4-labelings and grid embeddings of plane quadrangulations

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Abstract

We show that each quadrangulation on n vertices has a closed rectangle of influence drawing on the $(n-2) \times (n-2)$ grid. Further, we present a simple algorithm to obtain a straight-line drawing of a quadrangulation on the $\left\lceil \frac{n}{2} \right\rceil \times \left\lceil \frac{3n}{4} \right\rceil$ grid. This is not optimal but has the advantage over other existing algorithms that it is not needed to add edges to the quadrangulation to make it 4-connected. The algorithm is based on angle labeling and simple face counting in regions analogous to Schnyder's grid embedding for triangulation. This extends previous results on book embeddings for quadrangulations from Felsner, Huemer, Kappes, and Orden (2008). Our approach also yields a representation of a quadrangulation as a pair of rectangulations with a curious property.

Keywords: Embedding, labeling, quadrangulation, planar bipartite graph, rectangle of influence, rectangulation.

1 Introduction

Finding straight-line drawings of planar graphs on an integer grid is a prominent problem in graph drawing. A common requirement for such drawings is to guarantee a small size of the integer grid [8]. One classical algorithm for drawing triangulations, or maximal planar graphs, was given by Schnyder [15] and is based on labelings of the angles of a triangulation on n vertices, leading to a drawing on the $(n-2) \times (n-2)$ grid. Here we are interested in quadrangulations, or maximal bipartite planar graphs. Quadrangulations on n vertices have a straight-line embedding on an integer grid of size $(\left\lceil \frac{n}{2} \right\rceil - 1) \times \left\lfloor \frac{n}{2} \right\rfloor$. This was proved by Biedl and Brandenburg in [2] who showed that each quadrangulation can be augmented to a 4-connected plane graph by adding edges. Miura, Nakano and Nishizeki [13] showed that 4-connected plane graphs can be drawn on a grid of this size. Another algorithm for drawing quadrangulations on an integer grid of small size, due to Fusy [9], is based on so-called transversal structures which are related to angle labelings considered in the present work. Also the algorithm of Fusy requires to add edges to make the quadrangulation 4-connected. We present here a simple algorithm that does not need to add edges and also works for quadrangulations with connectivity 2. Our approach is in analogy to Schnyder's algorithm for embedding triangulations. We roughly outline the idea of Schnyder's proof and its adaption to quadrangulations.

Schnyder [15] showed that labeling the angles of a triangulation T with 3 colors, with special rules, gives a 3-coloring and 2-orientation of the edges of T such that the edges of each color form a directed tree. For each interior vertex of T, the three colored paths to the sinks of the respective trees divide T into three regions. Counting the number of faces in each region gives the coordinates of the interior vertex in the grid drawing. Felsner [4] extended this result to the class of 3-connected plane graphs. In [10] it was studied to adapt this method to quadrangulations Q: In this case, the angles of Q can be colored with 2 colors, which gives an analogous 2-coloring and 2-orientation of the edges of Q such that the edges of each color form a directed tree, and for each interior vertex the two colored paths to the respective sinks divide Q into two regions. In Felsner et al. [6] it is shown that counting the number of faces in a region of an interior vertex v of Q gives the coordinate of v in a book embedding of Q with two pages. In a book embedding with 2 pages the vertices of the graph are placed on a line (the spine of the book) and the two pages are halfplanes separated by this line; each edge is drawn on one page and no two edges cross. Each page in this book embedding for Q contains one of the two trees. Book embeddings of quadrangulations were also found in [7]. Whether this approach also gives a grid embedding for quadrangulations remained open.

We show here that labeling the angles of Q with 4 colors instead of 2 (which gives a 4-coloring and 2-orientation of the edges) allows to obtain a pair of book embeddings of Q such that the coordinates of a vertex v in the two book embeddings are the coordinates of v in the grid drawing of Q.

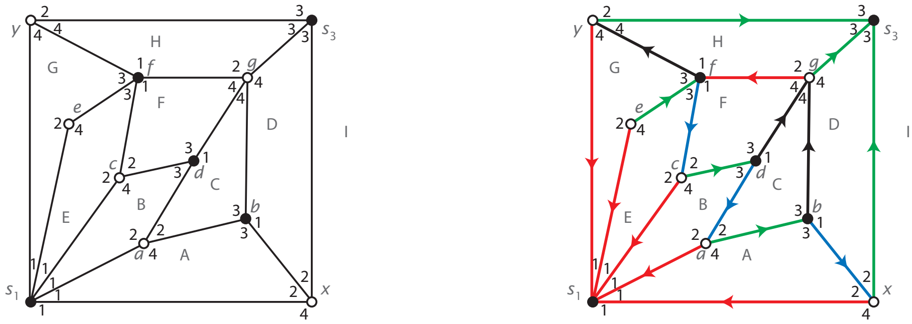
It turns out that this grid drawing for a quadrangulation Q is a closed rectangle of influence drawing on the $(n-2)\times(n-2)$ grid. In such a straight line drawing for each edge uv of Q, the closed axis-parallel rectangle with opposite corners u and v is empty. Rectangle of influence drawing have been studied for example in [1, 3, 11, 12, 14]. In particular, Biedl, Bretscher, and Meijer [3] showed that every planar graph on n vertices without separating triangle has a closed rectangle of influence drawing on the $(n-1)\times(n-1)$ grid. The rectangle of influence drawing that we obtain has the further property that edges of different colors are oriented in different directions (north-east, south-east, south-west, north-west). Since no two interior vertices lie on the same column or raw in the grid drawing, we can further reduce the size of the grid to at most $\left\lceil \frac{n}{2} \right\rceil \times \left\lceil \frac{3n}{4} \right\rceil$ by simple scaling.

