\mu_1)}, \ldots, \mu_{d_C}^{m_C(\mu_{d_C})}\},$$

with $\mu_0 > \mu_1 > \cdots > \mu_{d_C}$, and the set of different eigenvalues of C is denoted by $\mathrm{ev}_C \Gamma := \{\mu_0, \mu_1, \ldots, \mu_{d_C}\}$. Note that, since $\boldsymbol{e}_C$ is unitary, we have $\sum_{l=0}^{d_C} m_C(\lambda_l) = 1.$, or As we have done for the spectrum of Γ, in order to simplify notation we introduce the moment-like parameters

$$\pi_l(C) := \prod_{0 \leq h \leq d_C (h \neq l)} |\mu_h - \mu_l| \qquad (0 \leq l \leq d_C).$$

The set $\Gamma_k(C) = \{v \in V | \partial(v, C) = k\}$ of vertices at distance k from C is denoted by C_k. Thus, if C has eccentricity ε_C, $C_0 (= C), C_1, \ldots, C_{\varepsilon_C}$ is a partition of V. We denote by $\overline{C}$ the set of vertices at maximum distance from C, $\overline{C} = C_{\varepsilon_C}$, and we refer to it as its antipodal set. If there is no possible confusion, we write $D = \overline{C}$.

The polynomial $Z_C = \prod_{l=0}^{d_C}(x - \mu_l)$ is the monic polynomial with minimum degree such that $Z_C \boldsymbol{e}_C = \boldsymbol{0}$, and the polynomial

$$H_C = \frac{\|\boldsymbol{\nu}\|^2}{\pi_0(C)\|\boldsymbol{\rho} C\|^2} \prod_{l=1}^{d_C}(x - \mu_l)$$

satisfy $H_C \boldsymbol{\nu} = H_C(\lambda_0)\boldsymbol{\nu} = \frac{\|\boldsymbol{\nu}\|^2}{\|\boldsymbol{\rho} C\|^2}\boldsymbol{\nu}$. What is more, H_C is the unique polynomial of degree at most d_C satisfying $H_C \boldsymbol{\rho} C = \frac{\|\boldsymbol{\rho} C\|^2}{\|\boldsymbol{\nu}\|^2} H_C \boldsymbol{\nu} = \boldsymbol{\nu}$ and so, inspired by Hoffman [8], it is named *C-local Hoffman polynomial*. This allows us to conclude that the eccentricity of C and the number of C-local eigenvalues are related by $\varepsilon_C \leq d_C$; see [5]. In case of equality, $\varepsilon_C = d_C$, we say that C is *extremal*.

Proposition 2.1 *Let C be an extremal set and let D be its antipodal set. Then,* $\mathrm{ev}_C \Gamma \subset \mathrm{ev}_D \Gamma$ *and the C-multiplicities and D-multiplicities satisfy*

$$m_C(\mu_l) m_D(\mu_l) \geq \frac{\pi_0^2(C)}{\pi_l^2(C)} \frac{\|\boldsymbol{\rho} C\|^2 \|\boldsymbol{\rho} D\|^2}{\|\boldsymbol{\nu}\|^4} \qquad \text{for all } \mu_l \in \mathrm{ev}_C \Gamma,$$

where equality is equivalent to the linear dependence of the vectors $\boldsymbol{z}_C(\mu_l)$ and $\boldsymbol{z}_D(\mu_l)$.

Proof. Consider the interpolating polynomials associated with the local spectrum of C:

$$Z_l^C = \frac{(-1)^l}{\pi_l(C)} \prod_{0 \leq h \leq d_C\ (h \neq l)} (x - \mu_h) \qquad (0 \leq l \leq d_C),$$

verifying $Z_l^C(\mu_h) = \delta_{lh}$. Since both Z_l^C and H_C have degree d_C and their leader coefficients are, respectively, $\frac{(-1)^l}{\pi_l(C)}$ and $\frac{\|\boldsymbol{\nu}\|^2}{\pi_0(C)\|\boldsymbol{\rho} C\|^2}$, the polynomial

$$T = \pi_0(C)\frac{\|\boldsymbol{\rho} C\|^2}{\|\boldsymbol{\nu}\|^2} H_C - (-1)^l \pi_l(C) Z_l^C$$

has degree less than d_C. The extremal character of C gives that $\langle \boldsymbol{\rho} C, Z_l^C \boldsymbol{\rho} D\rangle = \langle Z_l^C \boldsymbol{\rho} C, \boldsymbol{\rho} D\rangle = \frac{(-1)^l}{\pi_l(C)}\langle x^{d_C}\boldsymbol{\rho} C, \boldsymbol{\rho} D\rangle \neq 0$. In particular, $Z_l^C \boldsymbol{\rho} D \neq \boldsymbol{0}$. Moreover, if $\lambda_l \in \mathrm{ev}_C \Gamma$,

$$\begin{aligned}
\langle \boldsymbol{\rho} C, Z_l^C \boldsymbol{\rho} D\rangle &= \langle \boldsymbol{\rho} C, \sum_{h=0}^{d} Z_l^C(\lambda_h)\boldsymbol{E}_h \boldsymbol{\rho} D\rangle \\
&= \langle \boldsymbol{\rho} C, \sum_{\lambda_h \notin \mathrm{ev}_C \Gamma} Z_l^C(\lambda_h)\boldsymbol{E}_h \boldsymbol{\rho} D + Z_l^C(\lambda_l)\boldsymbol{E}_l \boldsymbol{\rho} D\rangle \\
&= \langle \boldsymbol{\rho} C, Z_l^C(\lambda_l)\boldsymbol{E}_l \boldsymbol{\rho} D\rangle = \langle \boldsymbol{\rho} C, \boldsymbol{z}_D(\mu_l)\rangle,
\end{aligned}$$

and $\mathrm{ev}_C\,\Gamma \subset \mathrm{ev}_D\,\Gamma$.

Since T has degree less than $d_C = \varepsilon_C$, the vectors $T\boldsymbol{e}_C$ and $\boldsymbol{e}_D$ are orthogonal, giving:

$$\begin{aligned}
0 = \langle T\boldsymbol{e}_C, \boldsymbol{e}_D\rangle &= \pi_0(C)\frac{\|\boldsymbol{\rho}C\|^2}{\|\boldsymbol{\nu}\|^2}\langle H_C\boldsymbol{e}_C, \boldsymbol{e}_D\rangle - (-1)^l\pi_l(C)\langle Z_l^C\boldsymbol{e}_C, \boldsymbol{e}_D\rangle \\
&= \pi_0(C)\frac{\|\boldsymbol{\rho}C\|}{\|\boldsymbol{\rho}D\|\,\|\boldsymbol{\nu}\|^2}\langle H_C\boldsymbol{\rho}C, \boldsymbol{\rho}D\rangle - (-1)^l\pi_l(C)\langle \boldsymbol{z}_C(\mu_l), \boldsymbol{e}_D\rangle \\
&= \pi_0(C)\frac{\|\boldsymbol{\rho}C\|}{\|\boldsymbol{\rho}D\|\,\|\boldsymbol{\nu}\|^2}\langle \boldsymbol{\nu}, \boldsymbol{\rho}D\rangle - (-1)^l\pi_l(C)\langle \boldsymbol{z}_C(\mu_l), \boldsymbol{z}_D(\mu_l)\rangle \\
&= \pi_0(C)\frac{\|\boldsymbol{\rho}C\|\,\|\boldsymbol{\rho}D\|}{\|\boldsymbol{\nu}\|^2} - (-1)^l\pi_l(C)\|\boldsymbol{z}_C(\mu_l)\|\,\|\boldsymbol{z}_D(\mu_l)\|\cos\alpha_l^{(C,D)},
\end{aligned}$$

where $\alpha_l^{(C,D)}$ is the angle between the vectors $\boldsymbol{z}_C(\mu_l)$, $\boldsymbol{z}_D(\mu_l)$. Therefore,

$$\langle \boldsymbol{z}_C(\mu_l), \boldsymbol{z}_D(\mu_l)\rangle = (-1)^l\frac{\pi_0(C)}{\pi_l(C)}\,\frac{\|\boldsymbol{\rho}C\|\,\|\boldsymbol{\rho}D\|}{\|\boldsymbol{\nu}\|^2}, \tag{1}$$

and also:

$$\frac{\pi_0^2(C)}{\pi_l^2(C)}\,\frac{\|\boldsymbol{\rho}C\|^2\|\boldsymbol{\rho}D\|^2}{\|\boldsymbol{\nu}\|^4} = m_C(\mu_l)m_D(\mu_l)\cos^2\alpha_l^{(C,D)} \le m_C(\mu_l)m_D(\mu_l),$$

where the equality occurs if and only if $\alpha_l^{(C,D)}$ is 0 or π. □

Proposition 2.2 *Let C be an extremal set and D its antipodal set. Then, the following statements are equivalent:*

