SEMIVALUES: POWER, POTENTIAL AND MULTILINEAR EXTENSIONS*

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Abstract

The notions of power and potential, both defined for any semivalue, give rise to two endomorphisms of the vector space of all cooperative games on a given player set. Several properties of these linear mappings are stated and their action on unanimity games is emphasized. We also relate in both cases the multilinear extension of the image game to the multilinear extension of the original game.

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1 Introduction

The notion of *semivalue* was first introduced by Weber [24] for simple games (see also Einy [12]). The extension to all cooperative games, including an axiomatic characterization and a formula in terms of weighting coefficients, was provided by Dubey, Neyman and Weber [11]. Semivalues represent a generalization of the Shapley value [23] that includes the Banzhaf value defined by Owen [19]—an extension to all cooperative games of the original Banzhaf power index ([21], [2], [4])—. An alternative axiomatization can be found in Weber [25].

The concept of *total power* (simply *power*, in the sequel) was used by Feltkamp [13], as a synonymous of "sum of allocations", when providing a parallel axiomatization of the Shapley and Banzhaf values (see also Dubey and Shapley [10]). While its meaning for the Shapley value is almost trivial because of efficiency, it can be extended to any semivalue (Dragan [7]). The concept of *potential*, first introduced for the Shapley value by Hart and Mas–Colell ([16], [17]), and later on for the Banzhaf value by Dragan [5], also admits an extension to all semivalues (Dragan [7]), which coincides with the restriction to this class of values of a wider notion provided by Calvo and Santos [3], who axiomatically characterized the solution concepts that possess a potential.

Thus, these notions of power and potential are defined for any semivalue acting on the vector space of all cooperative games on a given player set N. If ψ is a semivalue and v is a game, both defined on N, the ψ–power of v is the sum of payoffs allocated

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by ψ to all players in v, while the ψ–potential works for ψ as a "potential" (in a rather physical sense) by allocating a single number to every subgame of v.

Power and potential have a common characteristic: they induce endomorphisms of the space of cooperative games; the image games assign to each coalition, respectively, the additive power got by all its players and the potential of the restricted game, always according to the referred semivalue. Both linear maps were introduced by Dragan [8] for any semivalue, and the image games were respectively called *power game* and *potential game*. Calvo and Santos [3] have also employed the concept of power game, under the name of "auxiliary game". In papers by Dragan [8] and Dragan and Martinez-Legaz [9], the power and potential maps have been used in an essential way, so that these two notions have turned out to be useful tools to study several properties related with semivalues.

The present work is focussed on these power and potential maps attached to any semivalue. We state the similar behavior of the unanimity games with respect to them in the sense that they are eigenvectors of both with specific eigenvalues depending on the considered semivalue. Some consequences are derived. In particular, we study the relationship with respect to the multilinear extension, a concept introduced by Owen [18] that has proven to be an interesting tool for value computation. Using the image games given by the power and potential maps, the computations can be extended to the power of the players and to the potential of the game and also of all its restricted games.

More precisely, the computation of allocations to the players can be obtained from the multilinear extension by using a common procedure for all semivalues. This method generalizes the results obtained for the Shapley and Banzhaf values by Owen ([18], [19]) but the steps in the procedure have changed, since the consideration of the potential game enables us to differentiate the contribution due to the game from the contribution due to the semivalue.

According to the above considerations, the paper is organized as follows. In Section 2 we provide preliminaries devoted to cooperative games and semivalues, paying special attention to the weighting coefficients that define each semivalue. The power map is considered in Section 3, where the results on unanimity games are provided and the relationship between the multilinear extensions of a game and its power game is stated. The main results are Theorem 3.4, Corollary 3.5 and Proposition 3.8. Section 4 presents a similar treatment for the potential map and includes a unifying viewpoint on the use of the multilinear extension technique. In this case, the main results are given in Theorem 4.3, Corollary 4.4, Proposition 4.5 and Remark 4.7. In Section 5, analogies and differences between both maps are emphasized. First, as to their matrix representation in the common diagonalization basis. Next, by discussing an equivalence notion between the image games based on the additivity of the difference game. Thirdly, by providing a splitting of the space into (common) invariant subspaces and determining and comparing, only in the binomial case, the kernel and the (full) invariant subspace of each map. Finally, the asymptotic behavior of the iterations of each map is described.

No conclusion section has been added since we feel that the results of the paper have been clearly stated in this (somewhat long) introduction.

2 Preliminaries

Let N be a finite set of *players* and 2^N be the set of its *coalitions* (subsets of N). A *cooperative game* (with transferable utility) on N is a function $v : 2^N \to \mathbb{R}$, which assigns a real number $v(S)$ to each coalition $S \subseteq N$ and satisfies $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T$. A game v is *additive* (or *inessential*) if $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$. Given a nonempty coalition $T \subseteq N$, the *restriction* to T of a given game v on N is the game $v_{|T}$ on T defined by $v_{|T}(S) = v(S)$ for all $S \subseteq T$.

Endowed with the natural operations for real–valued functions, denoted by $v + v'$ and λv for $\lambda \in \mathbb{R}$, the set of all cooperative games on N is a vector space $\mathcal{G}_N$. For every nonempty coalition $T \subseteq N$, the *unanimity game* u_T is defined on N by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, and it is easily checked that the set of all unanimity games is a basis for $\mathcal{G}_N$, so that $\dim(\mathcal{G}_N) = 2^n - 1$ if $n = |N|$. Each game $v \in \mathcal{G}_N$ can then be uniquely written as a linear combination of unanimity games, and its components are the *Harsanyi dividends* (Harsanyi [15]):

$$v = \sum_{T \subseteq N:\, T \neq \emptyset} \alpha_T u_T, \quad \text{where} \quad \alpha_T = \alpha_T(v) = \sum_{S \subseteq T} (-1)^{t-s} v(S) \tag{1}$$

and, as usual, $t = |T|$ and $s = |S|$. The additive games form a linear subspace of $\mathcal{G}_N$ that we denote as $\mathcal{AG}_N$ and is spanned by the set $\{u_{\{i\}} : i \in N\}$. Finally, every permutation θ of N induces a linear automorphism of $\mathcal{G}_N$ that leaves invariant $\mathcal{AG}_N$ and is defined by $(\theta v)(S) = v(\theta^{-1} S)$ for all $S \subseteq N$ and all $v \in \mathcal{G}_N$.