Quadrangulations Q are known to admit a touching segment representation: de Fraysseix, de Mendez and Pach [7] showed that one can assign vertical segments and horizontal segments to the vertices of Q such that two segments touch if and only if the two corresponding vertices of Q are adjacent. A different proof of this result, based on book embeddings of Q, is by Felsner et al. [5], who provided a bijection between the two trees of book embeddings of quadragulations and rectangulations of a diagonal point set. For a given point set S and an axis-parallel rectangle R that contains S in its interior, a rectangulation of S is a subdivision of R into rectangles by non-crossing axis-parallel segments, such that every segment contains a point and every point lies on a segment; in a diagonal point set the points are placed on the line $y = x$ or $y = n - x$.

The 4-labeling of a quadrangulation Q gives two book embeddings and therefore two rectangulations by [5]. This pair of rectangulations has the further nice property that in each rectangulation the boxes correspond isomorphically to the faces of Q (that is, the dual graphs are isomorphic), both rectangulations have the same fixed outer face, and each segment intersects the line with slope 1 in one rectangulation and intersects the line with slope -1 in the other one.

The work is organized as follows. In Section 2 we introduce the 4-labeling of a quadrangulation, we give the relation to binary labelings and obtain a pair of book embeddings on two pages for a quadrangulation. In Section 3 we show that this pair of book embeddings gives a closed rectangle of influence drawing and we show that the grid size of this drawing can be reduced. Finally in Section 4 we state the relation of a quadrangulation to a pair of rectangulations.

Using 4-labelings, which are in bijection with binary labelings from [6], allows us to get more insight into the combinatorial structure of quadrangulations. In Section 2 we refer to [6] and



(a) (b)

Figure 1: (a) 4-labeling of a plane quadrangulation of order 11. (b) The corresponding 4-edge-coloring and 2-orientation.

extensively make use of results therein since identifying pairs of colors in the 4-labeling brings us to the setting of binary labelings; this also significantly shortens the presentation.

2 4-labelings and their properties

In what follows we consider only quadrangulations, that is, planar graphs every face of which has degree 4. Quadrangulations can also be seen as maximal bipartite planar graphs. We will assume that the vertices are colored *black* and *white*, according to the two independent sets. In this section we give the definition and properties of 4-labelings of quadrangulations.

Definition 2.1 (4-labeling) *Let Q be a quadrangulation. A* 4-labeling *of Q is a labeling of the angles of Q on $\{1, 2, 3, 4\}$ satisfying:*

- *The two black vertices of the external face, say s_1 and s_3, are called* special vertices*. All the angles incident to s_i are labeled i.*
- *At every other black vertex, the incident labels form a non-empty interval of 1s and a non-empty interval of 3s, while the incident labels of every white vertex satisfy the same property, with 2s and 4s.*
- *At every edge, the incident labels coincide at one endpoint and differ at the other.*
- *The four angles in each internal face are labeled consecutively counter-clockwise from 1 to 4.*

We assume the vertices in the external face to be labeled s_1, x, s_3, y, in counter-clockwise order. The definition implies that the incident labels of x are all equal to 2, except for the external angle, which is labeled 4. Similarly, the incident labels of y are all equal to 4, except for the external angle, which is labeled 2. See Figure 1 (a) for an example of a quadrangulation with a 4-labeling.

A strong binary labeling of a quadrangulation [6], a binary labeling for short, is a labeling of the angles on $\{0, 1\}$, with two special vertices, s_0 and s_1, such that angles incident at s_i are labeled i; the incident labels at every non-special vertex form a non-empty interval of 0s and a non-empty interval of 1s; at every edge, the incident labels coincide at one endpoint and differ at the other; and each face has exactly one pair of adjacent 0-labels and one pair of adjacent 1-labels, in such a way that the edge incident to s_0 which has the outer face on its right, when traversed from its white end to its black end, has two 0-labels in the outer face.

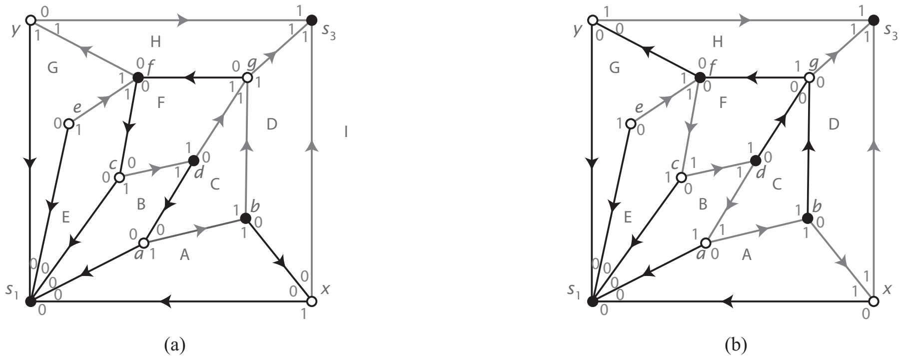


Figure 2: The binary labelings corresponding to the 4-labeling in Figure 1: (a) assigning 0 to 1 and 2, and 1 to 3 and 4; (b) assigning 0 to 1 and 4, and 1 to 2 and 3. The corresponding 2-edge-coloring and 2-orientations are also shown. In (a), the set of black arcs forms the tree T_0, and the set of grey arcs forms the tree T_1.

