SAKARYA UNIVERSITY
INSTITUTE OF SCIENCE AND TECHNOLOGY

SOME SEQUENCE SPACES DEFINED IN n-NORMED SPACES

Ph.D. THESIS

Şükran KONCA

Department	**:**	**MATHEMATICS**
Field of Science	**:**	**TOPOLOGY**
Supervisor	**:**	**Prof. Dr. Metin BAŞARIR**

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PREFACE

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LIST OF SYMBOLS AND ABBREVIATIONS

$\mathbb{N}$	: The set of natural numbers
$\mathbb{R}$	: The set of real numbers
$\mathbb{C}$	: The set of complex numbers
$\mathbb{R}^n$	: Euclidean n space
w	: The space of all sequences
c_0	: The space of all null sequences
c	: The space of all convergent sequences
l_∞	: The space of all bounded sequences
l^p	: The space of all p-summable sequences
$L^p(X)$	: The space of all Lebesque measurable functions on X
$C[a,b]$	: The space of all continous functions given on a closed interval $[a,b]$
$\dim X$	: Dimension of a space X
$x \perp y$	: x is orthogonal to y
$\langle x, y \rangle$	: Inner product of x and y
span M	: Span of a set M
sup	: Supremum
inf	: Infimum
$\varnothing$	: Empty set
Δ	: The difference matrix
Δ^μ	: The difference matrix order μ
$B(r,s)$	: The generalized difference matrix
B^μ	: The generalized difference matrix order μ

ÖZET

Anahtar Kelimeler: 2-Norm, n-Norm, Dizi Uzayı, Orlicz Fonksiyonu, Hemen Hemen Yakınsaklık, Genelleştirilmiş Fark Matrisi, Riesz Ortalama, Ağırlıklı İstatistiksel Yakınsaklık.

Bu tez çalışması beş bölümden oluşmaktadır. Birinci bölümde, bazı temel tanım ve teoremler verildi. İkinci bölümde, 2-norm ve n-norm kavramları ile ilgili bazı temel tanım ve teoremler verildi. İkinci bölümün bir kısmı, üçüncü bölüm ve dördüncü bölümler bu tezin orijinal kısmını oluşturmaktadır.

Üçüncü bölümde 2-normlu uzaylarla ilgili kısımlar bulunurken üçüncü bölümde n-normlu uzaylarla ilgili çalışmalar yer almaktadır. Üçüncü bölümde, iki alt başlık yer almaktadır. Bu bölümün ilk kısmında, yeni bir genelleştirilmiş $B^{\mu}_{(\eta)}$ fark matrisi tanımlanarak 2-normlu uzayda bazı $B^{\mu}_{(\eta)}$-fark istatistiksel yakınsak dizi uzayları tanıtıldı ve bazı topolojik özellikleri incelendi. Aynı bölümün ikinci kısmında ise, Riesz ortalama ile türetilen bazı yeni dizi uzayları tanıtıldı. Ayrıca, ağırlıklı hemen hemen istatistiksel yakınsaklık ve $[\tilde{R}, p_n]$-istatistiksel yakınsaklık kavramları tanıtılarak bu kavramlar arasındaki ilişki incelendi.

Dördüncü bölümün ilk kısmında, Lacunary dizisi ve Riesz ortalaması tanımları birleştirilerek n-normlu uzayda ağırlıklı hemen hemen lacunary istatistiksel yakınsaklık olarak adlandırılan yeni bir kavram tanıtıldı. Bu yeni kavramla hemen hemen lacunary istatistiksel yakınsaklık ve ağırlıklı hemen hemen istatiksel yakınsaklık arasındaki ilişki incelendi. Dördüncü bölümün ikinci kısmında, bir sonsuz matris, Orlicz fonksiyonu ve genelleştirilmiş B-fark matrisi kullanılarak bazı dizi uzayları tanıtıldı. Son kısmında ise reel lineer n-normlu uzayında Orlicz fonksiyonu yardımıyla, lacunary dizisi içeren bazı dizi uzayları tanıtılarak bu dizi uzaylarının bazı topolojik özellikleri incelendi.

Son bölümde ise elde edilen temel sonuçlar özetlendi.

SOME SEQUENCE SPACES DEFINED IN n-NORMED SPACES

SUMMARY

Key Words: 2-Norm, n-Norm, Sequence Space, Orlicz Function, Almost Convergence, Generalized Difference Matrix, Riesz Mean, Weighted Almost Lacunary Statistical Convergence.

This thesis contains five chapters. In the first chapter, some basic definitions and theorems are given. In the second chapter, some fundamental definitions and theorems related to the concepts of 2-normed space and n-normed space, are given. A part of the second chapter, the third and fourth chapters are original parts of this study. The third chapter is related to the concept of 2-normed space while the studies related with n-normed space are located in the fourth chapter.

The third chapter consists of two parts. In the first part of this chapter, a new generalized difference $B^{\mu}_{(\eta)}$ matrix is defined and some $B^{\mu}_{(\eta)}$-difference statistically convergent sequence spaces in 2-normed space are introduced. In the second part of it, some new sequence spaces derived by Riesz mean are introduced. Further, new concepts of statistical convergence which will be called weighted almost statistical convergence, $[\tilde{R}, p_n]$-statistical convergence in 2-normed space, are defined and some relations between them are investigated.

There are three parts in the fourth chapter. In the first part of it, we obtain a new concept of statistical convergence which is called weighted almost lacunary statistical convergence in n-normed space by combining both of the definitions of lacunary sequence and Riesz mean. We examine some connections between this notion with the concept of almost lacunary statistical convergence and weighted almost statistical convergence, where the base space is a real n-normed space. In the second part of this chapter, some new sequence spaces associated with multiplier sequence by using an infinite matrix, an Orlicz function and generalized B-difference operator on a real n-normed space are introduced. In the last part of it, some sequence spaces, involving lacunary sequence, in a real linear n-normed space are introduced.

In the last section of this thesis, the main results, which were obtained, are summarized.

CHAPTER 1. INTRODUCTION

In this section, review of the literature, some basic definitions and theorems, which are necessary throughout this thesis, are given.

1.1. Definitions and Preliminaries

Definition 1.1.1. [1] A vector (linear) space $(X,+,.)$ over a field F ($\mathbb{R}$ or $\mathbb{C}$) is a non-empty set X whose elements are called vectors, and in which two operations addition and scalar multiplication, are defined,

$$\begin{array}{ll} +: X\times X\to X & .: F\times X\to X \\ \quad (x,y)\to x+y & \quad (\lambda,x)\to \lambda.x \end{array}$$

such that for all $\lambda,\mu\in F$ and $x,y,z\in X$ with the following familiar algebraic properties:

i. $x+y=y+x$

ii. $(x+y)+z=x+(y+z)$

iii. There exists $\theta\in X$ such that $x+\theta=x$

iv. There exists $-x\in X$ such that $x+(-x)=\theta$

v. $1.x=x$

vi. $\lambda.(x+y)=\lambda.x+\lambda.y$

vii. $(\lambda+\mu).x=\lambda.x+\mu.x$

viii. $\lambda.(\mu.x)=(\lambda.\mu).x$

Definition 1.1.2. [2] Let $F=\mathbb{R}$ or $F=\mathbb{C}$

$$w=\{x=(x_k)|x:\mathbb{N}\to F,k\to x(k)=(x_k)\}$$

denotes the space of all sequences, then w together with co-ordinatewise addition and scalar multiplication defined by $((x_k),(y_k))\to(x_k+y_k)$ and $(\lambda,(x_k))\to(\lambda x_k)$ respectively, is a linear space over F.

Example 1.1.3. [3] The space of p-summable sequences l^p $(1\le p<\infty)$

$$l^p=\left\{x=(x_k)\in w:\sum_{k=1}^{\infty}|x_k|^p<\infty,1\le p<\infty\right\} \quad \textbf{(1.1.1)}$$

is a vector space with the algebraic operations defined as usual in connection with sequences, that is,

$$(\xi_1,\xi_2,...)+(\eta_1,\eta_2,...)=(\xi_1+\eta_1,\xi_2+\eta_2,...) \text{ and } \alpha(\xi_1,\xi_2,...)=(\alpha\xi_1,\alpha\xi_2,...).$$

In fact, $x=(\xi_j)\in l^p$ and $y=(\eta_j)\in l^p$ implies $x+y\in l^p$, as follows readily from the Minkowski inequality; also $\alpha x\in l^p$.

Example 1.1.4. [3] The space of all continuous real valued functions on $[a,b]$ which is called $C[a,b]$ is a vector space. Each point of this space is a continuous real valued function on $[a,b]$. The set of all these functions forms a real vector space with the algebraic operations defined in the usual way:

$$(x+y)(t)=x(t)+y(t) \text{ and } (\alpha x)(t)=\alpha x(t), \quad (\alpha\in\mathbb{R}).$$

In fact, $x+y$ and αx are continuous real-valued functions defined on $[a,b]$ if x and y are such functions and $\alpha\in\mathbb{R}$.

Definition 1.1.5. [4] A subset Y of a linear space X is said to be a linear subspace if $x_1 + x_2 \in Y$ whenever $x_1, x_2 \in Y$ and $\alpha x \in Y$ whenever $\alpha \in F$ and $x \in Y$.

Note that a linear subspace is itself a linear space.

Example 1.1.6. [2] $c_0 = \left\{ x = (x_k) \in w : \lim_{k\to\infty} x_k = 0 \right\}$,

$c = \left\{ x = \left(x_k\right) \in w : \lim_{k\to\infty} x_k = l,\ \exists l \in \mathbb{R} \right\}$,

$l_\infty = \left\{ x = (x_k) \in w : \sup_{k\in\mathbb{N}} |x_k| < \infty \right\}$,

The sequence spaces c_0, c, l_∞ are all linear with the co-ordinatewise operations as defined in w. Moreover, the spaces c_0, c, l_∞ are linear subspaces of w.

Another special subspace of any vector space X is $Y = \{0\}$.

Fact 1.1.7. [3] Let $p > 1$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$. p and q are then called conjugate exponents. The Hölder's inequality for sums is given as follows:

$$\sum_{k=1}^{\infty} |x_k y_k| \le \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |y_k|^q \right)^{\frac{1}{q}}. \qquad \textbf{(1.1.2)}$$

This inequality was given by O. Hölder in 1889. If $p = 2$, then $q = 2$ and (1.1.2) yields the Cauchy-Schwarz inequality for sums

$$\sum_{k=1}^{\infty} |x_k y_k| \le \sqrt{\sum_{k=1}^{\infty} |x_k|^2} \sqrt{\sum_{k=1}^{\infty} |y_k|^2}.$$

Fact 1.1.8. [3] Let $p \geq 1$, then the following inequality is called Minkowski inequality for sums:

$$\left(\sum_{k=1}^{\infty}|x_k + y_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty}|y_k|^p\right)^{\frac{1}{p}}.$$

Definition 1.1.9. [3] A linear combination of vectors $x_1,...,x_m$ of a vector space X is expression of the form

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m$$

where the coefficients $\alpha_1, \alpha_2, ..., \alpha_m$ are any scalars.

For any nonempty subset $M \subset X$ the set of all linear combinations of vectors of M is called the span of M, written span M.
Obviously, this is a subspace Y of X, and it is said that Y is spanned or generated by M.

Definition 1.1.10. [3] Linear independence and dependence of a given set M of vectors $x_1,...,x_r$ $(r \geq 1)$ in a vector space X are defined by means of the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_r x_r = 0, \qquad \textbf{(1.1.3)}$$

where $\alpha_1,...,\alpha_r$ are scalars. Clearly, equation (1.1.3) holds for $\alpha_1 = \alpha_2 = ... = \alpha_r = 0$. If this is the only r-tuple of scalars for which (1.1.3) holds, the set M is said to be linearly independent. M is said to be linearly dependent if M is not linearly independent, that is, if (1.1.3) also holds for some r-tuple of scalars, not all zero.

Remark 1.1.11. [3] An arbitrary subset M of X is said to be linearly independent if every nonempty finite subset of M is linearly independent. M is said to be linearly dependent if any finite subset of M is linearly dependent.

Result 1.1.12. [3] A motivation for this terminology results from the fact that if $M = \{x_1, ..., x_r\}$ is linearly dependent, at least one vector of M can be written as a linear combination of others; for instance, if (1.1.3) holds with an $\alpha_r \neq 0$, then M is linearly dependent and we may solve (1.1.3) for x_r to get

$$x_r = \beta_1 x_1 + ... + \beta_{r-1} x_{r-1} \qquad \left(\beta_j = \frac{-\alpha_j}{\alpha_r}, \quad j = 1, 2, ..., r-1 \right).$$

Definition 1.1.13. [3] A vector space X is said to be finite dimensional if there is a positive integer n such that X contains a linearly independent set of n vectors whereas any set of $n+1$ or more vectors of X is linearly dependent. n is called the dimension of X, written $n = \dim X$. By definition, $X = \{0\}$ is finite dimensional and $\dim X = 0$. If X is not finite dimensional, it is said to be infinite dimensional.

In analysis, infinite dimensional vector spaces are of greater interest than finite dimensional ones. For instance, $C[a,b]$ and l^p are infinite dimensional, whereas $\mathbb{R}^n$ and $\mathbb{C}^n$ are n-dimensional.

Definition 1.1.14. [3] If $\dim X = n$, a linearly independent n-tuple of vectors of X is called a basis for X (or a basis in X). If $\{e_1, ..., e_n\}$ is a basis for X, every $x \in X$ has a unique representation as a linear combination of the basis vectors:

$$x = \alpha_1 e_1 + ... + \alpha_n e_n .$$

Example 1.1.15. [3] For instance, a basis for $\mathbb{R}^n$ is

$$e_1 = (1, 0, ..., 0),\ e_2 = (0, 1, 0, ..., 0),\ \dots,\ e_n = (0, 0, ..., 0, 1).$$

More generally, if X is any vector space, not necessarily finite dimensional, and B is a linearly independent subset of X which spans X, then B is called a basis (or

Hamel basis) for X. Hence if B is a basis for X, then every nonzero $x \in X$ has a unique representation as a linear combination of (finitely many) elements of B with nonzero scalars as coefficients.

Remark 1.1.16. [3] Every vector space $X \neq \{0\}$ has a basis.

Theorem 1.1.17. [3] Let X be an n-dimensional vector space. Then any proper subspace Y of X has dimension less than n.

Definition 1.1.18. [1] A metric space is a pair (X, d), where X is a non-empty set and d is a metric on X (or distance function on X), that is, a function such that $d: X \times X \to \mathbb{R}$ satisfying the following conditions for all x, y and z in X

i. $d(x,y) \geq 0$,

ii. $d(x,y) = 0$ if and only if $x = y$,

iii. $d(x,y) = d(y,x)$,

iv. $d(x,y) \leq d(x,z) + d(z,y)$.

Example 1.1.19. [3] The set of all real numbers $\mathbb{R}$, is a metric space, taken with the usual metric defined by

$$d_1(x,y) = |x-y|.$$

Example 1.1.20. [3] The metric space $\mathbb{R}^n$, called the Euclidean space $\mathbb{R}^n$, is obtained by taking the set of all ordered n-tuples of real numbers, written $x = (\xi_1, \ldots, \xi_n)$, $y = (\eta_1, \ldots, \eta_n)$, etc., and the Euclidean metric defined by

$$d_2(x,y) = \sqrt{\sum_{i=1}^{n} (\xi_i - \eta_i)^2}.$$

Definition 1.1.21. [3] The norm on a real or complex vector space X is a real-valued function such that $\|.\|: X \to \mathbb{R}$, satisfying the following conditions:

$i.\ \|x\| \geq 0$, for $x \in X$ and $\|x\| = 0$ if and only if $x = \theta$,

$ii.\ \|\alpha x\| = |\alpha|\|x\|$, for $\alpha \in \mathbb{R}$ and $x \in X$,

$iii.\ \|x + y\| \leq \|x\| + \|y\|$, for $x, y \in X$.

The normed space is denoted by $(X, \|.\|)$.

A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\|, \quad (x, y \in X)$$

and is called the metric induced by the norm.

Every metric on a vector space can not be obtained from a norm. A counter example is the space of all bounded or unbounded sequences of complex numbers w. Its metric d defined by

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1 + |\xi_j - \eta_j|}$$

where $x = (\xi_j)$ and $y = (\eta_j)$ can not be derived from a norm. A metric d induced by a norm on a normed space X satisfies the followings

$i.\ d(x + a, y + a) = d(x, y)$

$ii.\ d(\alpha x, \alpha y) = |\alpha| d(x, y)$

for all $x, y, a \in X$ and every scalar α.

Example 1.1.22. [3] Euclidean space $\mathbb{R}^n$ is a normed space with norm defined by

$$\|x\| = \left(\sum_{j=1}^{n} |\xi_j|^2 \right)^{\frac{1}{2}}.$$

We note in particular that in $\mathbb{R}^3$ $\|x\| = |x| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$. The norm generalizes the elementary notion of the length $|x|$ of a vector.

Example 1.1.23. [3] The space $L^p[a,b]$ of p-th integrable functions on $[a,b]$, $(1 \le p < \infty)$, is a normed space with the norm given by

$$\|x\| = \left(\int_a^b |x(t)|^p \, dt \right)^{\frac{1}{p}}.$$

Definition 1.1.24. [3] A sequence (x_n) in a normed space X is convergent if X contains an x such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Definition 1.1.25. [3] A sequence (x_n) in a normed space X is Cauchy if for every $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that

$\|x_m - x_n\| < \varepsilon$, for all $m, n > N$.

If every Cauchy sequence in a normed space X is convergent to a $x \in X$, then X is said to be complete normed space, that is; Banach space.

Example 1.1.26. [3] The space l^p is a Banach space with the usual norm given by

$$\|x\| = \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{\frac{1}{p}}.$$

Example 1.1.27. [3] The space $C[a,b]$ is a Banach space with the norm given by

$$\|x\| = \max_{t \in [a,b]} |x(t)|.$$

Theorem 1.1.28. [3] A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

Definition 1.1.29. [3] If (x_k) is a sequence in a normed space X, we can associate with (x_k) the sequence (s_n) of partial sums

$$s_n = x_1 + x_2 + \ldots + x_n$$

where $n = 1, 2, \ldots$. If (s_n) is convergent, say $s_n \to s$, that is, $\|s_n - s\| \to 0$, then the series $\sum_{k=1}^{\infty} x_k$ is said to converge or to be convergent, s is called the sum of the series.

Definition 1.1.30. [3] If a normed space X contains a sequence (e_n) with the property that for every $x \in X$ there is a unique sequence of scalars (α_n) such that

$$\|x - (\alpha_1 e_1 + \ldots + \alpha_n e_n)\| \to 0 \text{ as } n \to \infty.$$

Then (e_n) is called a Schauder basis (or basis) for X. The series $\sum_{k=1}^{\infty} \alpha_k e_k$ which has the sum x is then called the expansion of x with respect to (e_n), and we write

$x = \sum_{k=1}^{\infty} \alpha_k e_k$.

Example 1.1.31. [3] l^p has a Schauder basis, namely (e_n), where $e_n = (\delta_{nj})$, that is, e_n is the sequence whose n^{th}-term is 1 and all other terms are zero; thus

$e_1 = (1,0,0,0,...)$
$e_2 = (0,1,0,0,...)$
………………..
$e_n = (0,0,...,0,1,0,...)$
………………..

Definition 1.1.32. [3].A norm $\|.\|$ on a vector space X is said to be equivalent to a norm $\|.\|_0$ on X if there are positive numbers a and b such that for all $x \in X$ we have

$a\|x\|_0 \le \|x\| \le b\|x\|_0$.

Equivalent norms on X define the same topology for X .

In a normed space we can add vectors and multiply vectors by scalars, just as in elementary vector algebra. Furthermore, the norm on such a space generalizes the elementary concept of the length of a vector. However, what is still missing in a general normed space, and what we would like to have if possible, is an analogue of the familiar dot product

$a.b = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$

and resulting formulas, notably

$$|a| = \sqrt{a.a}$$

and the condition for orthogonality (perpendicularity)

$$a.b = 0$$

which are important tools in many applications. Hence the question arises whether the dot product and orthogonality can be generalized to arbitrary vector spaces. In fact, this can be done and leads to inner product spaces and complete inner product spaces, called Hilbert spaces. Inner product spaces are special normed spaces. Historically they are older than general normed spaces. Their theory is richer and retains many features of Euclidean space, a central concept being orthogonality. In fact, inner product spaces are probably the most natural generalization of Euclidean space. The whole theory was initiated by the work of D. Hilbert [5] in 1912.

Definition 1.1.33. [3] An inner product space on X is a mapping of $X \times X$ into the scalar field K of X ; that is, with every pair of vectors x and y there is associated a scalar which is written and is called the inner product of x and y, such that for all vectors x, y, z and scalars α we have

$i.\ \langle x,x \rangle \geq 0,\ \langle x,x \rangle = 0$ if and only if $x = 0$,

$ii.\ \langle x,y \rangle = \overline{\langle y,x \rangle}$,

$iii.\ \langle \alpha x,y \rangle = \alpha \langle x,y \rangle$,

$iv.\ \langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$.

An inner product on X defines a norm on X given by

$$\|x\| = \sqrt{\langle x,x \rangle} \qquad \textbf{(1.1.4)}$$

and a metric on X given by

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y \rangle}.$$

Hence inner product spaces are normed spaces, and Hilbert spaces are Banach spaces. In (ii), the bar denotes complex conjugation. Consequently, if X is a real vector space, we simply have

$$\langle x,y \rangle = \langle y,x \rangle \quad \text{(symmetry).}$$

Definition 1.1.34. [1] A norm on an inner product space satisfies the parallelogram equality:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

If a norm does not satify the parallelogram law, it can not be obtained from an inner product by the use of (1.1.4). Not all normed spaces are inner product spaces.

Example 1.1.35. [3] The space l^2 is a Hilbert space with inner product defined by

$$\langle x,y \rangle = \sum_{j=1}^{\infty} \xi_j \overline{\eta_j}$$

where $x = (\xi_1, ..., \xi_n, ...)$ and $y = (\eta_1, ..., \eta_n, ...)$ in l^2 and the bar denotes complex conjugation. The norm is defined by

$$\|x\|_2 = \sqrt{\langle x,x \rangle} = \left(\sum_{j=1}^{\infty} |\xi_j|^2 \right)^{\frac{1}{2}}.$$

Example 1.1.36. [3] The space l^p with $p \neq 2$ is not an inner product space, hence is not a Hilbert space.

Definition 1.1.37. [3] An element x of an inner product space X is said to be orthogonal to an element $y \in X$ if

$$\langle x, y \rangle = 0 .$$

It is also said that x and y are orthogonal, and it is written $x \perp y$. Similarly, for subsets $A, B \subset X$ it is written $x \perp A$ if $x \perp a$ for all $a \in A$, and $A \perp B$ if $a \perp b$ for all $a \in A$ and all $b \in B$.

Definition 1.1.38. [6] On a normed space $(X, \|.\|)$, the functional $g : X^2 \to \mathbb{R}$ defined by the formula

$$g(x, y) := \frac{\|x\|}{2}\left(\lambda_+(x, y) + \lambda_-(x, y)\right),$$

where

$$\lambda_\pm(x, y) := \lim_{t \to \pm 0} t^{-1}\left(\|x + ty\| - \|x\|\right),$$

satisfies the following properties:

i. $g(x, x) = \|x\|^2$ for all $x \in X$,

ii. $g(\alpha x, \beta y) = \alpha\beta\, g(x, y)$ for all $x, y \in X$, $\alpha, \beta \in \mathbb{R}$,

iii. $g(x, x + y) = \|x\|^2 + g(x, y)$ for all $x, y \in X$,

iv. $|g(x, y)| \le \|x\|\|y\|$ for all $x, y \in X$.

If, in addition, the functional $g(x, y)$ is linear in $y \in X$, it is called a semi-inner product on X.

Example 1.1.39. [6] The functional

$$g(x,y) := \|x\|_p^{2-p} \sum_k |x_k|^{p-1} \operatorname{sgn}(x_k) y_k\,, \quad x=(x_k),\ y=(y_k)\in l^p \tag{1.1.5}$$

defines a semi-inner product on the space l^p, for $1\le p<\infty$, where $\|.\|_p$ is the usual norm on l^p.

Definition 1.1.40. [6] Using a semi-inner product g, one may define the notion of orthogonality on X. In particular, it can be defined

$$x\perp_g y \Leftrightarrow g(x,y)=0\,. \tag{1.1.6}$$

Note that since g is in general not commutative, $x\perp_g y$ does not imply that $y\perp_g x$. Further, one can also define the g-orthogonal projection of y on x by

$$y_x := \frac{g(x,y)}{\|x\|^2}x$$

and call $y-y_x$ the g-orthogonal complement of y on x. Notice here that $x\perp_g y-y_x$.

Definition 1.1.41. [4] A paranorm $g:X\to\mathbb{R}$, X being a linear space, satisfies $g(\theta)=0$, $g(x)=g(-x)$, $g(x+y)\le g(x)+g(y)$ and scalar multiplication is continuous, i.e. $\lambda_r\to\lambda$, $g(x^r-x)\to 0$ as $r\to\infty$ imply that $g(\lambda_r x^r-\lambda x)\to 0$ as $r\to\infty$ where λ_r, are scalars and (x^r), $x\in X$, where θ is the zero vector in the linear space X. A paranorm g for which $g(x)=0$ implies $x=\theta$ is called a total paranorm on X, and the pair (X,g) is called a total paranormed space.