By a *value* on $\mathcal{G}_N$ we will mean a map $f : \mathcal{G}_N \to \mathbb{R}^N$, which assigns to every game v on N a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

Following Dubey, Neyman and Weber's [11] axiomatic description, $\psi : \mathcal{G}_N \to \mathbb{R}^N$ is a *semivalue* iff it satisfies the following properties:

(i) *Linearity*: $\psi[\lambda v + \mu v'] = \lambda \psi[v] + \mu \psi[v']$ for all $v, v' \in \mathcal{G}_N$ and $\lambda, \mu \in \mathbb{R}$.
(ii) *Anonymity*: $\psi_{\theta i}[\theta v] = \psi_i[v]$ for all θ on N, $i \in N$, and $v \in \mathcal{G}_N$.
(iii) *Positivity*: if v is monotonic, then $\psi[v] \geq 0$.
(iv) *Projection property*: if v is additive, then $\psi_i[v] = v(\{i\})$ for all $i \in N$.

The same authors (loc. cit.) provided a useful characterization of semivalues by means of *weighting coefficients*: (a) for every *weighting vector* $(p_s)_{s=1}^n$ such that

$$\sum_{s=1}^{n} \binom{n-1}{s-1} p_s = 1 \quad \text{and} \quad p_s \geq 0 \ \text{ for all } s, \tag{2}$$

the expression

$$\psi_i[v] = \sum_{S \subseteq N:\, i \in S} p_s [v(S) - v(S \backslash \{i\})] \quad \text{for all } i \in N \text{ and all } v \in \mathcal{G}_N, \tag{3}$$

where $s = |S|$, defines a semivalue ψ; (b) conversely, every semivalue can be obtained in this way; (c) the correspondence $(p_s)_{s=1}^n \mapsto \psi$ is bijective.

Notice that the number of coalitions of size s that contain a given player $i \in N$ is $\binom{n-1}{s-1}$, so that conditions (2) give a probability distribution on the set of all coalitions containing i that depends only on cardinality. Therefore, $\psi_i[v]$ is the expected marginal contribution of player i in v to a random coalition this player is joining.

Well known examples of semivalues are the *Shapley value* φ (Shapley [23]), for which $p_s = 1/n\binom{n-1}{s-1}$, and the *Banzhaf value* β (Owen [19]), for which $p_s = 2^{1-n}$.

The Shapley value φ is the only *efficient* semivalue, in the sense that $\sum_{i\in N}\varphi_i[v] = v(N)$ for every $v \in \mathcal{G}_N$. It is worthy of mention that these two classical values are defined for each N. The same happens with the *binomial semivalues*, introduced by Puente [22] (see also Giménez [14] or Amer and Giménez [1]) as follows. Let $\alpha \in (0,1)$ and $p_{\alpha,s} = \alpha^{s-1}(1-\alpha)^{n-s}$ for $s = 1, 2, \ldots, n$. Then $(p_{\alpha,s})_{s=1}^n$ is a weighting vector and defines a semivalue that will be denoted as ψ_α and called the *α–binomial semivalue*. Using the convention that $0^0 = 1$, the definition makes sense also for $\alpha = 0$ and $\alpha = 1$, where we respectively get the *dictatorial index* $\psi_0 = \delta$ and the *marginal index* $\psi_1 = \mu$, introduced by Owen [20] and such that $\delta_i[v] = v(\{i\})$ and $\mu_i[v] = v(N) - v(N\backslash\{i\})$ for all $i \in N$ and all $v \in \mathcal{G}_N$. Of course, $\alpha = 1/2$ gives $\psi_{1/2} = \beta$, the Banzhaf value.

In fact, semivalues are defined on cardinalities rather than on specific player sets: this means that a weighting vector $(p_s)_{s=1}^n$ defines a semivalue ψ on all N such that $n = |N|$. When necessary, we shall write ψ^n for a semivalue on cardinality n and p_s^n for its weighting coefficients. This often matters since a semivalue $\psi = \psi^n$ on cardinality n gives rise to *induced semivalues* ψ^t for all cardinalities t such that $1 \le t \le n-1$, recurrently defined by their weighting coefficients, which are given by an expression obtained by Dubey et al. [11] and referred by Dragan ([6], [7]) as the *Pascal triangle (inverse) formula*:

$$p_s^t = p_s^{t+1} + p_{s+1}^{t+1} \quad \text{for} \quad 1 \le s \le t < n. \tag{4}$$

It is not difficult to check that the induced semivalues of the Shapley value (resp., the α–binomial semivalue) are all Shapley values (resp., α–binomial semivalues). By applying equation (4) successively, one gets the expression of the weighting coefficients of any induced semivalue in terms of the coefficients of the original semivalue, namely

$$p_s^t = \sum_{j=0}^{n-t} \binom{n-t}{j} p_{s+j}^n \quad \text{for} \quad 1 \le s \le t < n \tag{5}$$

and, inductively, a kind of (partially) inverse formula that provides the remaining weighting coefficients at all cardinality levels in terms of the last ones (p_t^t for all t):

$$p_1^1 = 1 \quad \text{and} \quad p_s^t = \sum_{c=0}^{t-s} (-1)^c \binom{t-s}{c} p_{s+c}^{s+c} \quad \text{for} \quad 1 \le s \le t-1 \quad \text{and} \quad 1 < t \le n. \tag{6}$$

We end this section by recalling one of the three basic notions that will be dealt with here, that of multilinear extension of a game, due to Owen [18]. The other two, power and potential, will be introduced in the corresponding sections.

The *multilinear extension* (MLE, in the sequel) of a game $v \in \mathcal{G}_N$ is the function $f_v : [0,1]^n \longrightarrow \mathbb{R}$ defined by

$$f_v(X_N) = \sum_{S\subseteq N} \prod_{i\in S} x_i \prod_{j\in N\backslash S} (1-x_j) v(S), \tag{7}$$

where X_N denotes the set of variables x_i for $i \in N$. The following properties directly derive from the definition:

(i) If $v, v' \in \mathcal{G}_N$ and $\lambda, \mu \in \mathbb{R}$, then $f_{\lambda v+\mu v'} = \lambda f_v + \mu f_{v'}$.

(ii) If $\emptyset \neq T \subseteq N$, then $f_{u_T}(X_N) = \prod_{i\in T} x_i$.

(iii) In general, if $v = \sum_{T\subseteq N:\, T\neq\emptyset} \alpha_T u_T$, then $f_v(X_N) = \sum_{T\subseteq N:\, T\neq\emptyset} \alpha_T \prod_{i\in T} x_i$.

(iv) If $v \in G_N$ and $\emptyset \neq T \subseteq N$, then $f_{v_{|T}}(X_T) = f_v(X_T, 0_{N\setminus T})$, where $(X_T, 0_{N\setminus T})$ denotes the set X_N with $x_i = 0$ for all $i \in N\setminus T$. Moreover, $\dfrac{\partial f_{v_{|T}}}{\partial x_i}(1_T) = v(T) - v(T\setminus\{i\})$ for each $i \in T$, where 1_T means $x_i = 1$ for all $i \in T$.