Proposition 2.2 *Strong binary labelings and* 4*-labelings of a quadrangulation are in bijection.*

Proof. Let Q be a quadrangulation with a 4-labeling. It is easy to verify that assigning 0 to angles labeled 1 and 2, and 1 to angles labeled 3 and 4 gives a binary labeling. This bijectively maps the set of 4-labelings onto the set of strong binary labelings. Indeed, given a binary labeling, the corresponding 4-labeling can be obtained by assigning angles labeled 0 to 1 and 2, and angles labeled 1 to 3 and 4, while preserving the counter-clockwise order of the labels in each face. ■

Assigning 0 to angles labeled 1 and 4, and 1 to angles labeled 3 and 2, gives a labeling which can be obtained from the strong binary labeling by reversing all the angle labels at white vertices. This labeling satisfies properties analogous to those satisfied by a strong binary labeling. Figure 2 shows the two binary labelings obtained from the 4-labeling in Figure 1.

It is worth noticing that the use of 4-labelings instead of binary labelings leads us to define an edge coloring with four colors instead of two. This gives us more insight into the underlying combinatorial structure and allows us to extend the results from [6] to obtain two book embeddings (instead of one) which will be used in the definition of a grid embedding of Q with good properties. An orientation in which every vertex but s_1 and s_3 has out-degree 2 is a 2-orientation.

Definition 2.3 (4-coloring and 2-orientation associated to a 4-labeling) *A* 4*-labeling of a quadrangulation* Q *defines a* 4*-coloring and a* 2*-orientation of the edges in the following way. An edge* uv *is assigned color* i *and oriented from* v *to* w *if and only if the two angles incident to* w *are labeled* i*. (See Figure 1 (b).)*

In all figures we use red for color 1, blue for color 2, green for color 3 and black for color 4.

Proposition 2.2 states the equivalence between strong labelings and 4-labelings. In fact, it is proved in [6] that strong labelings are in bijection with 2-orientations.

The next proposition is a direct consequence of the existence of strong binary labelings of a quadrangulation and Proposition 2.2. A direct proof can also be obtained following the lines in [10].

Proposition 2.4 *Every quadrangulation admits a 4-labeling.*

Given a quadrangulation Q with a 4-labeling, the following properties are satisfied. (See Figure 1 (b).)

Property 2.5 *Around each vertex the colors of the edges appear in clockwise increasing order from* 1 *to* 4*, according to the following properties:*

- *The vertex s_1 has only in-arcs, all of color* 1*; the vertex s_3 has only in-arcs, all of color* 3*.*
- *At every non-special black vertex there are exactly one out-arc of color* 2 *and one of color* 4*, and two (possibly empty) intervals of in-arcs of colors* 1 *and* 3*.*
- *At every white vertex there are exactly one out-arc of colors* 1 *and one of color* 3*, and two (possibly empty) intervals of in-arcs of colors* 2 *and* 4*.*

Proof. All the angles incident to the special vertex s_1 are labeled 1. This implies that every arc at s_1 is an in-arc, and it is colored 1. The same is true for s_3, with labels 3 and color 3.

Let v be a non-special black vertex. Its incident angles form a non-empty interval of 1s and a non-empty interval of 3s. The edges with two different angles at v will be out-arcs, one of them colored 2 and the other one colored 4. If there is more than one angle labeled 1, the edges with both incident angles in v labeled 1 will be in-arcs, colored 1. The same is true for angles labeled 3. Finally, since the orientation of angle labels in each face is counter-clockwise, the orientation of colors in the arcs at each vertex is clockwise.

The result for white vertices can be analogously proved. ■

Property 2.6 *There is no vertex with an in-arc and an out-arc of the same color. This implies that the set of edges of color i is the union of disjoint stars, each with arcs pointing to its center.*

Proof. Black vertices have out-arcs colored 2 and 4, and in-arcs colored 1 and 3, while white vertices have out-arcs colored 1 and 3, and in-arcs colored 2 and 4.

Since black and white vertices correspond to the two independent sets in Q, we have that there is no vertex with an in-arc and an out-arc of the same color. Clearly, this implies that Q has no monochromatic path and, thus, the set of edges of one color induces a union of stars. Moreover, at each star, arcs are oriented from the leaves to the center. ■

Property 2.7 *For every pair i, j, with $1 \leq i, j \leq 4$, i odd and j even, let us denote by T_{ij} the union of the sets of edges of colors i and j. Then, T_{ij} induces a directed tree with sink s_i, that spans all vertices but s_{4-i}.*

This property is a consequence of the results in [6]. Let $1 \leq i, j \leq 4$ with i odd and j even and consider the set T_{ij}. We have four possible sets: T_{12}, T_{34}, T_{14}, and T_{32}. If we define the binary labeling as in the proof of Proposition 2.2, then T_{12}, T_{34}, and the vertices s_1 and s_3 correspond to the sets T_0, T_1, and the vertices s_0 and s_1 in [6], respectively. Corollary 3.9 in [6] states that T_0 and T_1 are directed trees with sinks s_0 and s_1, respectively. To prove the result for T_{14} and T_{32}, we only need to notice that the results in [6] also hold for the binary labeling obtained by assigning label 0 to angles labeled 1 and 4, and label 1 to angles labeled 3 and 2.

Notice that we have two different partitions of Q into two disjoint directed trees, $Q = T_{12} \cup T_{34}$, and $Q = T_{14} \cup T_{32}$. T_{12} and T_{14} are rooted at s_1, and span all vertices of Q except s_3. T_{34} and T_{32} are rooted at s_3, and span all vertices of Q except s_1. Since black vertices have in-arcs of odd color, and white vertices have in-arcs of even color, we can identify colors i and j in T_{ij} without losing any information. By identifying two colors i, j (i odd, j even) and identifying the other two colors we are in the setting of [6].

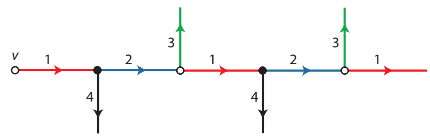


Figure 3: All outgoing edges of color k lie on the same side of path $P_{ij}(v)$.