Definition 1.1.42. [7] Let X and Y be two subsets of w. By (X,Y), we denote the class of all matrices of A such that $A_m(x)=\sum_{k=1}^{\infty} a_{mk}x_k$ converges for each $m\in\mathbb{N}$, the set of all natural numbers, and the sequence $Ax=(A_m(x))_{m=1}^{\infty}\in Y$ for all $x\in X$.

Theorem 1.1.43. [1] Let $A=(a_{mk})$ be an infinite matrix of complex numbers. Then A is said to be regular if and only if it satisfies the following well-known Silverman-Toeplitz conditions:

i. $\sup_m \sum_{k=1}^{\infty} |a_{mk}| < \infty$

ii. $\lim_{m\to\infty} a_{mk} = 0$, for each $k \in \mathbb{N}$,

iii. $\lim_{m\to\infty} \sum_{k=1}^{\infty} a_{mk} = 1$.

Definition 1.1.44. [8] Let A be a non-negative regular summability matrix. Then a sequence $x=(x_k)$ is said to be A-statistically convergent to a number ξ if $\delta_A(K) = \lim_{m\to\infty} \sum_{k=1}^{\infty} a_{mk}\chi_K(k) = 0$ or equivalently $\lim_{m\to\infty} \sum_{k\in K} a_{mk} = 0$ for every $\varepsilon > 0$ where $K = \{k \in \mathbb{N} : |x_k - \xi| \ge \varepsilon\}$ and $\chi_K(k)$ is the characteristic function of K. We denote this limit by $\mathrm{st}_A - \lim \mathrm{x} = \xi$.

Definition 1.1.45. [9] Let $\Lambda = (\lambda_k)$ be a sequence of nonzero scalars. Then for a sequence space E the multiplier sequence space $E_{(\Lambda)}$, which associated with multiplier sequence Λ, is defined as $E_{(\Lambda)} = \{(x_k) \in w : (\lambda_k x_k) \in E\}$.

Lemma 1.1.46. [10] Let $p=(p_k)$ be a positive sequence of real numbers with $\inf_k p_k = h,$ $\sup_k p_k = H,$ and $D = \max\{1, 2^{H-1}\}$. Then for all $a_k, b_k \in \mathbb{C}$ for all $k \in \mathbb{N},$ we have

$$|a_k + b_k|^{p_k} \le D\left(|a_k|^{p_k} + |b_k|^{p_k}\right) \text{ and } |\lambda|^{p_k} \le \max\left\{ |\lambda|^h, |\lambda|^H \right\} \text{ for } \lambda \in \mathbb{C}.$$

Definition 1.1.47. [11] A sequence space E is said to be solid (or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \le 1$ for all

$k \in \mathbb{N}$.

Lemma 1.1.48. [12] Every closed linear subspace F of an arbitrary linear normed space E, different from E, is a nowhere dense set in E.

Definition 1.1.49. [11] An Orlicz function is a function $M:[0,\infty)\to[0,\infty)$ which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\to\infty$ as $x\to\infty$. It is well known if M is a convex function then $M(\alpha x)\le\alpha M(x)$ with $0<\alpha<1$.

Definition 1.1.50. [13] By a lacunary sequence $\theta=(k_r)$ where $k_0=0$, we will mean an increasing sequence of non-negative integers with $k_r-k_{r-1}\to\infty$ as $r\to\infty$. The intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$. We write $h_r=k_r-k_{r-1}$. The ratio $\dfrac{k_r}{k_{r-1}}$ will be denoted by q_r.

Definition 1.1.51. [14] If K is a subset of natural numbers $\mathbb{N}$, and the set $K_n=\{j\in K: j\le n\}$ and $|K_n|$ will denote the cardinality of K_n. Natural density of K is given by $\delta(K):=\lim_n \dfrac{1}{n}|K_n|$, if it exists.

Definition 1.1.52. [15] The sequence $x=(x_j)$ is statistically convergent to ξ provided that for every $\varepsilon>0$ the set $K:=K(\varepsilon):=\left\{j\in\mathbb{N}:\left|x_j-\xi\right|\ge\varepsilon\right\}$ has natural density zero.

Definition 1.1.53. [16] Let (p_k) be a sequence of non-negative real numbers and $P_n=p_1+p_2+...+p_n$ for $n\in\mathbb{N}$. Then Riesz transformation of $x=(x_k)$ is defined as:

$$t_n := \frac{1}{P_n}\sum_{k=1}^{n} p_k x_k. \tag{1.1.7}$$

If the sequence (t_n) has a finite limit ξ then the sequence x is said to be (R, p_n)-convergent to ξ. Let us note that if $P_n \to \infty$ as $n \to \infty$ then Riesz transformation is a regular summability method, that is it transforms every convergent sequence to convergent sequence and preserves the limit.

If $p_k = 1$ for all $k \in \mathbb{N}$ in (1.1.7), then Riesz mean reduces to Cesaro mean C_1 of order one.

In general, statistical convergence of weighted mean is studied as a regular matrix transformations. In [17] and [18], the concept of statistical convergence is generalized by using Riesz summability method and it is called weighted statistical convergence.

Theorem 1.1.54. [19] A sequence x is almost convergent to a number ξ if and only if $t_{km}(x) \to \xi$ as $k \to \infty$, uniformly in m, where

$$t_{km}(x) = \frac{x_m + x_{m+1} + \ldots + x_{m+k-1}}{k}, \; k \in \mathbb{N}, \; m \geq 0. \tag{1.1.8}$$

We write f-$\lim x = \xi$ if x is almost convergent to ξ.

Theorem 1.1.55. [20] A sequence $x = (x_j)$ is strongly almost convergent to a number ξ if and only if $t_{km}(|x - \xi e|) \to 0$ as $k \to \infty$, uniformly in m, where $x - \xi e = (x_j - \xi)$ for all j and $e = (1,1,1,\ldots)$.

If x is strongly almost convergent to ξ, we write $[f]$-$\lim x = \xi$. It is easy to see that $[f] \subset f \subset l_\infty$ and each inclusion is proper.

The notion of difference sequence space was introduced by Kızmaz [21]. It was further generalized by Et and Çolak [22].

Definition 1.1.56. [22] $Z(\Delta^{\mu}) = \{x = (x_k) \in w : (\Delta^{\mu} x_k) \in Z\}$ for $Z = l_{\infty}$, c and c_0 where μ is a non-negative integer, $\Delta^{\mu} x_k = \Delta^{\mu-1} x_k - \Delta^{\mu-1} x_{k+1}, \Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$.

Dutta [23] introduced the following difference sequence spaces using a new difference operator.

Definition 1.1.57. [23] $Z(\Delta_{(\eta)}) = \{x = (x_k) \in w : \Delta_{(\eta)} x \in Z\}$ for $Z = l_{\infty}$, c and c_0 where $\Delta_{(\eta)} x = (\Delta_{(\eta)} x_k) = (x_k - x_{k-\eta})$ for all k, $\eta \in \mathbb{N}$.

Dutta [24] introduced the sequence spaces $\bar{c}\left(\|.,.\|, \Delta^{\mu}_{(\eta)}, p\right)$, $\bar{c_0}\left(\|.,.\|, \Delta^{\mu}_{(\eta)}, p\right)$, $l_{\infty}\left(\|.,.\|, \Delta^{\mu}_{(\eta)}, p\right)$, $m\left(\|.,.\|, \Delta^{\mu}_{(\eta)}, p\right)$ and $m_0\left(\|.,.\|, \Delta^{\mu}_{(\eta)}, p\right)$ where η, $\mu \in \mathbb{N}$ and $\Delta^{\mu}_{(\eta)} x = (\Delta^{\mu}_{(\eta)} x_k) =$ $(\Delta^{\mu-1}_{(\eta)} x_k - \Delta^{\mu-1}_{(\eta)} x_{k-\eta})$ and $\Delta^{0}_{(\eta)} x_k = x_k$ for all k, $\eta \in \mathbb{N}$ which is equivalent to the following binomial representation:

$$\Delta^{\mu}_{(\eta)} x_k = \sum_{v=0}^{\mu} (-1)^v \binom{\mu}{v} x_{k-\eta v}.$$

Definition 1.1.58. [25] The generalized difference matrix $B = (b_{mk})$, which is a generalization of $\Delta_{(1)}-$difference operator, is defined for all k, $m \in \mathbb{N}$ by

$$b_{mk}(r,s) = \begin{cases} r, & (k = m) \\ s, & (k = m-1) \\ 0, & (0 \le k < m-1) \text{ or } (k > m) \end{cases}.$$

Definition 1.1.59. [26] The generalized B^{μ}-difference operator is equivalent to the following binomial representation:

$$B^{\mu}x = B^{\mu}\left(x_k\right) = \sum_{v=0}^{\mu}\binom{n}{v} r^{\mu-v} s^{v} x_{k-v} .$$

Başarır and Kayıkçı [26] defined the matrix $B^{\mu} = (b_{mk}^{\mu})$ which reduced the difference matrix $\Delta_{(1)}^{\mu}$ in case $r = 1,\ s = -1$.

CHAPTER 2. THE CONCEPTS OF 2-NORMED SPACE AND n-NORMED SPACE

In this section, some fundamental definitions and theorems related to the concepts of 2-normed space and n-normed space, are given.

2.1. The Concept of 2-Norm and Relation with The Concept of 2-Metric

As well known, in the present mathematics, one of the most important notions is the notion of metrics, which is fundamental in geometry, analysis and others. We certainly admit the importance of the notion of metrics. However, we must recognize that the notion of metrics has a limitation. To pass the limitation, we need a new notion. One of the treatments is to consider a 2-metric space introduced by S. Gähler [27] which is based on the researches of K. Menger [28]. The notion of a metric is to be regarded as a generalization of the notion of the distance between two points. On the other hand, the notion of 2-metric spaces is obtained by a generalization of the notion of area. The area in the Euclidean plane is uniquely determined by given three points in the plane [29].

Definition 2.1.1. [27] Let $X \neq \varnothing$. We consider a mapping which is defined on the set of all triples of points (x, y, z) of X into the reals such that $\rho : X \times X \times X \to \mathbb{R}^{+} \cup \{0\}$ satisfies

i. There are three points x, y, z such that $\rho(x, y, z) \neq 0$,

ii. $\rho(x, y, z) = 0$ if and only if at least two points of three points are equal,

iii. $\rho(x, y, z) = \rho(x, z, y) = \rho(y, z, x) = \ldots$ ($\rho(x, y, z)$ is symmetric for x, y, z),

iv. $\rho(x, y, z) \leq \rho(x, y, w) + \rho(x, w, z) + \rho(w, y, z)$.

Then ρ is called a 2-metric on X and (X,ρ) is called a 2-metric space.

Example 2.1.2. [30] Every Euclidean space of finite dimension $d \geq 2$ has a 2-metric defined by

$$\rho(x,y,z) := \frac{1}{2}\left(\sum_{i<j}\begin{vmatrix} x_i & x_j & 1 \\ y_i & y_j & 1 \\ z_i & z_j & 1 \end{vmatrix}^2\right)^{\frac{1}{2}}$$

where x_i, y_i, z_i are the coordinates of x, y, z, respectively.

Definition 2.1.3. [30] For each positive real ε we define the ε-nbd (neighborhood) for two points a and b in X as the set $U_\varepsilon(a,b)$ of all points x in X such that $\rho(x,y,z) < \varepsilon$. Let V be the set of all intersections $\cap U_{\varepsilon_i}(a_i,b_i)$ of finitely many ε_i-nbds of arbitrary points a_i, b_i in X. $\{V\}$ forms a basis for the 2-metric topology of X. This topology is called the natural topology or the topology generated by the 2-metric ρ in X.

The totality of all set $W_\Sigma(a) = \cap U_{\varepsilon_i}(a,b_i)$ with arbitrary n and arbitrary pairs $\Sigma = \{(b_1,\varepsilon_1),(b_2,\varepsilon_2),\ldots,(b_n,\varepsilon_n)\}$ forms a complete system of neighborhoods of the point a.

Definition 2.1.4. [30] A 2-norm on a vector space X of d dimension, where $d \geq 2$, is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following conditions for all x, y, $z \in X$ and for any $\alpha \in \mathbb{R}$.

i. $\|x,y\| = 0$ if and only if x and y are linearly dependent,

ii. $\|x,y\| = \|y,x\|$,

iii. $\|\alpha x,y\| = |\alpha|\|x,y\|$,

iv. $\|x+y,z\| \le \|x,z\| + \|y,z\|$.

The pair $(X,\|\cdot,\cdot\|)$ is then called a 2-normed space. For any 2-normed space X, we put $\rho(x,y,z) = \|y-x, z-x\|$. Then the 2-normed space X becomes a 2-metric space.

Example 2.1.5. [31] Let $(X,\langle .,. \rangle)$ be an inner product space, equipped with the standard 2-norm

$$\|x,y\|_S := \begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{vmatrix}^{\frac{1}{2}}. \qquad \textbf{(2.1.1)}$$

Note that geometrically $\|x,y\|$ represents the area of the parallelogram spanned by x and y. The determinant is known as the Gramian of x and y. Euclidean 2-norm on $\mathbb{R}^2$ is given by

$$\|x,y\|_E = abs\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right), x = (x_1,x_2),\ y = (y_1,y_2) \in \mathbb{R}^2, \qquad \textbf{(2.1.2)}$$

where the subscript E is for Euclidean. The standard 2-norm is exactly same as the Euclidean 2-norm if $X = \mathbb{R}^2$.

For $X = \mathbb{R}^2$, from the equation (2.1.1) we obtain a better inequality $\|x,y\|_S \le \|x\|_S \|y\|_S$ which is a special case of Hadamard's inequality ([32]) where $\|x\|_S := \sqrt{\langle x,x\rangle}$ and the inner product $\langle .,. \rangle$ defined in Example 1.1.35.

Example 2.1.6. [33] Consider the space Z for l_∞, c and c_0. Let us define:

$$\|x, y\|_\infty = \sup_{i\in\mathbb{N}} \sup_{j\in\mathbb{N}} \left|x_i y_j - x_j y_i\right|, \tag{2.1.3}$$

where $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...) \in Z$. Then $\|.,.\|$ is a 2-norm on Z.

Definition 2.1.7 [34] Let $\{y, z\}$ be a linearly independent set on a 2-normed space $(X, \|.,.\|)$. A sequence (x_k) in X is called a Cauchy with respect to the set $\{y, z\}$ if $\lim_{i, j\to\infty} \|x_i - x_j, y\| = 0$ and $\lim_{i, j\to\infty} \|x_i - x_j, z\| = 0$.

Definition 2.1.8. [35] A sequence $x = (x_j)$ in 2-normed space $(X, \|.,.\|)$ is called a Cauchy sequence with respect to the $\|.,.\|$ if $\lim_{i, j\to\infty} \|x_i - x_j, z\| = 0$, for every nonzero $z \in X$.

There are two definitions of Cauchy sequences in 2-normed spaces. Definition 2.1.8 is clearly stronger than the Definition 2.1.7.

Definition 2.1.9. [34] A sequence $x = (x_j)$ in a linear 2-normed space X is called a convergent sequence, if there is an ξ in X such that $\lim_{j\to\infty} \|x_j - \xi, z\| = 0$ for every nonzero z in X.

Similar to the Definition 2.1.7 we have another definition of convergent sequences in 2-normed space, clearly weaker than the Definition 2.1.9. We will give the related details after the definitions of convergent and Cauchy sequences in n-normed spaces.

A linear 2-normed space in which every Cauchy sequence is convergent is called a 2-Banach space.

Example 2.1.10. [34] Let P_n denote the set of all real polynomials of degree $\leq n$ on the interval [0,1]. Define vector addition and scalar multiplication in the usual manner. Hence P_n is a linear space over the reals. Let $\{x_i\}_{i=0}^{2n}$ be $2n+1$ arbitrary but distinct fixed points in [0,1]. Let $f,\ g \in P_n$. Define $\|f,g\| = 0$ if f and g are linearly dependent, and define $\|f,g\| = \sum_{i=0}^{2n} |f(x_i)g(x_i)|$ if f and g are linearly independent. Then $(P_n, \|.,.\|)$ is a 2-Banach space.

On the other hand, there is a linear 2-normed space of dimension 3 which is not a 2-Banach space (such an example is given by A. White in [34]). But every 2-normed space of dimension 2 is a 2-Banach space when the underlying field is complete.

Definition 2.1.11. [36] A sequence $x = (x_j)$ is said to be statistically convergent to ξ if for every $\varepsilon > 0$ the set $K := \{ j \in \mathbb{N} : \|x_j - \xi, z\| \geq \varepsilon \}$ has natural density zero for each nonzero z in X, in other words $x = (x_j)$ statistically convergent to ξ in 2-normed space $(X, \|.,.\|)$ if $\lim_{j\to\infty} \frac{1}{j} \left| \{ j \in \mathbb{N} : \|x_j - \xi, z\| \geq \varepsilon \} \right| = 0$, for each nonzero z in X. For $\xi = 0$, we say this is statistically null.

2.2. The Concepts of 2-Inner Product and n-Inner Product

Along with the 2-norm, we have the standard 2-inner product $\langle .,.|.\rangle : X \times X \times X \to \mathbb{R}$ given by the formula

$$\langle x, y | z \rangle := \begin{vmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{vmatrix}.$$

Observe here that $\|x, z\| = \langle x, x | z \rangle^{\frac{1}{2}}$ [31].

Definition 2.2.1. [31] Let X be a real vector space of dimension $d \geq 2$. The real-valued function $\langle .,.|.\rangle$ which satisfies the following properties on X^3 is called a 2-inner product on X, and the pair $(X, \langle .,.|.\rangle)$ is called a 2-inner product space.

$i. \langle x, x|z \rangle \geq 0$; $\langle x, x|z \rangle = 0$ if and only if x and z are linearly dependent,

$ii. \langle x, y|z \rangle = \langle y, x|z \rangle$,

$iii. \langle x, x|z \rangle = \langle z, z|x \rangle$,

$iv. \langle \alpha x, y|z \rangle = \alpha \langle x, y|z \rangle$, for $\alpha \in \mathbb{R}$,

$v. \langle x_1 + x_2, y|z \rangle = \langle x_1, y|z \rangle + \langle x_2, y|z \rangle$.

The concept of 2-normed spaces was first introduced by Gähler [30], while that of 2-inner product spaces was developed by Diminnie, Gähler and White [37, 38]. Their generalization for $n \geq 2$ may be found in Misiak's works [39, 40].

Definition 2.2.2. [39] Let $n \geq 2$ be an integer and X be a real vector space of dimension $d \geq n$. A real-valued function $\langle .,.|.,...,. \rangle$ on X^{n+1} satisfying the following five properties:

$i. \langle z_1, z_1|z_2, ..., z_n \rangle \geq 0$; $\langle z_1, z_1|z_2, ..., z_n \rangle = 0$ if and only if $z_1, z_2, ..., z_n$ are linearly dependent,

$ii. \langle z_1, z_1|z_2, ..., z_n \rangle = \langle z_{i1}, z_{i1}|z_{i2}, ..., z_{in} \rangle$, for every permutation $(i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$,

$iii. \langle x, y|z_2, ..., z_n \rangle = \langle y, x|z_2, ..., z_n \rangle$,

$iv. \langle \alpha x, y|z_2, ..., z_n \rangle = \alpha \langle x, y|z_2, ..., z_n \rangle$, for $\alpha \in \mathbb{R}$,

$v. \langle x + x', y|z_2, ..., z_n \rangle = \langle x, y|z_2, ..., z_n \rangle + \langle x', y|z_2, ..., z_n \rangle$

is called an n-inner product on X, and the pair $(X, \langle .,.|.,...,. \rangle)$ is called an n-inner product space.

2.3. The Concepts of n-Norm and n-Normed Spaces

On an n-inner product space $\left(X,\langle .,.|.,...,.\rangle\right)$, the following function $\|z_1,z_2,...,z_n\|:=\sqrt{\langle z_1,z_1,|z_2,...,z_n\rangle}$ defines an n-norm, which enjoys the following four properties given in Definition 2.3.1 [41].

Definition 2.3.1. [33] Let $n\geq 2$ be an integer and X be a real vector space of dimension $d\geq n$ (d may be infinite). A real-valued function $\|.,...,.\|$ on X^n satisfying the following four properties

$i.$ $\|x_1,x_2,...,x_n\|=0$ if and only if $x_1,x_2,...,x_n$ are linearly dependent,

$ii.$ $\|x_1,x_2,...,x_n\|$ is invariant under permutation,

$iii.$ $\|\alpha x_1,x_2,...,x_n\|=|\alpha|\|x_1,x_2,...,x_n\|$, for any $\alpha\in\mathbb{R}$,

$iv.$ $\|x+x',x_2,...,x_n\|\leq\|x,x_2,...,x_n\|+\|x',x_2,...,x_n\|$,

is called an n-norm on X, and the pair $\left(X,\|.,...,.\|\right)$ is called an n-normed space.

For recent results on n-normed spaces and n-inner product spaces, see, for example [33], [39-52].

Example 2.3.2. [45] Any real inner product space $\left(X,\langle .,.\rangle\right)$ can be equipped with the standard n-norm $\|x_1,...,x_n\|:=\sqrt{\det\left(\langle x_i,x_j\rangle\right)}$, which can be interpreted as the volume of the n-dimensional parallelepiped spanned by $x_1,...,x_n\in X$. On $\mathbb{R}^n$, this n-norm can be simplified as $\|x_1,...,x_n\|:=\left|\det\left(x_i,x_j\right)\right|$ where $x_i=(x_{i1},...,x_{in})\in\mathbb{R}^n,\ i=1,...,n$.

Example 2.3.3. [44] Any inner product space $\left(X,\langle .,.\rangle\right)$ can be equipped with the standard n-inner product

$$\langle x, y | z_2, \ldots, z_n \rangle := \begin{vmatrix} \langle x, y \rangle & \langle x, z_2 \rangle & \cdots & \langle x, z_n \rangle \\ \langle z_2, y \rangle & \langle z_2, z_2 \rangle & \cdots & \langle z_2, z_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z_n, y \rangle & \langle z_n, z_2 \rangle & \cdots & \langle z_n, z_n \rangle \end{vmatrix}.$$

Observe here that the induced n-norm $\|x, z_2, \ldots, z_n\| := \sqrt{\langle x, x | z_2, \ldots, z_n \rangle}$ represents the volume of the n-dimensional parallelepiped spanned by $x, z_2, \ldots, z_n$.

Definition 2.3.4. [33] A sequence (x_k) in an n-normed space $(X, \|., \ldots, .\|)$ is said to be convergent to some $x \in X$ in the n-norm if for each $\varepsilon > 0$ there exists a positive integer $n_0 = n_0(\varepsilon)$ such that $\|x_k - x, y_2, \ldots, y_n\| < \varepsilon$ for all $k \geq n_0$ and for every nonzero $y_2, \ldots, y_n \in X$.

Similar to the 2-normed spaces, we have a new definition of Cauchy sequence for n-normed space as follows.

Definition 2.3.5. Let $A := \{a_1, \ldots, a_n\}$ be a linearly independent set on an n-normed space $(X, \|., \ldots, .\|)$. Then we say that a sequence (x_k) in X is said to be a Cauchy with respect to the set A if $\lim_{k,l \to \infty} \|x_k - x_l, a_{i_2}, \ldots, a_{i_n}\| = 0$, for $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$.

Definition 2.3.6. [33] A sequence (x_k) in an n-normed space $(X, \|., \ldots, .\|)$ is said to be a Cauchy with respect to the n-norm if $\lim_{k,l \to \infty} \|x_k - x_l, y_2, \ldots, y_n\| = 0$, for every nonzero $y_2, \ldots, y_n \in X$.

Definition 2.3.6 is clearly stronger than Definition 2.3.5. But in some cases, like finite dimensional case and the standard case the two definitions are equivalent. What is not clear is in the infinite dimensional case. But from the results in [42, 43] we understand that the two definitions are still equivalent in l^p and L^p spaces. We will show this for l^p, it can be done similarly for L^p. Now we need some lemmas which were given in [42].

Lemma 2.3.7. (Lemma 2.2, [42])

$\|x_1,...,x_n\|_p \leq (n!)^{1-\frac{1}{p}} \|x_1\|_p ... \|x_n\|_p$ holds for every $x_1,...,x_n \in l^p$.

Lemma 2.3.8. (Proposition 2.3, [42]) Let $\{a_1,...,a_n\}$ be a linearly independent set on l^p. Then the following function

$$\|x\|_p^* := \left[\sum_{\{i_2,...,i_n\}\subset\{1,...,n\}} \|x, a_{i_2}, ..., a_{i_n}\|_p^p \right]^{\frac{1}{p}}$$

defines a norm on l^p.

Lemma 2.3.9. (Proposition 2.5, [42]) Let $\{a_1,...,a_n\}$ be a linearly independent set on l^p. Then the norm $\|x\|_p^*$ is equivalent to the usual norm $\|x\|_p$ on l^p. Precisely, for every $x \in l^p$ we have

$$\frac{n\|a_1,...,a_n\|_p}{(2n-1)\left[\|a_1\|_p + ... + \|a_n\|_p\right]} \|x\|_p \leq \|x\|_p^* \leq (n!)^{1-\frac{1}{p}} \left[\sum_{\{i_2,...,i_n\}\subset\{1,...,n\}} \|a_{i_2}\|_p^p ... \|a_{i_n}\|_p^p \right]^{\frac{1}{p}} \|x\|_p .$$

By the following theorem, we will show that the Definition 2.3.6 and the Definition 2.3.5 are equivalent for l^p.