3 The power map

In this section, we study an endomorphism of the vector space of games on a given player set that is defined in terms of the power notion attached to each particular semivalue. As a consequence, it follows that the MLE of a game can be applied to computing the power of the players in the game and in all its restricted games.

Definition 3.1 (Dragan [7]) Let ψ be a semivalue on $\mathcal{G}_N$ defined by the weighting vector $(p_s)_{s=1}^n$. The *power* of (the players in) a given game $v \in \mathcal{G}_N$ according to semivalue ψ is the sum of allocations to all players, that is,

$$\Pi_\psi(v) = \sum_{i\in N} \psi_i[v] = \sum_{i\in N} \sum_{S\subseteq N:\, i\in S} p_s[v(S) - v(S\setminus\{i\})]. \tag{8}$$

Lemma 3.2 *Let ψ be a semivalue on $\mathcal{G}_N$ defined by the weighting vector $(p_s)_{s=1}^n$. Let $v \in G_N$. Then the power of (the players in) any restricted game $v_{|T}$ according to ψ^t is given by*

$$\Pi_{\psi^t}(v_{|T}) = tp_t^t v(T) + \sum_{U\subset T} \left[up_u^t - (t-u)p_{u+1}^t\right] v(U). \tag{9}$$

Proof. Since $v_{|T} \in \mathcal{G}_T$, equation (8) yields

$$\Pi_{\psi^t}(v_{|T}) = \sum_{i\in T} \sum_{U\subseteq T:\, i\in U} p_u^t[v(U) - v(U\setminus\{i\})].$$

For each $U \subseteq T$, $v(U)$ appears $u = |U|$ times with coefficient p_u^t and $t - u = |T\setminus U|$ times with coefficient $-p_{u+1}^t$. (For an alternative proof, see Dragan [7].) $\square$

Definition 3.3 (Dragan [7]) Let ψ be a semivalue on $\mathcal{G}_N$ defined by the weighting vector $(p_s)_{s=1}^n$. The *ψ–power map* is the endomorphism Π_ψ^* of $\mathcal{G}_N$ that transforms any game v in the game $\Pi_\psi^*(v)$ defined by

$$\Pi_\psi^*(v)(S) = \Pi_{\psi^s}(v_{|S}) \quad \text{for all } S \subseteq N. \tag{10}$$

Thus $\Pi_\psi^*(v)$, the *ψ–power game* of v, assigns to each coalition S the power allocated to the players of S in the restricted game $v_{|S}$ according to the induced semivalue ψ^s.

Theorem 3.4 *Let ψ be a semivalue on $\mathcal{G}_N$ defined by the weighting vector $(p_s)_{s=1}^n$. Every unanimity game u_T is an eigenvector for the linear mapping Π_ψ^* with eigenvalue tp_t^t, i.e.,*

$$\Pi_\psi^*(u_T) = tp_t^t\, u_T \quad \text{if } \emptyset \neq T \subseteq N. \tag{11}$$

Proof. For each $S \subseteq N$ we write $\Pi^*_\psi(u_T)(S)$ according to (9):

$$\Pi^*_\psi(u_T)(S) = \Pi_{\psi^s}(u_{T|S}) = sp^s_s u_T(S) + \sum_{U \subset S} \big[up^s_u - (s-u)p^s_{u+1}\big] u_T(U).$$

Now, we distinguish three cases.
(a) If $S \not\supseteq T$, then no coalition $U \subset S$ contains T and hence $\Pi^*_\psi(u_T)(S) = 0$.
(b) If $S = T$, then $\Pi^*_\psi(u_T)(T) = t\, p^t_t\, u_T(T) + 0 = t\, p^t_t$.
(c) Finally, if $S \supset T$, then

$$\begin{aligned}
\Pi^*_\psi(u_T)(S) &= sp^s_s + \sum_{u=t}^{s-1} \binom{s-t}{u-t} \big[up^s_u - (s-u)p^s_{u+1}\big] = \\
&= sp^s_s + tp^s_t + \sum_{u=t+1}^{s-1} \binom{s-t}{u-t} up^s_u - \sum_{u=t}^{s-2} \binom{s-t}{u-t}(s-u)p^s_{u+1} - (s-t)p^s_s = \\
&= tp^s_t + tp^s_s + \sum_{u=t+1}^{s-1} \binom{s-t}{u-t} up^s_u - \sum_{u=t+1}^{s-1} \binom{s-t}{u-t-1}(s-u+1)p^s_u = \\
&= t\sum_{u=t}^{s} \binom{s-t}{u-t} p^s_u + \sum_{u=t+1}^{s-1} \left[\binom{s-t}{u-t}(u-t) - \binom{s-t}{u-t-1}(s-u+1)\right] p^s_u.
\end{aligned}$$

According to expression (5) for induced weights, $\sum_{u=t}^{s} \binom{s-t}{u-t} p^s_u = p^t_t$ whereas the second sum vanishes. Therefore

$$\Pi^*_\psi(u_T)(S) = tp^t_t \quad \text{for all } S \supset T. \quad \square$$

Corollary 3.5 *Let ψ be a semivalue on $\mathcal{G}_N$ defined by the weighting vector $(p_s)_{s=1}^n$. The ψ–power map Π^*_ψ satisfies the following properties:*

(i) Π^*_ψ *is a bijection iff* $p^n_n > 0$.

(ii) Π^*_ψ *is the identity map in* G_N *iff* $\psi = \varphi$, *the Shapley value.*

(iii) If v *is an additive game then* $\Pi^*_\psi(v) = v$. *If* $p^t_t \neq 1/t$ *for* $t = 2, \ldots, n$ *then the converse is true.*

Proof. (i) ($\Leftarrow$) According to equation (5), if $p^n_n > 0$ then $p^t_t > 0$ for $t = 1, \ldots, n$. Then, all unanimity games are eigenvectors with positive eigenvalue and hence Π^*_ψ is a bijection. ($\Rightarrow$) Conversely, if the endomorphism is a bijection then no eigenvalue can vanish: in particular, $p^n_n > 0$.

(ii) ($\Leftarrow$) By efficiency of the Shapley value, $\Pi^*_\varphi(v)(S) = \Pi_{\varphi^s}(v_{|S}) = v(S)$ for all $v \in \mathcal{G}_N$ and all $S \subseteq N$ whence Π^*_φ is the identity map. ($\Rightarrow$) Conversely, assume Π^*_ψ is the identity map, i.e., $\Pi^*_\psi(v)(T) = v(T)$ for all $v \in \mathcal{G}_N$ and all $T \subseteq N$. By applying equation (9) to u_T it follows that $tp^t_t = 1$ for $t = 1, \ldots, n$ and $up^t_u - (t-u)p^t_{u+1} = 0$ for all u such that $1 \leq u < t \leq n$. By induction, it is easy to see that $p_s = p^n_s = (s-1)!(n-s)!/n!$ whenever $1 \leq s \leq n$, and this means that $\psi = \varphi$, the Shapley value.