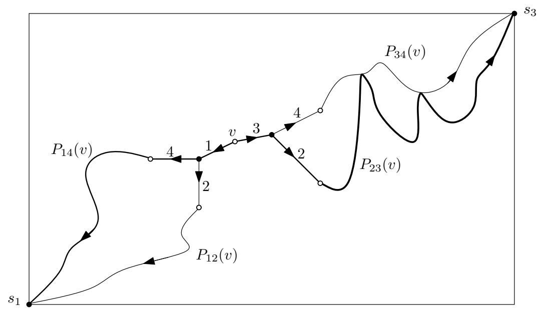


Figure 4: The four paths $P_{ij}(v)$, $1 \leq i, j \leq 4$ with i odd, j even, do not cross.

2.1 Book embeddings of a quadrangulation

In [6], the authors define a method to obtain a book embedding with 2 pages of a quadrangulation Q, given a binary labeling of Q. In this section the same method, which is inspired in Schnyder's method to obtain straight-line embeddings of triangulations on small grids [15], is used to obtain two different 2-book embeddings of a quadrangulation Q with a 4-labeling.

First we define some particular paths and regions in the plane.

- For each non-special vertex v and every pair i, j, with $1 \leq i, j \leq 4$, i odd and j even, the path $P_{ij}(v)$ is the directed path from v to the special vertex s_i in the tree T_{ij}. From [6] it follows that $P_{ij}(v)$ is chord-free and if $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$ (i and k odd) then $P_{ij}(v)$ and $P_{k\ell}(v)$ do not cross.
- For each non-special vertex v, the paths $P_{12}(v)$ and $P_{34}(v)$ split the quadrangulation into two regions $R_{12}(v)$ and $\overline{R_{12}}(v)$, where $R_{12}(v)$ is the region to the right of the path P_{12} and including both paths. Similarly, the paths $P_{14}(v)$ and $P_{32}(v)$ split the quadrangulation into two regions $R_{14}(v)$ and $\overline{R_{14}}(v)$, where $R_{14}(v)$ is the region to the left of the path P_{14} and including both paths. (See Figure 6.)

In the next two lemmas we state some properties of the paths $P_{ij}(v)$ and the regions $R_{1j}(v)$ that will be used for the definition of the two book embeddings of Q and in the proof of Theorem 3.1.

Lemma 2.8 *For each non-special vertex $v \notin \{s_1, s_3\}$, and for $1 \leq i, j, k, \ell \leq 4$, with i, k odd, j, ℓ even, the paths $P_{ij}(v)$ and $P_{k\ell}(v)$ do not cross.*

Proof. We just have to consider two paths $P_{ij}(v)$ and $P_{ik}(v)$ with $k \neq j$. Observe that along each of the paths the color of arcs alternates.

Property 2.5 implies that all outgoing edges of color k of vertices of $P_{ij}(v)$ lie on the same side of $P_{ij}(v)$, that is in the same region $R_{ij}(v)$ or $\overline{R_{ij}}(v)$. (See Figure 3.)

Furthermore, if the path $P_{ik}(v)$ meets the path $P_{ij}(v)$ in a vertex w and points towards w from the side containing the outgoing arcs of color k of $P_{ij}(v)$, then this arc towards w in $P_{ik}(v)$ can only have color i and the next arc on the path is of color k, thus staying on the same side of $P_{ij}(v)$. It remains to show that the path $P_{ik}(v)$ starts at the side of $P_{ij}(v)$ that contains the outgoing arcs of color k. This is again implied by Property 2.5. The paths $P_{ij}(v)$ and $P_{ik}(v)$ might share the first arc, as shown in Figure 4. ■

The following lemma from [6] also holds for the case of four colors instead of two.

Lemma 2.9 *Let u, v be distinct interior vertices. Let $m \in \{12, 14\}$, and $\overline{m}$ denote the two colors different from m. The following implications hold:*

- $u \in int(R_m(v)) \Rightarrow R_m(u) \subset R_m(v)$;
- $u \in P_m(v) \cup P_{\overline{m}}(v) \Rightarrow R_m(u) \subset R_m(v)$ *or* $R_m(v) \subset R_m(u)$.

Based on this lemma, and also similar to [6, 15], we can count the number of internal faces $f_m(v)$ contained in region $R_m(v)$, $m \in \{12, 14\}$, which yields a pair of book embeddings of a quadrangulation on two pages. More precisely, given v an internal vertex of Q, $f_{12}(v)$ is the number of internal faces of Q in $R_{12}(v)$ and $f_{14}(v)$ is the number of internal faces of Q in $R_{14}(v)$. For the vertices of the external face, we define $f_{12}(s_1) = f_{14}(s_1) = -1$, $f_{12}(y) = f_{14}(x) = 0$, $f_{12}(s_3) = f_{14}(s_3) = n - 2$, and $f_{12}(x) = f_{14}(y) = n - 3$.

Then, $v \rightarrow f_{12}(v) + 1$ and $v \rightarrow f_{14}(v) + 1$ both give orderings of the vertices of Q on $\{0, 1, \ldots, n-1\}$ which induce two different book embeddings of Q in two pages. (See Figure 5.)

The next proposition is a direct consequence of the results in [6].