Theorem 2.3.10. (x_k) is a Cauchy sequence in l^p according to Definition 2.3.6 if and only if there exists a linearly independent set $A = \{a_1,...,a_n\}$ such that (x_k) is Cauchy sequence in l^p with respect to the set A.

Proof. Assume that (x_k) is a Cauchy sequence in l^p according to Definition 2.3.6. Then $\lim_{k,l\to\infty}\|x_k - x_l, y_2,\dots,y_n\|_p = 0$, for every $y_2,\dots,y_n \in l^p$. Hence, there exists a linearly independent set $A=\{a_1,\dots,a_n\}$ on l^p, $\lim_{k,l\to\infty}\left\|x_k - x_l, a_{i_2},\dots,a_{i_n}\right\|_p = 0$, for any $\{i_2,\dots,i_n\}\subset\{1,\dots,n\}$. Thus, we obtain the Definition 2.3.5.

Now, suppose that (x_k) is a Cauchy sequence in l^p according to Definition 2.3.5. Then for $\{i_2,\dots,i_n\}\subset\{1,\dots,n\}$ we have $\lim_{k,l\to\infty}\left\|x_k - x_l, a_{i_2},\dots,a_{i_n}\right\|_p = 0$. Hence, we obtain

$$\begin{aligned}\lim_{k,l\to\infty}\|x_k - x_l\|_p^* &= \left[\lim_{k,l\to\infty}\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}}\left\|x_k - x_l, a_{i_2},\dots,a_{i_n}\right\|_p^p\right]^{\frac{1}{p}}\\ &= \left[\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}}\lim_{k,l\to\infty}\left\|x_k - x_l, a_{i_2},\dots,a_{i_n}\right\|_p^p\right]^{\frac{1}{p}}.\\ &= 0\end{aligned}$$

By Lemma 2.3.9, we then conclude that $\lim_{k,l\to\infty}\|x_k - x_l\|_p = 0$. Hence, for every $y_2,\dots,y_n \in l^p$, we have by Lemma 2.3.7

$$\lim_{k,l\to\infty}\|x_k - x_l, y_2,\dots,y_n\|_p \le (n!)^{1-\frac{1}{p}}\lim_{k,l\to\infty}\|x_k - x_l\|_p\|y_2\|_p\cdots\|y_n\|_p = 0.$$

Thus, we obtain

$\lim_{k,l\to\infty}\|x_k - x_l, y_2,\dots,y_n\|_p = 0$ for every $y_2,\dots,y_n \in l^p$. This completes the proof.

We obtain the following corollary by the inspire of the theorem above.

Corollary 2.3.11. Let $A:=\{a_1,\dots,a_n\}$ and $B:=\{b_1,\dots,b_n\}$ be linearly independent sets on l^p. (x_k) is a Cauchy sequence with respect to the set A if and only if (x_k) is a Cauchy sequence with respect to the set B.

Proof. Let (x_k) be a Cauchy sequence in l^p with respect to the set A. Then from Theorem 2.3.10, (x_k) is a Cauchy sequence in l^p according to Definition 2.3.6. Thus, we have from Theorem 2.3.10 that there exists a linearly independent set, i.e., say, $B=\{b_1,...,b_n\}$ such that (x_k) is a Cauchy sequence in l^p with respect to the set B. For the converse, change the position of A and B. Hence, we have the result.

Remark 2.3.12. By replacing the phrases "(x_k) is Cauchy" with "(x_k) converges to x" and "$x_k - x_l$" with "$x_k - x$", we see that the analogues of Definition 2.3.5, Definition 2.3.6, Theorem 2.3.10 and Corollary 2.3.11 hold for convergent sequences.

Gunawan et. al. [46] interested in computing the "volume" of the n-dimensional parallelepiped spanned by a linearly independent set of n vectors in a normed space. In the space l^p, which is given by (1.1.1), they used the known semi-inner product g given by (1.1.5) and obtained, in general, $n!$ ways of doing it, depending on the order of the vectors. Given a finite sequence of linearly independent vectors $x_1,...,x_n$ $(n \geq 2)$ in X, they constructed a left g-orthogonal sequence $x_1^*,...,x_n^*$ such that $x_1^* := x_1$ and, for $i=2,...,n$,

$$x_i^* := x_i - (x_i)_{S_{i-1}}, \tag{2.3.1}$$

where $S_{i-1} = \operatorname{span}\{x_1^*, x_2^*, ..., x_{i-1}^*\}$. Then $x_i^* \perp_g x_j^*$ for $i,j=1,...,n$ with $i<j$. They defined the "volume" of the n-dimensional parallelepiped spanned by $x_1,x_2,...,x_n \in X$ to be

$$V(x_1,...,x_n) := \prod_{i=1}^{n} \|x_i^*\|. \tag{2.3.2}$$

Due to the limitation of g, however, $V(x_1,...,x_n)$ may not be invariant under permutations of $(x_1,...,x_n)$. This is important to indicate the difference between the usual norm and n-norm. They also show that all resulting "volumes" satisfy one common inequality which can be seen in the following theorem.

Theorem 2.3.13. (Theorem 1, [44]) Let $\{x_1,...,x_n\}$ be a linearly independent set of vectors in l^p. For any permutation $(i_1,...,i_n)$ of $(1,...,n)$, define $V\left(x_{i_1},...,x_{i_n}\right)$ as in (2.3.2) by using the semi-inner product g in (1.1.5), with $x_1^* := x_{i_1}$ and so forth as in (2.3.1). Then we have

$$V\left(x_{i_1},...,x_{i_n}\right) \le (n!)^{\frac{1}{p}} \|x_1,...,x_n\|_p .$$

The following example illustrates the situation in l^1. Let $x_1 = (1,0,0,...)$ and $x_2 = (1,1,0,...)$. Put $x_1^* = x_1$ and

$$\begin{aligned} x_2^* &= x_2 - (x_2)_{x_1} = x_2 - \frac{g(x_1,x_2)}{\|x_1\|_1^2} x_1 \\ &= (1,1,0,...) - 1.(1,0,0,...) = (0,1,0,...) \end{aligned}$$

$$V(x_1,x_2) = \|x_1^*\|_1 \|x_2^*\|_1 = 1.1 = 1 .$$

But if we put $x_1^* = x_2$ and

$$\begin{aligned} x_2^* &= x_1 - (x_1)_{x_2} = x_1 - \frac{g(x_2,x_1)}{\|x_2\|_1^2} x_2 \\ &= (1,0,0,...) - \frac{2}{2^2}.(1,1,0,...) \\ &= (1,0,0,...) - \left(\frac{1}{2},\frac{1}{2},0,...\right) \\ &= \left(\frac{1}{2},-\frac{1}{2},0,...\right), \end{aligned}$$

then we have

$$V\left(x_2, x_1\right) = \left\|x_1^*\right\|_1 \left\|x_2^*\right\|_1 = 2.1 = 2 .$$

Meanwhile,

$$\begin{aligned}\left\|x_1, x_2\right\|_1 &= \frac{1}{2}\sum_j \sum_k \left\|\begin{matrix} x_{1j} & x_{1k} \\ x_{2j} & x_{2k} \end{matrix}\right\| \\ &= \frac{1}{2}\left(\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}\right) \\ &= \frac{1}{2}(1+1) = 1 .\end{aligned}$$

Hence, we see that

$V\left(x_{i_1}, x_{i_2}\right) \le 2\left\|x_1, x_2\right\|_1$ for each permutation $\left(i_1, i_2\right)$ of $\left(1,2\right)$.

CHAPTER 3. SOME SEQUENCE SPACES IN 2-NORMED SPACE

3.1. Some Generalized Difference Statistically Convergent Sequence Spaces in 2-Normed Space

In this section, a new generalized difference matrix $B^{\mu}_{(\eta)}$ is defined and some $B^{\mu}_{(\eta)}$-difference statistically convergent sequence spaces in a real linear 2-normed space are introduced. Also some topological properties of these spaces are investigated.

By $w(\|.,.\|)$, $c(\|.,.\|)$, $c_o(\|.,.\|)$, $\bar{c}(\|.,.\|)$, $\bar{c}_0(\|.,.\|)$, $l_\infty(\|.,.\|)$, $m(\|.,.\|)$ and $m_0(\|.,.\|)$ we denote the spaces of all, convergent, null, statistically convergent, statistically null, bounded, bounded statistically convergent and bounded statistically null X valued sequence spaces, where $(X,\|.,.\|)$ is a real 2-normed space. By $\theta=(0,0,...)$ we mean the zero element of X.

In this section, we define the generalized difference matrix $B^{\mu}_{(\eta)}$ and we introduce difference sequence spaces $\bar{c}\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$, $\bar{c}_0\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$, $m\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$, $m_0\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$, $c\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$, $c_0\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$, $l_\infty\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$, $W\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$, which are defined on a real linear 2-normed space. We investigate some topological properties of the spaces $\bar{c}_0\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$, $\bar{c}\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$, $m\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$, and $m_0\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$ including linearity, existence of paranorm and solidity. Further, we show that the sequence spaces $m\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$ and $m_0\left(B^{\mu}_{(\eta)},p,\|.,.\|\right)$ are complete paranormed spaces where the base space is a 2-Banach space. Moreover, we give some inclusion relations.

By the notation $x_k \xrightarrow{\text{stat}} \xi$, we mean that x_k is statistically convergent to ξ, throughout this thesis. Let m,n be non-negative integers and $p=(p_k)$ be a sequence of strictly positive real numbers. Then we define new sequence spaces as follows:

$$\bar{c}\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right) = \left\{ \begin{array}{l} x=(x_k)\in w(\|.,.\|) : \left\|B^{\mu}_{(\eta)}x_j-\xi, z\right\|^{p_k} \xrightarrow{\text{stat}} 0, \\ \text{for every nonzero } z\in X \text{ and for some } \xi\in X \end{array} \right\},$$

$$\bar{c}_0\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right) = \left\{ \begin{array}{l} x=(x_k)\in w(\|.,.\|) : \left\|B^{\mu}_{(\eta)}x_k, z\right\|^{p_k} \xrightarrow{\text{stat}} 0, \\ \text{for every nonzero } z\in X \end{array} \right\},$$

$$l_\infty\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right) = \left\{ \begin{array}{l} x=(x_k)\in w(\|.,.\|) : \sup\limits_{k\geq 1}\left(\left\|B^{\mu}_{(\eta)}x_k, z\right\|^{p_k}\right) < \infty, \\ \text{for every nonzero } z\in X \end{array} \right\},$$

$$c\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right) = \left\{ \begin{array}{l} x=(x_k)\in w(\|.,.\|) : \lim\limits_{k\to\infty}\left\|B^{\mu}_{(\eta)}x_k-\xi, z\right\|^{p_k} = 0, \\ \text{for every nonzero } z\in X \text{ and for some } \xi\in X \end{array} \right\},$$

$$c_0\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right) = \left\{ \begin{array}{l} x=(x_k)\in w(\|.,.\|) : \lim\limits_{k\to\infty}\left\|B^{\mu}_{(\eta)}x_k, z\right\|^{p_k} = 0, \\ \text{for every nonzero } z\in X \end{array} \right\},$$

$$W\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right) = \left\{ \begin{array}{l} x=(x_k)\in w(\|.,.\|) : \lim\limits_{j\to\infty}\frac{1}{j}\sum\limits_{k=1}^{j}\left\|B^{\mu}_{(\eta)}x_k-\xi, z\right\|^{p_k} = 0, \\ \text{for every nonzero } z\in X \text{ and for some } \xi\in X \end{array} \right\},$$

$$m\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right) = \bar{c}\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)\cap l_\infty\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$$

and

$$m_0\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right) = \bar{c}_0\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)\cap l_\infty\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right),$$

where $B^{\mu}_{(\eta)}x = \left(B^{\mu}_{(\eta)}x\right)_k = rB^{\mu-1}_{(\eta)}x_k + sB^{\mu-1}_{(\eta)}x_{k-\eta}$ and $B^{0}_{(\eta)}x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the binomial representation as follows:

$$B^{\mu}_{(\eta)}x_k = \sum_{v=0}^{\mu} \binom{\mu}{v} r^{\mu-v} s^{v} x_{k-\eta v}.$$

In this representation, we obtain the matrix $B^{\mu}_{(1)}$ defined in [26] for $\mu > 1$ and in [25] for $\mu = 1$.

i. If we take $\mu = 0$ and $p_k = 1$ for all $k \in \mathbb{N}$, then the sequence spaces above are reduced to $\bar{c}(\|.,.\|)$, $\bar{c}_0(\|.,.\|)$, $l_\infty(\|.,.\|)$, $c(\|.,.\|)$, $c_0(\|.,.\|)$, $W(\|.,.\|)$, $m(\|.,.\|)$ and $m_0(\|.,.\|)$, respectively.

ii. If we take $r = 1$, $s = -1$, then the sequence spaces $\bar{c}\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$, $\bar{c}_0\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$, $l_\infty\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$, $W\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$, $m\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$, $m_0\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$ are reduced to $\bar{c}\left(\Delta^{\mu}_{(\eta)}, p, \|.,.\|\right)$, $\bar{c}_0\left(\Delta^{\mu}_{(\eta)}, p, \|.,.\|\right)$, $l_\infty\left(\Delta^{\mu}_{(\eta)}, p, \|.,.\|\right)$, $W\left(\Delta^{\mu}_{(\eta)}, p, \|.,.\|\right)$, $m\left(\Delta^{\mu}_{(\eta)}, p, \|.,.\|\right)$ and $m_0\left(\Delta^{\mu}_{(\eta)}, p, \|.,.\|\right)$, respectively, which are studied in [24].

iii. By taking $p_k = 1$ for all $k \in \mathbb{N}$, then these sequence spaces are denoted by $\bar{c}\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$, $\bar{c}_0\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$, $l_\infty\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$, $c\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$, $c_0\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$, $W\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$, $m\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$ and $m_0\left(B^{n}_{(m)}, \|.,.\|\right)$, respectively.

iv. If we take $r = 1$, $s = -1$, $p_k = 1$ for all $k \in \mathbb{N}$, then these sequence spaces are denoted by $\bar{c}\left(\Delta^{\mu}_{(\eta)}, \|.,.\|\right)$, $\bar{c}_0\left(\Delta^{\mu}_{(\eta)}, \|.,.\|\right)$, $l_\infty\left(\Delta^{\mu}_{(\eta)}, \|.,.\|\right)$, $c\left(\Delta^{\mu}_{(\eta)}, \|.,.\|\right)$, $c_0\left(\Delta^{\mu}_{(\eta)}, \|.,.\|\right)$, $W\left(\Delta^{\mu}_{(\eta)}, \|.,.\|\right)$, $m\left(\Delta^{\mu}_{(\eta)}, \|.,.\|\right)$ and $m_0\left(\Delta^{\mu}_{(\eta)}, p, \|.,.\|\right)$, respectively.

Theorem 3.1.1. Let $p=(p_k)$ be a sequence of strictly positive real numbers. Then the sequence spaces $Z\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$ are linear spaces where $Z=\overline{c}, \overline{c}_0, l_\infty, W, m, m_0$.

Proof. We prove the theorem only for the space $\overline{c}\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$. The others can be proved similarly. Let $x=(x_k),\ y=(y_k)\in\overline{c}\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$. Then there exist $\xi_1, \xi_2 \in X$ such that for every nonzero $z\in X$

$$\left\|B^{\mu}_{(\eta)}x_k-\xi_1, z\right\|^{p_k} \xrightarrow{\text{stat}} 0 \text{ and } \left\|B^{\mu}_{(\eta)}y_k-\xi_2, z\right\|^{p_k} \xrightarrow{\text{stat}} 0.$$

Let α, β be scalars. Then we have for every nonzero $z\in X$

$$\begin{aligned}
&\left\|B^{\mu}_{(\eta)}\left((\alpha x_k+\beta y_k)-(\alpha\zeta_1+\beta\zeta_2)\right), z\right\|^{p_k} \\
&=\left\|\alpha\left(B^{\mu}_{(\eta)}x_k-\zeta_1\right)-\beta\left(B^{\mu}_{(\eta)}y_k-\zeta_2\right), z\right\|^{p_k} \\
&\le\left(|\alpha|\left\|B^{\mu}_{(\eta)}x_k-\zeta_1, z\right\|+|\beta|\left\|B^{\mu}_{(\eta)}y_k-\zeta_2, z\right\|\right)^{p_k} \\
&\le D\max\left(|\alpha|^h, |\alpha|^H\right)\left\|B^{\mu}_{(\eta)}x_k-\zeta_1, z\right\|^{p_k}+D\max\left(|\beta|^h, |\beta|^H\right)\left\|B^{\mu}_{(\eta)}y_k-\zeta_2, z\right\|^{p_k} \xrightarrow{\text{stat}} 0,
\end{aligned}$$

as $k\to\infty$ where $h=\inf_k p_k$, $H=\sup_k p_k$ and $D=\max\left(1, 2^{H-1}\right)$. Hence the sequence space $\overline{c}\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$ is a linear space.

Theorem 3.1.2. For any two sequences $p=(p_k)$ and $t=(t_k)$ of positive real numbers and for any two 2-norms $\|.,.\|_1$ and $\|.,.\|_2$ on X we have
$Z\left(B^{\mu}_{(\eta)}, p, \|.,.\|_1\right)\cap Z\left(B^{\mu}_{(\eta)}, p, \|.,.\|_2\right)\neq\varnothing$, where $Z=\overline{c}, \overline{c}_0, m, m_0$.

Proof. The proof follows from the fact that the zero element belongs to each of the sequence spaces involved in the intersection.

Theorem 3.1.3. Let $\left(X,\|.,.\|\right)$ be a 2-Banach space. Then the spaces $m\left(B_{(\eta)}^{\mu}, p, \|.,.\|\right)$, $m_0\left(B_{(\eta)}^{\mu}, p, \|.,.\|\right)$ are paranormed sequence spaces, paranormed by

$$g\left(x\right)=\sup_{k\in\mathbb{N},\theta\neq z\in X}\left\|B_{(\eta)}^{\mu}x_k, z\right\|^{\frac{p_k}{H'}},$$

where $H'=\max\left\{1,H\right\}$ and $H=\sup_k p_k$, $h=\inf_k p_k$.

Proof. We will prove the theorem for the sequence space $m_0\left(B_{(\eta)}^{\mu}, p, \|.,.\|\right)$. It can be proved for the space $m\left(B_{(\eta)}^{\mu}, p, \|.,.\|\right)$ similarly.

Clearly $g(-x)=g(x)$ and $g(\theta)=0$. From the following inequality, we have

$$g\left(x+y\right)=\sup_{k\in\mathbb{N},\theta\neq z\in X}\left\|B_{(\eta)}^{\mu}\left(x_k+y_k\right), z\right\|^{\frac{p_k}{H'}}$$

$$\leq \sup_{k\in\mathbb{N},\,\theta\neq z\in X}\left\|B_{(\eta)}^{\mu}x_k, z\right\|^{\frac{p_k}{H'}}+\sup_{k\in\mathbb{N},\,\theta\neq z\in X}\left\|B_{(\eta)}^{\mu}y_k, z\right\|^{\frac{p_k}{H'}}.$$

This implies that $g\left(x+y\right)\leq g\left(x\right)+g\left(y\right)$.

To prove the continuity of scalar multiplication, assume that $\left(x^n\right)$ be any sequence of the points in $m_0\left(B_{(\eta)}^{\mu}, p, \|.,.\|\right)$ such that $g\left(x^n-x\right)\to 0$ and $\left(\lambda_n\right)$ be any sequence of scalars such that $\lambda_n\to\lambda$. Since the inequality $g\left(x^n\right)\leq g\left(x\right)+g\left(x^n-x\right)$ holds by subadditivity of g, then $\left(g\left(x^n\right)\right)$ is bounded. Thus we have

$$g\left(\lambda_n x^n-\lambda x\right)=\sup_{k\in\mathbb{N},\,\theta\neq z\in X}\left\|B_{(\eta)}^{\mu}\lambda_n x_k^n-\lambda x_k, z\right\|^{\frac{p_k}{H'}}$$

$$\leq \left(\max\left\{ |\lambda_n - \lambda|^h, |\lambda_n - \lambda|^H \right\} \right)^{\frac{1}{H'}} \sup_{k\in\mathbb{N},\, \theta\neq z\in X} \left\| B^{\mu}_{(\eta)} x_k, z \right\|^{\frac{p_k}{H'}}$$

$$+ \left(\max\left\{ |\lambda|^h, |\lambda|^H \right\} \right)^{\frac{1}{H'}} \sup_{k\in\mathbb{N},\, \theta\neq z\in X} \left\| B^{\mu}_{(\eta)} \left(x^n_k - x \right), z \right\|^{\frac{p_k}{H'}}$$

$$= \left(\max\left\{ |\lambda_n - \lambda|^h, |\lambda_n - \lambda|^H \right\} \right)^{\frac{1}{H'}} g\left(x^n \right)$$

$$+ \left(\max\left\{ |\lambda|^h, |\lambda|^H \right\} \right)^{\frac{1}{H'}} g\left(x^n - x \right)$$

which tends to zero as $n \to \infty$. Hence, g is a paranorm on the sequence space $m_0\left(B^{\mu}_{(\eta)}, p, \|.,.\| \right)$.

To prove that $m_0\left(B^{\mu}_{(\eta)}, p, \|.,.\| \right)$ is complete, assume that $\left(x^i\right)$ is a Cauchy sequence in $m_0\left(B^{\mu}_{(\eta)}, p, \|.,.\| \right)$. Then for a given ε $\left(0 < \varepsilon < 1\right)$, there exists a positive integer N_0 such that $g\left(x^i - x^j\right) < \varepsilon$, for all $i, j \geq N_0$. This implies that

$$\sup_{k\in\mathbb{N},\, \theta\neq z\in X} \left\| B^{\mu}_{(\eta)} x^i_k - B^{\mu}_{(\eta)} x^j_k, z \right\|^{\frac{p_k}{H'}} < \varepsilon ,$$

for all $i, j \geq N_0$. It follows that for every nonzero $z \in X$,

$$\left\| B^{\mu}_{(\eta)} x^i_k - B^{\mu}_{(\eta)} x^j_k, z \right\| < \varepsilon ,$$

for each $k \geq 1$ and for all $i, j \geq N_0$. Hence $\left(B^{\mu}_{(\eta)} x^i_k \right)$ is a Cauchy sequence in X for all $k \in \mathbb{N}$. Since X is a 2-Banach space, $\left(B^{\mu}_{(\eta)} x^i_k \right)$ is convergent in X for all $k \in \mathbb{N}$, so we write $\left(B^{\mu}_{(\eta)} x^i_k \right) \to \left(B^{\mu}_{(\eta)} x_k \right)$ as $i \to \infty$. Now we have for all $i, j \geq N_0$,

$$\sup_{k\in\mathbb{N},\,\theta\neq z\in X}\left\|B^{\mu}_{(\eta)}\left(x^i_k - x^j_k\right), z\right\|^{\frac{p_k}{H'}} < \varepsilon$$

$$\Rightarrow \lim_{j\to\infty}\left\{\sup_{k\in\mathbb{N},\,\theta\neq z\in X}\left\|B^{\mu}_{(\eta)}\left(x^i_k - x^j_k\right), z\right\|^{\frac{p_k}{H'}}\right\} < \varepsilon$$

$$\Rightarrow \sup_{k\in\mathbb{N},\,\theta\neq z\in X}\left\|B^{\mu}_{(\eta)}\left(x^i_k - x_k\right), z\right\|^{\frac{p_k}{H'}} < \varepsilon$$

for all $i \geq N_0$. It follows that $\left(x^i - x\right) \in m_0\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$. Since $\left(x^i\right) \in m_0\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$ and $m_0\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$ is a linear space, so we have $x = x^i - \left(x^i - x\right) \in m_0\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$. This completes the proof.

Theorem 3.1.4. *i.* If $Z_1 \subset Z_2$, then $Z_1\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right) \subset Z_2\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$ and the inclusion is strict, where Z_1 and Z_2 are equal to c, c_0, l_∞.

ii. If $\mu_1 < \mu_2$, then $Z\left(B^{\mu_1}_{(\eta)}, p, \|.,.\|\right) \subset Z\left(B^{\mu_2}_{(\eta)}, p, \|.,.\|\right)$ and the inclusion is strict, where $Z = c, c_0, l_\infty$.

Proof. The parts of proof $Z_1\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right) \subset Z_2\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$ and $Z\left(B^{\mu_1}_{(\eta)}, p, \|.,.\|\right) \subset Z\left(B^{\mu_2}_{(\eta)}, p, \|.,.\|\right)$ are easy. To show the inclusions are strict, choose $Z_1 = c_0$, $Z_2 = c$, $x = (x_k) = \left(k^2, k^2\right)$ and consider the 2-norm $\|.,.\|_E$ as given in (2.1.2), let $p_k = 1$ for all $k \in \mathbb{N}$, $\eta = 1$, $\mu = 2$, $r = 1$, $s = -1$, then $x \in c\left(B^2_{(1)}, \|.,.\|\right)$ but $x \notin c_0\left(B^2_{(1)}, \|.,.\|\right)$. If we choose $Z = c$, $x = (x_k) = \left(k^2, k^2\right)$ and $p_k = 1$ for all $k \in \mathbb{N}$, $\eta = 1$, $\mu = 2$ $r = 1$, $s = -1$, then $x \in c\left(B^2_{(1)}, \|.,.\|\right)$ but $x \notin c\left(B^1_{(1)}, \|.,.\|\right)$. These complete the proofs of parts (i) and (ii) of the theorem, respectively.