(*a*) *For every $\mu_l \in \mathrm{ev}_C\,\Gamma$, we have*

$$m_C(\mu_l)m_D(\mu_l) = \frac{\pi_0^2(C)}{\pi_l^2(C)}\,\frac{\|\boldsymbol{\rho}C\|^2\|\boldsymbol{\rho}D\|^2}{\|\boldsymbol{\nu}\|^4}.$$

(*b*) *The projection of the vector $\tilde{\boldsymbol{m}}_D = (\|\boldsymbol{z}_D(\mu_0)\|, \|\boldsymbol{z}_D(\mu_1)\|, \ldots, \|\boldsymbol{z}_D(\mu_{\varepsilon_C})\|)$ over the vector $\boldsymbol{m}_C = (\|\boldsymbol{z}_C(\mu_0)\|, \|\boldsymbol{z}_C(\mu_1)\|, \ldots, \|\boldsymbol{z}_C(\mu_{\varepsilon_C})\|)$ is*

$$\frac{\|\boldsymbol{\rho}C\|\,\|\boldsymbol{\rho}D\|}{\|\boldsymbol{\nu}\|^2}\sum_{l=0}^{\varepsilon_C}\frac{\pi_0(C)}{\pi_l(C)},$$

or, equivalently,

$$\left(\sum_{l=0}^{\varepsilon_C} m_D(\mu_l)\right)\cos^2\alpha^{(C,D)} = \left(\sum_{l=0}^{\varepsilon_C}\frac{\pi_0(C)}{\pi_l(C)}\right)^2\frac{\|\boldsymbol{\rho}C\|^2\,\|\boldsymbol{\rho}D\|^2}{\|\boldsymbol{\nu}\|^4},$$

where $\alpha^{(C,D)}$ is the angle between the two vectors.

(*c*) *There exists a polynomial* $p \in \mathbb{R}_{\varepsilon_C}[x]$ *such that*

$$\boldsymbol{\rho} D = p\boldsymbol{\rho} C + \boldsymbol{z}, \quad \textit{where} \quad \boldsymbol{z} \in \bigoplus_{\lambda_l \in \mathrm{ev}_D \Gamma \setminus \mathrm{ev}_C \Gamma} \mathcal{E}_l.$$

(*d*) *For every* $\mu_l \in \mathrm{ev}_C \Gamma$, *we have*

$$\frac{\|\boldsymbol{\rho} D\|^2}{\sum_{l=0}^{\varepsilon_C} m_D(\mu_l)} = \|\boldsymbol{\nu}\|^2 \left(\sum_{l=0}^{\varepsilon_C} \frac{m_C(\mu_0)\pi_0^2(C)}{m_C(\mu_l)\pi_l^2(C)} \right)^{-1}.$$

Proof. By adding up for $l = 0, 1, \ldots, \varepsilon_C$ the inequalities given in Proposition 2.1 we obtain:

$$\langle \boldsymbol{m}_C, \tilde{\boldsymbol{m}}_D \rangle = \|\tilde{\boldsymbol{m}}_D\| \cos \alpha^{(C,D)} = \sum_{l=0}^{\varepsilon_C} \|\boldsymbol{z}_C(\mu_l)\| \, \|\boldsymbol{z}_D(\mu_l)\| \geq \frac{\|\boldsymbol{\rho} C\| \, \|\boldsymbol{\rho} D\|}{\|\boldsymbol{\nu}\|^2} \sum_{l=0}^{\varepsilon_C} \frac{\pi_0(C)}{\pi_l(C)},$$

giving the equivalence between (*a*) and (*b*).

Suppose that (*a*) holds. Then, given $\mu_l \in \mathrm{ev}_C \Gamma$, the vectors $\boldsymbol{z}_D(\mu_l)$, $\boldsymbol{z}_C(\mu_l)$ are proportional. More precisely, by (1), there exist $\xi_l > 0$ such that $\boldsymbol{z}_D(\mu_l) = (-1)^l \xi_l \boldsymbol{z}_C(\mu_l)$. Let p be the unique polynomial in $\mathbb{R}_{\varepsilon_C}[x]$ such that $p(\mu_l) = (-1)^l \frac{\|\boldsymbol{\rho} D\|}{\|\boldsymbol{\rho} C\|} \xi_l$ for all $\mu_l \in \mathrm{ev}_C \Gamma$. We have

$$\begin{aligned} \boldsymbol{E}_l \boldsymbol{\rho} D &= \|\boldsymbol{\rho} D\| \boldsymbol{z}_D(\mu_l) = (-1)^l \|\boldsymbol{\rho} D\| \xi_l \boldsymbol{z}_C(\mu_l) \\ &= (-1)^l \frac{\|\boldsymbol{\rho} D\|}{\|\boldsymbol{\rho} C\|} \xi_l \boldsymbol{E}_l \boldsymbol{\rho} C = p(\mu_l) \boldsymbol{E}_l \boldsymbol{\rho} C = \boldsymbol{E}_l p \boldsymbol{\rho} C. \end{aligned}$$

Thus the vector $\boldsymbol{z} = \boldsymbol{\rho} D - p\boldsymbol{\rho} C \in \bigoplus_{\lambda_l \in \mathrm{ev}_D \Gamma \setminus \mathrm{ev}_C \Gamma} \mathcal{E}_l$ and (*c*) is obtained. Conversely, assuming that (*c*) holds, by projecting on $\mathcal{E}_l$ we obtain $\|\boldsymbol{\rho} D\| \boldsymbol{z}_D(\mu_l) = p(\mu_l) \|\boldsymbol{\rho} C\| \boldsymbol{z}_C(\mu_l)$ and Proposition 2.1 gives (*a*).

Finally we prove the equivalence between (*c*) and (*d*). The existence of the polynomial p in (*c*) is equivalent to the linear dependence of the vectors $\boldsymbol{z}_D(\mu_l)$ and $\boldsymbol{z}_C(\mu_l)$ for all $\mu_l \in \mathrm{ev}_C \Gamma$, and Proposition 2.1 ensures us that

$$m_C(\mu_l) m_D(\mu_l) = \frac{\pi_0^2(C)}{\pi_l^2(C)} \frac{\|\boldsymbol{\rho} C\|^2 \| \, \|\boldsymbol{\rho} D\|^2}{\|\boldsymbol{\nu}\|^4} \qquad (0 \leq l \leq d_C).$$

Hence, in this case,

$$\sum_{l=0}^{\varepsilon_C} m_D(\mu_l) = \frac{\|\boldsymbol{\rho} C\|^2 \|\boldsymbol{\rho} D\|^2}{\|\boldsymbol{\nu}\|^4} \sum_{l=0}^{\varepsilon_C} \frac{\pi_0^2(C)}{m_C(\mu_l)\pi_l^2(C)} = \frac{\|\boldsymbol{\rho} D\|^2}{\|\boldsymbol{\nu}\|^2} \sum_{l=0}^{\varepsilon_C} \frac{m_C(\mu_0)\pi_0^2(C)}{m_C(\mu_l)\pi_l^2(C)}, \tag{2}$$

and the proof is concluded. □

Corollary 2.3 *The polynomial* p *described in Proposition 2.2(c) satisfies the following properties:*

(a) *$p \in \mathbb{R}_{\varepsilon_C}[x]$ is unique, has degree ε_C and all its roots are real, different and interlace the eigenvalues $\mu_0, \mu_1, \ldots, \mu_{\varepsilon_C}$.*

(b) *Its value at μ_0 is:*

$$p(\mu_0) = \frac{\|\boldsymbol{\rho} D\|^2}{\|\boldsymbol{\rho} C\|^2} = \frac{\|\boldsymbol{\nu}\|^2}{\|\boldsymbol{\rho} C\|^2} \left(\sum_{l=0}^{\varepsilon_C} m_D(\mu_l) \right) \left(\sum_{l=0}^{\varepsilon_C} \frac{m_C(\mu_0)\pi_0^2(C)}{m_C(\mu_l)\pi_l^2(C)} \right)^{-1}.$$

(c) *Given $q \in \mathbb{R}_{\varepsilon_C - 1}[x]$, we have:*

$$\sum_{l=0}^{\varepsilon_C} m_C(\mu_l)p(\mu_l)q(\mu_l) = 0 \quad \text{and} \quad \sum_{l=0}^{\varepsilon_C} m_C(\mu_l)p^2(\mu_l) = \left(\sum_{l=0}^{\varepsilon_C} m_D(\mu_l) \right) p(\mu_0).$$