Proposition 2.10 *The two book embeddings defined by f_{12} and f_{14} satisfy:*

- *The arcs of one page correspond to the edges of one of the trees in the tree decomposition of Q defined by the 4-labeling. More precisely, in the book embedding defined by f_{12}, the tree T_{12} is in one page, while T_{34} is in the other. In the book embedding defined by f_{14}, the tree T_{14} is in one page, while T_{32} is in the other.*
- *In both book embeddings, two edges of different color in the same page, have opposite orientations.*

In fact, the trees T_{12} and T_{34} obtained by the 4-labeling of a quadrangulation Q correspond to the pair of trees T_0 and T_1 obtained by a binary labeling of Q in [6], and denoted as twin-alternating pairs of trees in [5]. (See Figure 5.)

We remark that at first sight the two orders of vertices along the spines of the two book embeddings do not seem to be related, but this orders reveal a very nice correspondence when representing the quadrangulation as rectangulations in Section 4.

3 Embedding on the grid

Theorem 3.1 *Given a quadrangulation Q and a 4-labeling of Q, we draw Q by placing vertex v in the point of coordinates (i, j), with $0 \leq i, j \leq n - 1$, if v is in position i in the first book embedding associated to the labeling and j in the second one.*

This embedding satisfies:

1. *No two vertices are in the same vertical line or horizontal line.*

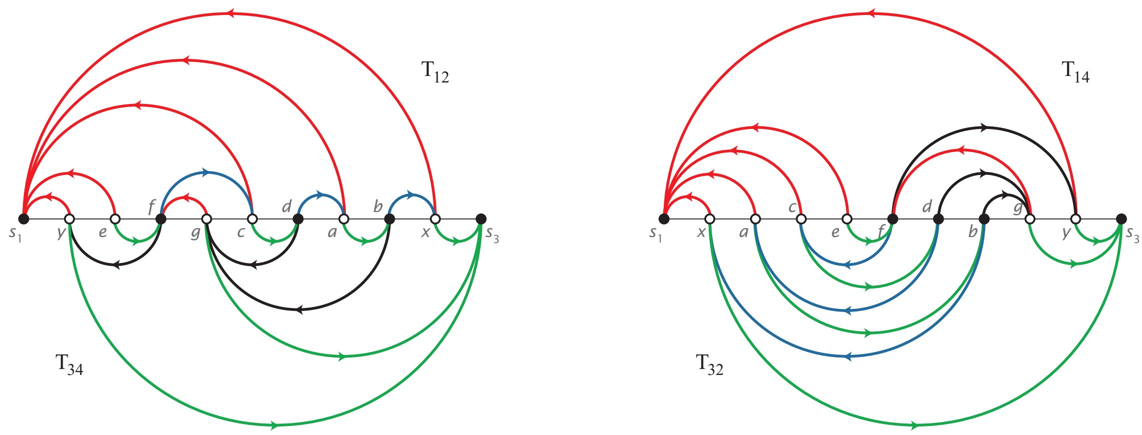


Figure 5: The two book embeddings corresponding to the graph and the 4-labeling in Figure 1.

2. *Arcs of color* 1 *point in direction south-west, arcs of color* 2 *in direction south-east, arcs of color* 3 *in direction north-east, and arcs of color* 4 *in direction north-west.*

3. *No two edges cross and any two disjoint edges are separated by a vertical line or a horizontal line.*

To prove the theorem we will use the following lemma. Its proof makes use of Lemmas 2.8 and 2.9.

Lemma 3.2 *For any two edges ab and cd of a quadrangulation Q and for at least one of $m \in \{12, 14\}$ we have* $\max\{f_m(c), f_m(d)\} < \min\{f_m(a), f_m(b)\}$ *or* $\max\{f_m(a), f_m(b)\} < \min\{f_m(c), f_m(d)\}$.

Proof. Assume the arc ab is oriented from a to b and consider the eight paths $P_{ij}(a)$ and $P_{ij}(b)$ (i odd, j even), which divide the quadrangulation into several regions, as shown in Figure 6. The vertices c and d are contained in one of this regions, they also possibly lie on the boundary of a region.

First, consider the case when neither c nor d lies on a path $P_{ij}(a)$ or $P_{ij}(b)$. Note that the statement holds if both vertices of the edge cd lie on the same side of the paths $P_{12}(a)$ and $P_{12}(b)$ (respectively $P_{14}(a)$ and $P_{14}(b)$) by Lemma 2.9. It might be that the vertices c and d lie on different sides of the paths $P_{12}(a)$ and $P_{12}(b)$, namely to the left of $P_{12}(a)$ and to the right of $P_{12}(b)$, in which case they belong to a critical region as shown in Figure 7. Similarly, c and d might be in the critical region to the right of $P_{14}(a)$ and to the left of $P_{14}(b)$. Note that this critical regions for $P_{12}(a)$ and $P_{14}(a)$ only share the two outgoing arcs from vertex a. This follows from Property 2.5 and Lemma 2.8. Therefore c and d cannot lie in both critical regions and by Lemma 2.9 $\max\{f_m(c), f_m(d)\} < \min\{f_m(a), f_m(b)\}$ or $\max\{f_m(a), f_m(b)\} < \min\{f_m(c), f_m(d)\}$ for at least one $m \in \{12, 14\}$.