(iii) For any additive game v the projection property implies $\Pi^*_\psi(v) = v$. Now, let us suppose that $\Pi^*_\psi(v)(S) = v(S)$ for all $S \subseteq N$. We will prove that $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$ (equivalent to the additivity of v) by induction on $s = |S|$. If $s = 2$ then $S = \{i_1, i_2\}$ and formula (9) gives

$$(1 - 2p^2_2)v(\{i_1, i_2\}) = \big(p^2_1 - p^2_2\big)\big[v(\{i_1\}) + v(\{i_2\})\big] = (1 - 2p^2_2)\big[v(\{i_1\}) + v(\{i_2\})\big].$$

Condition $p_2^2 \neq 1/2$ implies that $v(\{i_1, i_2\}) = v(\{i_1\}) + v(\{i_2\})$. Now, let $2 < k \leq n$, assume that $v(S) = \sum_{i \in S} v(\{i\})$ whenever $|S| < k$, and consider a coalition $S = \{i_1, \ldots, i_k\}$ of cardinality k. Then

$$(1 - kp_k^k)v(\{i_1, \ldots, i_k\}) = \sum_{U \subset \{i_1, \ldots, i_k\}} \big[up_u^k - (k-u)p_{u+1}^k\big]v(U) =$$

$$= \sum_{u=1}^{k-1} \binom{k-1}{u-1} \big[up_u^k - (k-u)p_{u+1}^k\big]\big[v(\{i_1\}) + \cdots + v(\{i_k\})\big].$$

The sum appearing as a coefficient of $[v(\{i_1\}) + \cdots + v(\{i_k\})]$ is

$$p_1^k + \sum_{u=2}^{k-1} \binom{k-1}{u-1} up_u^k - \sum_{u=2}^{k} \binom{k-1}{u-2}(k-u+1)p_u^k =$$

$$= p_1^k + \sum_{u=2}^{k-1} \left[\binom{k-1}{u-1}u - \binom{k-1}{u-2}(k-u+1)\right]p_u^k - (k-1)p_k^k =$$

$$= p_1^k + \sum_{u=2}^{k-1} \binom{k-1}{u-1} p_u^k + p_k^k - kp_k^k = 1 - kp_k^k,$$

and therefore $(1-kp_k^k)v(\{i_1, \ldots, i_k\}) = (1-kp_k^k)\big[v(\{i_1\}) + \cdots + v(\{i_k\})\big]$. As $p_k^k \neq 1/k$ for $k = 3, \ldots, n$, it follows that $v(S) = v(\{i_1\}) + \cdots + v(\{i_k\})$. $\square$

Remark 3.6 Condition $p_n^n > 0$ is necessary and sufficient for Π_ψ^* to be an automorphism of $\mathcal{G}_N$, and this means that the ψ–power game allows us to regenerate the original game in all cases. On the contrary, whenever $p_n^n = 0$ there are pairs of distinct games $v, v' \in \mathcal{G}_N$ sharing a ψ–power game, i.e., such that $\Pi_\psi^*(v) = \Pi_\psi^*(v')$. Given $v \in \mathcal{G}_N$, the set of games having the same ψ–power game as v is the additive coset $v + \ker(\Pi_\psi^*)$. The subspace $\ker(\Pi_\psi^*)$ is spanned by the unanimity games u_T such that $p_t^t = 0$. It is easy to see that $p_t^t = 0$ for some t with $2 \leq t \leq n$ is equivalent to $p_s^s = 0$ for all s such that $t \leq s \leq n$, and also to $p_s^n = 0$ for all such s. Then, if p_{t-1}^n, where $1 < t \leq n$, is the last nonnull weighting coefficient of ψ, then the subspace $\ker(\Pi_\psi^*)$ is spanned by the unanimity games u_S with $t \leq s \leq n$, so that $\dim \ker(\Pi_\psi^*) = \sum_{s=t}^n \binom{n}{s}$. The extreme case corresponds to the semivalue defined by $p_2^2 = 0$, i.e., $p_2^n = \cdots = p_n^n = 0$ or, equivalently, $p_1^n = 1$: this is the dictatorial index. In this case, $\dim \ker(\Pi_\psi^*) = 2^n - n - 1$.

Remark 3.7 Conditions $p_t^t \neq 1/t$ for $t = 2, \ldots, n$ are all necessary in Corollary 3.5(iii). In order to see it, let us look at the Banzhaf value β. From equation (11) it follows that $\Pi_\beta^*(u_{\{1,2\}}) = u_{\{1,2\}}$ although the unanimity game $u_{\{1,2\}}$ is not additive.

For every semivalue ψ, the additive games are all invariant under the ψ–power map Π_ψ^*. In general, the subspace of games invariant under Π_ψ^* (eigenvectors with eigenvalue equal to 1) is spanned by the unanimity games u_T such that $p_t^t = 1/t$, and therefore $\dim \ker(\Pi_\psi^* - id) = \sum_{t:\, p_t^t = 1/t} \binom{n}{t}$, where id denotes the identity map. An extreme case corresponds to the semivalue such that $p_t^t = 1/t$ for all t from 1 to n: the Shapley value. In this case, the subspace of invariant games coincides with $\mathcal{G}_N$.

The next property provides an easy procedure to obtain the MLE of the power game $\Pi^*_\psi(v)$: it suffices to add, as a factor, the eigenvalue tp_t^t to each product of t variables in the MLE of the original game v.

Proposition 3.8 *Let ψ be a semivalue on $\mathcal{G}_N$, defined by the weighting vector $(p_s)_{s=1}^n$, and let the MLE of $v \in \mathcal{G}_N$ be given by equation (7). Then:*

(i) The MLE of game $\Pi^*_\psi(v)$ is

$$f_{\Pi^*_\psi(v)}(X_N) = \sum_{T\subseteq N:\, T\neq\emptyset} tp_t^t \alpha_T \prod_{i\in T} x_i.$$

(ii) The ψ^t–power of any restricted game $v_{|T}$ is $\Pi_{\psi^t}(v_{|T}) = f_{\Pi^*_\psi(v)}(1_T, 0_{N\setminus T})$, where $(1_T, 0_{N\setminus T})$ means $x_i = 1$ if $i \in T$ and $x_i = 0$ otherwise.