Proof. *(a)* Using (1), the computation

$$\begin{aligned}
(-1)^l \frac{\pi_0(C)}{\pi_l(C)} \frac{\|\boldsymbol{\rho} C\| \, \|\boldsymbol{\rho} D\|}{\|\boldsymbol{\nu}\|^2} &= \langle \boldsymbol{z}_C(\mu_l), \boldsymbol{z}_D(\mu_l) \rangle = \langle \boldsymbol{e}_C, \boldsymbol{E}_l \boldsymbol{e}_D \rangle \\
&= \frac{1}{\|\boldsymbol{\rho} C\| \, \|\boldsymbol{\rho} D\|} \langle \boldsymbol{\rho} C, \boldsymbol{E}_l \boldsymbol{\rho} D \rangle \\
&= \frac{1}{\|\boldsymbol{\rho} C\| \, \|\boldsymbol{\rho} D\|} \langle \boldsymbol{\rho} C, \boldsymbol{E}_l p \boldsymbol{\rho} C \rangle \\
&= \frac{1}{\|\boldsymbol{\rho} C\| \, \|\boldsymbol{\rho} D\|} p(\mu_l) \langle \boldsymbol{\rho} C, \boldsymbol{E}_l \boldsymbol{\rho} C \rangle \\
&= \frac{\|\boldsymbol{\rho} C\|}{\|\boldsymbol{\rho} D\|} p(\mu_l) \langle \boldsymbol{z}_C(\mu_l), \boldsymbol{z}_C(\mu_l) \rangle = m_C(\mu_l) \frac{\|\boldsymbol{\rho} C\|}{\|\boldsymbol{\rho} D\|} p(\mu_l),
\end{aligned}$$

gives

$$p(\mu_l) = (-1)^l \frac{\pi_0(C)}{m_C(\mu_l)\pi_l(C)} \frac{\|\boldsymbol{\rho} D\|^2}{\|\boldsymbol{\nu}\|^2} \qquad \text{for all } \mu_l \in \operatorname{ev} C, \tag{3}$$

thus, the polynomial $p \in \mathbb{R}_{\varepsilon_C}[x]$ is unique and the alternance of the sign over $\operatorname{ev}_C \Gamma$ guaranties that their roots interlace its elements.

(b) From Proposition 2.2 *(c)* we get $\|\boldsymbol{\rho} D\|^2 = \langle p \boldsymbol{\rho} C, \boldsymbol{\nu} \rangle = \langle \boldsymbol{\rho} C, p \boldsymbol{\nu} \rangle = p(\mu_0) \langle \boldsymbol{\rho} C, \boldsymbol{\nu} \rangle = p(\mu_0) \|\boldsymbol{\rho} C\|^2$. This, together with Proposition 2.2 *(d)*, gives the equalities.

(c) Using *(b)* and (3),

$$\begin{aligned}
\sum_{l=0}^{\varepsilon_C} m_C(\mu_l) p^2(\mu_l) &= \sum_{l=0}^{\varepsilon_C} m_C(\mu_l) \frac{\pi_0^2(C)}{m_C^2(\mu_l)\pi_l^2(C)} \frac{\|\boldsymbol{\rho} D\|^4}{\|\boldsymbol{\nu}\|^2} \\
&= \frac{\|\boldsymbol{\rho} D\|^4}{\|\boldsymbol{\rho} C\|^4} \sum_{l=0}^{\varepsilon_C} \frac{m_C^2(\mu_0)\pi_0^2(C)}{m_C(\mu_l)\pi_l^2(C)} \\
&= \frac{\|\boldsymbol{\rho} D\|^2}{\|\boldsymbol{\rho} C\|^2} \sum_{l=0}^{\varepsilon_C} m_D(\mu_l) = \left(\sum_{l=0}^{\varepsilon_C} m_D(\mu_l) \right) p(\mu_0).
\end{aligned}$$

The polynomials

$$Z_l^C = \frac{(-1)^l}{\pi_l(C)} \prod_{0\le h\le \varepsilon_C\ (h\neq l)} (x-\mu_h) \qquad (0\le l\le \varepsilon_C),$$

verifying $Z_l^C(\mu_h) = \delta_{lh}$, allow us to write every polynomial $q\in \mathbb{R}_{\varepsilon_C}[x]$ as $q=\sum_{l=0}^{\varepsilon_C} q(\mu_l)Z_l^C$. In particular, $\sum_{l=0}^{\varepsilon_C} \mu_l^k Z_l^C = x^k$, $k=0,1,\ldots,\varepsilon_C$. Equating the coefficients of degree ε_C we obtain

$$\sum_{l=0}^{\varepsilon_C} (-1)^l \frac{\mu_l^k}{\pi_l(C)} = \delta_{k\varepsilon_C} \qquad (0\le k\le \varepsilon_C).$$

Then,

$$\sum_{l=0}^{\varepsilon_C} (-1)^l \frac{q(\mu_l)}{\pi_l(C)} = 0 \qquad \text{for all } q\in \mathbb{R}_{\varepsilon_C-1}[x], \tag{4}$$

and

$$\sum_{l=0}^{\varepsilon_C} m_C(\mu_l)p(\mu_l)q(\mu_l) = \pi_0(C)\frac{\|\boldsymbol{\rho} D\|^2}{\|\boldsymbol{\nu}\|^2}\sum_{l=0}^{\varepsilon_C}(-1)^l\frac{q(\mu_l)}{\pi_l(C)} = 0. \qquad \square$$

Corollary 2.4 *Let $C\subset V$ be an extremal set with $\mathrm{sp}_C\,\Gamma = \{\mu_0,\mu_1,\ldots,\mu_{d_C}\}$ and let D be its antipodal set. If the statements of Proposition 2.2 hold, then the angle between the vectors $\boldsymbol{m}_C = (\|\boldsymbol{z}_C(\mu_0)\|,\|\boldsymbol{z}_C(\mu_1)\|,\ldots,\|\boldsymbol{z}_C(\mu_{d_C})\|)$, $\tilde{\boldsymbol{m}}_D = (\|\boldsymbol{z}_D(\mu_0)\|,\|\boldsymbol{z}_D(\mu_1)\|,\ldots,\|\boldsymbol{z}_D(\mu_{d_C})\|)$ satisfy*

$$\cos\alpha^{(C,D)} = \frac{\sum_{l=0}^{\varepsilon_C}\frac{1}{\pi_l(C)}}{\sqrt{\sum_{l=0}^{\varepsilon_C}\frac{1}{m_C(\mu_l)\pi_l^2(C)}}}.$$

3 C-local pseudo-distance-regularity in combinatorial sense

The notion of pseudo-distance-regularity was first introduced in [7] as a generalization for non-regular graphs of the distance-regularity. More precisely, in this section we are interested in C-local pseudo-distance-regularity, which, when restricted to regular graphs, is equivalent to the fact that C is a completely regular code. For a more exhaustive study of this property see [5], where the authors obtain several characterizations which, in particular, yield new characterizations for completely regular codes.

Given a set of vertices of a graph Γ, C, with eccentricity ε_C, we associate to it the functions $a, b, c : V \longrightarrow [0, \lambda_0]$ defined for $i \in C_k$ by

$$c(i) = \begin{cases} 0 & (k = 0); \\ \dfrac{1}{\nu_i} \displaystyle\sum_{j \in \Gamma(i) \cap C_{k-1}} \nu_j & (1 \leq k \leq \varepsilon_C). \end{cases}$$

$$a(i) = \frac{1}{\nu_i} \sum_{j \in \Gamma(i) \cap C_k} \nu_j \quad (0 \leq k \leq \varepsilon_C).$$

$$b(i) = \begin{cases} \dfrac{1}{\nu_i} \displaystyle\sum_{j \in \Gamma(i) \cap C_{k+1}} \nu_j & (0 \leq k \leq \varepsilon_C - 1); \\ 0 & (k = \varepsilon_C). \end{cases}$$

Since $\boldsymbol{\nu}$ is an eigenvector of eigenvalue λ_0,

$$c(i) + a(i) + b(i) = \frac{1}{\nu_i} \sum_{j \in \Gamma(i)} \nu_j = \lambda_0 \quad \text{for all } i \in V,$$

that is, the sum over the three functions a, b, c, is constant and their images are all in $[0, \lambda_0]$. In other words, by assigning weight ν_i to each vertex i, the average weight degree becomes constant and the graph is "regularized". Note that, since every vertex in C_k must be adjacent to a vertex of C_{k-1}, the function c is strictly positive over $V \setminus C_0$. We say that C is a *flowing set* when the associated function b is strictly positive over $V \setminus C_{\varepsilon_C}$.

Lemma 3.1 *Let $C \in V$ be a set of vertices with eccentricity ε_C and let D be its antipodal set. Then, C is a flowing set if and only if $\varepsilon_C = \varepsilon_D = \varepsilon$ and the corresponding distance partitions, $C_0(= C), C_1, \ldots, C_\varepsilon$ and $D_0(= D), D_1, \ldots, D_\varepsilon$, satisfy $D_k = C_{\varepsilon-k}$, $0 \leq k \leq \varepsilon$.*

Proof. The condition suffices to guaranty that C is a flowing set since it implies that the function b corresponding to C coincides with the function c corresponding to D. Conversely, if C is a flowing set, every vertex in C_k is at distance $\varepsilon_C - k$ from D and then $C_k \subset D_{\varepsilon_C - k}$, $0 \leq k \leq \varepsilon_C$. From this we get

$$V = C_0 \cup C_1 \cup \cdots \cup C_{\varepsilon_C} \subset D_{\varepsilon_C} \cup D_{\varepsilon_C - 1} \cup \cdots \cup D_0 \subset V$$

and, since C_k (respectively, D_k), $1 \leq k \leq \varepsilon_C$, do not intersect each other, $\varepsilon_C = \varepsilon_D = \varepsilon$ and $D_k = C_{\varepsilon-k}$, $0 \leq k \leq \varepsilon$. $\square$

Note that, by symmetry, the previous lemma establishes that C is a flowing set if and only if D is.