It remains to consider the case when at least one of c and d lies on the boundary of a region. We focus on the case when vertex c lies on the path $P_{14}(a)$, the other cases can be treated similarly. If c is a black vertex, then its outgoing arc of color 2 points inside $R_{14}(a)$, which implies that $f_{14}(c) < f_{14}(a) < f_{14}(b)$. Further, $f_{12}(c) < f_{12}(a) < f_{12}(b)$. Now, if d is contained in $int(R_{14}(a))$ then $f_{14}(d) < f_{14}(a) < f_{14}(b)$ as claimed. Otherwise, d is contained in $R_{12}(a)$ and $f_{12}(d) < f_{12}(a) < f_{12}(b)$. Finally, if c is a white vertex, then its outgoing arc of color 3 points inside $\overline{R_{14}}(a) \subset int(R_{12}(a))$, which implies that $f_{12}(c) < f_{12}(a) < f_{12}(b)$. If d lies in the interior of $R_{12}(a)$ then also $f_{12}(d) < f_{12}(a)$. If d lies on the path $P_{12}(a)$ then arc cd is oriented

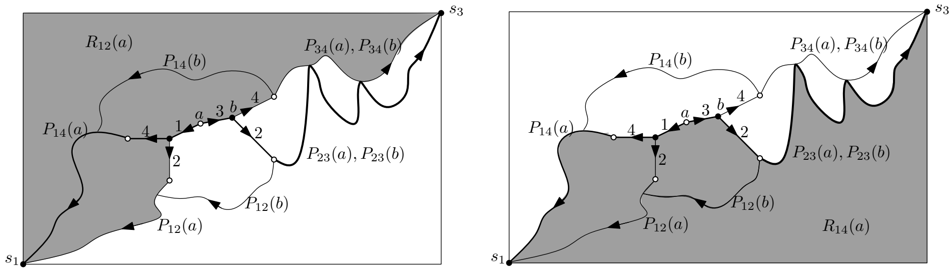


Figure 6: The regions $R_{12}(a)$ and $R_{14}(a)$.

towards c and points inside $R_{12}(a)$. Thus, we also obtain $f_{12}(d) < f_{12}(a) < f_{12}(b)$, as claimed. ■

Proof of Theorem 3.1. Property 1 of this theorem follows immediately from the two book embeddings. That is, each vertex appears exactly once in each of the two book embeddings and any two vertices have different positions along the spine of a book. Thus, no two vertices (i, j) and (k, ℓ) in the grid drawing have a common coordinate.

For proving Property 2 let us examine the edges of color 1. In each of the two book embeddings the arcs of color 1 are oriented towards the vertex s_1. Thus, in the grid drawing these arcs point to the left (since they point to the left in one book embedding with vertices drawn with increasing x-coordinate from s_1 to s_3) and downwards (since they point downwards in the other book embedding, with vertices drawn with increasing y-coordinate from s_1 to s_3). Thus, arcs of color 1 point in direction south-west. It is straightforward to verify the directions of arcs for the other colors.

The proof of Property 3 follows almost immediately from Lemma 3.2, which implies that any two edges ab, cd of the quadrangulation are separated in at least one of the two-book embeddings, i.e., the vertices a and b both appear before c and d, or both after c and d, in the vertex ordering along the spine of the book. Thus, they are also separated in the grid drawing. This separation property also implies that no two edges cross. ■

A (closed) *rectangle of influence drawing* of a graph is a straight line drawing where for each edge uv the (closed) axis-parallel rectangle with opposite corners u and v is empty. Property 3 of Theorem 3.1 also implies the following corollary.

Corollary 3.3 *The drawing given in Theorem 3.1 is a (closed) rectangle of influence drawing.*

This drawing has size $(n-2) \times (n-2)$, when moving vertex $(0, 0)$ to position $(1, 1)$ and vertex $(n-1, n-1)$ to position $(n-2, n-2)$. Rectangle of influence drawings for planar graphs have been widely studied, see e.g., [1, 3, 11, 14]. In particular, a different algorithm for obtaining a closed rectangle of influence drawing was given in [3].

3.1 Reducing the grid size

Using that in the drawing given in Theorem 3.1 no two vertices lie on the same axis-parallel line, we can further reduce the grid size:

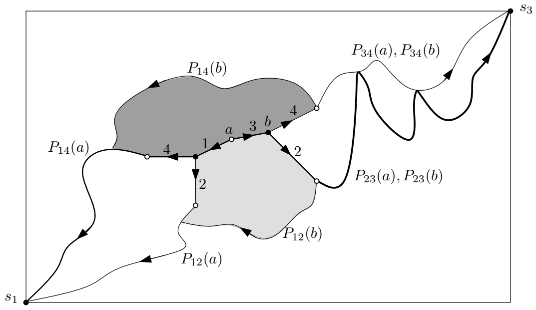


Figure 7: The two critical regions.

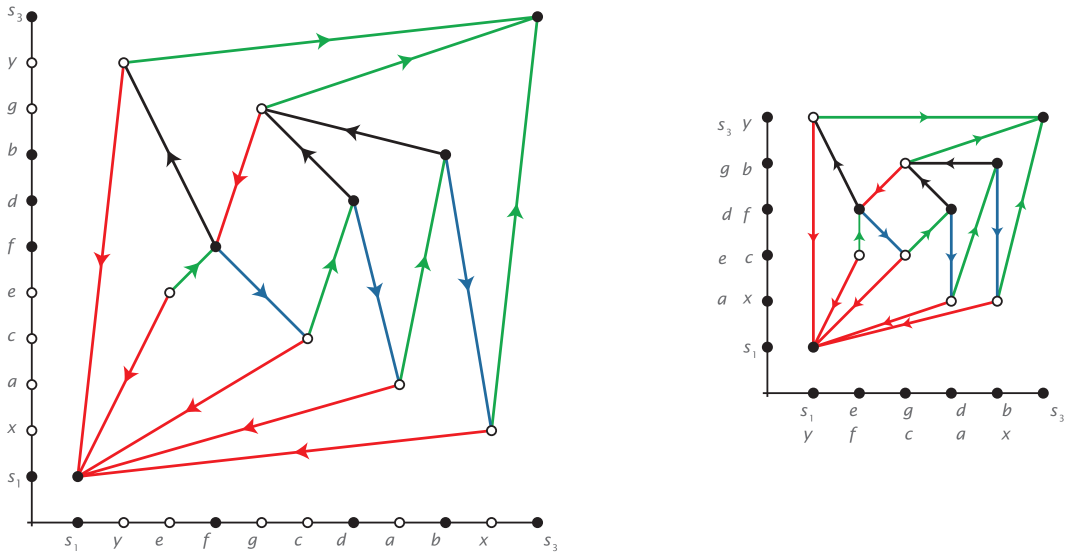


Figure 8: The 11×11 grid embedding corresponding to the graph and the 4-labeling in Figure 1, and its reduction to 6×6 by mapping coordinates (i, j) onto $\lceil \frac{i}{2} \rceil, \lfloor \frac{j}{2} \rfloor$.