Theorem 3.1.5. *i.* $c\left(B^{\mu}_{(\eta)}, \|.,.\|\right) \subset \bar{c}\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$ and the inclusion is strict.

ii. $\bar{c}(\|.,.\|) \subset \bar{c}\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$ and the inclusion is strict.

iii. $\bar{c}\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$ and $l_{\infty}\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$ overlap but neither one contains the other.

Proof. *i.* It is clear that $c\left(B^{\mu}_{(\eta)}, \|.,.\|\right) \subset \bar{c}\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$. To show that the inclusion is strict, choose the sequence $x = (x_k)$ such that,

$$B^{\mu}_{(\eta)} x_k = \begin{cases} \left(0, \sqrt{k}\right), & k = n^2 \\ (0,0), & k \neq n^2 \end{cases} \qquad \textbf{(3.1.1)}$$

where $n \in \mathbb{N} - \{0\}$, and consider the 2-norm $\|.,.\|_E$ as given in (2.1.2). Then we obtain $B^{\mu}_{(\eta)} x_k \in \bar{c}(\|.,.\|)$, but $B^{\mu}_{(\eta)} x_k \notin c(\|.,.\|)$. That is, $x_k \in \bar{c}\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$, but $x_k \notin c\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$.

ii. It is easy to see that $\bar{c}(\|.,.\|) \subset \bar{c}\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$. To show that the inclusion is strict, let us take $x = (x_k) = (k,k)$ and consider the 2-norm $\|.,.\|_E$ as given in (2.1.2), $\eta = 1$, $\mu = 1$, $r = 1$, $s = -1$, then $x \in \bar{c}\left(B^{1}_{(1)}, \|.,.\|\right)$ but $x \notin \bar{c}(\|.,.\|)$.

iii. Since the sequence $x = \theta$ belongs to each of the sequence spaces, the overlapping part of the proof is obvious. For the other part of the proof, consider the sequence defined by (3.1.1) and the 2-norm $\|.,.\|_E$ as given in (2.1.2). Then $x \in \bar{c}\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$, but $x \notin l_{\infty}\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$. Conversely if we choose $\left(B^{\mu}_{(\eta)} x_k\right) = \left(\bar{1}, \bar{0}, \bar{1}, \bar{0}, \ldots\right)$ where $\bar{k} = (k,k)$ for all $k = 0, 1$, then $B^{\mu}_{(\eta)} x_k \in l_{\infty}(\|.,.\|)$ but $B^{\mu}_{(\eta)} x_k \notin \bar{c}(\|.,.\|)$. That is, $x \in l_{\infty}\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$ but $x \notin \bar{c}\left(B^{\mu}_{(\eta)}, \|.,.\|\right)$.

Theorem 3.1.6. The space $Z\left(B^{\mu}_{(\eta)}, p, \|.,.\|\right)$ is not solid in general, where $Z = \bar{c}$, $\bar{c}_0$, m, m_0.

Proof. To show that the space is not solid in general, consider the following examples.

Example 3.1.7. Let $\eta = 3$, $\mu = 1$, $r = 1$, $s = -1$ and consider the 2-norm $\|.,.\|_\infty$ as given by (2.1.3). Let $p_k = 5$ for all $k \in \mathbb{N}$. Consider the sequence (x_k), where $x_k = (x_k^i)$ is defined by $(x_k^i) = (k, k, k, ...)$ for each fixed $k \in \mathbb{N}$. Then $x_k \in Z(B_{(3)}^1, p, \|.,.\|)$ for $Z = \bar{c}$, m. Then $x_k \in Z(B_{(3)}^1, p, \|.,.\|)$ for $Z = \bar{c}$, m. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin Z(B_{(3)}^1, p, \|.,.\|)$ for $Z = \bar{c}$, m. Thus $Z(B_{(3)}^1, p, \|.,.\|)$ for $Z = \bar{c}$, m is not solid in general.

Example 3.1.8. Let $\eta = 3$, $\mu = 1$, $r = 1$, $s = -1$ and consider the 2-norm $\|.,.\|_\infty$ as given by (2.1.3). $p_k = 1$ for all odd k and $p_k = 2$ for all even k. Consider the sequence (x_k), where $x_k = (x_k^i)$ is defined by $(x_k^i) = (3, 3, ...)$ for each fixed $k \in \mathbb{N}$. Then $x_k \in Z(B_{(3)}^1, p, \|.,.\|)$ for $Z = \bar{c}_0$, $\bar{m}_0$. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k) \notin Z(B_{(3)}^1, p, \|.,.\|)$ for $Z = \bar{c}_0$, $\bar{m}_0$. Thus $Z(B_{(3)}^1, p, \|.,.\|)$ for $Z = \bar{c}_0$, $\bar{m}_0$ is not solid in general.

Theorem 3.1.9. The spaces $m_0\left(B_{(\eta)}^\mu, p, \|.,.\|\right)$ and $m\left(B_{(\eta)}^\mu, p, \|.,.\|\right)$ are nowhere dense subsets of $l_\infty\left(B_{(\eta)}^\mu, p, \|.,.\|\right)$.

Proof. From the Theorem 3.1.3, it follows that $m_0\left(B_{(\eta)}^\mu, p, \|.,.\|\right)$ and $m\left(B_{(\eta)}^\mu, p, \|.,.\|\right)$ are closed subspaces of $l_\infty\left(B_{(\eta)}^\mu, p, \|.,.\|\right)$. Since the inclusion relations $m_0\left(B_{(\eta)}^\mu, p, \|.,.\|\right) \subset l_\infty\left(B_{(\eta)}^\mu, p, \|.,.\|\right)$, $m\left(B_{(\eta)}^\mu, p, \|.,.\|\right) \subset l_\infty\left(B_{(\eta)}^\mu, p, \|.,.\|\right)$ are strict, the spaces $m_0\left(B_{(\eta)}^\mu, p, \|.,.\|\right)$ and $m\left(B_{(\eta)}^\mu, p, \|.,.\|\right)$ are nowhere dense subsets of $l_\infty\left(B_{(\eta)}^\mu, p, \|.,.\|\right)$ by Lemma 1.1.48.

Theorem 3.1.10. Let $p=(p_k)$ be a sequence of non-negative bounded real numbers such that $\inf_k p_k > 0$. Then $W(B^n_{(m)}, p, \|.,.\|) \cap l_\infty(B^n_{(m)}, p, \|.,.\|) \subset m(B^n_{(m)}, p, \|.,.\|)$.

Proof. Let $(x_k) \in W\left(B^\mu_{(\eta)}, p, \|.,.\|\right) \cap l_\infty\left(B^\mu_{(\eta)}, p, \|.,.\|\right)$. Then for a given $\varepsilon > 0$, we have

$$\frac{1}{j}\sum_{k=1}^{j}\left\|B^\mu_{(\eta)}x_k - L, z\right\|^{p_k} \geq \frac{1}{j}\sum_{\substack{k=1 \\ \left\|B^\mu_{(\eta)}x_k - L, z\right\|^{p_k} \geq \varepsilon}}^{j}\left\|B^\mu_{(\eta)}x_k - L, z\right\|^{p_k}$$

$$\geq \varepsilon\frac{1}{j}\left|\left\{k \leq j : \left\|B^\mu_{(\eta)}x_k - L, z\right\|^{p_k} \geq \varepsilon\right\}\right|.$$

If we take the limit for $j \to \infty$, it follows that $(x_k) \in c\left(B^\mu_{(\eta)}, p, \|.,.\|\right)$ from the inequality above. Since $(x_k) \in l_\infty\left(B^\mu_{(\eta)}, p, \|.,.\|\right)$, we have the result.

3.2. Some Sequence Spaces Derived by Riesz Mean in a Real 2-Normed Space

In this part of this chapter, we introduce some new sequence spaces derived by Riesz mean and the notions of almost and strongly almost convergence in a real 2-normed space. Some topological properties of these spaces are investigated. Further, new concepts of statistical convergence which will be called weighted almost statistical convergence, almost statistical convergence and $[\tilde{R}, p_n]$-statistical convergence in a real 2-normed space, are defined. Also, some relations between these concepts are investigated.

Let A and B be any sequence spaces. We use the notation $A_{reg} \subset B_{reg}$ to mean if the sequence x converges to the limit ξ in A, then the sequence x converges to the same limit in B.

Now, we introduce some new sequence spaces derived by weighted mean and notions of almost and strongly almost convergence in a real 2-normed space as follows:

$$\left[\tilde{R}, p_n, \|.,.\|\right] = \left\{ \begin{array}{l} x \in w(\|.,.\|) : \lim_{n\to\infty} \left\| \frac{1}{P_n} \sum_{k=1}^{n} p_k t_{km}(x - \xi e), z \right\| = 0, \\ \text{uniformly in } m, \text{ for some } \xi \text{ and for every nonzero } z \in X \end{array} \right\},$$

$$\left(\tilde{R}, p_n, \|.,.\|\right) = \left\{ \begin{array}{l} x \in w(\|.,.\|) : \lim_{n\to\infty} \frac{1}{P_n} \sum_{k=1}^{n} p_k \left\| t_{km}(x - \xi e), z \right\| = 0, \\ \text{uniformly in } m, \text{ for some } \xi \text{ and for every nonzero } z \in X \end{array} \right\},$$

$$\left|\tilde{R}, p_n, \|.,.\|\right| = \left\{ \begin{array}{l} x \in w(\|.,.\|) : \lim_{n\to\infty} \frac{1}{P_n} \sum_{k=1}^{n} p_k t_{km}\left(\|x - \xi e, z\|\right) = 0, \\ \text{uniformly in } m, \text{ for some } \xi \text{ and for every nonzero } z \in X \end{array} \right\},$$

where $t_{km}(x)$ is defined as in (1.1.8).

If we take $m = 0$ then the sequence spaces $\left[\tilde{R}, p_n, \|.,.\|\right]$, $\left(\tilde{R}, p_n, \|.,.\|\right)$, $\left|\tilde{R}, p_n, \|.,.\|\right|$ reduce to the sequence spaces $\left[C,1,\|.,.\|\right]$, $\left(C,1,\|.,.\|\right)$, $\left|C,1,\|.,.\|\right|$, respectively as follows:

$$\left[C,1,\|.,.\|\right] = \left\{ \begin{array}{l} x \in w(\|.,.\|) : \lim_{n\to\infty} \left\| \frac{1}{P_n} \sum_{k=1}^{n} p_k t_{k0}(x - \xi e), z \right\| = 0, \\ \text{for some } \xi \text{ and for every nonzero } z \in X \end{array} \right\},$$

$$\left(C,1,\|.,.\|\right) = \left\{ \begin{array}{l} x \in w(\|.,.\|) : \lim_{n\to\infty} \frac{1}{P_n} \sum_{k=1}^{n} p_k \left\| t_{k0}(x - \xi e), z \right\| = 0, \\ \text{for some } \xi \text{ and for every nonzero } z \in X \end{array} \right\},$$

$$\left|C,1,\|.,.\|\right| = \left\{ \begin{array}{l} x \in w(\|.,.\|) : \lim_{n\to\infty} \frac{1}{P_n} \sum_{k=1}^{n} p_k t_{k0}\left(\|x - \xi e, z\|\right) = 0, \\ \text{for some } \xi \text{ and for every nonzero } z \in X \end{array} \right\}.$$

Let Z be any sequence space. If $x \in Z$ and $x_j \to \xi$ as $j \to \infty$, then x is said to be Z-convergent to ξ.

Now, we define a new type of statistical convergence and investigate some inclusion relations.

Definition 3.2.1. A sequence x is said to be weighted almost statistically convergent to ξ if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{P_n} \left| \left\{ k \le P_n : p_k \left\| t_{km}\left(x - \xi e\right), z \right\| \ge \varepsilon \right\} \right| = 0, \text{ uniformly in } m,$$

for every nonzero $z \in X$. By $\left(S_{\tilde{R}}, \|.,.\|\right)$, we denote the set of all weighted almost statistically convergent sequences in a 2-normed space.

In the definition above, if we take $p_k = 1$ for all $k \in \mathbb{N}$ then we obtain the definition of almost statistical convergence. That is, x is called almost statistically convergent to ξ if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left\| t_{km}\left(x - \xi e\right), z \right\| \ge \varepsilon \right\} \right| = 0,$$

uniformly in m, for every nonzero $z \in X$. We denote the set of all almost statistically convergent sequences in a 2-normed space by $\left(S, \|.,.\|\right)$.

Theorem 3.2.2. If the sequence x is $\left(\tilde{R}, p_n, \|.,.\|\right)$-convergent to ξ then the sequence x is weighted almost statistically convergent to ξ.

Proof. Let the sequence x be $\left(\tilde{R}, p_n, \|.,.\|\right)$-convergent to ξ and

$$K_{nm}(\varepsilon)=\left\{k\le P_n : p_k\left\|t_{km}\left(x-\xi e\right),z\right\|\ge\varepsilon\right\}.$$

Then for a given $\varepsilon>0$, we have

$$\begin{aligned}\frac{1}{P_n}\sum_{k=1}^{n}p_k\left\|t_{km}\left(x-\xi e\right),z\right\| &\ge \frac{1}{P_n}\sum_{\substack{k=1\\ k\in K_{nm}(\varepsilon)}}^{n}p_k\left\|t_{km}\left(x-\xi e\right),z\right\|\\ &\ge \varepsilon\frac{1}{P_n}\left|\left\{k\le P_n : p_k\left\|t_{km}\left(x-\xi e\right),z\right\|\ge\varepsilon\right\}\right|\end{aligned}$$

for each $m\ge 0$ and for every nonzero $z\in X$. Hence we obtain that the sequence x is weighted almost statistically convergent to ξ by taking the limit as $n\to\infty$.

Now, we give a new definition which will be used in the next theorem.

Definition 3.2.3. A sequence x is said to be $\left[\tilde{R},p_n\right]$-statistically convergent to ξ if for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n : \left\|\omega_{nm}\left(x-\xi e\right),z\right\|\ge\varepsilon\right\}\right|=0,$$

uniformly in m, for every nonzero $z\in X$, where $\omega_{nm}\left(x-\xi e\right)=\frac{1}{P_n}\sum_{k=1}^{n}p_k t_{km}\left(x-\xi e\right)$.

By $\left(S_{[\tilde{R},p_n]}\right)$, we denote the set of all $\left[\tilde{R},p_n\right]$-statistically convergent sequences in 2-normed space.

Theorem 3.2.4. Let $P_n\to\infty$ as $n\to\infty$ and $p_k\left\|t_{km}\left(x-\xi e\right),z\right\|\le M$ for all $k\in\mathbb{N}$, for each $m\ge 0$ and for every nonzero $z\in X$. Then the following statements are true:

i. $\left(S_{\tilde{R}},\|.,.\|\right)_{\text{reg}}\subset\left(\tilde{R},p_n,\|.,.\|\right)_{\text{reg}}$, *ii.* $\left(S_{\tilde{R}},\|.,.\|\right)_{\text{reg}} \subset \left(S_{[\tilde{R},p_n]}\right)_{\text{reg}}$.

Proof. *i.* Let x be convergent to ξ in $\left(S_{\tilde{R}},\|.,.\|\right)$ and let us take

$$K_{nm}(\varepsilon)=\left\{k \le P_n : p_k \left\|t_{km}(x-\xi e),z\right\| \ge \varepsilon\right\}.$$

Since $p_k \left\|t_{km}(x-\xi e),z\right\| \le M$ for all $k \in \mathbb{N}$, for each $m \ge 0$, for every nonzero $z \in X$ and $P_n \to \infty$ as $n \to \infty$, then for a given $\varepsilon > 0$ we have

$$\begin{aligned}
\frac{1}{P_n}\sum_{k=1}^{n} p_k \left\|t_{km}(x-\xi e),z\right\| &= \frac{1}{P_n}\sum_{\substack{k=1\\ k\in K_{nm}(\varepsilon)}}^{n} p_k \left\|t_{km}(x-\xi e),z\right\| \\
&\quad + \frac{1}{P_n}\sum_{\substack{k=1\\ k\notin K_{nm}(\varepsilon)}}^{n} p_k \left\|t_{km}(x-\xi e),z\right\| \\
&\le M\frac{1}{P_n}\left|\left\{k \le P_n : p_k \left\|t_{km}(x-\xi e),z\right\| \ge \varepsilon\right\}\right| + \frac{n}{P_n}\varepsilon \\
&\le M\frac{1}{P_n}\left|\left\{k \le P_n : p_k \left\|t_{km}(x-\xi e),z\right\| \ge \varepsilon\right\}\right| + \varepsilon
\end{aligned}$$

for each $m \ge 0$ and for every nonzero $z \in X$. Since ε is arbitrary, we have $x \in \left(\tilde{R}, p_n, \|.,.\|\right)$ by taking the limit as $n \to \infty$.

ii. Let x be convergent to ξ in $\left(S_{\tilde{R}},\|.,.\|\right)$, then $lim_n \to \infty \frac{1}{P_n}\left|K_{nm}(\varepsilon)\right| = 0$ where $K_{nm}(\varepsilon)=\left\{k \le P_n : p_k \left\|t_{km}(x-\xi e),z\right\| \ge \varepsilon\right\}$. Then for each $m \ge 0$ and for every nonzero $z \in X$ we have

$$\left\|\omega_{nm}(x-\xi e),z\right\| = \left\|\frac{1}{P_n}\sum_{k=1}^{n} p_k t_{km}(x-\xi e),z\right\|$$

$$= \left\| \frac{1}{P_n} \left(\sum_{\substack{k=1 \\ k \in K_{nm(\varepsilon)}}}^{n} + \sum_{\substack{k=1 \\ k \notin K_{nm(\varepsilon)}}}^{n} \right) p_k t_{km}(x - \xi e), z \right\|$$

$$\leq \frac{1}{P_n} \sum_{\substack{k=1 \\ k \in K_{nm(\varepsilon)}}}^{n} p_k \left\| t_{km}(x - \xi e), z \right\| + \frac{1}{P_n} \sum_{\substack{k=1 \\ k \notin K_{nm(\varepsilon)}}}^{n} p_k \left\| t_{km}(x - \xi e), z \right\|$$

$$\leq \frac{M}{P_n} \left| K_{nm(\varepsilon)} \right| + \frac{n}{P_n} \varepsilon$$

which leads us by taking the limit as $n \to \infty$, uniformly in m that we get x converges to ξ in $\left[\tilde{R}, p_n, \|.,.\|\right]$. Hence, we can say that the sequence x is $\left[\tilde{R}, p_n\right]$-statistically convergent to ξ. This completes the proof.

Now, we introduce a new sequence space as follows.

$$\left(\tilde{R}, p_n, \|.,.\|, q_n\right) = \left\{ \begin{array}{l} x : \lim\limits_{n \to \infty} \frac{1}{P_n} \sum\limits_{k=1}^{n} p_k \left\| t_{km}(x - \xi e), z \right\|^{q_k} = 0, \text{uniformly in } m, \\ \text{for some } \xi \text{ and for every nonzero } z \in X \end{array} \right\},$$

where (q_k) is a bounded sequence of strictly positive real numbers with $h = \inf_k q_k$ and $H = \sup_k q_k$. If (q_k) is constant, then $\left(\tilde{R}, p_n, \|.,.\|, q_n\right)$ reduces to $\left(\tilde{R}, p_n, \|.,.\|, q\right)$. If we take $q_k = 1$ for all $k \in \mathbb{N}$, then we get the sequence space $\left(\tilde{R}, p_n, \|.,.\|\right)$ which is defined in the beginning of this part.

Theorem 3.2.5. Let $P_n \to \infty$ as $n \to \infty$ and (q_k) be a bounded sequence of strictly positive real numbers with $h = \inf_k q_k$, $H = \sup_k q_k < \infty$ and $H' = \max(1, H)$. Then $\left(\tilde{R}, p_n, \|.,.\|, q_n\right)$ is a linear topological space paranormed (need not be total) by

$$g(x) = \sup_{\substack{n \geq 1, m \geq 1 \\ \theta \neq z \in X}} \left(\frac{1}{P_n} \sum_{k=1}^{n} p_k \left\| t_{km}(x), z \right\|^{q_k} \right)^{\frac{1}{H'}},$$

and $\left(\tilde{R}, p_n, \|.,.\|, q\right)$ is a seminormed sequence space by

$$\|x\| = \sup_{\substack{n\geq 1,\, m\geq 1 \\ \theta\neq z\in X}} \left(\frac{1}{P_n} \sum_{k=1}^{n} p_k \left\| t_{km}\left(x\right), z\right\|^{q} \right)^{\frac{1}{q}}.$$

Proof. It is easy to see that $\left(\tilde{R}, p_n, \|.,.\|, q_n\right)$ is a linear space with coordinatewise addition and scalar multiplication. We will prove that $g(x)$ is a paranorm on $\left(\tilde{R}, p_n, \|.,.\|, q_n\right)$. We omit the proof the case $q_k = q \geq 1$ for all $k \in \mathbb{N}$ which $\|x\|$ is a seminorm. Clearly $g(\theta) = 0$, $g(x) = g(-x)$ and g is subadditive. To prove the continuity of scalar multiplication, assume that (x^r) be any sequence of the points in $\left(\tilde{R}, p_n, \|.,.\|, q_n\right)$ such that $g(x^r - x) \to 0$ as $r \to \infty$ and (λ_r) be any sequence of scalars such that $\lambda_r \to \lambda$ as $r \to \infty$. Since the inequality

$$g(x^r) \leq g(x) + g(x^r - x)$$

holds by subadditivity of g, $g(x^r)$ is bounded. Thus, by using Minkowski's inequality for $q_k \geq 1$ we have

$$\begin{aligned} g(\lambda_r x^r - \lambda x) &= \sup_{\substack{n\geq 1, m\geq 1 \\ \theta\neq z\in X}} \left(\frac{1}{P_n} \sum_{k=1}^{n} p_k \left\| t_{km}\left(\lambda_r x^r - \lambda x\right), z\right\|^{q_k} \right)^{\frac{1}{M}} \\ &\leq \left(\max\left\{ |\lambda_r - \lambda|^h, |\lambda_r - \lambda|^H \right\} \right)^{\frac{1}{M}} g(x^r) \\ &\quad + \max\left(|\lambda|^h, |\lambda|^H \right)^{\frac{1}{M}} g(x^r - x) \end{aligned}$$

which tends to zero as $r \to \infty$. Moreover, the result holds for $0 < q_k < 1$ by using Lemma 1.1.46. This proves the fact that g is a paranorm on $\left(\tilde{R}, p_n, \|.,.\|, q_n\right)$.

Theorem 3.2.6. If the following conditions hold, then $\left(\tilde{R}, p_n, \|.,.\|, q\right)_{reg} \subset \left(S_{\tilde{R}}, \|.,.\|\right)_{reg}$.

i. $0 < q < 1$ and $0 \le \left\|t_{km}\left(x - \xi e\right), z\right\| < 1$.

ii. $1 \le q < \infty$ and $1 \le \left\|t_{km}\left(x - \xi e\right), z\right\| < \infty$.

Proof. Let a sequence x be $\left(\tilde{R}, p_n, \|.,.\|, q\right)$-convergent to the limit ξ. Since $p_k \left\|t_{km}\left(x - \xi e\right), z\right\|^q \ge p_k \left\|t_{km}\left(x - \xi e\right), z\right\|$ for case (i) and (ii), then we have

$$\begin{aligned}
\frac{1}{P_n}\sum_{k=1}^{n} p_k \left\|t_{km}\left(x - \xi e\right), z\right\|^q &\ge \frac{1}{P_n}\sum_{k=1}^{n} p_k \left\|t_{km}\left(x - \xi e\right), z\right\| \\
&\ge \frac{1}{P_n}\sum_{\substack{k=1 \\ k \in K_{nm(\varepsilon)}}}^{n} p_k \left\|t_{km}\left(x - \xi e\right), z\right\| \\
&\ge \varepsilon \frac{1}{P_n}\left|K_{nm}(\varepsilon)\right|
\end{aligned}$$

for each $m \ge 0$ and for every nonzero $z \in X$. We get the result if we take the limit as $n \to \infty$. That is, $lim_{n\to\infty} \frac{1}{P_n}\left|K_{nm}(\varepsilon)\right| = 0$ where

$$K_{nm}\left(\varepsilon\right) = \left\{k \le P_n : p_k \left\|t_{km}\left(x - \xi e\right), z\right\| \ge \varepsilon\right\}.$$

Hence x converges to ξ in $\left(S_{\tilde{R}}, \|.,.\|\right)$. This completes the proof.

Theorem 3.2.7. Let $p_k \left\|t_{km}\left(x - \xi e\right), z\right\| \le M$ for all $k \in \mathbb{N}$, for each $m \ge 0$, for every nonzero $z \in X$ and $P_n \to \infty$ as $n \to \infty$. If the following conditions hold, then $\left(S_{\tilde{R}}, \|.,.\|\right)_{reg} \subset \left(\tilde{R}, p_n, \|.,.\|, q\right)_{reg}$.

i. $0 < q < 1$ and $1 \leq \left\| t_{km}\left(x - \xi e\right), z \right\| < \infty$.

ii. $1 \leq q < \infty$ and $0 \leq \left\| t_{km}\left(x - \xi e\right), z \right\| < 1$.