Definition 3.2 *A graph* Γ *is* C-local pseudo-distance-regular (*or* pseudo-distance -regular around C) in combinatorial sense *when the functions c, a and b associated to C are constant over every C_k, $k = 0, 1, \ldots, \varepsilon_C$.*

Its clear that if a graph Γ is C-local pseudo-distance-regular in combinatorial sense, then C is a flowing set. In this case, from Lemma 3.1 we have that $\overline{D} = C$ and the distance partitions associated to C and D coincide. Moreover, in this case, Γ is also D-local pseudo-distance-regular with the roles of the functions b and c interchanged.

In a C-local pseudo-distance-regular graph, we indicate by c_k, a_k and b_k the values of c, a and b, respectively, over a vertex of C_k, and we refer to them as the *pseudo-intersection numbers of* C. Note that when Γ is a regular graph and C consists of a single vertex, the above numbers become the usual intersection numbers.

3.1 Some characterizations of C-local pseudo-distance-regularity

In [5], several quasi-spectral characterizations of pseudo-distance-regularity around a vertex set are given. The authors obtain their results through a sequence of orthogonal polynomials constructed from the C-local spectrum. In order to introduce these polynomials, let us first define, in the quotient ring $\mathbb{R}[x]/(Z_C)$, the following *C-local scalar product*:

$$\langle p, q\rangle_C := \langle p\boldsymbol{e}_C, q\boldsymbol{e}_C\rangle = \sum_{l=0}^{d_C} m_C(\mu_l)p(\mu_l)q(\mu_l).$$

A family of polynomials $r_0, r_1, \ldots, r_{d_C}$ is an orthogonal system with respect to the C-local scalar product when $\deg r_k = k$ and $\langle r_k, r_h\rangle_C = \delta_{kh}$, $0 \le k, h \le d_C$. Then, the family of *C-local predistance polynomials*, $\{p_k^C\}_{0\le k\le d_C}$ is the unique orthogonal system with respect to the C-local scalar product such that $\|p_k^C\|_C^2 = p_k^C(\lambda_0)$, $k = 0, 1, \ldots, d_C$; see [2].

As mentioned, several characterizations of C-local pseudo-distance-regularity can be obtained in terms of these polynomials which, in this case, are called *C-local distance polynomials*; see [5].

Theorem 3.3 *A graph $\Gamma = (V, E)$ is pseudo-distance-regular around a set $C \subset V$, with eccentricity ε_C, if and only if there exist a sequence of polynomials $r_0, r_1, \ldots, r_{\varepsilon_C}$, with $\deg r_k = k$, such that $\boldsymbol{\rho} C_k = r_k\boldsymbol{\rho} C$ for any $0 \le k \le \varepsilon_C$. Moreover, in this case, $\varepsilon_C = d_C$ and the the polynomials $\{r_k\}_{0\le k\le d_C}$ are the C-local (pre)distance polynomials.* □

Moreover, for an extremal set C, $\varepsilon_C = d_C$, the C-local pseudo-distance-regularity can be characterized in terms of only the highest degree C-local predistance polynomial.

Theorem 3.4 *Let $\Gamma = (V, E)$ be a graph containing an extremal set $C \subset V$. Let $\overline{C}$ denote the antipodal set of C. Then the following statements are equivalent:*

(*a*) *Γ is C-local pseudo-distance-regular in combinatorial sense.*

(*b*) $p_{d_C}^C\boldsymbol{\rho} C = \boldsymbol{\rho}\overline{C}$.

(*c*) $p_{d_C}^C(\lambda_0) = \frac{\|\boldsymbol{\rho}\overline{C}\|^2}{\|\boldsymbol{\rho} C\|^2}$. □

The results of the above two theorem will be proved in the next section, by using an algebraic (or Terwilliger-like) approach to pseudo-distance-regularity around C.

4 C-local pseudo-distance-regularity in algebraic sense

Let $C \subset V$ be a set of vertices of a simple connected graph $\Gamma = (V, E)$. For each $k = 0, 1, \ldots, \varepsilon_C$, let $\mathcal{E}_k^\star$ be the vector space having $\{\boldsymbol{e}_i\}_{i\in C_k}$ as a basis. Denote by $\boldsymbol{E}_k^\star$ the projection $\mathcal{V} \to \mathcal{E}_k^\star$. As a generalization of the subconstituent algebras defined in [11], also known as Terwilliger algebras, we consider the algebra $\mathcal{T}_C$ generated by the linear operators $\{\boldsymbol{A}, \boldsymbol{E}_0^\star, \boldsymbol{E}_1^\star, \ldots, \boldsymbol{E}_{\varepsilon_C}^\star\}$. A $\mathcal{T}_C$-module W is a subspace of $\mathcal{V}$ which is invariant under the action of $\mathcal{T}_C$, that is, $\mathcal{T}_C W = W$.

In the context of association schemes, Terwilliger [11] defined a *thin* module as a module W satisfying $\dim \boldsymbol{E}_k^\star W \leq 1$ for every k. As commented in [3, 5], if we consider a single vertex i, the notion of $\{i\}$-local pseudo-distance-regularity is equivalent to the thin character of the primary T_i-module, that is, the unique irreducible module containing $\boldsymbol{\rho}\{i\} = \nu_i \boldsymbol{e}_i$. With the aim of generalize this definition to any subset of vertices, let us consider a vector $\boldsymbol{w}_C \in \mathcal{E}_0^\star$ and $W_C := \mathcal{T}_C \boldsymbol{w}_C \subset \mathcal{V}$, the minimum $\mathcal{T}_C$-module containing $\boldsymbol{w}_C$. The definition of C-local pseudo-distance-regularity in algebraic sense will require the subspaces $\boldsymbol{E}_k^\star W_C$, $k = 0, 1, \ldots, d_C$, to be one-dimensional. Let us first study some conditions that $\boldsymbol{w}_C$ must satisfy. Let $\boldsymbol{w}_C = \sum_{i\in C} \xi_i \boldsymbol{e}_i$. Since $\boldsymbol{E}_k = \frac{(-1)^k}{\pi_k} \prod_{0\leq l\leq d\,(l\neq k)} (\boldsymbol{A} - \lambda_l \boldsymbol{I}) \in T_C$, $k = 0, 1, \ldots, d_C$, we have

$$
\begin{aligned}
\boldsymbol{E}_k^\star \boldsymbol{E}_0 \boldsymbol{w}_C &= \sum_{i\in C} \xi_i \boldsymbol{E}_k^\star \boldsymbol{E}_0 \left(\frac{\nu_i}{\|\boldsymbol{\nu}\|^2} \boldsymbol{\nu} + \boldsymbol{z}_i(\lambda_1) + \ldots + \boldsymbol{z}_i(\lambda_d) \right) \\
&= \sum_{i\in C} \frac{\xi_i \nu_i}{\|\boldsymbol{\nu}\|^2} \boldsymbol{E}_k^\star \boldsymbol{\nu} = \left(\sum_{i\in C} \frac{\xi_i \nu_i}{\|\boldsymbol{\nu}\|^2} \right) \boldsymbol{\rho} C_k \,.
\end{aligned}
$$

Thus if $\dim \boldsymbol{E}_k^\star W_C = 1$, the vector $\boldsymbol{\rho} C_k$ will constitute a basis of $\boldsymbol{E}_k^\star W_C$. In particular, $\boldsymbol{w}_C = \boldsymbol{E}_0^\star \boldsymbol{I} \boldsymbol{w}_C$ is linearly dependent with $\boldsymbol{\rho} C_0$. Thus, the generalization for a set of vertices of the definition of Terwilliger for a single vertex must be:

Definition 4.1 *A graph* Γ *is* C-local pseudo-distance-regular in algebraic sense *when* $\dim \boldsymbol{E}_k^\star W_C = 1$*,* $k = 0, 1, \ldots, \varepsilon_C$*, where* W_C *is the* $\mathcal{T}_C$*-module* $W_C := \mathcal{T}_C \boldsymbol{\rho} C$*.*

This definition generalizes also, for any graph, the one given in [10] for a set of vertices in a distance-regular graph.

From the previous comments, if Γ is C-local pseudo-distance-regular in algebraic sense, $\boldsymbol{E}_k^\star T \boldsymbol{\rho} C \in \langle \boldsymbol{\rho} C_k \rangle$ for every $T \in \mathcal{T}_C$ and $k = 0, 1, \ldots, \varepsilon_C$. The following result gives a characterization of C-local pseudo-distance-regularity in algebraic sense, which coincides with the one of Theorem 3.3. This proves the equivalence between combinatorial and algebraic C-local pseudo-distance-regularity.