Proof. (i) Since all unanimity games are eigenvectors of Π^*_ψ, we have

$$\Pi^*_\psi(v) = \sum_{T\subseteq N:\, T\neq\emptyset} \alpha_T \Pi^*_\psi(u_T) = \sum_{T\subseteq N:\, T\neq\emptyset} tp_t^t \alpha_T u_T$$

and the expression of the MLE of game $\Pi^*_\psi(v)$ readily follows from property (iii) of the MLE (see the end of Section 2).

(ii) It derives from $\Pi_{\psi^t}(v_{|T}) = \Pi^*_\psi(v)(T) = f_{\Pi^*_\psi(v)}(1_T, 0_{N\setminus T})$ for all $T \subseteq N$. $\square$

Example 3.9 Let us consider the *weighted majority game* $[68; 46, 42, 23, 15, 9]$, i.e., the 5–person (simple) game v defined by $v(S) = 1$ if $\sum_{i\in S} w_i \geq q$ (*winning* coalitions) and $v(S) = 0$ otherwise, where $q = 68$ is the *quota* and the *weights* are $w_1 = 46$, $w_2 = 42$, and so on up to $w_5 = 9$ (incidentally, this game describes the Catalonia Parliament during Legislature 2003–2007). The set of minimal winning coalitions in this game is $W^m(v) = \{\{1,2\},\{1,3\},\{1,4,5\},\{2,3,4\},\{2,3,5\}\}$ and its MLE is

$$f_v(x_1,x_2,x_3,x_4,x_5) = x_1x_2 + x_1x_3 - x_1x_2x_3 + x_1x_4x_5 + x_2x_3x_4 + \\ + x_2x_3x_5 - \sum_{i<j<k<l} x_ix_jx_kx_l + 2x_1x_2x_3x_4x_5.$$

Let us consider the semivalue $\psi = \psi^5$ defined by the weighting vector

$$(p_s^5)_{s=1}^5 = (5/48, 1/12, 1/16, 1/24, 1/48).$$

Notice that semivalue ψ^5 weights the marginal contributions by means of coefficients that decrease a constant amount whenever the coalition size increases a unit (we say that such a semivalue is an *arithmetic* semivalue with increase $-1/48$). We first compute the last weighting coefficients for its induced semivalues:

$$p_5^5 = 1/48, \quad p_4^4 = 1/16, \quad p_3^3 = 1/6, \quad p_2^2 = 5/12 \quad \text{and} \quad p_1^1 = 1.$$

According to Proposition 3.8, the MLE of the power game $\Pi^*_\psi(v)$ can be obtained from the MLE of v, and we get

$$f_{\Pi^*_\psi(v)}(x_1,x_2,x_3,x_4,x_5) = \frac{5}{6}\,[x_1x_2 + x_1x_3] + \frac{1}{2}\,[-x_1x_2x_3 + x_1x_4x_5 + x_2x_3x_4 + \\ + x_2x_3x_5] - \frac{1}{4}\sum_{i<j<k<l} x_ix_jx_kx_l + \frac{5}{24}\,x_1x_2x_3x_4x_5.$$

Now, the power assigned by semivalue ψ to (the players of) the original game or anyone of its restricted games can be easily computed. For instance,

$$\Pi_\psi(v) = \Pi^*_\psi(v)(N) = f_{\Pi^*_\psi(v)}(1,1,1,1,1) = 13/8,$$

and, if e.g. $T = \{1,2,3\}$,

$$\Pi_{\psi^3}(v_{|T}) = \Pi^*_\psi(v)(T) = f_{\Pi^*_\psi(v)}(1,1,1,0,0) = 7/6.$$

4 The potential map

The results obtained for power in the preceding section will have here their respective counterparts for potential.

Definition 4.1 (Dragan [7]) Let ψ be a semivalue on $\mathcal{G}_N$ defined by the weighting vector $(p_s)_{s=1}^n$. The *potential* of (the semivalue ψ for) the restriction of a given game $v \in \mathcal{G}_N$ to a nonempty coalition $T \subseteq N$ is defined by

$$P_\psi(v_{|T}) = \sum_{S \subseteq T} p_s^t v(S). \tag{12}$$

The characteristic of a potential is the following: if $i \in T \subseteq N$ and $v \in \mathcal{G}_N$, then

$$\begin{array}{ll} \text{(i)} & P_\psi(v_{|T}) - P_\psi(v_{|T\setminus\{i\}}) = \psi_i^t[v_{|T}] \quad \text{if } |T| > 1, \\ \text{(ii)} & P_\psi(v_{|\{i\}}) = v(\{i\}) = \psi_i^1[v_{|\{i\}}] \quad \text{if } |T| = 1. \end{array}$$

Definition 4.2 (Dragan [7]) Let ψ be a semivalue on $\mathcal{G}_N$ defined by the weighting vector $(p_s)_{s=1}^n$. The *ψ–potential map* is the endomorphism P^*_ψ of $\mathcal{G}_N$ that transforms any game v in the game $P^*_\psi(v)$ defined by

$$P^*_\psi(v)(S) = P_\psi(v_{|S}) \quad \text{for all } S \subseteq N. \tag{13}$$

Thus $P^*_\psi(v)$, the *ψ–potential game* of v, assigns to each coalition S the potential of the semivalue ψ for the restricted game $v_{|S}$.

Theorem 4.3 *Let ψ be a semivalue on $\mathcal{G}_N$ defined by the weighting vector $(p_s)_{s=1}^n$. Every unanimity game u_T is an eigenvector for the linear mapping P^*_ψ with eigenvalue p_t^t, i.e.,*

$$P^*_\psi(u_T) = p_t^t\, u_T \quad \textit{if } \emptyset \neq T \subseteq N. \tag{14}$$

Proof. For each $S \subseteq N$ we write $P^*_\psi(u_T)(S)$ according to (12):

$$P^*_\psi(u_T)(S) = P_\psi(u_{T|S}) = \sum_{U \subseteq S} p_u^s u_T(U).$$

If $S \not\supseteq T$, then no coalition $U \subseteq S$ contains T and hence $P^*_\psi(u_T)(S) = 0$. Moreover, $P^*_\psi(u_T)(T) = p_t^t$. If $S \supset T$, then

$$P^*_\psi(u_T)(S) = \sum_{U \subseteq S:\, T \subseteq U} p_u^s = \sum_{u=t}^{s} \binom{s-t}{u-t} p_u^s = \sum_{j=0}^{s-t} \binom{s-t}{j} p_{t+j}^s = p_t^t. \quad \square$$

Corollary 4.4 *Let ψ be a semivalue on $\mathcal{G}_N$ defined by the weighting vector $(p_s)_{s=1}^n$. The ψ–potential map P_ψ^* satisfies the following properties:*

(i) P_ψ^* is a bijection iff $p_n^n > 0$.