Proof. Assume that x converges to ξ in $\left(S_{\tilde{R}}, \|.,.\|\right)$ and $P_n \to \infty$ as $n \to \infty$. Then for $\varepsilon > 0$, we have $\delta(K_{nm}(\varepsilon)) = 0$ where

$$K_{nm}(\varepsilon) = \left\{ k \leq P_n : p_k \left\| t_{km}\left(x - \xi e\right), z \right\| \geq \varepsilon \right\}.$$

Since $p_k \left\| t_{km}\left(x - \xi e\right), z \right\| \leq M$ for all $k \in \mathbb{N}$, for each $m \geq 0$ and for every nonzero $z \in X$, then we have

$$\begin{aligned} \frac{1}{P_n} \sum_{k=1}^{n} p_k \left\| t_{km}\left(x - \xi e\right), z \right\|^q &\leq \frac{1}{P_n} \sum_{\substack{k=1 \\ k \notin K_{nm}(\varepsilon)}}^{n} p_k \left\| t_{km}\left(x - \xi e\right), z \right\|^q \\ &+ \frac{1}{P_n} \sum_{\substack{k=1 \\ k \in K_{nm}(\varepsilon)}}^{n} p_k \left\| t_{km}\left(x - \xi e\right), z \right\|^q \\ &= T_n + T_n' \end{aligned}$$

for each $m \geq 0$ and for every nonzero $z \in X$, where

$$T_n = \frac{1}{P_n} \sum_{\substack{k=1 \\ k \notin K_{nm}(\varepsilon)}}^{n} p_k \left\| t_{km}\left(x - \xi e\right), z \right\|^q \text{ and } T'_n = \frac{1}{P_n} \sum_{\substack{k=1 \\ k \in K_{nm}(\varepsilon)}}^{n} p_k \left\| t_{km}\left(x - \xi e\right), z \right\|^q.$$

For $k \notin K_{nm}(\varepsilon)$, we have

$$T_n = \frac{1}{P_n} \sum_{\substack{k=1 \\ k \notin K_{nm}(\varepsilon)}}^{n} p_k \left\| t_{km}\left(x - \xi e\right), z \right\|^q < \frac{n}{P_n} \varepsilon \leq \varepsilon$$

for each $m \geq 0$ and for every nonzero $z \in X$. If $k \in K_{nm}(\varepsilon)$, then

$$T'_n = \frac{1}{P_n} \sum_{\substack{k=1 \\ k \in K_{nm(\varepsilon)}}}^{n} p_k \left\| t_{km}\left(x - \xi e\right), z \right\|^q \leq \frac{1}{P_n} \sum_{\substack{k=1 \\ k \in K_{nm(\varepsilon)}}}^{n} p_k \left\| t_{km}\left(x - \xi e\right), z \right\| \leq \frac{M}{P_n} \left| K_{nm}\left(\varepsilon\right) \right|$$

for each $m \geq 0$ and for every nonzero $z \in X$. If we take the limit as $n \to \infty$, since $\delta(K_{nm}(\varepsilon)) = 0$ then x converges to ξ in $\left(\tilde{R}, p_n, \|.,.\|, q\right)$. This completes the proof.

CHAPTER 4. SOME SEQUENCE SPACES IN n-NORMED SPACE

In this section, some sequence spaces are introduced and some topological properties related with these spaces are given.

4.1. On Some Spaces of Almost Lacunary Convergent Sequences Derived by Riesz Mean and Weighted Almost Lacunary Statistical Convergence in a Real n-Normed Space

In this subsection, we introduce some new spaces of almost convergent sequences derived by Riesz mean and lacunary sequence in a real n-normed space. By combining both of the definitions of lacunary sequence and Riesz mean, we obtain a new concept of statistical convergence which will be called weighted almost lacunary statistical convergence in a real n-normed space. We examine some connections between this notion with the concept of almost lacunary statistical convergence and weighted almost statistical convergence, where the base space is a real n-normed space.

Let $(X,\|.,...,.\|)$ be an n-normed space and $(w,\|.,...,.\|)$, $(l_\infty,\|.,...,.\|)$ be the set of all sequences and all bounded sequences in n-normed space, respectively.

We need some new notations, which will be used throughout this chapter, by combining both of the definitions of lacunary sequence and Riesz mean:

Let $\theta=(k_r)$ be a lacunary sequence, (p_k) be a sequence of positive real numbers such that $H_r:=\sum_{k\in I_r} p_k$, $P_{k_r}:=\sum_{k\in(0,k_r]} p_k$, $P_{k_{r-1}}:=\sum_{k\in(0,k_{r-1}]} p_k$, $Q_r:=\dfrac{P_{k_r}}{P_{k_r-1}}$, $P_0=0$ and the intervals determined by θ and (p_k) are denoted by $I_r'=\left(P_{k_{r-1}},P_{k_r}\right]$. It is easy to see that $H_r = P_{k_r} - P_{k_{r-1}}$. If we take $p_k = 1$ for all $k \in \mathbb{N}$, then H_r, P_{k_r}, $P_{k_{r-1}}$, Q_r and I_r' reduce to h_r, k_r, k_{r-1}, q_r and I_r, respectively.

If $\theta = (k_r)$ is a lacunary sequence and $P_r \to \infty$ as $r \to \infty$, then $\theta' = (P_{k_r})$ is a lacunary sequence, that is, $P_0 = 0$, $0 < P_{k_{r-1}} < P_{k_r}$ and $H_r = P_{k_r} - P_{k_{r-1}} \to \infty$ as $r \to \infty$.

Throughout the paper, we take $P_r \to \infty$ as $r \to \infty$.

We define the following sets as follows:

$$F = \left\{ \begin{array}{l} x \in l_\infty \left(\|.,...,.\|\right) : \lim\limits_{k\to\infty} \|t_{km}(x - \xi e), z_1, ..., z_{n-1}\| = 0, \text{ uniformly in } m, \\ \text{for every nonzero } z_1, ..., z_{n-1} \in X \text{ and for some } \xi \end{array} \right\}$$

and

$$\left[F\right] = \left\{ \begin{array}{l} x \in l_\infty \left(\|.,...,.\|\right) : \lim\limits_{k\to\infty} t_{km} \left(\|x - \xi e, z_1, ..., z_{n-1}\|\right) = 0, \text{ uniformly in } m, \\ \text{for every nonzero } z_1, ..., z_{n-1} \in X \text{ and for some } \xi \end{array} \right\},$$

where $t_{km}(x)$ is defined as in (1.1.8). We write $F - \lim x = \xi$ if x is almost convergent to ξ in n-normed space and $\left[F\right] - \lim x = \xi$ if x is strongly almost convergent to ξ in n-normed space. Taking advantages of (iii) and (iv) conditions of 2-norm, it is easy to see that the inclusions $\left[F\right] \subset F \subset l_\infty \left(\|.,...,.\|\right)$ hold.

Now, we define some new sequence spaces in a real n-normed space as follows:

$$\left[\tilde{R}, p_r, \theta\right]_n = \left\{ \begin{array}{l} x : \lim\limits_{r\to\infty} \left\| \dfrac{1}{H_r} \sum\limits_{k \in I_r} p_k t_{km} \left(x - \xi e\right), z_1, ..., z_{n-1} \right\| = 0, \text{ uniformly in } m, \\ \text{for every nonzero } z_1, ..., z_{n-1} \in X \text{ and for some } \xi \end{array} \right\},$$

$$\left(\tilde{R},p_r,\theta\right)_n=\left\{\begin{array}{l}x:\lim\limits_{r\to\infty}\dfrac{1}{H_r}\sum\limits_{k\in I_r}p_k\left\|t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|=0,\text{ uniformly in }m,\\ \text{for every nonzero }z_1,...,z_{n-1}\in X\text{ and for some }\xi\end{array}\right\},$$

$$\left|\tilde{R},p_r,\theta\right|_n=\left\{\begin{array}{l}x:\lim\limits_{r\to\infty}\dfrac{1}{H_r}\sum\limits_{k\in I_r}p_k t_{km}\left(\left\|x-\xi e,z_1,...,z_{n-1}\right\|\right)=0,\text{ uniformly in }m,\\ \text{for every nonzero }z_1,...,z_{n-1}\in X\text{ and for some }\xi\end{array}\right\}.$$

The following results are obtained for some special cases:

i. If we take $m=0$, then the sequence spaces $\left[\tilde{R},p_r,\theta\right]_n$, $\left(\tilde{R},p_r,\theta\right)_n$, $\left|\tilde{R},p_r,\theta\right|_n$ reduce to the sequence spaces $\left[C_1,\theta\right]_n$, $\left(C_1,\theta\right)_n$, $\left|C_1,\theta\right|_n$, respectively as follows:

$$\left[C_1,\theta\right]_n=\left\{\begin{array}{l}x:\lim\limits_{r\to\infty}\left\|\dfrac{1}{H_r}\sum\limits_{k\in I_r}p_k t_{k0}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|=0,\\ \text{for every nonzero }z_1,...,z_{n-1}\in X\text{ and for some }\xi\end{array}\right\},$$

$$\left(C_1,\theta\right)_n=\left\{\begin{array}{l}x:\lim\limits_{r\to\infty}\dfrac{1}{H_r}\sum\limits_{k\in I_r}p_k\left\|t_{k0}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|=0,\\ \text{for every nonzero }z_1,...,z_{n-1}\in X\text{ and for some }\xi\end{array}\right\},$$

$$\left|C_1,\theta\right|_n=\left\{\begin{array}{l}x:\lim\limits_{r\to\infty}\dfrac{1}{H_r}\sum\limits_{k\in I_r}p_k t_{k0}\left(\left\|x-\xi e,z_1,...,z_{n-1}\right\|\right)=0,\text{ uniformly in }m,\\ \text{for every nonzero }z_1,...,z_{n-1}\in X\text{ and for some }\xi\end{array}\right\}.$$

ii. If we take $p_k=1$ for all $k\in\mathbb{N}$, then the sequence spaces above reduce to the following spaces:

$$\left[w_\theta\right]_n=\left\{\begin{array}{l}x:\lim\limits_{r\to\infty}\left\|\dfrac{1}{h_r}\sum\limits_{k\in I_r}p_k t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|=0,\text{ uniformly in }m,\\ \text{for every nonzero }z_1,...,z_{n-1}\in X\text{ and for some }\xi\end{array}\right\},$$

$$\left(w_{\theta}\right)_n=\left\{\begin{array}{l} x:\lim\limits_{r\to\infty}\dfrac{1}{h_r}\sum\limits_{k\in I_r}p_k\left\|t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|=0,\ \text{uniformly in } m, \\ \text{for every nonzero } z_1,...,z_{n-1}\in X \text{ and for some } \xi \end{array}\right\},$$

$$\left|w_{\theta}\right|_n=\left\{\begin{array}{l} x:\lim\limits_{r\to\infty}\dfrac{1}{h_r}\sum\limits_{k\in I_r}p_k t_{km}\left(\left\|x-\xi e,z_1,...,z_{n-1}\right\|\right)=0,\ \text{uniformly in } m, \\ \text{for every nonzero } z_1,...,z_{n-1}\in X \text{ and for some } \xi \end{array}\right\}.$$

iii. Let us choose $\theta=(k_r)=2^r$ for $r>0$, then these sequence spaces given above reduce to the following spaces:

$$\left[\tilde{R},p_r\right]_n=\left\{\begin{array}{l} x:\lim\limits_{r\to\infty}\left\|\dfrac{1}{P_r}\sum\limits_{k=1}^{r}p_k t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|=0,\ \text{uniformly in } m, \\ \text{for every nonzero } z_1,...,z_{n-1}\in X \text{ and for some } \xi \end{array}\right\},$$

$$\left(\tilde{R},p_r\right)_n=\left\{\begin{array}{l} x:\lim\limits_{r\to\infty}\dfrac{1}{P_r}\sum\limits_{k=1}^{r}p_k\left\|t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|=0,\ \text{uniformly in } m, \\ \text{for every nonzero } z_1,...,z_{n-1}\in X \text{ and for some } \xi \end{array}\right\},$$

$$\left|\tilde{R},p_r\right|_n=\left\{\begin{array}{l} x:\lim\limits_{r\to\infty}\dfrac{1}{P_r}\sum\limits_{k=1}^{r}p_k t_{km}\left\|x-\xi e,z_1,...,z_{n-1}\right\|=0,\ \text{uniformly in } m, \\ \text{for every nonzero } z_1,...,z_{n-1}\in X \text{ and for some } \xi \end{array}\right\}.$$

iv. If we select $\theta=(k_r)=2^r$ for $r>0$ and the base space as $\left(X,\|.,.\|\right)$ then these sequence spaces above reduce to the sequence spaces which can be seen in Subsection 3.2.

v. If we choose $p_k=1$ for all $k\in\mathbb{N}$ and $\theta=(k_r)=2^r$ for $r>0$, then these sequence spaces above reduce to the sequence spaces $\left[C_1\right],\left(C_1\right),\left|C_1\right|$.

Now, we give the following theorem to demonstrate some inclusion relations among the sequence spaces $[C_1,\theta]$, (C_1,θ), $|C_1,\theta|$, $\left[\tilde{R},p_r,\theta\right]_n$, $\left(\tilde{R},p_r,\theta\right)_n$, $\left|\tilde{R},p_r,\theta\right|_n$ with the spaces F and $[F]$.

Theorem 4.1.1. The following statements are true:

i. $[F]\subset\left|\tilde{R},p_r,\theta\right|_n\subset\left(\tilde{R},p_r,\theta\right)_n\subset\left[\tilde{R},p_r,\theta\right]_n\subset[C_1,\theta]_n$,

ii. $[F]\subset F\subset\left(\tilde{R},p_r,\theta\right)_n\subset\left[\tilde{R},p_r,\theta\right]_n\subset[C_1,\theta]_n$,

iii. $[F]\subset\left|\tilde{R},p_r,\theta\right|_n\subset|C_1,\theta|\subset(C_1,\theta)_n\subset[C_1,\theta]_n$.

Proof. We give the proof only for (i). The proofs of (ii) and (iii) can be done, similarly. So we omit them. Let $x\in[F]$ and $[F]$-$\lim x=\xi$. Then $t_{km}\left(\|x-\xi,z_1,...,z_{n-1}\|\right)\to 0$ as $k\to\infty$, uniformly in m, for every nonzero $z_1,...,z_{n-1}\in X$. Since $H_r\to\infty$ as $r\to\infty$, then its weighted lacunary mean also converges to ξ as $r\to\infty$, uniformly in m. This proves that $x\in\left|\tilde{R},p_r,\theta\right|_n$ and $[F]$-$\lim x=\left|\tilde{R},p_r,\theta\right|_n$-$\lim x=\xi$. Also since

$$\begin{aligned}\left\|\frac{1}{H_r}\sum_{k\in I_r} p_k t_{km}(x-\xi e),z_1,...,z_{n-1}\right\| &\le \frac{1}{H_r}\sum_{k\in I_r} p_k\left\|t_{km}(x-\xi e),z_1,...,z_{n-1}\right\| \\ &\le \frac{1}{H_r}\sum_{k\in I_r} p_k t_{km}\left(\|x-\xi e,z_1,...,z_{n-1}\|\right)\end{aligned}$$

then it follows that $[F]\subset\left|\tilde{R},p_r,\theta\right|_n\subset\left(\tilde{R},p_r,\theta\right)_n\subset\left[\tilde{R},p_r,\theta\right]_n$ and $[F]$-$\lim x=\left|\tilde{R},p_r,\theta\right|_n$-$\lim x=\left(\tilde{R},p_r,\theta\right)_n$-$\lim x=\left|\tilde{R},p_r,\theta\right|_n$-$\lim x=\xi$. Since uniform convergence of $\left\|\frac{1}{H_r}\sum_{k\in I_r} p_k t_{km}(x-\xi e),z_1,...,z_{n-1}\right\|$ for every nonzero $z_1,...,z_{n-1}\in X$ with respect to m as $r\to\infty$ implies convergence for $m=0$ it follows that

$\left[\tilde{R},p_r,\theta\right]_n \subset \left[C_1,\theta\right]_n$ and $\left[\tilde{R},p_r,\theta\right]_n$-$\lim x = \left[C_1,\theta\right]_n$-$\lim x = \xi$. This completes the proof.

Theorem 4.1.2. Let $\theta = (k_r)$ be a lacunary sequence and $\liminf_r Q_r > 1$. Then $\left(\tilde{R},p_r\right)_n \subseteq \left(\tilde{R},p_r,\theta\right)_n$ with $\left(\tilde{R},p_r\right)_n$-$\lim x = \left(\tilde{R},p_r,\theta\right)_n$-$\lim x = \xi$.

Proof. Suppose that $\liminf_r Q_r > 1$, then there exists a $\delta > 0$ such that $Q_r \geq 1+\delta$ for sufficiently large values of r, which implies that $\frac{H_r}{P_{k_r}} \geq \frac{\delta}{1+\delta}$. If $x \in \left(\tilde{R},p_r\right)_n$ with $\left(\tilde{R},p_r\right)_n$-$\lim x = \xi$, then for sufficiently large values of r, we have

$$
\begin{aligned}
&\frac{1}{P_{k_r}}\sum_{k=1}^{k_r} p_k \left\| t_{km}\left(x-\xi e\right), z_1,\ldots,z_{n-1} \right\| \\
&= \frac{1}{P_{k_r}}\left(\sum_{k=1}^{k_{r-1}} p_k \left\| t_{km}\left(x-\xi e\right), z_1,\ldots,z_{n-1} \right\| + \sum_{k=k_{r-1}+1}^{k_r} p_k \left\| t_{km}\left(x-\xi e\right), z_1,\ldots,z_{n-1} \right\| \right) \\
&\geq \frac{H_r}{P_{k_r}}\left(\frac{1}{H_r}\sum_{k\in I_r} p_k \left\| t_{km}\left(x-\xi e\right), z_1,\ldots,z_{n-1} \right\| \right) \\
&\geq \frac{\delta}{1+\delta}\cdot\frac{1}{H_r}\sum_{k\in I_r} p_k \left\| t_{km}\left(x-\xi e\right), z_1,\ldots,z_{n-1} \right\|
\end{aligned}
$$

for each $m \geq 0$ and for every nonzero $z_1,\ldots,z_{n-1} \in X$. Then, it follows that $x \in \left(\tilde{R},p_r,\theta\right)_n$ with $\left(\tilde{R},p_r,\theta\right)_n$-$\lim x = \xi$, by taking the limit as $r \to \infty$. This completes the proof.

Theorem 4.1.3. Let $\theta = (k_r)$ be a lacunary sequence with $\limsup_r Q_r < \infty$. Then $\left(\tilde{R},p_r,\theta\right)_n \subseteq \left(\tilde{R},p_r\right)_n$ with $\left(\tilde{R},p_r,\theta\right)_n$-$\lim x = \left(\tilde{R},p_r\right)_n$-$\lim x = \xi$.

Proof. Let $x \in \left(\tilde{R},p_r,\theta\right)_n$ with $\left(\tilde{R},p_r,\theta\right)_n$-$\lim x = \xi$. Then for $\varepsilon > 0$, there exists q_0 such that for every $q > q_0$

$$L_q = \frac{1}{H_q} \sum_{k \in I_q} p_k \left\| t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1} \right\| < \varepsilon \ , \quad \textbf{(4.1.1)}$$

for each $m \geq 0$ and for every nonzero $z_1, ..., z_{n-1} \in X$, that is, we can find some positive constant M such that

$$L_q \leq M \text{ for all } q . \quad \textbf{(4.1.2)}$$

$\lim \sup_r Q_r < \infty$ implies that there exist some positive number K such that

$$Q_r \leq K \text{ for all } r \geq 1 . \quad \textbf{(4.1.3)}$$

Therefore for $k_{r-1} < r \leq k_r$, we have by (4.1.1), (4.1.2) and (4.1.3)

$$\begin{aligned}
&\frac{1}{P_r} \sum_{k=1}^{r} p_k \left\| t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1} \right\| \\
&\leq \frac{1}{P_{k_{r-1}}} \sum_{k=1}^{k_r} p_k \left\| t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1} \right\| \\
&= \frac{1}{P_{k_{r-1}}} \left(\sum_{k \in I_1} p_k \left\| t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1} \right\| + \sum_{k \in I_2} p_k \left\| t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1} \right\| + ... \right. \\
&\quad \left. + \sum_{k \in I_{q_0}} p_k \left\| t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1} \right\| + ... + \sum_{k \in I_r} p_k \left\| t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1} \right\| \right) \\
&= \frac{1}{P_{k_{r-1}}} \left(L_1 H_1 + L_2 H_2 + ... + L_{q_0} H_{q_0} + L_{q_0+1} H_{q_0+1} + ... + L_r H_r \right) \\
&\leq \frac{M}{P_{k_{r-1}}} \left(H_1 + H_2 + ... + H_{q_0} \right) + \frac{\varepsilon}{P_{k_{r-1}}} \left(H_{q_0+1} + ... + H_r \right) \\
&= \frac{M}{P_{k_{r-1}}} \left(P_{k_1} - P_{k_0} + P_{k_2} - P_{k_1} + ... + P_{k_{q0}} - P_{k_{q0-1}} \right) + \frac{\varepsilon}{P_{k_{r-1}}} \left(P_{k_{q0+1}} - P_{k_{q0}} + ... + P_{k_r} - P_{k_{r-1}} \right) \\
&= M \frac{P_{k_{q0}}}{P_{k_{r-1}}} + \varepsilon \frac{P_{k_r} - P_{k_{q0}}}{P_{k_{r-1}}} \\
&\leq M \frac{P_{k_{q0}}}{P_{k_{r-1}}} + \varepsilon K ,
\end{aligned}$$

for each $m \geq 0$ and for every nonzero $z_1,...,z_{n-1} \in X$. Since $P_{k_{r-1}} \to \infty$ as $r \to \infty$, we get $x \in \left(\tilde{R}, p_r\right)_n$ with $\left(\tilde{R}, p_r\right)_n$-$\lim x = \xi$. This completes the proof.

Corollary 4.1.4. Let $1<\liminf_r Q_r \leq \limsup_r Q_r < \infty$. Then $\left(\tilde{R}, p_r, \theta\right)_n = \left(\tilde{R}, p_r\right)_n$ and $\left(\tilde{R}, p_r, \theta\right)_n$-$\lim x = \left(\tilde{R}, p_r\right)_n$-$\lim x = \xi$.

Proof. It follows from Theorem 4.1.2 and Theorem 4.1.3.

In the following theorem, we give the relations between the sequence spaces $\left(w_\theta\right)_n$ and $\left(\tilde{R}, p_r\right)_n$.

Theorem 4.1.5. *i*. If $p_k < 1$ for all $k \in \mathbb{N}$, then $\left(w_\theta\right)_n \subseteq \left(\tilde{R}, p_r\right)_n$ and $\left(w_\theta\right)_n$-$\lim x = \left(\tilde{R}, p_r\right)_n$-$\lim x = \xi.$

ii. If $p_k > 1$ for all $k \in \mathbb{N}$ and $\left(\frac{H_r}{h_r}\right)$ is upper-bounded, then $\left(\tilde{R}, p_r\right)_n \subseteq \left(w_\theta\right)_n$ and $\left(\tilde{R}, p_r\right)_n$-$\lim x = \left(w_\theta\right)_n$-$\lim x = \xi$.