Theorem 4.2 *A graph Γ is C-local pseudo-distance-regular in algebraic sense if and only if there exist polynomials $p_0, p_1, \ldots, p_{\varepsilon_C}$ in $\mathbb{R}_{\varepsilon_C}[x]$ such that $p_k \boldsymbol{\rho} C = \boldsymbol{\rho} C_k$, $k = 0, 1, \ldots, \varepsilon_C$.*

Proof. Suppose that Γ is C-local pseudo-distance-regular in algebraic sense. Given $r \in \mathbb{R}_{\varepsilon_C}[x]$ and $k = 0, 1, \ldots, \varepsilon_C$, consider $\xi_k(r) \in \mathbb{R}$ such that $\boldsymbol{E}_k^\star r \boldsymbol{\rho} C = \xi_k(r) \boldsymbol{\rho} C_k$. We have that the map

$$\mathbb{R}_{\varepsilon_C}[x] \xrightarrow{\Theta} \mathbb{R}^{\varepsilon_C + 1} \qquad \text{defined by} \quad \Theta r := (\xi_0(r), \xi_1(r), \ldots, \xi_{\varepsilon_C}(r)) \tag{5}$$

is linear. If $r \in \mathbb{R}_{\varepsilon_C}[x]$ satisfies $\Theta r = \mathbf{0}$ then $\boldsymbol{E}_k^\star r \boldsymbol{\rho} C = \mathbf{0}$ for every k and $r \boldsymbol{\rho} C = \left(\sum_{k=0}^{\varepsilon_C} \boldsymbol{E}_k^\star\right) r \boldsymbol{\rho} C = \mathbf{0}$. Consequently, r will vanish over all the $d_C + 1$ elements of $\operatorname{ev}_C \Gamma$, and, since $\deg r \leq \varepsilon_C \leq d_C$, we conclude that $r = 0$. This proves that Θ is an isomorphism, and by considering the polynomial $p_k \in \mathbb{R}_{\varepsilon_C}[x]$ such that

$$\Theta p_k = (0, \ldots, \overset{(k)}{1}, \ldots, 0),$$

we have that $p_k \boldsymbol{\rho} C = \boldsymbol{\rho} C_k$, $k = 0, \ldots, \varepsilon_C$.

Conversely, let us now show that the existence of such polynomials implies the C-local pseudo-distance-regularity. With this aim, consider the polynomial $q = p_0 + p_1 + \cdots + p_{\varepsilon_C} \in \mathbb{R}_{\varepsilon_C}[x]$ satisfying $q \boldsymbol{\rho} C = \sum_{k=0}^{\varepsilon_C} \boldsymbol{\rho} C_k = \boldsymbol{\nu}$. Thus, $q(\mu_0) = \frac{\|\boldsymbol{\nu}\|^2}{\|\boldsymbol{\rho} C\|^2}$ and $q(\mu_l) = 0$, $l = 1, \ldots, d_C$, giving $d_C \leq \deg q \leq \varepsilon_C \leq d_C$, so that C is extremal ($\varepsilon_C = d_C$). Moreover, $q = \frac{\|\boldsymbol{\nu}\|^2}{\pi_0(C) \|\boldsymbol{\rho} C\|^2} (x - \mu_1) \cdots (x - \mu_{\varepsilon_C}) = H_C$, the C-local Hoffman polynomial.

The hypothesis guaranties that the polynomials p_k, $k = 0, 1, \ldots, \varepsilon_C$, constitute a basis of $\mathbb{R}_{\varepsilon_C}[x]$, identified with $\mathbb{R}[x]/(Z_C)$. Define $\gamma_{hk}^l \in \mathbb{R}$ by

$$p_h p_k = \sum_{l=0}^{\varepsilon_C} \gamma_{hk}^l p_l \qquad (0 \leq h, k \leq \varepsilon_C).$$

Every element of $\boldsymbol{E}_k^\star \mathcal{T}_C \boldsymbol{\rho} C$ can be seen as a linear combination of vectors $T_r T_{r-1} \cdots T_1 \boldsymbol{\rho} C$, where $T_l = \boldsymbol{E}_{t_l}^\star p_{s_l}$, $1 \leq l \leq r$ and $t_r = k$. We can suppose that $s_1 = t_1$ (since, otherwise, we get the zero vector). Then,

$$\begin{aligned}
T_1 \boldsymbol{\rho} C &= \boldsymbol{E}_{t_1}^\star p_{s_1} \boldsymbol{\rho} C = \boldsymbol{E}_{t_1}^\star \boldsymbol{\rho} C_{s_1} = \boldsymbol{\rho} C_{s_1} = p_{s_1} \boldsymbol{\rho} C, \\
T_2 T_1 \boldsymbol{\rho} C &= \boldsymbol{E}_{t_2}^\star p_{s_2} p_{s_1} \boldsymbol{\rho} C = \boldsymbol{E}_{t_2}^\star \left(\sum_{l=0}^{\varepsilon_C} \gamma_{s_2 s_1}^l p_l \right) \boldsymbol{\rho} C = \boldsymbol{E}_{t_2}^\star \sum_{l=0}^{\varepsilon_C} \gamma_{s_2 s_1}^l \boldsymbol{\rho} C_l = \gamma_{t_1 s_2}^{t_2} \boldsymbol{\rho} C_{t_2} \\
&= \gamma_{t_1 s_2}^{t_2} p_{t_2} \boldsymbol{\rho} C
\end{aligned}$$

and, iterating, we get

$$T_r \cdots T_1 \boldsymbol{\rho} C = \gamma_{t_1 s_2}^{t_2} \cdots \gamma_{t_{r-1} s_r}^{t_r} p_{t_r} \boldsymbol{\rho} C = \gamma_{t_1 s_2}^{t_2} \cdots \gamma_{t_{r-1} s_r}^{t_r} \boldsymbol{\rho} C_k \, .$$

Hence, $\dim \boldsymbol{E}_k^\star W_C = 1$, $k = 0, 1, \ldots, \varepsilon_C$, and Γ is C-local pseudo-distance-regular in algebraic sense. $\square$

In particular, notice that we have shown that the condition of being extremal, $\varepsilon_C = d_C$, is necessary for having C-local pseudo-distance-regularity. Moreover, the polynomials of Theorem 4.2 satisfy the following properties:

Corollary 4.3 *Let $\Gamma = (V, E)$ be a graph and $C \subset V$ such that Γ is C-local pseudo-distance-regular in algebraic sense. For every $k = 0, 1, \ldots \varepsilon_C(= d_C)$, the polynomial $p_k \in \mathbb{R}_{\varepsilon_C}[x]$ satisfying $p_k \boldsymbol{\rho} C = \boldsymbol{\rho} C_k$ is unique, it has degree k, and coincides with the C-local predistance polynomial, $p_k = p_k^C$.*

Proof. The unicity is provided by the fact that the map Θ defined in (5) is an isomorphism. In particular, this gives that $p_0 = 1$. Now, consider $1 \le k \le d_C$, if $\deg p_k < k$ a contradiction arises: $\|\boldsymbol{\rho} C_k\|^2 = \langle p_k \boldsymbol{\rho} C, \boldsymbol{\rho} C_k\rangle = 0$. Let s, $1 \le s \le \varepsilon_C - 1$, be the maximum integer such that $\deg p_s > s$. There exist $\xi_{s+1}, \ldots, \xi_{\varepsilon_C} \in \mathbb{R}$ such that the polynomial $q = p_s + \xi_{s+1} p_{s+1} + \cdots + \xi_{\varepsilon_C} p_{\varepsilon_C}$ has degree less or equal to s. Consider l such that $\xi_l \ne 0$.

$$\begin{aligned}\langle q\boldsymbol{\rho} C, \boldsymbol{\rho} C_l\rangle &= \langle p_s \boldsymbol{\rho} C, \boldsymbol{\rho} C_l\rangle + \sum_{h=s+1}^{\varepsilon_C} \xi_h \langle p_h \boldsymbol{\rho} C, \boldsymbol{\rho} C_l\rangle \\ &= \langle \boldsymbol{\rho} C_s, \boldsymbol{\rho} C_l\rangle + \sum_{h=s+1}^{\varepsilon_C} \xi_h \langle \boldsymbol{\rho} C_h, \boldsymbol{\rho} C_l\rangle = \xi_l \|\boldsymbol{\rho} C_l\|^2 \ne 0\,.\end{aligned}$$

On the other hand, since $\deg q \le s < s+1 \le l$, we get $\langle q\boldsymbol{\rho} C, \boldsymbol{\rho} C_l\rangle = 0$, which is impossible. So it does not exists such an index s and $\deg p_k = k$ for every $0 \le k \le \varepsilon_C$. Finally, the polynomials $\{p_k\}_{0 \le k \le \varepsilon_C}$ are orthogonal:

$$\langle p_k, p_h\rangle_C = \langle p_k \boldsymbol{e}_C, p_h \boldsymbol{e}_C\rangle = \frac{1}{\|\boldsymbol{\rho} C\|^2}\langle \boldsymbol{\rho} C_k, \boldsymbol{\rho} C_h\rangle = 0 \quad \text{for } k \ne h,$$

and they have norm:

$$\begin{aligned}\|p_k\|_C^2 &= \frac{1}{\|\boldsymbol{\rho} C\|^2}\langle p_k \boldsymbol{\rho} C, p_k \boldsymbol{\rho} C\rangle = \frac{1}{\|\boldsymbol{\rho} C\|^2}\langle \boldsymbol{\rho} C_k, \boldsymbol{\rho} C_k\rangle \\ &= \frac{1}{\|\boldsymbol{\rho} C\|^2}\langle \boldsymbol{\nu}, p_k \boldsymbol{\rho} C\rangle = \frac{1}{\|\boldsymbol{\rho} C\|^2}\langle p_k \boldsymbol{\nu}, \boldsymbol{\rho} C\rangle = \frac{p_k(\mu_0)}{\|\boldsymbol{\rho} C\|^2}\langle \boldsymbol{\nu}, \boldsymbol{\rho} C\rangle = p_k(\mu_0).\end{aligned}$$

Consequently, they are the C-local predistance polynomials $\{p_k^C\}_{0 \le k \le d_C}$, as claimed. □

The following result gives another characterization of C-local pseudo-distance-regularity, which is proved by using the algebraic approach.