(ii) P_ψ^* is the identity map in G_N iff $\psi = \mu$, the marginal index.

(iii) If v is an additive game then $P_\psi^*(v) = v$. If $p_n^n \neq 1$ then the converse is true.

Proof. (i) It is very similar to the proof of Corollary 3.5(i).

(ii) $P_\psi^*(v) = v$ for all $v \in \mathcal{G}_N$ is equivalent to $P_\psi^*(u_T) = u_T$ for all nonempty $T \subseteq N$, and also to $p_t^t = 1$ whenever $1 \leq t \leq n$. And this set of conditions is merely equivalent to $p_n^n = 1$, i.e., to the fact that ψ is the marginal index.

(iii) Every additive game v is a linear combination of unanimity games u_T with $|T| = 1$; since these games are eigenvectors for P_ψ^* with eigenvalue 1, it follows that $P_\psi^*(v) = v$. Now, let us suppose that $P_\psi^*(v)(S) = v(S)$ for all $S \subseteq N$. We will prove that $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$ by induction on $s = |S|$. For $s = 2$, $S = \{i_1, i_2\}$ and formula (12) gives

$$v(\{i_1, i_2\}) = P_\psi^*(v)(\{i_1, i_2\}) = p_2^2 v(\{i_1, i_2\}) + p_1^2[v(\{i_1\}) + v(\{i_2\})]$$

and therefore

$$(1 - p_2^2)v(\{i_1, i_2\}) = (1 - p_2^2)[v(\{i_1\}) + v(\{i_2\})].$$

Condition $p_n^n \neq 1$ implies that $p_s^s \neq 1$ for $s = 2, \ldots, n-1$, and hence $v(\{i_1, i_2\}) = v(\{i_1\}) + v(\{i_2\})$. Now, let $2 < k \leq n$, assume that $v(S) = \sum_{i \in S} v(\{i\})$ whenever $|S| < k$, and consider a coalition $S = \{i_1, \ldots, i_k\}$ of cardinality k. Then

$$(1 - p_k^k)v(\{i_1, \ldots, i_k\}) = \sum_{U \subset \{i_1, \ldots, i_k\}} p_u^k v(U) = \sum_{u=1}^{k-1} \binom{k-1}{u-1} p_u^k \big[v(\{i_1\}) + \cdots + v(\{i_k\})\big].$$

Since

$$\sum_{u=1}^{k-1} \binom{k-1}{u-1} p_u^k = \sum_{u=1}^{k} \binom{k-1}{u-1} p_u^k - p_k^k = 1 - p_k^k \neq 0,$$

we conclude that $v(S) = v(\{i_1\}) + \cdots + v(\{i_k\})$. $\square$

The next property allows us to compute the allocations given by any semivalue on any game v by using convenient modifications of the MLE of v. From the Harsanyi dividends $\alpha_T = \alpha_T(v)$, given in equation (1), we define the following amounts for any nonempty coalition $T \subseteq N$, any $i \in T$ and each $u = 1, 2, \ldots, t = |T|$:

$$\gamma_i^u(v, T) = \sum_{\substack{U \subseteq T: \\ i \in U,\ |U| = u}} \alpha_U.$$

Proposition 4.5 *Let ψ be a semivalue on $\mathcal{G}_N$, defined by the weighting vector $(p_s)_{s=1}^n$, and let the MLE of $v \in \mathcal{G}_N$ be given by equation (7). Then:*

(i) The MLE of game $P_\psi^*(v)$ is

$$f_{P_\psi^*(v)}(X_N) = \sum_{T \subseteq N:\ T \neq \emptyset} p_t^t \alpha_T \prod_{i \in T} x_i.$$

(ii) If $\emptyset \neq T \subseteq N$, the allocation to player $i \in T$ in the restricted game $v_{|T}$ given by the induced semivalue ψ^t is

$$\psi_i^t[v_{|T}] = \left.\frac{\partial f_{P_\psi^*(v)}(X_T, 0_{N\setminus T})}{\partial x_i}\right|_{1_T} = \sum_{u=1}^{t} p_u^u \gamma_i^u(v,T). \tag{15}$$

Proof. (i) Since all unanimity games are eigenvectors of P_ψ^*, we have

$$P_\psi^*(v) = \sum_{T\subseteq N:\, T\neq\emptyset} \alpha_T P_\psi^*(u_T) = \sum_{T\subseteq N:\, T\neq\emptyset} p_t^t \alpha_T u_T,$$

and the expression of the MLE of game $P_\psi^*(v)$ follows from property (iii) of the MLE.

(ii) Let $|T| > 1$. On one hand, and according to characteristic (i) in Definition 4.1, we have

$$\psi_i^t[v_{|T}] = P_\psi(v_{|T}) - P_\psi(v_{|T\setminus\{i\}}) = P_\psi^*(v)(T) - P_\psi^*(v)(T\setminus\{i\}).$$

On the other hand, $f_{P_\psi^*(v)}(X_T, 0_{N\setminus T})$ is the MLE of the restriction of game $P_\psi^*(v)$ to T. Using property (iv) of the MLE, the partial derivative of this MLE with respect to a variable x_i, evaluated at 1_T, gives the marginal contribution of player i to coalition T, that is, $P_\psi^*(v)(T) - P_\psi^*(v)(T\setminus\{i\})$, and the first equality follows at once. As for the second equality, it suffices to see that the computation of the partial derivative with respect to variable x_i in $f_{P_\psi^*(v)}(X_T, 0_{N\setminus T})$ leads us to consider only the terms of those U such that $i \in U \subseteq T$. Then, when we replace the remaining variables with 1, each $\alpha_U(v)$ appears with the corresponding factor p_u^u according to the number u of variables in each initial product.