Proof. *i*. If $p_k < 1$ for all $k \in \mathbb{N}$, then $H_r < h_r$ for all $r \in \mathbb{N}$. So, there exist an M_1, a constant, such that $0 < M_1 \leq \frac{H_r}{h_r} < 1$ for all $r \in \mathbb{N}$. Let $x \in \left(w_\theta\right)_n$ with $\left(w_\theta\right)_n$-$\lim x = \xi$, then for an arbitrary $\varepsilon > 0$ we have

$$\frac{1}{H_r}\sum_{k\in I_r} p_k \left\|t_{km}\left(x-\xi e\right), z_1,...,z_{n-1}\right\| \leq \frac{1}{M_1}\frac{1}{h_r}\sum_{k\in I_r}\left\|t_{km}\left(x-\xi e\right), z_1,...,z_{n-1}\right\|,$$

for each $m \geq 0$ and for every nonzero $z_1,...,z_{n-1} \in X$. Therefore, we get the result by taking the limit as $r \to \infty$.

ii. Let $p_k > 1$ for all $k \in \mathbb{N}$, then $H_r > h_r$ for all $r \in \mathbb{N}$. Suppose that $\left(\frac{H_r}{h_r}\right)$ is upper-bounded, then there exists an M_2 constant such that $1 < \frac{H_r}{h_r} \leq M_2 < \infty$ for all $r \in \mathbb{N}$. Let $x \in \left(\tilde{R}, p_r\right)_n$ and $\left(\tilde{R}, p_r\right)_n$-$\lim x = \xi$. So the result is obtained by taking the limit as $r \to \infty$ for each $m \geq 0$ and for every nonzero $z_1, ..., z_{n-1} \in X$, from the following inequality:

$$\frac{1}{h_r} \sum_{k \in I_r} \left\| t_{km} \left(x - \xi e \right), z_1, ..., z_{n-1} \right\| \leq M_2 \frac{1}{H_r} \sum_{k \in I_r} p_k \left\| t_{km} \left(x - \xi e \right), z_1, ..., z_{n-1} \right\|.$$

Now, we define a new concept of statistical convergence in n-normed space, which will be called weighted almost lacunary statistical convergence:

Definition 4.1.6. The weighted almost lacunary density of $K \subseteq \mathbb{N}$ is denoted by $\delta_{(\tilde{R},\theta)}(K) = \lim_{r \to \infty} \frac{1}{H_r} \left| K_r(\varepsilon) \right|$ if the limit exists. We say that the sequence $x = (x_j)$ is said to be weighted almost lacunary statistically convergent to ξ if for every $\varepsilon > 0$, the set $K_r(\varepsilon) = \left\{ k \in I_r' : p_k \left\| t_{km} \left(x - \xi e \right), z_1, ..., z_{n-1} \right\| \geq \varepsilon \right\}$ has weighted lacunary density zero, i.e.,

$$\lim_{r \to \infty} \frac{1}{H_r} \left| \left\{ k \in I_r' : p_k \left\| t_{km} \left(x - \xi e \right), z_1, ..., z_{n-1} \right\| \geq \varepsilon \right\} \right| = 0 \qquad \textbf{(4.1.4)}$$

uniformly in *m*, for every nonzero $z_1, ..., z_{n-1} \in X$. In this case, we write $\left(S_{(\tilde{R},\theta)}, n\right)$-$\lim x = \xi$. By $\left(S_{(\tilde{R},\theta)}, n\right)$ we denote the set of all weighted almost lacunary statistically convergent sequences in n-normed space.

i. If we take $p_k = 1$ for all $k \in \mathbb{N}$ in (4.1.4) then we obtain the definition of almost lacunary statistical convergence in n-normed space, that is, x is called almost lacunary statistically convergent to ξ if for every $\varepsilon > 0$, the set $K_\theta(\varepsilon) = \left\{k \in I_r : \left\| t_{km}(x - \xi e), z_1, ..., z_{n-1} \right\| \geq \varepsilon \right\}$ has lacunary density zero, i.e.

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| t_{km}(x - \xi e), z_1, ..., z_{n-1} \right\| \geq \varepsilon \right\} \right| = 0 \qquad \textbf{(4.1.5)}$$

uniformly in *m*, for every nonzero $z_1, ..., z_{n-1} \in X$. In this case, we write $(S_\theta, n) - \lim_j x_j = \xi$. By (S_θ, n) we denote the set of all weighted almost lacunary statistically convergent sequences in n-normed space.

ii. Let us choose $\theta = (k_r)$ for $r > 0$ then the definition of weighted almost lacunary statistical convergence which is given in (4.1.4) is reduced to the definition of weighted almost statistically convergence, that is, x is called weighted almost statistically convergent to ξ if for every $\varepsilon > 0$, the set

$$K_{P_r}(\varepsilon) = \left\{ k \leq P_r : p_k \left\| t_{km}(x - \xi e), z_1, ..., z_{n-1} \right\| \geq \varepsilon \right\}$$

has weighted density zero, i.e.,

$$\lim_{r\to\infty} \frac{1}{P_r} \left| \left\{ k \leq P_r : \left\| t_{km}(x - \xi e), z_1, ..., z_{n-1} \right\| \geq \varepsilon \right\} \right| = 0 \qquad \textbf{(4.1.6)}$$

uniformly in *m*, for every nonzero $z_1, ..., z_{n-1} \in X$. In this case, we write $(S_{\tilde{R}}, n)$-$\lim x = \xi$. By $(S_{\tilde{R}}, n)$ we denote the set of all weighted almost lacunary statistically convergent sequences in n-normed space.

iii. Let us choose $\theta = (k_r)$ for $r > 0$ and $p_k = 1$ for all $k \in \mathbb{N}$, then the definition of weighted almost lacunary statistical convergence which is given in (4.1.4) reduces to the definition of almost statistically convergence.

Theorem 4.1.7. If the sequence x is $\left(\tilde{R}, p_r, \theta\right)_n$-convergent to ξ then the sequence x is weighted almost lacunary statistically convergent to ξ.

Proof. Let the sequence x be $\left(\tilde{R}, p_r, \theta\right)_n$-convergent to ξ and

$$K_{rm}(\varepsilon) = \left\{k \in I_r' : p_k \left\|t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1}\right\| \geq \varepsilon\right\}.$$

Then for a given $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{H_r}\sum_{k \in I_r} p_k \left\|t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1}\right\| &\geq \frac{1}{H_r}\sum_{\substack{k \in I_r \\ k \in K_{rm}(\varepsilon)}} p_k \left\|t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1}\right\| \\ &\geq \varepsilon \frac{1}{H_r}\left|K_{rm}\left(\varepsilon\right)\right|,\end{aligned}$$

for each $m \geq 0$ and for every nonzero $z \in X$. Hence we obtain that the sequence x is weighted almost statistically convergent to ξ by taking the limit as $r \to \infty$.

Theorem 4.1.8. Let $p_k \left\|t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1}\right\| \leq M$ for all $k \in \mathbb{N}$, for each $m \geq 0$ and for every nonzero $z_1, ..., z_{n-1} \in X$. Then $\left(S_{(\tilde{R},\theta)}, n\right) \subset \left(\tilde{R}, p_r, \theta\right)_n$ with $\left(S_{(\tilde{R},\theta)}, n\right)$-$\lim x = \left(\tilde{R}, p_r, \theta\right)_n$-$\lim x = \xi$.

Proof. Let x be convergent to ξ in $\left(S_{(\tilde{R},\theta)}, n\right)$ and let us take

$$K_{rm}\left(\varepsilon\right) = \left\{k \in I_r' : p_k \left\|t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1}\right\| \geq \varepsilon\right\}.$$

Since $p_k \left\|t_{km}\left(x - \xi e\right), z_1, ..., z_{n-1}\right\| \leq M$ for all $k \in \mathbb{N}$, for each $m \geq 0$, for every nonzero $z_1, ..., z_{n-1} \in X$ and $H_r \to \infty$ as $r \to \infty$, then for a given $\varepsilon > 0$ we have

$$\frac{1}{H_r}\sum_{k\in I_r} p_k \left\|t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\| = \frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\in K_{rm(\varepsilon)}}} p_k \left\|t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|$$

$$+\frac{1}{H_r}\sum_{\substack{k\in I_r\\ k\notin K_{rm(\varepsilon)}}} p_k \left\|t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|$$

$$\leq M\frac{1}{H_r}\left|K_{rm(\varepsilon)}\right|+\frac{h_r}{H_r}\varepsilon$$

$$\leq M\frac{1}{H_r}\left|K_{rm(\varepsilon)}\right|+\varepsilon\ ,$$

for each $m\geq 0$ and for every nonzero $z_1,...,z_{n-1}\in X$. Since ε is arbitrary, we have $x\in\left(\tilde{R},p_r,\theta\right)_n$ by taking the limit as $r\to\infty$.

Theorem 4.1.9. The following statements are true:

i. If $p_k\leq 1$ for all $k\in\mathbb{N}$, then $\left(S_\theta,n\right)\subset\left(S_{\left(\tilde{R},\theta\right)},n\right)$.

ii. Let $p_k\geq 1$ for all $k\in\mathbb{N}$ and $\left(\dfrac{H_r}{h_r}\right)$ be upper-bounded, then $\left(S_{\left(\tilde{R},\theta\right)},n\right)\subset\left(S_\theta,n\right)$.

Proof. *i*. If $p_k\leq 1$ for all $k\in\mathbb{N}$, then $H_r\leq h_r$ for all $r\in\mathbb{N}$. So, there exist M_1 and M_2 constants such that $0<M_1\leq\dfrac{H_r}{h_r}\leq M_2\leq 1$ for all $r\in\mathbb{N}$. Let $x\in\left(S_\theta,n\right)$ with $\left(S_\theta,n\right)$-$\lim x=\xi$, then for an arbitrary $\varepsilon>0$ we have

$$\frac{1}{H_r}\left|\left\{k\in I_r':p_k\left\|t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|\geq\varepsilon\right\}\right|$$

$$=\frac{1}{H_r}\left|\left\{P_{k_{r-1}}<k\leq P_{k_r}:p_k\left\|t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|\geq\varepsilon\right\}\right|$$

$$\leq\frac{1}{M_1}\frac{1}{h_r}\left|\left\{P_{k_{r-1}}\leq k_{r-1}<k\leq P_{k_r}\leq k_r:\left\|t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|\geq\varepsilon\right\}\right|$$

$$=\frac{1}{M_1}\frac{1}{h_r}\left|\left\{k_{r-1}<k\leq k_r:\left\|t_{km}\left(x-\xi e\right),z_1,...,z_{n-1}\right\|\geq\varepsilon\right\}\right|$$

$$=\frac{1}{M_1}\frac{1}{h_r}\left|\left\{k\in I_r:\left\|t_{km}\left(x-\xi e\right),z_1,\ldots,z_{n-1}\right\|\geq\varepsilon\right\}\right| ,$$

for each $m\geq 0$ and for every nonzero $z_1,\ldots,z_{n-1}\in X$. Hence, we obtain the result by taking the limit as $r\to\infty$.

ii. Let $\left(\frac{H_r}{h_r}\right)$ be upper-bounded, then there exist M_1 and M_2 constants such that $1\leq M_1\leq\frac{H_r}{h_r}\leq M_2<\infty$ for all $r\in\mathbb{N}$. Suppose that $p_k\geq 1$ for all $k\in\mathbb{N}$, then $H_r\geq h_r$ for all $r\in\mathbb{N}$. Let $x\in\left(\tilde{R},p_r\right)_n$ and $\left(\tilde{R},p_r\right)_n$-$\lim x=\xi$, then for an arbitrary $\varepsilon>0$ we have

$$\begin{aligned}
&\frac{1}{h_r}\left|\left\{k\in I_r:\left\|t_{km}\left(x-\xi e\right),z_1,\ldots,z_{n-1}\right\|\geq\varepsilon\right\}\right|\\
&=\frac{1}{h_r}\left|\left\{k_{r-1}<k\leq k_r:\left\|t_{km}\left(x-\xi e\right),z_1,\ldots,z_{n-1}\right\|\geq\varepsilon\right\}\right|\\
&\leq M_2\frac{1}{H_r}\left|\left\{k_{r-1}\leq P_{k_{r-1}}<k\leq k_r\leq P_{k_r}:p_k\left\|t_{km}\left(x-\xi e\right),z_1,\ldots,z_{n-1}\right\|\geq\varepsilon\right\}\right|\\
&=M_2\frac{1}{H_r}\left|\left\{P_{k_{r-1}}<k\leq P_{k_r}:p_k\left\|t_{km}\left(x-\xi e\right),z_1,\ldots,z_{n-1}\right\|\geq\varepsilon\right\}\right|\\
&=M_2\frac{1}{H_r}\left|\left\{k\in I_r':p_k\left\|t_{km}\left(x-\xi e\right),z_1,\ldots,z_{n-1}\right\|\geq\varepsilon\right\}\right|
\end{aligned}$$

for each $m\geq 0$ and for every nonzero $z_1,\ldots,z_{n-1}\in X$. Hence, the result is obtained by taking the limit as $r\to\infty$.

Theorem 4.1.10. For any lacunary sequence θ, if $\liminf_r Q_r>1$ then $\left(S_{\tilde{R}},n\right)\subset\left(S_{(\tilde{R},\theta)},n\right)$ and $\left(S_{\tilde{R}},n\right)$-$\lim x=\left(S_{(\tilde{R},\theta)},n\right)$-$\lim x=\xi$.

Proof. Suppose that $\liminf_r Q_r>1$, then there exists a $\delta>0$ such that $Q_r\geq 1+\delta$

for sufficiently large values of r, which implies that $\frac{H_r}{P_{k_r}} \geq \frac{\delta}{1+\delta}$. If $x \in \left(S_{\tilde{R}}, n\right)$ with $\left(S_{\tilde{R}}, n\right)$-$\lim x = \xi$, then for every $\varepsilon > 0$ and for sufficiently large values of r, we have

$$\begin{aligned}
&\frac{1}{P_{k_r}}\left|\left\{k \leq P_{k_r} : p_k \left\|t_{km}\left(x-\xi e\right), z_1, \ldots, z_{n-1}\right\| \geq \varepsilon\right\}\right| \\
&\geq \frac{1}{P_{k_r}}\left|\left\{P_{k_{r-1}} < k \leq P_{k_r} : p_k \left\|t_{km}\left(x-\xi e\right), z_1, \ldots, z_{n-1}\right\| \geq \varepsilon\right\}\right| \\
&= \frac{H_r}{P_{k_r}}\left(\frac{1}{H_r}\left|\left\{P_{k_{r-1}} < k \leq P_{k_r} : p_k \left\|t_{km}\left(x-\xi e\right), z_1, \ldots, z_{n-1}\right\| \geq \varepsilon\right\}\right|\right) \\
&\geq \frac{\delta}{1+\delta}\left(\frac{1}{H_r}\left|\left\{k \in I_r' : p_k \left\|t_{km}\left(x-\xi e\right), z_1, \ldots, z_{n-1}\right\| \geq \varepsilon\right\}\right|\right),
\end{aligned}$$

for each $m \geq 0$ and for every nonzero $z_1, \ldots, z_{n-1} \in X$. Hence, we get the result by taking the limit as $r \to \infty$.

Theorem 4.1.11. Let $\theta = (k_r)$ be a lacunary sequence with $\limsup_r Q_r < \infty$, then $\left(S_{(\tilde{R},\theta)}, n\right) \subset \left(S_{\tilde{R}}, n\right)$ and $\left(S_{\tilde{R}}, n\right)$-$\lim x = \left(S_{(\tilde{R},\theta)}, n\right)$-$\lim x = \xi$.

Proof. If $\limsup_r Q_r < \infty$, then there is a $K > 0$ such that $Q_r \leq K$ for all $r \in \mathbb{N}$. Suppose that $x \in \left(S_{(\tilde{R},\theta)}, n\right)$ with $\left(S_{(\tilde{R},\theta)}, n\right)$-$\lim x = \xi$ and let

$$N_r := \left|\left\{k \in I_r' : p_k \left\|t_{km}\left(x-\xi e\right), z_1, \ldots, z_{n-1}\right\| \geq \varepsilon\right\}\right|. \qquad \textbf{(4.1.7)}$$

By (4.1.7), given $\varepsilon > 0$, there is a $r_0 \in \mathbb{N}$ such that $\frac{N_r}{H_r} < \varepsilon$ for all $r > r_0$. Now, let $M := \max\left\{N_r : 1 \leq r \leq r_0\right\}$ and let r be any integer satisfying $k_{r-1} < r \leq k_r$, then we can write

$$\frac{1}{P_r}\left|\left\{k \le P_r : p_k \left\|t_{km}\left(x-\xi e\right), z_1, \ldots, z_{n-1}\right\| \ge \varepsilon\right\}\right|$$
$$\le \frac{1}{P_{k_{r-1}}}\left|\left\{P_{k_{r-1}} < k \le P_{k_r} : p_k \left\|t_{km}\left(x-\xi e\right), z_1, \ldots, z_{n-1}\right\| \ge \varepsilon\right\}\right|$$
$$= \frac{1}{P_{k_{r-1}}}\left(N_1 + N_2 + \ldots + N_{r_0} + N_{r_0+1} + \ldots + N_r\right)$$
$$\le \frac{M.r_0}{P_{k_{r-1}}} + \frac{1}{P_{k_{r-1}}}\varepsilon\left(H_{r_0+1} + \ldots + H_r\right)$$
$$= \frac{M.r_0}{P_{k_{r-1}}} + \varepsilon\frac{\left(P_{k_r} - P_{k_{r_0}}\right)}{P_{k_{r-1}}}$$
$$\le \frac{M.r_0}{P_{k_{r-1}}} + \varepsilon Q_r \le \frac{M.r_0}{P_{k_{r-1}}} + \varepsilon K$$

which completes the proof by taking the limit as $r \to \infty$.

Corollary 4.1.12. Let $1 < \liminf_r Q_r \le \limsup_r Q_r < \infty$. Then $\left(S_{(\tilde{R},\theta)}, n\right) = \left(S_{\tilde{R}}, n\right)$ and $\left(S_{\tilde{R}}, n\right)$-$\lim x = \left(S_{(\tilde{R},\theta)}, n\right)$-$\lim x = \xi$.

Proof. It follows from Theorem 4.1.10 and Theorem 4.1.11.

4.2. Generalized Difference Sequence Spaces Associated with Multiplier Sequence on a Real n-Normed Space

In this section, some new sequence spaces associated with multiplier sequence by using an infinite matrix, an Orlicz function and generalized B-difference operator on a real n-normed space are introduced. Some topological properties of these spaces are examined. A new concept which will be called $(B_\Lambda^\mu)^n$-statistical A-convergence in an n-normed space, is defined and some inclusion connections between the sequence space $W\left(A, B_\Lambda^\mu, p, \|.,\ldots,.\|\right)$ and the set of all $(B_\Lambda^\mu)^n$-statistical A-convergent sequences are established.

Let $A=(a_{mk})$ be an infinite matrix of non-negative real numbers, let $p=(p_k)$ be a bounded sequence of positive real numbers for all $k\in\mathbb{N}$ and $\Lambda=(\Lambda_k)$ be a sequence of nonzero scalars. Further, let M be an Orlicz function and $(X,\|.,...,.\|)$ be an n-normed space. We denote the space of all X-valued sequence spaces by $w(\|.,...,.\|)$ and $x=(x_k)\in w(\|.,...,.\|)$ by $x=(x_k)$ for brevity. We define the following sequence spaces:

$$W_0\left(A,B_\Lambda^\mu,M,p,\|.,...,.\|\right)=\left\{\begin{array}{l} x=(x_k):\lim\limits_{m\to\infty}\sum\limits_{k=1}^{\infty}a_{mk}\left[M\left(\left\|\dfrac{B_\Lambda^\mu x_k}{\rho},z_1,...,z_{n-1}\right\|\right)\right]^{p_k}=0, \\ \text{for every nonzero } z_1,...,z_{n-1}\in X \end{array}\right\},$$

$$W\left(A,B_\Lambda^\mu,M,p,\|.,...,.\|\right)=\left\{\begin{array}{l} x=(x_k):\lim\limits_{m\to\infty}\sum\limits_{k=1}^{\infty}a_{mk}\left[M\left(\left\|\dfrac{B_\Lambda^\mu x_k-\xi}{\rho},z_1,...,z_{n-1}\right\|\right)\right]^{p_k}=0 \\ \text{for every nonzero } z_1,...,z_{n-1}\in X \text{ and for some } \xi \end{array}\right\},$$

$$W_\infty\left(A,B_\Lambda^\mu,M,p,\|.,...,.\|\right)=\left\{\begin{array}{l} x=(x_k):\sup\limits_{m}\sum\limits_{k=1}^{\infty}a_{mk}\left[M\left(\left\|\dfrac{B_\Lambda^\mu x_k}{\rho},z_1,...,z_{n-1}\right\|\right)\right]^{p_k}<\infty, \\ \text{for every nonzero } z_1,...,z_{n-1}\in X \end{array}\right\},$$

where $B_\Lambda^\mu x_k=\sum\limits_{v=0}^{\mu}\binom{\mu}{v}r^{\mu-v}s^v x_{k-v}\Lambda_{k-v}$ and $\mu,k\in\mathbb{N}$. If we consider some special cases of the spaces above, the followings are obtained:

i. If we take $\mu=0$, then the spaces above reduce to $W(A,\Lambda,M,p,\|.,...,.\|)$, $W_0(A,\Lambda,M,p,\|.,...,.\|)$, $W_\infty(A,\Lambda,M,p,\|.,...,.\|)$, respectively.

ii. If we take $r=1$, $s=-1$ then the spaces above reduce to the spaces $W(A,\Delta_\Lambda^\mu,M,p,\|.,...,.\|)$, $W_0(A,\Delta_\Lambda^\mu,M,p,\|.,...,.\|)$, $W_\infty(A,\Delta_\Lambda^\mu,M,p,\|.,...,.\|)$.

iii. If $M(x)=x$ then the above spaces are denoted by $W\left(A,B_{\Lambda}^{\mu},p,\|.,\ldots,.\|\right)$, $W_0\left(A,B_{\Lambda}^{\mu},p,\|.,\ldots,.\|\right)$, $W_{\infty}\left(A,B_{\Lambda}^{\mu},p,\|.,\ldots,.\|\right)$, respectively.

iv. If $p_k=1$ for all $k\in\mathbb{N}$ and $\Lambda=(\Lambda_k)=(1,1,1,\ldots)$ then the spaces above are denoted by $W\left(A,B^{\mu},M,\|.,\ldots,.\|\right)$, $W_0\left(A,B^{\mu},M,\|.,\ldots,.\|\right)$, $W_{\infty}\left(A,B^{\mu},M,\|.,\ldots,.\|\right)$, respectively.

v. If $M(x)=x$ and $p_k=1$ for all $k\in\mathbb{N}$, then the spaces above are denoted by $W\left(A,B_{\Lambda}^{\mu},\|.,\ldots,.\|\right)$, $W_0\left(A,B_{\Lambda}^{\mu},\|.,\ldots,.\|\right)$, $W_{\infty}\left(A,B_{\Lambda}^{\mu},\|.,\ldots,.\|\right)$, respectively.

vi. If we take $A=C_1$, i.e., the Cesaro matrix, then the spaces above reduce to the spaces $W\left(B_{\Lambda}^{\mu},M,p,\|.,\ldots,.\|\right)$, $W_0\left(B_{\Lambda}^{\mu},M,p,\|.,\ldots,.\|\right)$, $W_{\infty}\left(B_{\Lambda}^{\mu},M,p,\|.,\ldots,.\|\right)$.

vii. If we take $A=(a_{mk})$ is de la Valée Poussin mean, i.e.,

$$a_{mk}=\begin{cases}\dfrac{1}{\lambda_m}, & k\in I_m=\left[m-\lambda_m+1,m\right]\\ 0, & \text{otherwise,}\end{cases} \tag{4.2.1}$$

where λ_m is a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{m+1}\le\lambda_m+1$, $\lambda_1=1$, then the spaces above are denoted by $W_0\left(\lambda,B_{\Lambda}^{\mu},M,p,\|.,\ldots,.\|\right)$, $W\left(\lambda,B_{\Lambda}^{\mu},M,p,\|.,\ldots,.\|\right)$, $W_{\infty}\left(\lambda,B_{\Lambda}^{\mu},M,p,\|.,\ldots,.\|\right)$.

viii. By a lacunary sequence $\theta=(k_m)$, $m=0,1,..$ where $k_0=0$, we mean an increasing sequence of non-negative integers with $h_m=(k_m-k_{m-1})\to\infty$ as $m\to\infty$. The intervals determined by θ are denoted by $I_m=\left(k_{m-1},k_m\right]$. Let

$$a_{mk} = \begin{cases} \dfrac{1}{h_m}, & k_{m-1} < k \le k_m \\ 0, & \text{otherwise.} \end{cases} \qquad \textbf{(4.2.2)}$$

Then we obtain the spaces $W(\theta, B_\Lambda^\mu, M, p, \|.,...,.\|)$, $W_0(\theta, B_\Lambda^\mu, M, p, \|.,...,.\|)$ and $W_\infty(\theta, B_\Lambda^\mu, M, p, \|.,...,.\|)$, respectively.

ix. If we take $A = (a_{mk})$ is Nörlund mean, i.e.,

$$a_{mk} = \begin{cases} \dfrac{p_{m-k}}{P_m}, & 0 < k \le m \\ 0, & k > m \end{cases} \qquad \textbf{(4.2.3)}$$

where (p_k) is a sequence of positive real numbers and $P_m = p_1 + p_2 + ... + p_m$, then the spaces above are denoted by $W(\bar{N}, B_\Lambda^\mu, M, p, \|.,...,.\|)$, $W_0(\bar{N}, B_\Lambda^\mu, M, p, \|.,...,.\|)$ and $W_\infty(\bar{N}, B_\Lambda^\mu, M, p, \|.,...,.\|)$, respectively.

x. Let the matrix $A = (a_{mk})$ be Riesz mean, i.e.,

$$a_{mk} = \begin{cases} \dfrac{p_k}{P_m}, & 0 < k \le m \\ 0, & k > m \end{cases} \qquad \textbf{(4.2.4)}$$

where (p_k) is a sequence of positive real numbers and $P_m = p_1 + p_2 + ... + p_m$, then we obtain the sequence spaces $W(R, B_\Lambda^\mu, M, p, \|.,...,.\|)$, $W_0(R, B_\Lambda^\mu, M, p, \|.,...,.\|)$ and $W_\infty(R, B_\Lambda^\mu, M, p, \|.,...,.\|)$, respectively.

xi. If we take $A = I$, where I is an identity matrix and $p_k = 1$ for all then the spaces above reduce to the sequence spaces $c(B_\Lambda^\mu, M, \|.,...,.\|)$, $c_0(B_\Lambda^\mu, M, \|.,...,.\|)$ and

$l_\infty\left(B_\Lambda^\mu, M, \|.,...,.\|\right)$, respectively.

xii. If we take $A = I$, where I is an identity matrix, $M(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$ then we denote the spaces above by the sequence spaces $c\left(B_\Lambda^\mu, \|.,...,.\|\right)$, $c_0\left(B_\Lambda^\mu, \|.,...,.\|\right)$ and $l_\infty\left(B_\Lambda^\mu, \|.,...,.\|\right)$.