Theorem 4.4 *Let $\Gamma = (V, E)$ be a graph with vertex subset $C \in V$ having eccentricity ε_C and local eigenvalues $\mathrm{ev}_C \Gamma = \{\mu_0, \mu_1, \ldots, \mu_{d_C}\}$. Let us consider the distance partition $V = C_0 \cup C_1 \cup \cdots \cup C_{\varepsilon_C}$ given by the distance to C, and the spectral decomposition $\boldsymbol{\rho} C = \hat{\boldsymbol{z}}_C(\mu_0) + \hat{\boldsymbol{z}}_C(\mu_1) + \cdots + \hat{\boldsymbol{z}}_C(\mu_{d_C})$. Then, Γ is C-local pseudo-distance-regular in algebraic sense if and only if the subspaces of $\mathcal{V}$ generated by $\boldsymbol{\rho} C_0, \boldsymbol{\rho} C_1, \ldots, \boldsymbol{\rho} C_{\varepsilon_C}$ and by $\hat{\boldsymbol{z}}_C(\mu_0), \hat{\boldsymbol{z}}_C(\mu_1), \ldots, \hat{\boldsymbol{z}}_C(\mu_{d_C})$ coincide.*

Proof. Let $R = \langle \boldsymbol{\rho} C_0, \boldsymbol{\rho} C_1, \ldots, \boldsymbol{\rho} C_{\varepsilon_C} \rangle$ and $S = \langle \hat{\boldsymbol{z}}_C(\mu_0), \hat{\boldsymbol{z}}_C(\mu_1), \ldots, \hat{\boldsymbol{z}}_C(\mu_{d_C}) \rangle$. Note that, since the involved vectors are linearly independent, $\dim R = \varepsilon_C$ and $\dim S = d_C$.

Suppose that Γ is C-local pseudo-distance-regular in algebraic sense. Theorem 4.2 guaranties that C is extremal, $d_C = \varepsilon_C$, and there exist polynomials $p_0, p_1, \ldots, p_{\varepsilon_C}$ in $\mathbb{R}_{\varepsilon_C}[x]$ such that $p_k \boldsymbol{\rho} C = \boldsymbol{\rho} C_k$, $k = 0, 1, \ldots, \varepsilon_C$. Given h, $0 \le h \le \varepsilon_C$, we have

$$\hat{\boldsymbol{z}}_C(\mu_h) = \boldsymbol{E}_h \boldsymbol{\rho} C = \left(\sum_{k=0}^{\varepsilon_C} \boldsymbol{E}_k^\star \right) \boldsymbol{E}_h \boldsymbol{\rho} C = \sum_{k=0}^{\varepsilon_C} \boldsymbol{E}_k^\star \boldsymbol{E}_h \boldsymbol{\rho} C = \sum_{k=0}^{\varepsilon_C} a_{hk} \boldsymbol{\rho} C_k, \tag{6}$$

where $a_{hk} \in \mathbb{R}$, thus $\hat{\boldsymbol{z}}_C(\mu_h) \in S$ and $R = S$.

Suppose now that $R = S$. In particular, $\varepsilon_C = d_C$ and C is extremal. For every k, $k = 0, 1, \ldots \varepsilon_C$, there are $b_{kh} \in \mathbb{R}$, $h = 0, 1, \ldots, \varepsilon_C$, satisfying

$$\boldsymbol{\rho} C_k = \sum_{h=0}^{\varepsilon_C} b_{kh} \hat{\boldsymbol{z}}_C(\mu_h).$$

Define $p_k \in \mathbb{R}_{\varepsilon_C}[x]$ as the unique polynomial such that $p_k(\mu_h) = b_{kh}$ for every $h = 0, 1, \ldots, \varepsilon_C$. Then

$$\boldsymbol{\rho} C_k = \sum_{h=0}^{\varepsilon_C} b_{kh} \hat{\boldsymbol{z}}_C(\mu_h) = \sum_{h=0}^{\varepsilon_C} p_k(\mu_h) \hat{\boldsymbol{z}}_C(\mu_h) = p_k \sum_{h=0}^{\varepsilon_C} \hat{\boldsymbol{z}}_C(\mu_h) = p_k \boldsymbol{\rho} C_0, \tag{7}$$

and Γ is C-local pseudo-distance-regular in algebraic sense. □

Consider the vector space $\mathcal{V}_C := \{ q \boldsymbol{\rho} C \,:\, \forall q \in \mathbb{R}[x] \}$. Since $\{ Z_k^C \}_{0 \le k \le d_C}$ is a basis of $\mathbb{R}_{d_C}[x]$, $\mathcal{V}_C = \langle \hat{\boldsymbol{z}}_C(\mu_0), \hat{\boldsymbol{z}}_C(\mu_1), \ldots, \hat{\boldsymbol{z}}_C(\mu_{d_C}) \rangle$. Taking in mind that $d_C \ge \varepsilon_C$, the next corollary is obtained.

Corollary 4.5 *Γ is C-local pseudo-distance-regular in algebraic sense if and only if*

$$q \boldsymbol{\rho} C \in \langle \boldsymbol{\rho} C_0, \boldsymbol{\rho} C_1, \ldots, \boldsymbol{\rho} C_{\varepsilon_C} \rangle \qquad \forall q \in \mathbb{R}[x]\,,$$

or, equivalently, if and only if there exists a basis $\mathcal{B}$ of $\mathbb{R}_{d_C}[x]$ such that

$$b \boldsymbol{\rho} C \in \langle \boldsymbol{\rho} C_0, \boldsymbol{\rho} C_1, \ldots, \boldsymbol{\rho} C_{\varepsilon_C} \rangle \quad \text{for all } b \in \mathcal{B}\,.$$

An interesting application of this corollary is the following characterization of a completely regular code (for other characterizations, see e.g. [6, 9]).

Theorem 4.6 *Let $\Gamma = (V, E)$ be a regular graph. Then $C \subset V$ is a completely regular code if and only if, for any given nonnegative integers $\ell \le d_C$ and $k \le \varepsilon_C$, the number of ℓ-walks between (the vertices of) C and $i \in C_k$ does not depend on the vertex i.*

Proof. In Corollary 4.5 take the canonical basis $\mathcal{B} = \{1, x, x^2, \dots, x^{d_C}\}$ of $\mathbb{R}_{d_C}[x]$. Then, there exist constants α_h, $0 \le h \le \varepsilon_C$, such that $x^\ell \boldsymbol{\rho} C = \sum_{h=0}^{\varepsilon_C} \alpha_h \boldsymbol{\rho} C_h$. Hence,

$$\begin{aligned}(x^\ell \boldsymbol{\rho} C)_i &= \left(\boldsymbol{A}^\ell \textstyle\sum_{j\in C} \boldsymbol{e}_j\right)_i = \sum_{j\in C} (\boldsymbol{A}^\ell)_{ji} \\ &= \left(\textstyle\sum_{h=0}^{\varepsilon_C} \alpha_h \boldsymbol{\rho} C_h\right)_i = \sum_{h=0}^{\varepsilon_C} \alpha_h \left(\textstyle\sum_{j\in C_h} \boldsymbol{e}_j\right)_i = \sum_{h=0}^{\varepsilon_C} \alpha_h \delta_{hk} = \alpha_k.\end{aligned}$$

From this, we get the result. $\square$

As the authors of [5] established in the study of the C-local pseudo-distance regularity from a combinatorial point of view, the conditions of Theorem 4.2 can be apparently relaxed by restricting them to the set of vertices at maximum distance from C, provided that C is extremal.