Finally, let $|T| = 1$. Then $T = \{i\}$ and $\psi_i^1[v_{\{i\}}] = v(\{i\}) = p_1^1 v(\{i\})$. □

Example 4.6 We go back to Example 3.9 and use the same semivalue as there. The MLE of the ψ–potential game $P_\psi^*(v)$ is

$$f_{P_\psi^*(v)}(x_1,x_2,x_3,x_4,x_5) = \frac{5}{12}\,[x_1x_2 + x_1x_3] + \frac{1}{6}\,[-x_1x_2x_3 + x_1x_4x_5 + x_2x_3x_4 + \\ + x_2x_3x_5] - \frac{1}{16}\sum_{i<j<k<l} x_ix_jx_kx_l + \frac{1}{24}\,x_1x_2x_3x_4x_5.$$

The payoffs given by ψ in v can be computed from the MLE. To this end, we apply the second equality in equation (15) for $T = N$. Thus,

$$\begin{aligned}
&\psi_1[v] = (1, 5/12, 1/6, 1/16, 1/48)\cdot(0,2,0,-4,2) = 5/8,\\
&\psi_2[v] = \psi_3[v] = (1, 5/12, 1/6, 1/16, 1/48)\cdot(0,1,1,-4,2) = 3/8,\\
&\psi_4[v] = \psi_5[v] = (1, 5/12, 1/6, 1/16, 1/48)\cdot(0,0,2,-4,2) = 1/8.
\end{aligned}$$

We can also compute allocations on restricted games. For instance, for game $v_{|T}$ where $T = \{1,2,3\}$, we first write the restricted MLE

$$f_{P_\psi^*(v)}(X_{\{1,2,3\}}, 0_{\{4,5\}}) = \frac{5}{12}(x_1x_2 + x_1x_3) - \frac{1}{6}\,x_1x_2x_3$$

and find therefore

$$\psi^3[v_{|T}] = (2/3, 1/4, 1/4).$$

Remark 4.7 The procedure for computing semivalue allocations from the MLE of a game consists, in general, of two steps. In the case of the Banzhaf value (Owen [19]), for each player i we first obtain the partial derivative with respect to variable x_i and replace next all the other variables with $1/2$. For products where variable x_i appears, this action replaces each product of k variables in the MLE with the amount $1/2^{k-1}$, which is nothing but p_k^k, the last weighting coefficient of the induced Banzhaf value on cardinality k.

A similar procedure is stated in Amer and Giménez [1] for all binomial semivalues. In this case, after partial differentiation we replace the remaining variables with α so that, again, each product of k variables in the MLE of the game has been replaced with α^{k-1}, the last weighting coefficient of the induced semivalue ψ_α^k.

The procedure for the Shapley value is not different. According to Owen [18], after partial differentiation with respect to x_i we replace the remaining variables with a unique variable and integrate between 0 and 1. Each product of k variables in the MLE containing variable x_i gives rise to amount $1/k$, which is also the last weighting coefficient of the (induced) Shapley value on cardinality k.

These three situations exhibit a common procedure: first, the marginal contributions appear (after partial differentiation) and, second, they are weighted according to the selected solution (after introducing $1/2$, α, or a variable with integration).

Instead, the consideration of the modified MLE $f_{P_\psi^*(v)}$ obtained from the initial one has changed the procedure: first, we weight the terms of the initial MLE and, second, we obtain players' marginal contributions by partial differentiation. This new procedure has an advantage with respect to the traditional method: the allocations given by any semivalue are available since the weighting coefficients p_k^k can be always obtained. In addition, we can split the allocations into two components, as indicated in Proposition 4.5(ii): for every nonempty $T \subseteq N$,

$$\psi_i^t[v_{|T}] = \sum_{u=1}^{t} p_u^u \gamma_i^u(v,T).$$

Here, the coefficients p_u^u are related to the semivalue, whereas the amounts $\gamma_i^u(v,T)$ depend only on the considered game. They are combined in equation (15) by means of the inner product in $\mathbb{R}^t$.

5 Comparing the maps

In this final section we consider together the power and potential maps attached to a given semivalue ψ. We first show their matrices in the basis of the unanimity games. Incidentally, we would like to mention that these matrices might be used to supply alternative proofs for Corollaries 3.5 and 4.4. For e.g. $n = 3$, this basis is

$$\mathcal{B} = \{u_{\{1\}}, u_{\{2\}}, u_{\{3\}}, u_{\{1,2\}}, u_{\{1,3\}}, u_{\{2,3\}}, u_{\{1,2,3\}}\}$$

and the matrices are

$$\Pi_\psi^* : \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 2p_2^2 & & & \\ & & & & 2p_2^2 & & \\ & & & & & 2p_2^2 & \\ & & & & & & 3p_3^3 \end{pmatrix}, \qquad P_\psi^* : \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & p_2^2 & & & \\ & & & & p_2^2 & & \\ & & & & & p_2^2 & \\ & & & & & & p_3^3 \end{pmatrix}.$$

Then these maps commute: $\Pi^*_\psi \circ P^*_\psi = P^*_\psi \circ \Pi^*_\psi$ for any semivalue ψ.

In the particular case where $\psi = \psi_\alpha$ (α–binomial semivalue), with $p^t_{\alpha,t} = \alpha^{t-1}$ for $1 \le t \le n$, these matrices reduce to

$$\Pi^*_{\psi_\alpha}: \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 2\alpha & & & \\ & & & & 2\alpha & & \\ & & & & & 2\alpha & \\ & & & & & & 3\alpha^2 \end{pmatrix}, \qquad P^*_{\psi_\alpha}: \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & \alpha & & \\ & & & & & \alpha & \\ & & & & & & \alpha^2 \end{pmatrix}.$$

Coming back to the general case of ψ (always for $n = 3$), it follows that the maps can be given the following formulas in terms of the Harsanyi dividends $\alpha_T = \alpha_T(v)$ (let us use here a slightly simplified notation): if

$$v = \big[\alpha_1 u_{\{1\}} + \alpha_2 u_{\{2\}} + \alpha_3 u_{\{3\}}\big] + \big[\alpha_{1,2} u_{\{1,2\}} + \alpha_{1,3} u_{\{1,3\}} + \alpha_{2,3} u_{\{2,3\}}\big] + \big[\alpha_{1,2,3} u_{\{1,2,3\}}\big],$$

then

$$\begin{aligned} \Pi^*_\psi(v) =& \big[\alpha_1 u_{\{1\}} + \alpha_2 u_{\{2\}} + \alpha_3 u_{\{3\}}\big] + \\ &+ 2p^2_2\big[\alpha_{1,2} u_{\{1,2\}} + \alpha_{1,3} u_{\{1,3\}} + \alpha_{2,3} u_{\{2,3\}}\big] + 3p^3_3\big[\alpha_{1,2,3} u_{\{1,2,3\}}\big], \\ P^*_\psi(v) =& \big[\alpha_1 u_{\{1\}} + \alpha_2 u_{\{2\}} + \alpha_3 u_{\{3\}}\big] + \\ &+ p^2_2\big[\alpha_{1,2} u_{\{1,2\}} + \alpha_{1,3} u_{\{1,3\}} + \alpha_{2,3} u_{\{2,3\}}\big] + p^3_3\big[\alpha_{1,2,3} u_{\{1,2,3\}}\big]. \end{aligned}$$

Let us pay attention to the difference $\Pi^*_\psi(v) - P^*_\psi(v)$ for any $v \in \mathcal{G}_N$. One might say, in general, that games $v, v' \in \mathcal{G}_N$ are *equivalent* iff $v - v' \in \mathcal{AG}_N$: this would be interpreted in the sense that these games are not very distinct since the difference game is an additive (i.e., inessential) game. Then $\Pi^*_\psi(v)$ and $P^*_\psi(v)$ are equivalent from this viewpoint iff $\Pi^*_\psi(v) = P^*_\psi(v)$. This occurs iff either $v \in \mathcal{AG}_N$ or $\psi = \delta$ (in which case the equality holds for all $v \in \mathcal{G}_N$). Thus, if v is not additive and ψ is not the dictatorial index then the difference between the ψ–power and ψ–potential games of v could be considered as essential.