Theorem 4.2.1. $W_0\left(A, B_\Lambda^\mu, M, p, \|.,...,.\|\right)$, $W\left(A, B_\Lambda^\mu, M, p, \|.,...,.\|\right)$ and $W_\infty\left(A, B_\Lambda^\mu, M, p, \|.,...,.\|\right)$ are linear spaces.

Proof. We consider only $W\left(A, B_\Lambda^\mu, M, p, \|.,...,.\|\right)$. Others can be treated similarly. Let $x, y \in W\left(A, B_\Lambda^\mu, M, p, \|.,...,.\|\right)$ and α, β be scalars, suppose that $x \to \xi_1$ and $y \to \xi_2$. Then there exists $|\alpha|\rho_1 + |\beta|\rho_2 > 0$ such that,

$$\sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_\Lambda^\mu(\alpha x_k + \beta y_k) - (\alpha\xi_1 + \beta\xi_2)}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_k}$$

$$\le \sum_{k=1}^{\infty} a_{mk}\left[M\left(\frac{|\alpha|\rho_1}{|\alpha|\rho_1 + |\beta|\rho_2}\left\|\frac{B_\Lambda^\mu x_k - \xi_1}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right.\right.$$

$$\left.\left. + \frac{|\beta|\rho_2}{|\alpha|\rho_1 + |\beta|\rho_2}\left\|\frac{B_\Lambda^\mu y_k - \xi_2}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_k}$$

$$\le \sum_{k=1}^{\infty} a_{mk}\left[\frac{|\alpha|\rho_1}{|\alpha|\rho_1 + |\beta|\rho_2} M\left(\left\|\frac{B_\Lambda^\mu x_k - \xi_1}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right.$$

$$\left. + \frac{|\beta|\rho_2}{|\alpha|\rho_1 + |\beta|\rho_2} M\left(\left\|\frac{B_\Lambda^\mu y_k - \xi_2}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_k}$$

$$\le D\sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_\Lambda^\mu x_k - \xi_1}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_k}$$

$$+ D\sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_\Lambda^\mu y_k - \xi_2}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_k},$$

which leads us by taking limit as $m \to \infty$ that we get $(\alpha x + \beta y) \in W\left(A, B_\Lambda^\mu, M, p, \|.,...,.\|\right)$.

Theorem 4.2.2. For any two sequences $p = (p_k)$ and $q = (q_k)$ of positive real numbers and for any two n-norms $\|.,...,.\|_1$, $\|.,...,.\|_2$ on X, the following holds:

$Z(A, B_\Lambda^\mu, M, p, \|.,\ldots,.\|_1) \cap Z(A, B_\Lambda^\mu, M, q, \|.,\ldots,.\|_2) \neq \emptyset$, where $Z = W$, W_0 and W_∞.

Proof. Since the zero element belongs to each of the above classes of sequences, thus the intersection is non-empty.

Theorem 4.2.3. Let $A = (a_{mk})$ be a non-negative matrix and $p = (p_k)$ be a bounded sequence of positive real numbers. Then for any fixed $m \in \mathbb{N}$ the sequence space $W_\infty\left(A, B_\Lambda^\mu, M, p, \|.,...,.\|\right)$ is a paranormed space with respect to the paranorm defined by

$$g_m(x) = \inf \left\{ \begin{array}{l} \rho^{\frac{p_m}{H}} : \left(\sum_{k=1}^{\infty} a_{mk} \left[M\left(\left\| \frac{B_\Lambda^\mu x_k}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} \right)^{\frac{1}{H}} < \infty, \\ \text{for every nonzero } z_1, ..., z_{n-1} \in X \text{ and for some } \rho > 0 \end{array} \right\}.$$

Proof. That $g_m(\theta) = 0$ and $g_m(-x) = g_m(x)$ are easy to prove. So, we omit them. Let us take $x = (x_k)$ and $y = (y_k)$ in $W_\infty\left(A, B_\Lambda^\mu, M, p, \|.,...,.\|\right)$. Let

$$A(x) = \left\{ \rho > 0 : \sum_{k=1}^{\infty} a_{mk} \left[M\left(\left\| \frac{B_\Lambda^\mu x_k}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} < \infty \right\},$$

$$A(y) = \left\{ \rho > 0 : \sum_{k=1}^{\infty} a_{mk} \left[M\left(\left\| \frac{B_\Lambda^\mu y_k}{\rho}, z_1, ..., z_{n-1} \right\| \right) \right]^{p_k} < \infty \right\},$$

for every nonzero $z_1,...,z_{n-1} \in X$. Let $\rho_1 \in A(x)$ and $\rho_2 \in A(y)$, then we have

$$\left(\sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_{\Lambda}^{\mu}\left(x_k+y_k\right)}{\left(\rho_1+\rho_2\right)}, z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} < \infty$$

by using Minkowski's inequality for $p=(p_k)>1$. Thus,

$$\begin{aligned} g_m(x+y) &= \inf\left\{\left(\rho_1+\rho_2\right)^{\frac{p_m}{H}} : \rho_1 \in A(x), \rho_2 \in A(y)\right\} \\ &\leq \inf\left\{\rho_1^{\frac{p_m}{H}} : \rho_1 \in A(x)\right\} + \inf\left\{\rho_2^{\frac{p_m}{H}} : \rho_2 \in A(y)\right\} \\ &= g_m(x)+g_m(y). \end{aligned}$$

We also get $g_m(x+y) \leq g_m(x)+g_m(y)$ for $0 < p_k \leq 1$, by Lemma 1.1.46. Hence, we complete the proof of this condition of paranorm. Finally, we show that the scalar multiplication is continuous. Whenever $\alpha \to 0$ and x is fixed imply $g_m(\alpha x) \to 0$. Also, whenever $x \to \theta$ and α is any number imply $g_m(\alpha x) \to 0$. By using the definition of the paranorm, for every nonzero $z_1,...,z_{n-1} \in X$ we have

$$g_m(\alpha x) = \inf\left\{\rho^{\frac{p_m}{H}} : \left(\sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_{\Lambda}^{\mu}\left(\alpha x_k\right)}{\rho}, z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} < \infty\right\}.$$

Then

$$g_m(\alpha x) = \inf\left\{\left(\alpha\sigma\right)^{\frac{p_m}{H}} : \left(\sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_{\Lambda}^{\mu} x_k}{\sigma}, z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} < \infty\right\},$$

where $\sigma = \dfrac{\rho}{\alpha}$. Since $|\alpha|^{p_k} \le \max\left\{|\alpha|^h, |\alpha|^H\right\}$ therefore $|\alpha|^{\frac{p_k}{H}} \le \left(\max\left\{|\alpha|^h, |\alpha|^H\right\}\right)^{\frac{1}{H}}$.

Then the required proof follows from the following inequality:

$$
\begin{aligned}
g_m(\alpha x) &\le \left(\max\left\{|\alpha|^h, |\alpha|^H\right\}\right)^{\frac{1}{H}} \\
&\quad .\inf\left\{\sigma^{\frac{p_m}{H}} : \left(\sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_\Lambda^\mu x_k}{\sigma}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} < \infty\right\} \\
&= \left(\max\left\{|\alpha|^h, |\alpha|^H\right\}\right)^{\frac{1}{H}} g_m(x).
\end{aligned}
$$

Theorem 4.2.4. Let M, M_1, M_2 be Orlicz functions. Then the followings hold.

i. Let $0 < h \le p_k \le 1$. Then $Z(A, B_\Lambda^\mu, M, p, \|\cdot,\ldots,\cdot\|) \subseteq Z(A, B_\Lambda^\mu, M, \|\cdot,\ldots,\cdot\|)$ where $Z = W, W_0$.

ii. Let $1 < p_k \le H < \infty$. Then $Z(A, B_\Lambda^\mu, M, \|\cdot,\ldots,\cdot\|) \subseteq Z(A, B_\Lambda^\mu, M, p, \|\cdot,\ldots,\cdot\|)$ where $Z = W, W_0$.

iii. $W_0(A, B_\Lambda^\mu, M_1, p, \|.,\ldots,.\|) \cap W_0(A, B_\Lambda^\mu, M_2, p, \|.,\ldots,.\|) \subseteq W_0(A, B_\Lambda^\mu, M_1 + M_2, p, \|.,\ldots,.\|)$.

Proof. i. We give the proof for the sequence space $W_0(A, B_\Lambda^\mu, M, p, \|.,\ldots,.\|)$ only. The other can be proved by a similar argument. Let $(x_k) \in W_0(A, B_\Lambda^\mu, M, p, \|\cdot,\ldots,\cdot\|)$ and $0 < h \le p_k \le 1$, then

$$
\sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_\Lambda^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right] \le \sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_\Lambda^\mu x_k}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_k}.
$$

Hence, we have the result by taking the limit as $m \to \infty$. This completes the proof.

ii. Let $1 < p_k \le H < \infty$ and $(x_k) \in W_0(A, B_\Lambda^\mu, M, \|\cdot,\ldots,\cdot\|)$. Then for each $0 < \varepsilon < 1$

there exists a positive integer M_0 such that

$$\sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_{\Lambda}^{\mu}x_k}{\rho},z_1,...,z_{n-1}\right\|\right)\right]<\varepsilon<1,$$

for all $m>M_0$. This implies that

$$\sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_{\Lambda}^{\mu}x_k}{\rho},z_1,...,z_{n-1}\right\|\right)\right]^{p_k}\leq\sum_{k=1}^{\infty} a_{mk}\left[M\left(\left\|\frac{B_{\Lambda}^{\mu}x_k}{\rho},z_1,...,z_{n-1}\right\|\right)\right].$$

Hence we have the result.

iii. Let $x=(x_k)\in W_0\left(A,B_{\Lambda}^{\mu},M_1,p,\|.,...,.\|\right)\cap W_0\left(A,B_{\Lambda}^{\mu},M_2,p,\|.,...,.\|\right)$. Then by the following inequality the result follows:

$$\sum_{k=1}^{\infty} a_{mk}\left[(M_1+M_2)\left(\left\|\frac{B_{\Lambda}^{\mu}x_k}{\rho},z_1,...,z_{n-1}\right\|\right)\right]^{p_k}$$

$$\leq D\sum_{k=1}^{\infty} a_{mk}\left[M_1\left(\left\|\frac{B_{\Lambda}^{\mu}x_k}{\rho},z_1,...,z_{n-1}\right\|\right)\right]^{p_k}$$

$$+D\sum_{k=1}^{\infty} a_{mk}\left[M_2\left(\left\|\frac{B_{\Lambda}^{\mu}x_k}{\rho},z_1,...,z_{n-1}\right\|\right)\right]^{p_k}.$$

If we take the limit as $m\to\infty$ then we get $(x_k)\in W_0(A,B_{\Lambda}^{\mu},M_1+M_2,p,\|.,\ldots,.\|)$. This completes the proof.

Theorem 4.2.5. $Z(A,B_{\Lambda}^{\mu-1},M,p,\|.,\ldots,.\|)\subset Z(A,B_{\Lambda}^{\mu},M,p,\|.,\ldots,.\|)$ and the inclusion is strict for $\mu\geq 1$. In general $Z(A,B_{\Lambda}^{j},M,p,\|.,\ldots,.\|_1)\subset Z(A,B_{\Lambda}^{\mu},M,p,\|.,\ldots,.\|)$ for $j=0,1,2,...,\mu-1$ and the inclusions are strict, where $Z=W$, W_0 and W_∞.

Proof. We give the proof for $W_0(A,B_\Lambda^{\mu-1},M,p,\|.,\ldots,.\|)$ only. The others can be proved by a similar argument. Let $x=(x_k)$ be any element in the space $W_0(A,B_\Lambda^{\mu-1},M,p,\|.,\ldots,.\|)$ then there exists $\rho=|r|\rho_1+|s|\rho_2>0$ such that

$$\lim_{m\to\infty}\sum_{k=1}^{\infty}a_{mk}\left[M\left(\left\|\frac{B_\Lambda^{\mu-1}x_k}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}=0.$$

Since M is non-decreasing and convex, it follows that

$$\begin{aligned}
&\sum_{k=1}^{\infty}a_{mk}\left[M\left(\left\|\frac{B_\Lambda^{\mu}x_k}{|r|\rho_1+|s|\rho_2},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\\
&=\sum_{k=1}^{\infty}a_{mk}\left[M\left(\left\|\frac{rB_\Lambda^{\mu-1}x_k+sB_\Lambda^{\mu-1}x_{k-1}}{|r|\rho_1+|s|\rho_2},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\\
&\le D\sum_{k=1}^{\infty}a_{mk}\left[M\left(\left\|\frac{B_\Lambda^{\mu-1}x_k}{\rho_1},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}\\
&+D\sum_{k=1}^{\infty}a_{mk}\left[M\left(\left\|\frac{B_\Lambda^{\mu-1}x_{k-1}}{\rho_2},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_k}.
\end{aligned}$$

The result holds by taking the limit as $m\to\infty$.

In the following example we show that the inclusion given in the theorem above is strict.

Example 4.2.6. Let $M(x)=x$, $p_k=1$ for all $k\in\mathbb{N}$, $\Lambda=(\Lambda_k)=(1,1,\ldots)$, $A=C_1$, i.e., the Cesaro matrix, $r=1,s=-1$ where $B_\Lambda^{\mu}x_k=\sum_{v=0}^{\mu}\binom{\mu}{v}r^{\mu-v}s^{v}x_{k-v}\Lambda_{k-v}$ for all $r,s\in\mathbb{R}-\{0\}$. Consider the sequence $x=(x_k)=(k^{\mu-1})$. Then $x=(x_k)$ belongs to $W_0(B^{\mu},M,p,\|.,\ldots,.\|)$ but does not belong to $W_0(B^{\mu-2},M,p,\|.,\ldots,.\|)$.

Theorem 4.2.7. Let $A=(a_{mk})$ be a non-negative regular matrix and $p=(p_k)$ be such that $0<h\le p_k\le H<\infty$. Then $l_\infty(B_\Lambda^\mu, M, \|.,...,.\|)\subseteq W_\infty(A, B_\Lambda^\mu, M, p, \|.,...,.\|)$.

Proof. Let $l_\infty\left(B_\Lambda^\mu, M, \|.,...,.\|\right)$. Then there exists $T_0>0$ such that

$$\left[M\left(\left\|\frac{B_\Lambda^\mu x_k}{\rho}, z_1,..., z_{n-1}\right\|\right)\right]\le T_0$$

for all $k\in\mathbb{N}$ and for every nonzero $z_1,..., z_{n-1}\in X$. Since $A=(a_{mk})$ is a non-negative regular matrix, we have the following inequality by the case (i) of Silverman-Toeplitz conditions.

$$\sup_m\sum_{k=1}^{\infty}a_{mk}\left[M\left(\left\|\frac{B_\Lambda^\mu x_k}{\rho}, z_1,..., z_{n-1}\right\|\right)\right]^{p_k}\le\max\left\{T_0^h, T_0^H\right\}\sup_m\sum_{k=1}^{\infty}a_{mk}<\infty .$$

Hence $l_\infty\left(B_\Lambda^\mu, M, \|.,...,.\|\right)\subseteq W_\infty\left(A, B_\Lambda^\mu, M, p, \|.,...,.\|\right)$.

Now, we introduce and study a new concept of $(B_\Lambda^\mu)^n$-statistical A-convergence in an n-normed space as follows:

Definition 4.2.8. Let $\left(X, \|.,...,.\|\right)$ be an n-normed space and let $A=(a_{mk})$ be a non-negative regular matrix. A real sequence $x=(x_k)$ is said to be $(B_\Lambda^\mu)^n$-statistically A-convergent to a number ξ, if $\delta_{A\left(B_\Lambda^\mu\right)^n}(K)=\lim_{m\to\infty}\sum_{k=1}^{\infty}a_{mk}\chi_K(k)=0$ or equivalently $\lim_{m\to\infty}\sum_{k\in K}a_{mk}=0$ for each $\varepsilon>0$ and for every nonzero $z_1,..., z_{n-1}\in X$ where $K=\{k\in\mathbb{N}:\left\|B_\Lambda^\mu x_k-L, z_1,..., z_{n-1}\right\|\geqslant\varepsilon\}$ and χ_K is the characteristic function of K.
In this case we write $(B_\Lambda^\mu)^n$stat-A-$\lim x=\xi$. $S\left(A(B_\Lambda^\mu)^n\right)$ denotes the set of all $(B_\Lambda^\mu)^n$-statistically A-convergent sequences.

If we consider some special cases of the matrix, then we have the following:

i. If $A = C_1$, the Cesaro matrix, then the definition reduces to $(B_\Lambda^\mu)^n$-statistical convergence.

ii. If $A = (a_{mk})$ is de la Vallee Poussin mean which is given by (4.2.1) then the definition reduces to $(B_\Lambda^\mu)^n$-statistical λ-convergence.

iii. If we take $A = (a_{mk})$ as in (4.2.2), then the definition reduces to $(B_\Lambda^\mu)^n$-statistical lacunary convergence.

iv. If we take $A = (a_{mk})$ as in (4.2.3), then the definition reduces to $(B_\Lambda^\mu)^n$-Nörlund statistical convergence.

v. If we take $A = (a_{mk})$ as in (4.2.4), then the definition reduces to $(B_\Lambda^\mu)^n$-Riesz statistical convergence.

Theorem 4.2.9. Let $p = (p_k)$ be a sequence of non-negative bounded real numbers such that $\inf_k p_k > 0$. Then $W\left(A, B_\Lambda^\mu, p, \|.,...,.\|\right) \subset S\left(A(B_\Lambda^\mu)^n\right)$.

Proof. Assume that $x = (x_k) \in W\left(A, B_\Lambda^\mu, p, \|.,...,.\|\right)$. So we have for every nonzero $z_1,..., z_{n-1} \in X$

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} a_{mk} \left\|B_\Lambda^\mu x_k - L, z_1, ..., z_{n-1}\right\|^{p_k} = 0 .$$

Let $\varepsilon > 0$ and $K = \left\{k \in \mathbb{N} : \left\|B_\Lambda^\mu x_k - L, z_1,..., z_{n-1}\right\| \geqslant \varepsilon\right\}$. We obtain the following:

$$\sum_{k=1}^{\infty} a_{mk} \left\|B_\Lambda^\mu x_k - L, z_1, ..., z_{n-1}\right\|^{p_k}$$

$$= \sum_{k \in K} a_{mk} \left\| B_\Lambda^\mu x_k - L, z_1, ..., z_{n-1} \right\|^{p_k}$$

$$+ \sum_{k \notin K} a_{mk} \left\| B_\Lambda^\mu x_k - L, z_1, ..., z_{n-1} \right\|^{p_k}$$

$$\geq \min\left\{ \varepsilon^h, \varepsilon^H \right\} \sum_{k \in K} a_{mk} .$$

If we take the limit as $m \to \infty$, then we get $x \in S\left(A(B_\Lambda^\mu)^n \right)$. This completes the proof.

Theorem 4.2.10. Let $p = (p_k)$ be a sequence of non-negative bounded real numbers such that $\inf_k p_k > 0$. Then $S(A(B_\Lambda^\mu)^n) \subset W\left(A, B_\Lambda^\mu, p, \|.,...,.\| \right)$.

Proof. Suppose that $x = (x_k) \in l_\infty\left(B_\Lambda^\mu, \|.,...,.\| \right) \cap S\left(A(B_\Lambda^\mu)^n \right)$. Then there exists an integer T such that $\left\| B_\Lambda^\mu x_k - \zeta, z_1, ..., z_{n-1} \right\| \leq T$ for all $k > 0$ and for every nonzero $z_1, ..., z_{n-1} \in X$ and $\lim_{m \to \infty} \sum_{k \in K} a_{mk} = 0$, where

$$K = \left\{ k \in \mathbb{N} : \left\| B_\Lambda^\mu x_k - \zeta, z_1, ..., z_{n-1} \right\| \geq \varepsilon \right\}.$$

Then we can write

$$\sum_{k=1}^{\infty} a_{mk} \left\| B_\Lambda^\mu x_k - \zeta, z_1, ..., z_{n-1} \right\|^{p_k}$$

$$= \sum_{k \notin K} a_{mk} \left\| B_\Lambda^\mu x_k - \zeta, z_1, ..., z_{n-1} \right\|^{p_k}$$

$$+ \sum_{k \in K} a_{mk} \left\| B_\Lambda^\mu x_k - \zeta, z_1, ..., z_{n-1} \right\|^{p_k}$$

$$\leq \max\left\{ \varepsilon^h, \varepsilon^H \right\} \sum_{k \notin K} a_{mk} + \max\left\{ T^h, T^H \right\} \sum_{k \in K} a_{mk} .$$

Since $A = (a_{mk})$ is a non-negative regular matrix, then we have

$$1 = \lim_{m\to\infty} \sum_{k=1}^{\infty} a_{mk}$$

$$= \lim_{m\to\infty} \sum_{k\notin K} a_{mk} + \lim_{m\to\infty} \sum_{k\in K} a_{mk} .$$

Hence,

$$\lim_{m\to\infty} \sum_{k\notin K} a_{mk} = 1 .$$

Thus

$$\lim_{m\to\infty} \sum_{k=1}^{\infty} a_{mk} \left\| B_{\Lambda}^{\mu} x_k - \zeta, z_1, \ldots, z_{n-1} \right\|^{p_k}$$

$$\leq \varepsilon' \lim_{m\to\infty} \sum_{k=1}^{\infty} a_{mk} + T' \lim_{m\to\infty} \sum_{k\in K} a_{mk} < \varepsilon' .$$

where $\max\left\{\varepsilon^h, \varepsilon^H\right\} = \varepsilon'$ and $\max\left\{T^h, T^H\right\} = T'$. Hence $x_k \in W\left(A, B_{\Lambda}^{\mu}, p, \|.,\ldots,.\|\right)$.

4.3. Some Topological Properties of Sequence Spaces Involving Lacunary Sequence in a Real n-Normed Space

Now, we define some new sequence spaces involving lacunary sequence in n-normed spaces. Let θ be a lacunary sequence and M be any Orlicz function. Then we denote by $l(p, \theta, M, \|.,\ldots,.\|)$ the sequence space involving lacunary sequence defined by as the set of all $x \in w(\|.,\ldots,.\|)$ such that

$$l\left(p, \theta, M, \|.,\ldots,.\|\right) = \left\{ \begin{array}{c} x = (x_k) : \sum_{r=1}^{\infty} \left[\frac{1}{h_r} \sum_{k\in I_r} M\left(\left\| \frac{x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_r} < \infty, \\ \text{for some } \rho > 0 \text{ and for every nonzero } z_1, \ldots, z_{n-1} \in X \end{array} \right\}. \qquad \textbf{(4.3.1)}$$

If $M(x) = x$ then we get the sequence space $l\left(p, \theta, \|.,\ldots,.\|\right)$ as follows

$$l\left(p,\theta,\|.,\ldots,.\|\right)=\left\{\begin{array}{l} x=(x_k):\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k,z_1,\ldots,z_{n-1}\|\right)^{p_r}<\infty, \\ \text{for every nonzero } z_1,\ldots,z_{n-1}\in X \end{array}\right\}.$$

If $p_r = p$ for all r, $M(x) = x$, then the sequence space which is given by (4.3.1) reduces to

$$l_p\left(\theta,\|.,\ldots,.\|\right)=\left\{\begin{array}{l} x=(x_k):\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\|x_k,z_1,\ldots,z_{n-1}\|\right)^{p}<\infty, \\ \text{for every nonzero } z_1,\ldots,z_{n-1}\in X \end{array}\right\}.$$

We denote $l(p,\theta,M,\|.,\ldots,.\|)$ by $l(\theta,\|.,\ldots,.\|)$ where $M(x)=x$ and $p_r = p = 1$ for all r. In the special case where $\theta = \left(2^r\right)$, we have $\operatorname{ces}\left[p,M,\|.,\ldots,.\|\right] = l\left(p,\theta,M,\|.,\ldots,.\|\right)$.

Theorem 4.3.1. Let $1 \le p_r < \infty$ and $(X,\|.,\ldots,.\|)$ be an n-Banach space. For any Orlicz function M and a bounded sequence $p=(p_r)$ of strictly positive real numbers $l(p,\theta,M,\|.,\ldots,.\|)$ is a linear paranormed space by

$$g_r(x)=\inf\left\{\begin{array}{l} \rho^{\frac{p_r}{H'}}:\left(\sum_{r=1}^{\infty}\left[\frac{1}{h_r}\sum_{k\in I_r}M\left(\left\|\frac{x_k}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_r}\right)^{\frac{1}{H'}}<\infty, \\ \text{for every nonzero } z_1,\ldots,z_{n-1}\in X, \text{ for some } \rho>0 \end{array}\right\},$$

where $H' = \max(1,H)$.