Theorem 4.7 *Let $\Gamma = (V, E)$ be a graph and let $C \subset V$ be an extremal set. Let $C_0, C_1, \dots, C_{\varepsilon_C}$ be the distance partition of V given by the distance to C. Then, Γ is C-local pseudo-distance-regular in algebraic sense if and only if there exists a polynomial $p \in \mathbb{R}_{\varepsilon_C}[x]$ such that $p\boldsymbol{\rho} C = \boldsymbol{\rho} C_{\varepsilon_C}$.*

Proof. The necessity of the condition follows from Theorem 4.2. To proves sufficiency, let $\mathrm{ev}_C \Gamma = \{\mu_0, \mu_1, \dots, \mu_{\varepsilon_C}\}$ and $\hat{\boldsymbol{z}}_C(\mu_k) = \boldsymbol{E}_k \boldsymbol{\rho} C$, $0 \le k \le \varepsilon_C$. In particular $\boldsymbol{z}_C(\mu_0) = \frac{\|\boldsymbol{\rho} C\|^2}{\|\boldsymbol{\nu}\|^2} \boldsymbol{\nu}$. We claim that $\boldsymbol{\rho} C_k \in \langle \hat{\boldsymbol{z}}_C(\mu_0), \hat{\boldsymbol{z}}_C(\mu_1), \dots, \hat{\boldsymbol{z}}_C(\mu_{\varepsilon_C})\rangle$, $k = 0, \dots, \varepsilon_C$, thus, by applying Theorem 4.4 the result arises.

Note that for $k = 0$ the claim is trivially satisfied and the case $k = \varepsilon_C$ is guarantied by the hypothesis: $\boldsymbol{\rho} C_{\varepsilon_C} = p\boldsymbol{\rho} C = \sum_{l=0}^{\varepsilon_C} p(\mu_l) \hat{\boldsymbol{z}}_C(\mu_l)$. Let D be the subset of vertices at distance ε_C from C, $D = \overline{C}$. From $\boldsymbol{\rho} D = p\boldsymbol{\rho} C$ we get $p(\mu_0) = \frac{\|\boldsymbol{\rho} D\|^2}{\|\boldsymbol{\rho} C\|^2}$ and, since $\partial(u, v) \ge \varepsilon_C$ for every $u \in C$ and $v \in D$, $\varepsilon_D \ge \varepsilon_C$. Moreover, the equality $p\boldsymbol{\rho} C = p(\mu_0)\boldsymbol{z}_C(\mu_0) + \cdots + p(\mu_{d_C})\boldsymbol{z}_C(\mu_{d_C}) = \boldsymbol{\rho} D$ gives $d_C \ge d_D$. All together, we have $\varepsilon_D \ge \varepsilon_C = d_C \ge d_D \ge \varepsilon_D$, thus $(\varepsilon :=)\, \varepsilon_D = \varepsilon_C$ and $(\mathcal{M} :=)\, \mathrm{ev}_C \Gamma = \mathrm{ev}_D \Gamma$. Note that $p \in \mathbb{R}_\varepsilon[x]$ must have degree ε and has an inverse p^{-1} in the ring $\mathbb{R}_\varepsilon[x]/(Z)$, being $Z := \prod_{l=0}^{\varepsilon}(x - \mu_l)$.

Consider the normalized weight functions on $\mathcal{M}$ given by the C-local and D-local multiplicities: $m_C(\mu_l) = \frac{\|\boldsymbol{E}_l \boldsymbol{\rho} C\|^2}{\|\boldsymbol{\rho} C\|^2}$ and $m_D(\mu_l) = \frac{\|\boldsymbol{E}_l \boldsymbol{\rho} D\|^2}{\|\boldsymbol{\rho} D\|^2}$. From $\boldsymbol{E}_l \boldsymbol{\rho} D = \boldsymbol{E}_l p \boldsymbol{\rho} C = p(\mu_l) \boldsymbol{E}_l \boldsymbol{\rho} C$, we have

$$m_D(\mu_l) = \frac{\|\boldsymbol{E}_l \boldsymbol{\rho} D\|^2}{\|\boldsymbol{\rho} D\|^2} = p^2(\mu_l) \frac{\|\boldsymbol{E}_l \boldsymbol{\rho} C\|^2}{\|\boldsymbol{\rho} C\|^2} \frac{\|\boldsymbol{\rho} C\|^2}{\|\boldsymbol{\rho} D\|^2} = \frac{p^2(\mu_l)}{p(\mu_0)} m_C(\mu_l)\,.$$

The orthogonal systems corresponding to the C-local predistance polynomials, $\{p_k\}_{0\le k\le\varepsilon}$, and D-local predistance polynomials, $\{\overline{p}_k\}_{0\le k\le\varepsilon}$, are related (in $\mathbb{R}[x]/\mathcal{I}$) by $\overline{p}_k = p_\varepsilon^{-1} p_{\varepsilon-k}$, $0 \le k \le \varepsilon$, where, as we have already seen, $p_\varepsilon = p$. (The existence of p_ε^{-1} is assured by

Corollary 2.3(a).) Indeed,

$$\begin{aligned}
\langle \overline{p}_k, \overline{p}_h \rangle_D &= \sum_{l=0}^{\varepsilon} m_D(\mu_l)\overline{p}_k(\mu_l)\overline{p}_h(\mu_l) \\
&= \frac{1}{p(\mu_0)} \sum_{l=0}^{\varepsilon} m_C(\mu_l) p^2(\mu_l) p^{-1}(\mu_l) p_{\varepsilon-k}(\mu_l) p^{-1}(\mu_l) p_{\varepsilon-h}(\mu_l) = \\
&= \frac{1}{p(\mu_0)} \sum_{l=0}^{\varepsilon} m_C(\mu_l) p_{\varepsilon-k}(\mu_l) p_{\varepsilon-h}(\mu_l) = \frac{1}{p(\mu_0)} \langle p_{\varepsilon-k}, p_{\varepsilon-h} \rangle_C \\
&= \delta_{kh} p^{-1}(\mu_0) p_{\varepsilon-k}(\mu_0) = \delta_{kh} \overline{p}_k(\mu_0).
\end{aligned}$$

Given $1 \le k \le \varepsilon - 1$, let us consider the set $S_k = \{r + ps : r \in \mathbb{R}_{k-1}[x], s \in \mathbb{R}_{\varepsilon-k-1}[x]\}$. Then, for any $q \in S_k$, we have:

$$\langle q\boldsymbol{\rho}C, \boldsymbol{\rho}C_k \rangle = \langle r\boldsymbol{\rho}C, \boldsymbol{\rho}C_k \rangle + \langle s\boldsymbol{\rho}D, \boldsymbol{\rho}C_k \rangle = 0. \tag{8}$$

Note also that

$$\begin{aligned}
\mathbb{R}_\varepsilon[x] &= \langle p_0, \ldots, p_{k-1}, p_k, p_{k+1}, \ldots, p_\varepsilon \rangle = \langle p_0, \ldots, p_{k-1}, p_k, p_\varepsilon \overline{p}_{\varepsilon-k-1}, \ldots, p_\varepsilon \overline{p}_0 \rangle \\
&= S_k \perp_C \langle p_k \rangle.
\end{aligned} \tag{9}$$

Consider the principal idempotents $\boldsymbol{E}_l^C$, $l = 0, 1, \ldots, \varepsilon$ corresponding to the members of the C-spectrum, that is $\boldsymbol{E}_l^C$ is the projection onto the eigenspace corresponding to μ_l. The polynomials

$$Z_l^C = \frac{(-1)^l}{\pi_l(C)} \prod_{0 \le h \le \varepsilon\, h \ne l} (x - \mu_h)$$

satisfy, in $\bigoplus_{l=0}^{\varepsilon} \operatorname{Ker}(\boldsymbol{A} - \mu_l \boldsymbol{I})$, that $Z_l^C(\boldsymbol{A}) = \boldsymbol{E}_l^C$. Using (8) and (9), we get $Z_l^C = q_{lk} + \xi_{lk} p_k$, and, since $\xi_{lk} p_k(\mu_0) = \langle \xi_{lk} p_k, p_k \rangle_C = \langle Z_l^C, p_k \rangle_C = m_C(\mu_l) p_k(\mu_l)$,

$$Z_l^C = q_{lk} + m_C(\mu_l) \frac{p_k(\mu_l)}{p_k(\mu_0)} p_k\,, \quad q_{lk} \in S_k \qquad (0 \le l, k \le \varepsilon),$$

In particular,

$$Z_0^C = q_{0k} + m_C(\mu_0) p_k = q_{0k} + \frac{\|\boldsymbol{\rho}C\|^2}{\|\boldsymbol{\nu}\|^2} p_k\,, \quad \text{amb } q_{0k} \in S_k \qquad (0 \le k \le \varepsilon) \tag{10}$$