Corollary 3.4 *The drawing given in Theorem 3.1 can be reduced to a grid of size at most*

$$\left\lceil \frac{n}{2} \right\rceil \times \left\lceil \frac{3n}{4} \right\rceil$$

such that each row and each column contains at most two vertices and no two edges cross.

Proof. Consider four mappings which map the vertices of the $n \times n$ grid drawing of Theorem 3.1 to a grid of size $\left\lceil \frac{n}{2} \right\rceil \times \left\lceil \frac{n}{2} \right\rceil$:

$$g_1(i,j) = (\left\lfloor \tfrac{i}{2} \right\rfloor, \left\lfloor \tfrac{j}{2} \right\rfloor), \quad g_2(i,j) = (\left\lceil \tfrac{i}{2} \right\rceil, \left\lfloor \tfrac{j}{2} \right\rfloor), \quad g_3(i,j) = (\left\lceil \tfrac{i}{2} \right\rceil, \left\lceil \tfrac{j}{2} \right\rceil), \quad g_4(i,j) = (\left\lfloor \tfrac{i}{2} \right\rfloor, \left\lceil \tfrac{j}{2} \right\rceil).$$

Observe that two vertices of the grid drawing are mapped to the same point by some g_i if and only if they are consecutive vertices in both book embeddings. More specifically, for some i, j,

- $g_1(v) = g_1(w)$ iff $v = (2i, 2j)$ and $w = (2i+1, 2j+1)$, or $v = (2i, 2j+1)$ and $w = (2i+1, 2j)$;
- $g_2(v) = g_2(w)$ iff $v = (2i-1, 2j)$ and $w = (2i, 2j+1)$, or $v = (2i, 2j)$ and $w = (2i-1, 2j+1)$;
- $g_3(v) = g_3(w)$ iff $v = (2i, 2j)$ and $w = (2i-1, 2j-1)$, or $v = (2i, 2j-1)$ and $w = (2i-1, 2j)$;

- $g_4(v) = g_4(w)$ iff $v = (2i, 2j)$ and $w = (2i+1, 2j-1)$, or $v = (2i, 2j-1)$ and $w = (2i+1, 2j)$.

Thus, we can partition the $n-1$ pairs of consecutive vertices in one of the book embeddings into five classes, where a pair belongs to class C_i if g_i maps this pair of vertices to the same point, and it belongs to class C_5 if none of the g_i maps this pair to the same point. It follows that at least one of the classes $C_i, 1 \leq i \leq 4$ contains at most $\left\lfloor \frac{n}{4} \right\rfloor$ pairs.

Let us assume that C_1 contains at most $\left\lfloor \frac{n}{4} \right\rfloor$ pairs; the other three cases can be treated the same way. We now define a mapping h which maps the vertices of the quadrangulation of the $n \times n$ grid drawing of Theorem 3.1 to a grid of size $\left\lceil \frac{3n}{4} \right\rceil \times \left\lceil \frac{n}{2} \right\rceil$ as follows:

$$h(i, j) = (\left\lfloor \frac{i}{2} \right\rfloor + k_i, \left\lfloor \frac{j}{2} \right\rfloor),$$

where $k_i = |\{v = (v_x, v_y) \text{ such that } v_x < i \text{ and } g_1(v) = g_1(v_x + 1, z)\}|$, that is, k_i is the number of pairs of consecutive points ordered by x-coordinate that appear before i in this order and that are mapped to the same point by g_1.

First, observe that since $|C_1| \leq \left\lfloor \frac{n}{4} \right\rfloor$ and $\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{4} \right\rfloor \leq \left\lceil \frac{n}{2} + \frac{n}{4} \right\rceil$ the vertices of the grid drawing are mapped to the grid of size at most $\left\lceil \frac{n}{2} + \frac{n}{4} \right\rceil \times \left\lceil \frac{n}{2} \right\rceil$. We now show that h is an injective mapping. Consider two vertices $v = (v_x, v_y)$ and $w = (w_x, y_w)$ with $v_x < w_x$. Note that k_i does not decrease as i increases. That is, if $v_x < w_x$ in the grid drawing, then $k_{v_x} \leq k_{w_x}$. Therefore $v_x < v_w$ implies $h(v_x) \leq h(w_x)$, and $v_y < w_y$ implies $h(v_y) \leq h(w_y)$ because of the monotonicity of the floor-function. If g_1 maps v and w to the same point, then v and w form a pair of consecutive vertices in both book embeddings and k_{v_x} and k_{w_x} differ by 1 (namely by the pair v, w). Consequently, $h(v)$ and $h(w)$ have different x-coordinate. Thus, assume that g_1 maps v and w to different points. If $\left\lfloor \frac{v_y}{2} \right\rfloor \neq \left\lfloor \frac{w_y}{2} \right\rfloor$ then $h(v)$ and $h(w)$ have different y-coordinates. Otherwise $\left\lfloor \frac{v_x}{2} \right\rfloor < \left\lfloor \frac{w_x}{2} \right\rfloor$ and since $k_{v_x} \leq k_{w_x}$, $h(v)$ and $h(w)$ have different x-coordinate. Hence, h is injective.