Proof. It is easy to see that for any Orlicz function M and $p=(p_r)$ of strictly positive real numbers $l(p,\theta,M,\|.,\ldots,.\|)$ is a linear space, so we omit it. The conditions (i)-(iii) of definition paranorm are clearly hold. We prove the scalar multiplication is continuous. Let λ be any number and by using the definition of the paranorm,

$$g_r(\lambda x) = \inf\left\{ \begin{array}{l} \rho^{\frac{p_r}{H'}} : \left(\sum_{r=1}^{\infty} \left[\frac{1}{h_r} \sum_{k \in I_r} M\left(\left\| \frac{\lambda x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_r} \right)^{\frac{1}{H'}} < \infty, \\ \text{for every nonzero } z_1, \ldots, z_{n-1} \in X, \text{ for some } \rho > 0 \end{array} \right\}.$$

Then

$$g_r(\lambda x) = \inf\left\{ \left(|\lambda| \sigma \right)^{\frac{p_r}{H'}} : \left(\sum_{r=1}^{\infty} \left[\frac{1}{h_r} \sum_{k \in I_r} M\left(\left\| \frac{x_k}{\sigma}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_r} \right)^{\frac{1}{H'}} < \infty \right\},$$

where $\sigma = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_r} \le \max\left(1, |\lambda|^{H'}\right)$ therefore $|\lambda|^{\frac{p_r}{H'}} \le \left(\max\left(1, |\lambda|^{H'}\right) \right)^{\frac{1}{H'}}$. Hence

$$g_r(\lambda x) \le \left(\max\left(1, |\lambda|^{H'}\right) \right)^{\frac{1}{H'}} .\inf\left\{ \sigma^{\frac{p_r}{H'}} : \left(\sum_{r=1}^{\infty} \left[\frac{1}{h_r} \sum_{k \in I_r} M\left(\left\| \frac{x_k}{\sigma}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_r} \right)^{\frac{1}{H'}} < \infty \right\}$$

$$= \left(\max\left(1, |\lambda|^{H'}\right) \right)^{\frac{1}{H'}} g_r(x).$$

which converges to zero as $g_r(x)$ converges to zero in $l\left(p, \theta, M, \|., \ldots, .\|\right)$. Now suppose $\lambda_r \to 0$ and x is in $l\left(p, \theta, M, \|., \ldots, .\|\right)$. Then there exists $\rho > 0$ such that

$$g_r(x) = \inf\left\{ \begin{array}{l} \rho^{\frac{p_r}{H'}} : \left(\sum_{r=1}^{\infty} \left[\frac{1}{h_r} \sum_{k \in I_r} M\left(\left\| \frac{x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_r} \right)^{\frac{1}{H'}} < \infty, \\ \text{for every nonzero } z_1, \ldots, z_{n-1} \in X \end{array} \right\}.$$

Now

$$g_r(\lambda x) = \inf\left\{\rho^{\frac{p_r}{H'}} : \left(\sum_{r=1}^{\infty}\left[\frac{1}{h_r}\sum_{k\in I_r} M\left(\left\|\frac{\lambda x_k}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_r}\right)^{\frac{1}{H'}} < \infty\right\} \to 0$$

as $\lambda \to 0$, for every nonzero $z_1, \ldots, z_{n-1} \in X$ and for some $\rho > 0$.

Let (x^i) be any Cauchy sequence in $l(p,\theta,M,\|.,\ldots,.\|)$, and let s and x_0 be fixed such that $M(sx_0) \geq 1$. Then for each $\frac{\varepsilon}{sx_0} > 0$ there exists a positive integer N such that $g_r(x^i - x^j) < \frac{\varepsilon}{sx_0}$, for all $i,\ j \geq N$. Since $g_r(x^i - x^j)$ is positive so we can substitute ρ for $g_r(x^i - x^j)$. From the definition of the paranorm we get

$$\left(\sum_{r=1}^{\infty}\left[\frac{1}{h_r}\sum_{k\in I_r} M\left(\left\|\frac{x_k^i - x_k^j}{g_r(x^i - x^j)}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_r}\right)^{\frac{1}{H'}} < \infty,$$

for all $i,\ j \geq N$ and for every nonzero $z_1, \ldots, z_{n-1} \in X$. Thus

$$\sum_{r=1}^{\infty}\left[\frac{1}{h_r}\sum_{k\in I_r} M\left(\left\|\frac{x_k^i - x_k^j}{g_r(x^i - x^j)}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_r} < \infty,$$

for all $i,\ j \geq N$. Since $1 \leq p_r < \infty$, we have

$$\frac{1}{h_r}\sum_{k\in I_r} M\left(\left\|\frac{x_k^i - x_k^j}{g_r(x^i - x^j)}, z_1, \ldots, z_{n-1}\right\|\right) \to 0, \text{ as } r \to \infty.$$

Since $\sum_{k\in I_r} M\left(\left\|\frac{x_k^i - x_k^j}{g_r(x^i - x^j)}, z_1, \ldots, z_{n-1}\right\|\right)$ is bounded then it follows that

$$M\left(\left\|\frac{x_k^i - x_k^j}{g_r(x^i - x^j)}, z_1, \ldots, z_{n-1}\right\|\right) \leq 1$$

for sufficiently large values of r. Since $M\left(\frac{sx_0}{2}\right) \geq 1$, we obtain that

$$M\left(\left\|\frac{x_k^i - x_k^j}{g_r(x^i - x^j)}, z_1, \ldots, z_{n-1}\right\|\right) \leq 1 \leq M\left(\frac{sx_0}{2}\right).$$

Since M is non-decreasing and convex function, then we have

$$\left\|\frac{x_k^i - x_k^j}{g_r(x^i - x^j)}, z_1, \ldots, z_{n-1}\right\| \leq \frac{sx_0}{2}.$$

Hence

$$\begin{aligned}\left\|x_k^i - x_k^j, z_1, \ldots, z_{n-1}\right\| &\leq \frac{sx_0}{2} . g_r(x^i - x^j) \\ &< \frac{sx_0}{2} . \frac{\varepsilon}{sx_0} = \frac{\varepsilon}{2},\end{aligned}$$

for all $i, j \geq N$. Since X is an n-Banach space, then (x^i) is convergent in X for all $i \geq N$. Using the continuity of functions M and $\|.,\ldots,.\|$ and taking the limit as $j \to \infty$ we have,

$$\left(\sum_{r=1}^{\infty}\left[\frac{1}{h_r}\sum_{k \in I_r} M\left(\left\|\frac{x_k^i - x_k}{\rho}, z_1, \ldots, z_{n-1}\right\|\right)\right]^{p_r}\right)^{\frac{1}{H'}} < \infty,$$

for every nonzero $z_1, \ldots, z_{n-1} \in X$. Taking the infimum of such ρ' s we get for every nonzero $z_1, \ldots, z_{n-1} \in X$ and for all $i \geq N$,

$$\inf\left\{\rho^{\frac{p_r}{H'}} : \left(\sum_{r=1}^{\infty}\left[\frac{1}{h_r}\sum_{k\in I_r} M\left(\left\|\frac{x_k}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_r}\right)^{\frac{1}{H'}} < \infty\right\} < \varepsilon .$$

The sequence space $l(p,\theta,M,\|.,...,.\|)$ is a linear space and $(x^i)\in l(p,\theta,M,\|.,...,.\|)$ then we have $x = x^i - (x^i - x) \in l(p,\theta,M,\|.,...,.\|)$. This completes the proof.

Theorem 4.3.2. $l(p,\theta,M_1,\|.,...,.\|)\cap l(p,\theta,M_2,\|.,...,.\|)\subset l(p,\theta,M_1+M_2,\|.,...,.\|)$.

Proof. Let $x\in l(p,\theta,M_1,\|.,...,.\|)\cap l(p,\theta,M_2,\|.,...,.\|)$ then,

$$\sum_{r=1}^{\infty}\left[\frac{1}{h_r}\sum_{k\in I_r} M_1\left(\left\|\frac{x_k}{\rho_1}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_r} < \infty,$$

for some $\rho_1 > 0$ and for every nonzero $z_1,..., z_{n-1}\in X$, and

$$\sum_{r=1}^{\infty}\left[\frac{1}{h_r}\sum_{k\in I_r} M_2\left(\left\|\frac{x_k}{\rho_2}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_r} < \infty,$$

for some $\rho_2 > 0$ and for every nonzero $z_1,..., z_{n-1}\in X$. Let ρ as; $\rho = \max(\rho_1,\rho_2)$. Since

$$\left[\frac{1}{h_r}\sum_{k\in I_r}(M_1+M_2)\left(\left\|\frac{x_k}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_r}$$

$$=\left[\frac{1}{h_r}\sum_{k\in I_r} M_1\left(\left\|\frac{x_k}{\rho}, z_1, ..., z_{n-1}\right\|\right)+\frac{1}{h_r}\sum_{k\in I_r} M_2\left(\left\|\frac{x_k}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_r}$$

$$\le K\left\{\left[\frac{1}{h_r}\sum_{k\in I_r} M_1\left(\left\|\frac{x_k}{\rho_1}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_r}+\left[\frac{1}{h_r}\sum_{k\in I_r} M_2\left(\left\|\frac{x_k}{\rho_2}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_r}\right\},$$

then we get,

$$\sum_{r=1}^{\infty}\left[\frac{1}{h_r}\sum_{k\in I_r}(M_1+M_2)\left(\left\|\frac{x_k}{\rho_1},z_1,...,z_{n-1}\right\|\right)\right]^{p_r}$$

$$\leq K\left\{\sum_{r=1}^{\infty}\left[\frac{1}{h_r}\sum_{k\in I_r}M_1\left(\left\|\frac{x_k}{\rho_1},z_1,...,z_{n-1}\right\|\right)\right]^{p_r}\right.$$

$$\left.+\sum_{r=1}^{\infty}\left[\frac{1}{h_r}\sum_{k\in I_r}M_2\left(\left\|\frac{x_k}{\rho_2},z_1,...,z_{n-1}\right\|\right)\right]^{p_r}\right\}<\infty.$$

Hence $x\in l\left(p,\theta,M_1+M_2,\|.,...,.\|\right)$.

CHAPTER 5. CONCLUSIONS AND RECOMMENDATIONS

In mathematics one of the most important notions is the notion of norm, which is fundamental in geometry, in analysis and others. The notion of a norm is to be regarded as a generalization of the notion of the distance. In a normed space $(X, \|.\|)$ we know how to measure lengths. How do we measure areas or volumes? This is not always easy. If we have an inner product we can measure volumes of n-dimensional parallelepipeds by the determinant

$$\begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \ldots & \langle x_n, x_n \rangle \end{vmatrix}$$

which is known as Gramian of linearly independent vectors $x_1, ..., x_n$ in $(X, \langle ., . \rangle)$, or we need orthogonality. However, we need inner product or at least semi-inner product to define orthogonality. If we have a semi-inner product we can also measure the volume of n-dimensional parallelepipeds. Using a semi inner product g, one may define the notion of orthogonality on X. In general, given a vector $y \in X$ and a subspace $S = \operatorname{span}\{x_1, ..., x_n\}$ of X, we can define the g-orthogonal projection of y on S. Next, given a finite sequence of linearly independent vectors $x_1, ..., x_n$ in X, we can construct a left g-orthogonal sequence $x_1^*, ..., x_n^*$ as in [6]. Having done so, we may define the volume of the n-dimensional parallelepiped spanned by $x_1, ..., x_n$ in X to be $V(x_1, ..., x_n) := \prod_{i=1}^{n} \|x_i^*\|$ ([46]). The volume formula (defined in a semi-inner product space) is not invariant with respect to permutation. Thus there is a limitation with such a formula. But if we don't have any inner product or semi-inner product we can not compute the volume. We must recognize that the notion of norm has a limitation. To pass the limitation, we need a new notion. One of the treatments is to consider the 2-normed space introduced by S. Gähler [30]. By this way we can compute the area of parallelogram spanned by two vectors. It was generalized to n-normed space by Misiak [39] to compute the volume of the n-dimensional parallelepiped spanned by linearly independent n vectors.

Now, consider the 2-normed space $(X, \|.,.\|)$. We know how to measure the areas. One question arises: How can we measure the lengths? At first, this question was asked by S. Gähler. He defined $\|x\|^* := \|x,a\| + \|x,b\|$ where $\{a,b\}$ is linearly independent set and $\dim(X) > 1$. By this way, for $X = \mathbb{R}^2$ the derived norm $\|.\|^*$ is equivalent to the usual norm $\|.\|$. $x_n \to x \ (n \to \infty) \Leftrightarrow \|x_n - x\|^* \to 0$ if and only if $\|x_n - x\| \to 0$. $(\mathbb{R}^2, \|.,.\|)$ has the same topology as $(\mathbb{R}^2, \|.\|)$. Later, Gunawan [32] derived a norm for the same purpose in a 2-normed space $(X, \|.,.\|)$ of dimension ≥ 2 choosing an arbitrary linearly independent set $\{a_1, a_2\}$ in X and with respect to $\{a_1, a_2\}$, he defined a norm $\|.\|_p^*$ on X by $\|x\|_p^* := \left(\|x,a_1\|^p + \|x,a_2\|^p\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$. Actually, for 2-normed space l^p, we choose, for convenience $a_1 = (1,0,0,...)$ and $a_2 = (0,1,0...)$, and define $\|.\|_p^*$ with respect to $\{a_1, a_2\}$ as above, then we have; the derived norm $\|.\|_p^*$ is equivalent to the usual norm $\|.\|_p$ on l^p. Precisely, we have $\|x\|_p \leq \|x\|_p^* \leq 2^{\frac{1}{p}} \|x\|_p$ for all $x \in l^p$. Indeed, it was not a goal, however, it was a result of how to measure distance.

It is correct if we know they are equivalent then the proofs in n-normed space or in 2-normed space can be done easily. But a few years ago, this was not known by mathematicians. This shows the importance of the equivalence and helps to understand the structure of the n-normed space. If we want to study n-normed space, we should stop to discuss why they are equivalent, and let’s to study something else. That could be interesting, because we don’t have only one vector, we have pairs in 2-

normed space and n vectors in n-normed space, something to explore it. For example, for C[a,b] we still don't know whether we can take arbitrary linearly independent set like l^p and L^p. But for C[a,b] the equivalence is obtained for only some specific vectors. So, the equivalence is true for only specific choice. We don't know the equivalence for arbitrary vectors. This is an open problem also to explore.

In this section, the results obtained from the previous sections of thesis will be also summarized. A part of the second section, the third and fourth sections of this thesis equipped with original works.

In the first part of third section, we define the generalized difference matrix $B^{\mu}_{(\eta)}$ and introduce difference sequence spaces $\bar{c}\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$, $\bar{c}_0\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$, $m\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$, $m_0\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$, $c\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$, $c_0\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$, $l_{\infty}\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$, $W\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$ which are defined on a real linear 2-normed space. We investigate some topological properties of the spaces $\bar{c}\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$, $\bar{c}_0\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$, $m\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$, $m_0\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$ including linearity, existence of paranorm and solidity. Further, we show that the sequence spaces $m\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$ and $m_0\left(B^{\mu}_{\eta}, p, \|.,.\|\right)$ are complete paranormed spaces where the base space is a 2-Banach space. Moreover, we give some inclusion relations ([53]).

In the second part of Chapter 3, we introduce some new sequence spaces derived by Riesz mean and the notions of almost and strongly almost convergence in a real 2-normed space. Some topological properties of these spaces are investigated. Further, new concepts of statistical convergence which will be called weighted almost statistical convergence and $[\tilde{R}, p_n]$-statistical convergence in a real 2-normed space, are defined. Also, some relations between these concepts are investigated ([54]).

There are three parts in the fourth chapter. In the first part of it, we obtain a new concept of statistical convergence which will be called weighted almost lacunary statistical convergence in a real n-normed space by combining both of the definitions of lacunary sequence and Riesz mean. We examine some connections between this notion with the concept of almost lacunary statistical convergence and weighted almost statistical convergence, where the base space is a real n-normed space ([55]). In the second part of this chapter, some new sequence spaces associated with multiplier sequence by using an infinite matrix, an Orlicz function and generalized B-difference operator on a real n-normed space are introduced ([56]). In the last part of it, some sequence spaces, involving lacunary sequence, in a real linear n-normed space are introduced ([57]). In the last section of this thesis, the main results, which were obtained, are summarized.

SOURCES

[1] MADDOX, I. J., Elements of Functional Analysis, Cambridge Univ. Press., Cambridge, 1970.

[2] BOOS, J., PETER, C., Classical and Modern Methods in Summability, Oxford University Press, 2000.

[3] KREYSZIG, E., Introductory Functional Analysis with Applications, University of Windsor, Canada, 1978.

[4] MURSALEEN, M., Elements of Metric Spaces, Anamaya Publishers, New Delhi, 2011.

[5] HILBERT, D., Grundzüge Einer Allgemeinen Theorie der Linearen Integralgleichungen, New York: Chelsea, 1912, Repr. 1953.

[6] MILICIC, P.M., On the Gramm-Schmidt Projection in Normed Spaces, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., 4: 89-96, 1993.

[7] MALKOWSKY, E., RAKOCEVIC, V., An Introduction into the Theory of Sequence Spaces and Measures of Noncompactness. Zb. Rad. (Beogr.), 9 (17): 143–234, 2010.

[8] FREEDMAN, A. R., SEMBER, J. J., Densities and Summability. Pacific J. Math., 95: 293–305, 1981.

[9] ET, M., GÖKHAN, A., ALTINOK, H., On Statistical Convergence of Vector-valued Sequences Associated with Multiplier Sequences. Ukrainian Math. J., 58 (1): 139–146, 2006.

[10] MADDOX, I. J. Spaces of Strongly Summable Sequences. Quart. J. Math., 18 (2): 345–355, 1967.

[11] KAMTHAN, P. K., GUPTA, M., Sequence Spaces and Series. Marcel Dekker, New York, 1981.

[12] GOFFMAN, C., Functions of Finite Baire Type. Amer. Math. Monthly, 67: 164–165, 1960.

[13] FREEDMAN, A. R., SEMBER, J. J., RAPHAEL, M., Some Cesaro-Type Summability Spaces. Proc. London Math. Soc., 37 (3): 508–520, 1978.

[14] NIVEN, I., ZUCKERMAN, H. S., An Introduction to the Theory of Numbers, 4th Ed., John Wiley and sons, New York, 1980.

[15] FAST, H., Sur la Convergence Statistique, Colloq. Math., 2: 241-244, 1951.

[16] PETERSEN, G. M., Regular Matrix Transformations. McGraw-Hill Publishing Company Limited, London, New York, 1966.

[17] KARAKAYA, V., CHISHTI, T. A., Weighted Statistical Convergence. Iranian J. Sci. Technol. Trans., A Sci. 33 (A3): 219–223, 2009.

[18] MURSALEEN, M., KARAKAYA, V., ERTÜRK, M., GÜRSOY, F., Weighted Statistical Convergence and Its Application to Korovkin Type Approximation Theorem. Appl. Math. Comput., 218: 9132–9137, 2012.

[19] LORENTZ, G. G., A Contribution to the Theory of Divergent Sequences. Acta Math., 80 (1): 167–190, 1948.

[20] MADDOX, I. J., A New Type of Convergence. Math. Proc. Camb. Phil. Soc., 83: 61–64, 1978.

[21] KIZMAZ, H. On Certain Sequence Spaces. Canadian Math. Bull., 24 (2): 169–176, 1981.

[22] ET, M., ÇOLAK, R., On Generalized Difference Sequence Spaces. Soochow J. Math., 21 (4): 377–386, 1995.

[23] DUTTA, H., On Some Difference Sequence Spaces. Pacific J. Sci. Tech., 10 (2): 243–247, 2009.

[24] DUTTA, H., Some Statistically Convergent Difference Sequence Spaces Defined Over Real 2-Normed Linear Space. Appl. Sci., 12: 37–47, 2010.

[25] ALTAY, B., BAŞAR, F., On the Fine Spektrum of the Generalized Difference Operator $B(r,s)$ Over the Sequence Spaces c_0 and c. Int. J. Math. Math. Sci., 18: 3005–3013, 2005.

[26] BAŞARIR, M., KAYIKÇI, M. On the Generalized B^m-Riesz Difference Sequence Space and β-Property. J. Ineq. Appl., 2009, Article ID 385029, 18 pages, 2009.

[27] GAHLER, S., 2-Metrische Räume und Ihre Topologısche Struktur, Math. Nachr., 26: 115-148, 1963.

[28] MENGER, K., Untersuchungen Ueber Allgeine Metrik, Math. Ann., 100: 75-163, 1928.

[29] ISEKI, K., Mathematics on 2-Normed Spaces, Bull. Korean. Math. Soc., 13 (2): 127-135, 1976.

[30] GAHLER, S., Lineare 2-Normierte Räume, Math. Nachr., 28: 1-43, 1965.

[31] GUNAWAN, H., Orthogonality in 2-Normed Spaces, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. 17: 1-8, 2006.

[32] GUNAWAN, H., The Space of p-Summable Sequences and its Natural n-Norm, Bull. Austral. Math. Soc., 64: 137-147, 2001.

[33] GUNAWAN, H., MASHADI, M., On n-Normed Spaces, Int. J. Math Math Sci., 27 (10): 631-639, 2001.

[34] WHITE, A.G., 2-Banach Spaces, Math. Nachr., 42 (1-3): 43-60, 1969.

[35] IDRIS, M., EKARIANI, S., GUNAWAN, H., On the Space of p-Summable Sequences, Mat. Bech., 65 (1): 58-63, 2013.

[36] GÜRDAL, M., PEHLİVAN, S., Statistical Convergence in 2-Normed Spaces. Southeast Asian Bull. Math., 33: 257–264, 2009.

[37] DIMINNIE, C., GAHLER, S., WHITE, A., 2-Inner Product Spaces, Demonstratio Math., 6: 525-536, 1973.

[38] DIMINNIE, C., GAHLER, S., WHITE, A., 2-Inner Product Spaces II, Demonstratio Math., 10: 169-188, 1977.

[39] MISIAK, A., n-Inner Product Spaces, Math. Nachr., 140: 299-319, 1989.

[40] MISIAK, A., Orthogonality and Orthonormality in n-Inner Product Spaces, Math. Nachr., 143: 249-261, 1989.

[41] GUNAWAN, H., Inner Product on n-Inner Product Spaces, Soochow J. Math., 28 (4): 2002, 389-398.

[42] EKARIANI, S., GUNAWAN, H., IDRIS, M., A Contractive Mapping Theorem for the n-Normed Space of p-Summable Sequences, J. Math. Analysis, 4: 1-7, 2013.

[43] EKARIANI, S., GUNAWAN, H., LINDIARNI, J., On the n-Normed Space of p-Integrable Functions, submitted.

[44] GUNAWAN, H., On Convergence in n-Inner Product Spaces, Bull. Malaysian Math. Sc. Soc., 25: 11-16, 2002.

[45] GUNAWAN, H., On n-Inner Products, n-Norms, and the Cauchy-Schwarz Inequality, Sci. Math. Japan., 55: 53-60, 2002.

[46] GUNAWAN, H., SETYA-BUDHI, W., MASHADI, GEMAWATI, S., On Volumes of n-Dimensional Parallelepipeds in l^p Spaces, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., 16: 48-54, 2005.

[47] GOZALI, S.G., GUNAWAN, H., NESWAN, O., On n-Norms and Bounded n-Linear Functionals in a Hilbert Space, Ann. Funct. Anal., 1: 72-79, 2010.

[48] PANGALELA, Y.E.P., GUNAWAN, H., The n-Dual Space of p-Summable Sequences, appear in Math. Bohemica, 2013.

[49] BATKUNDE, H., GUNAWAN, H., PANGALELA, Y.E.P., Bounded Linear Functionals on the n-Normed Space of p-Summable Sequences, Acta Univ. M. Belii Ser. Math., 2013: 66-75, ISSN 1338-7111, 2013.

[50] SAVAŞ, E., Sequence Spaces in n-Normed Space Defined by Ideal Convergence and an Orlicz Function. Abstr. Appl. Anal., 2011: Article ID 741382, 9 pages, 2011.

[51] SAVAŞ, E., Strongly Summable Sequence Spaces in n-Normed Spaces Defined by Ideal Convergence and an Orlicz Function. Appl. Math. Comput., 217 (1): 271–276, 2010.

[52] SAVAŞ, E., On Generalized A-Difference Strongly Summable Sequence Spaces Defined by Ideal Convergence on a Real n-Normed Space. J. Ineq. Appl. 2012:87, 9 pages, 2012.

[53] BAŞARIR, M., KONCA, Ş., KARA, E.E., Some Generalized Difference Statistically Convergent Sequence Spaces in 2-Normed Space. J. Ineq. Appl., 2013:177, 9 pages, 2013.

[54] BAŞARIR, M., KONCA, Ş., Some Sequence Spaces Derived by Riesz Mean in a Real 2-Normed Space, Iran. J. Sci. Tech. Trans. A: Science, 38 (1), 25-33, 2014.

[55] KONCA, Ş., BAŞARIR, M., On Some Spaces of Almost Lacunary Convergent Sequences Derived by Riesz Mean and Weighted Almost Lacunary Statistical Convergence in a Real n-Normed Space, J. Ineq. Appl., 2014:81, 2014. Doi. 10.1186\1029-242X-2014-81.

[56] KONCA, Ş., BAŞARIR, M., Generalized Difference Sequence Spaces Associated with a Multiplier Sequence on a Real n-Normed Space, J. Ineq. Appl., 2013:335, 12 pages, 2013. Doi. 10.1 186\1029-242X-2013-335.

[57] KONCA, Ş., ÖZTÜRK, M., Some Topological and Geometric Properties of Sequence Spaces Involving Lacunary Sequence in n-Normed Spaces, AIP Conf. Proc. 1479 (957): 957–967, 2012. doi:10.1063/1.4756302.

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