Using (9) again we obtain

$$\begin{aligned}
m_C(\mu_l) \frac{p_k(\mu_l)}{p_k(\mu_0)} \langle p_k \boldsymbol{\rho}C, \boldsymbol{\rho}C_k \rangle &= \langle Z_l^C \boldsymbol{\rho}C, \boldsymbol{\rho}C_k \rangle = \langle \boldsymbol{z}_C(\mu_l), \boldsymbol{E}_l^C \boldsymbol{\rho}C_k \rangle \\
&= \|\boldsymbol{\rho}C\| \sqrt{m_C(\mu_l)} \|\boldsymbol{E}_l^C \boldsymbol{\rho}C_k\| \cos \alpha_{lk},
\end{aligned}$$

where α_{lk} is the angle between $\boldsymbol{E}_l^C\boldsymbol{\rho}C_k$ and $\boldsymbol{z}_C(\mu_l)$, giving the inequalities

$$\begin{aligned} m_C(\mu_l)\frac{p_k^2(\mu_l)}{p_k^2(\mu_0)}\langle p_k\boldsymbol{\rho}C, \boldsymbol{\rho}C_k\rangle^2 &= \|\boldsymbol{\rho}C\|^2\|\boldsymbol{E}_l^C\boldsymbol{\rho}C_k\|^2\cos^2\alpha_{lk} \\ &\leq \|\boldsymbol{\rho}C\|^2\|\boldsymbol{E}_l^C\boldsymbol{\rho}C_k\|^2 \qquad (0\leq l,k\leq\varepsilon). \end{aligned} \tag{11}$$

By adding up the previous inequalities for $l=0,1,\ldots,d_C$, we obtain

$$\frac{1}{p_k(\mu_0)}\langle p_k\boldsymbol{\rho}C, \boldsymbol{\rho}C_k\rangle^2 \overset{(8a)}{\leq} \|\boldsymbol{\rho}C\|^2\sum_{l=0}^{\varepsilon}\|\boldsymbol{E}_l^C\boldsymbol{\rho}C_k\|^2 \overset{(8b)}{\leq} \|\boldsymbol{\rho}C\|^2\|\boldsymbol{\rho}C_k\|^2 \qquad (0\leq k\leq\varepsilon) \tag{12}$$

From (10) and (12) we obtain

$$\begin{aligned} \frac{\|\boldsymbol{\rho}C\|^4}{\|\boldsymbol{\nu}\|^4}\|\boldsymbol{\rho}C_k\|^4 &= \frac{\|\boldsymbol{\rho}C\|^4}{\|\boldsymbol{\nu}\|^4}\langle\boldsymbol{\nu}, \boldsymbol{\rho}C_k\rangle^2 = \langle\boldsymbol{E}_0^C\boldsymbol{\rho}C, \boldsymbol{\rho}C_k\rangle^2 \\ &= \frac{\|\boldsymbol{\rho}C\|^4}{\|\boldsymbol{\nu}\|^4}\langle p_k\boldsymbol{\rho}C, \boldsymbol{\rho}C_k\rangle^2 \leq \frac{\|\boldsymbol{\rho}C\|^4}{\|\boldsymbol{\nu}\|^4}p_k(\mu_0)\|\boldsymbol{\rho}C\|^2\|\boldsymbol{\rho}C_k\|^2 \end{aligned}$$

giving that

$$\|\boldsymbol{\rho}C_k\|^2 \leq \|\boldsymbol{\rho}C\|^2 p_k(\mu_0) \qquad (0\leq k\leq\varepsilon) \tag{13}$$

and, by adding up for $k=0,1,\ldots,\varepsilon$,

$$\|\boldsymbol{\nu}\|^2 \leq \|\boldsymbol{\rho}C\|^2\sum_{k=0}^{\varepsilon}p_k(\mu_0) = \|\boldsymbol{\rho}C\|^2 H_C(\mu_0) = \|\boldsymbol{\nu}\|^2,$$

we conclude that (13), (12b) and (11) are all equalities. Thus,

$$p_k(\mu_0) = \frac{\|\boldsymbol{\rho}C_k\|^2}{\|\boldsymbol{\rho}C\|^2}, \qquad \mathrm{ev}_{C_k}\Gamma\subset\mathrm{ev}_C\Gamma \qquad (0\leq k\leq\varepsilon),$$

and, for $0\leq l,k\leq\varepsilon$, there exist ψ_{lk} such that $\boldsymbol{E}_l^C\boldsymbol{\rho}C_k=\psi_{lk}\boldsymbol{z}_C(\mu_l)$. Finally, the expressions

$$\boldsymbol{\rho}C_k = \sum_{l=0}^{\varepsilon}\boldsymbol{E}_l^C\boldsymbol{\rho}C_k = \sum_{l=0}^{\varepsilon}\psi_{lk}\boldsymbol{z}_C(\mu_l) \in \langle\boldsymbol{z}_C(\mu_0), \boldsymbol{z}_C(\mu_1),\ldots,\boldsymbol{z}_C(\mu_\varepsilon)\rangle \qquad (0\leq k\leq\varepsilon)$$

yield that Γ is C-local pseudo-distance-regular in algebraic sense. □

As a consequence, we have the following quasi-spectral characterization of C-local pseudo-distance-regularity .

Theorem 4.8 *Let $\Gamma=(V,E)$ be a graph with an extremal set $C\subset V$ and C-local spectrum $\mathrm{sp}_C\Gamma=\{\mu_0^{m_C(\mu_0)},\mu_1^{m_C(\mu_1)},\ldots,\mu_{d_C}^{m_C(\mu_{d_C})}\}$. Let $\overline{C}$ be its antipodal set. Then, Γ is C-local pseudo-distance-regular if and only the highest degree C-local predistance polynomial satisfies*

$$p_{d_C}^C(\mu_0) = \frac{\|\boldsymbol{\rho}\overline{C}\|^2}{\|\boldsymbol{\rho}C\|^2} = \frac{\|\boldsymbol{\nu}\|^2}{\|\boldsymbol{\rho}C\|^2}\left(\sum_{l=0}^{\varepsilon_C}m_{\overline{C}}(\mu_l)\right)\left(\sum_{l=0}^{\varepsilon_C}\frac{m_C(\mu_0)\pi_0^2(C)}{m_C(\mu_l)\pi_l^2(C)}\right)^{-1}.$$

Proof. Let $d = d_C$, $N_{d-1} = \bigcup_{k=0}^{d-1} C_k = V \setminus \overline{C}$, $p_d = p_{d_C}^C$, and $q_{d-1} = \sum_{k=0}^{d-1} p_k^C = H_C - p_d$. Then, as $\deg q_{d-1} = d-1$, we have:

$$\langle \boldsymbol{\rho} N_{d-1}, q_{d-1}\boldsymbol{\rho} C\rangle = \langle \boldsymbol{\nu} - \boldsymbol{\rho}\overline{C}, q_{d-1}\boldsymbol{\rho} C\rangle = \langle \boldsymbol{\nu}, q_{d-1}\boldsymbol{\rho} C\rangle = \langle q_{d-1}\boldsymbol{\nu}, \boldsymbol{\rho} C\rangle = q_{d-1}(\mu_0)\|\boldsymbol{\rho} C\|^2,$$

and

$$\|q_{d-1}\boldsymbol{\rho} C\|^2 = \|\boldsymbol{\rho} C\|^2 \|q_{d-1}\|_C^2 = \|\boldsymbol{\rho} C\|^2 \left\| \sum_{k=0}^{d-1} p_k^C \right\|_C^2 = \|\boldsymbol{\rho} C\|^2 \sum_{k=0}^{d-1} \|p_k^C\|_C^2 = \|\boldsymbol{\rho} C\|^2 q_{d-1}(\mu_0).$$

Hence, the Cauchy-Schwarz inequality gives:

$$\langle \boldsymbol{\rho} N_{d-1}, q_{d-1}\boldsymbol{\rho} C\rangle^2 = q_{d-1}^2(\mu_0)\|\boldsymbol{\rho} C\|^4 \le \|\boldsymbol{\rho} N_{d-1}\|^2 \|\boldsymbol{\rho} C\|^2 q_{d-1}(\mu_0)$$

so that

$$q_{d-1}(\mu_0) \le \frac{\|\boldsymbol{\rho} N_{d-1}\|^2}{\|\boldsymbol{\rho} C\|^2} \qquad \Leftrightarrow \qquad p_d(\mu_0) \ge \frac{\|\boldsymbol{\rho}\overline{C}\|^2}{\|\boldsymbol{\rho} C\|^2},$$

where we have used that $q_{d-1}(\mu_0) = \frac{\|\boldsymbol{\nu}\|^2}{\|\boldsymbol{\rho} C\|^2} - p_d(\mu_0)$ and $\|\boldsymbol{\rho} N_{d-1}\|^2 = \|\boldsymbol{\nu}\|^2 - \|\boldsymbol{\rho}\overline{C}\|^2$. Moreover, if the above inequalities are equalities, the corresponding vectors are colinear, $q_{d-1}\boldsymbol{\rho} C = \alpha \boldsymbol{\rho} N_{d-1}$ with $\alpha \in \mathbb{R}$. But, taking square norms, $\|q_{d-1}\boldsymbol{\rho} C\|^2 = \|\boldsymbol{\rho} C\|^2 \|q_{d-1}\|_C^2 = \|\boldsymbol{\rho} C\|^2 q_{d-1}(\mu_0) = \alpha^2 \|\boldsymbol{\rho} N_{d-1}\|^2$, we have that $\alpha = 1$, and the resulting vector equality is equivalent to $p_d \boldsymbol{\rho} C = \alpha \boldsymbol{\rho}\overline{C}$. Consequently, Corollary 2.3(b) and Theorem 4.7 give the result. □

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