Now, let us denote as $\mathcal{G}^t_N$, for $1 \le t \le n$, the subspace spanned by $\{u_T : |T| = t\}$. Notice that $\mathcal{G}^1_N = \mathcal{AG}_N$. We have a splitting of $\mathcal{G}_N$ as a direct sum of these subspaces,

$$\mathcal{G}_N = \mathcal{G}^1_N \oplus \mathcal{G}^2_N \oplus \cdots \oplus \mathcal{G}^n_N,$$

which are, moreover, invariant under both Π^*_ψ and P^*_ψ: these endomorphisms act proportionally to each other on each such subspace and, in particular, both of them coincide on $\mathcal{G}^1_N$ with the identity map. As for the kernel and the subspace of invariant games of each map, different possibilities arise according to the semivalue ψ we are using. For simplicity, we will restrict the discussion to the binomial semivalues ψ_α.

If $1 \in H \subseteq \{1, \ldots, n\}$, let Q^H be the projection from $\mathcal{G}_N$ onto $\bigoplus_{h \in H} \mathcal{G}^h_N$, defined by

$$Q^H(v) = \sum_{T \subseteq N:\ |T| \in H} \alpha_T u_T \quad \text{if} \quad v = \sum_{T \subseteq N:\ T \neq \emptyset} \alpha_T u_T.$$

Then we have:

(a) If $\alpha = 0$ then $\psi_0 = \delta$, the dictatorial index. In this case $\Pi^*_\delta = P^*_\delta = Q^{\{1\}}$ and:

(1) The common kernel is the greatest one: $\ker(\Pi^*_\delta) = \ker(P^*_\delta) = \bigoplus_{t=2}^n \mathcal{G}^t_N$.

(2) The common subspace of invariant games is the smallest: $\ker(\Pi^*_\delta - id) = \ker(P^*_\delta - id) = \mathcal{G}^1_N$.

(b) If $\alpha = 1$ then $\psi_1 = \mu$, the marginal index. In this case $P^*_\mu = id$ and:

(1) The common kernel is the smallest one: $\ker(\Pi^*_\mu) = \ker(P^*_\mu) = \{0\}$.

(2) The subspaces of invariant games are, respectively, the smallest and the greatest one: $\ker(\Pi^*_\mu - id) = \mathcal{G}^1_N$ and $\ker(P^*_\mu - id) = \mathcal{G}_N$.

(c) If $0 < \alpha < 1$ (in particular, $\alpha = 1/2$ gives rise to $\psi_{1/2} = \beta$, the Banzhaf value) then:

(1) $\ker(\Pi^*_{\psi_\alpha}) = \ker(P^*_{\psi_\alpha}) = \{0\}$.

(2) The subspaces of invariant games are as follows. While $\ker(P^*_{\psi_\alpha} - id) = \mathcal{G}^1_N$ in all cases, $\ker(\Pi^*_{\psi_\alpha} - id)$ contains $\mathcal{G}^1_N$ but may well include other subspaces. For instance, if $n \geq 2$, $\ker(\Pi^*_\beta - id) = \mathcal{G}^1_N \oplus \mathcal{G}^2_N$; in general, if $\alpha = t^{\frac{1}{1-t}}$ for some t among $2, \ldots, n$ then $\mathcal{G}^t_N$ lies also in $\ker(\Pi^*_{\psi_\alpha} - id)$.

Finally, as to the asymptotic behavior when compounding each endomorphism with itself, for ψ in general we have ($n \geq 2$):

$$\text{(a)} \quad \lim_{m\to\infty} \left(\Pi^*_\psi\right)^m = \begin{cases} \nexists & \text{if} \quad p^t_t > 1/t \ \text{ for some } \ t \geq 2, \\ Q^{\{1\}} & \text{if} \quad p^t_t < 1/t \ \text{ for all } \ t \geq 2, \\ Q^H & \text{otherwise, where } H = \{t : p^t_t = 1/t\}, \end{cases}$$

$$\text{(b)} \quad \lim_{m\to\infty} \left(P^*_\psi\right)^m = \begin{cases} Q^{\{1\}} & \text{if} \quad p^t_t < 1 \ \text{ for } \ t = 2, \ldots, n, \\ id & \text{if} \quad p^t_t = 1 \ \text{ for } \ t = 2, \ldots, n, \end{cases}$$

and for ψ_α in particular:

$$\text{(a')} \quad \lim_{m\to\infty} \left(\Pi^*_{\psi_\alpha}\right)^m = \begin{cases} \nexists & \text{if} \quad \alpha > 1/2, \\ Q^{\{1,2\}} & \text{if} \quad \alpha = 1/2, \\ Q^{\{1\}} & \text{if} \quad \alpha < 1/2, \end{cases}$$

$$\text{(b')} \quad \lim_{m\to\infty} \left(P^*_{\psi_\alpha}\right)^m = \begin{cases} Q^{\{1\}} & \text{if} \quad \alpha < 1, \\ id & \text{if} \quad \alpha = 1. \end{cases}$$

The third case of (a) includes the Shapley value φ, for which $p^t_t = 1/t$ for all t, thus $H = \{1, 2, \ldots, n\}$, the sequence is constant and $Q^H = id$; and also, e.g., the semivalue defined for $n = 5$ by $(p_s)_{s=1}^5 = (0.60, 0, 0, 0.05, 0.20)$, which gives $Q^{\{1,4,5\}}$ as a limit. The second case of (b) corresponds to the marginal index μ, and the sequence is constant again. The second case of (a') is the Banzhaf value. The first case of (b') includes the dictatorial index δ (constant sequence) and the Banzhaf value, while the second case corresponds once more to the marginal index μ.

Two elementary points are useful to derive the above classification: (i) if $(p_s)_{s=1}^n$ is the weighting vector defining ψ, then all p^t_t for $t \geq 2$ are simultaneously either $= 1$ or < 1; and (ii) if $2\alpha \leq 1$ then $t\alpha^{t-1} < 1$ for all $t > 2$.

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