To see that no row or column contains more than two vertices in the reduced grid, observe that for vertices v and w at distance at least two in one coordinate, $h(v)$ and $h(w)$ have distance at least 1 in this coordinate, because of the floor-function and because k_i is not decreasing. It remains to show that the mapping h preserves planarity of the quadrangulation. By Property 3 of Theorem 3.1 any two edges ab and cd are separated by an axis-parallel line. Assume that $a_x < b_x < c_x < d_x$ and that ab and cd are separated by a vertical line. Thus, $h(a_x) \leq h(b_x) \leq h(c_x) \leq h(d_x)$ which precludes a crossing. Analogously, if the edges ab and cd are separated by a horizontal line, we can assume that $a_y < b_y < c_y < d_y$ which implies $h(a_y) \leq h(b_y) \leq h(c_y) \leq h(d_y)$ which again precludes a crossing. ■

In Figure 8 the grid size can even be reduced to $\left\lceil \frac{n}{2} \right\rceil \times \left\lceil \frac{n}{2} \right\rceil$ by choosing $g_2(i, j)$ instead of $g_1(i, j)$.

4 Rectangulations

For a given point set S and an axis-parallel rectangle R that contains S in its interior, a *rectangulation* of S is a subdivision of R into rectangles by non-crossing axis-parallel segments, such that every segment contains a point and every point lies on a segment.

In [5] rectangulations of diagonal point sets, that is, sets of points with coordinates $(i, n-i)$, are considered. This rectangulations are in close relation to quadrangulations: the trees T_{12} and T_{34} obtained by the 4-labeling constitute a twin-alternating pair of trees [5]. One of the

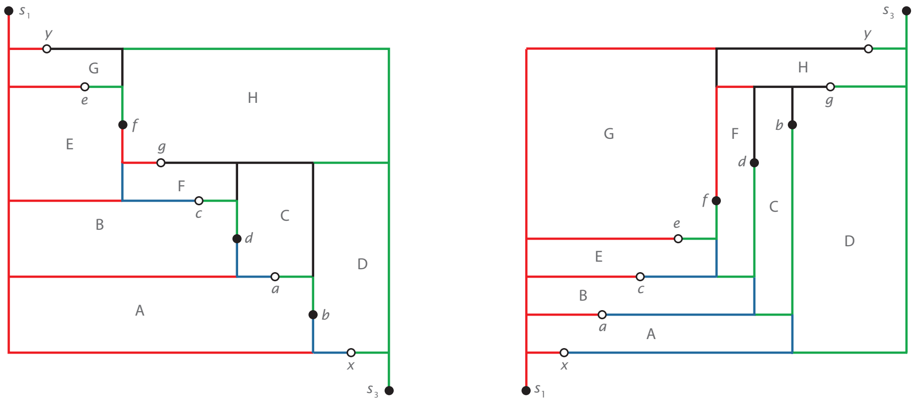


Figure 9: The quadrangulation of Figure 1 represented by a pair of rectangulations.

results in [5] states that twin-alternating pairs of trees and rectangulations of a diagonal point set are in bijection. In particular, given the book embedding of the trees T_{12} and T_{34}, the order of vertices along the spine of the book corresponds to the order of vertices on the diagonal set in the rectangulation, starting at s_1. We assume the sides of the outer rectangle R correspond to the vertices s_1, y, s_3, x and S is formed by the remaining vertices of the quadrangulation Q. Figure 9 shows an example, where also vertices s_1 and s_3 are drawn on the diagonal line.

This rectangulation has the further property that white vertices correspond to horizontal segments, and black vertices correspond to vertical segments. Such a representation of a quadrangulation also is a contact graph of segments [7]: two segments of the rectangulation touch if and only if the corresponding two vertices are adjacent in the quadrangulation.

Now, the pair of trees T_{14} and T_{32} also corresponds to a rectangulation of a diagonal point set, with vertices placed on a diagonal line according to the order of vertices in the book embedding of T_{14} and T_{32}. When drawing the vertices on a diagonal point set with coordinates (i, i), that is, on the line $y = x$ instead of $y = n - x$ we obtain the following curious property:

Theorem 4.1 *Every quadrangulation (with outer face s_1, y, s_3, x) has a representation by a pair of rectangulations, such that each edge intersects the line $y = x$ in one rectangulation and intersects the line $y = n - x$ in the other one, and such that the two rectangulations are isomorphic drawings with fixed outer face s_1, y, s_3, x in clockwise order and s_1 being the left side of the bounding rectangle.*

Proof. By drawing the vertices of one book embedding on the line $y = n - x$ with increasing x-coordinates in the order $s_1, y, \ldots, x, s_3$ and drawing the vertices of the other book embedding on the line $y = x$ in the order $s_1, x, \ldots, y, s_3$ we obtain two rectangulations with common outer face s_1, y, s_3, x in clockwise order. Note that s_1 and s_3 correspond to vertical segments, and x and y to horizontal segments. Further, every segment intersects the line $y = x$, respectively $y = n - x$ in the other rectangulation. Since adjacent vertices in Q correspond to touching segments in the rectangulation, a face of Q forms a rectangle in the rectangulation. Further, adjacent faces of Q correspond to adjacent rectangles in the rectangulation by construction of the book embedding in Section 2.1 and of the rectangulation [5]. That is, when considering the book embedding, the spine of the book embedding crosses each face of the quadrangulation and two faces are adjacent in the book embedding if they share an arc. This translates to adjacency of rectangles in the rectangulations via the bijection in [5]. ■

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