A New Approach to the Spectral Excess Theorem for Distance-Regular Graphs

M.A. Fiol[a], S. Gago[b], E. Garriga[a]

[a]*Dept. Matemàtica Aplicada IV, Universitat Politècnica de Catalunya, Jordi Girona 1–3, Mòdul C3, Campus Nord, 08034 Barcelona, Catalonia, Spain*
[b]*Dept. Matemàtica Aplicada IV, Universitat Politècnica de Catalunya, CIMNE C3, Av. Canal Olímpic s/n, 08860 Castelldefels, Catalonia, Spain*

Abstract

The Spectral Excess Theorem provides a quasi-spectral characterization for a (regular) graph Γ with $d+1$ different eigenvalues to be distance-regular graph, in terms of the mean (d-1)-excess of its vertices. The original approach, due to Fiol and Garriga in 1997, was obtained in a wide context from a local point of view, so giving a characterization of the so-called pseudo-distance-regularity around a vertex. In this paper we present a new simple method based in a global point of view, and where the mean degree of the distance-d graph Γ_d plays an essential role.

Key words: Distance-regular graphs, Eigenvalues, Excess

1. Introduction

Throughout all this paper $\Gamma = (V, E)$ denotes a (simple and finite) connected graph of order n. For any vertex $u \in V$, $\Gamma(u)$ denotes the set of its adjacent vertices, and $\delta(u) := |\Gamma(u)|$ stands for its degree. The distance between two vertices u, v is represented by $\partial(u, v)$. The eccentricity of a vertex

Email addresses: `fiol@ma4.upc.edu` (M.A. Fiol), `sgago@ma4.upc.edu` (S. Gago), `egarriga@ma4.upc.edu` (E. Garriga)

u is denoted by $\varepsilon(u) := \max_{v\in V} \partial(u,v)$ and the diameter of the graph is $D(\Gamma) := \max_{u\in V} \varepsilon(u) = \max_{u,v\in V} \partial(u,v)$.

Let $\Gamma_k(u)$ be the set of vertices at distance k from u, for $0 \leq k \leq \varepsilon(u)$, and let Γ_k be the graph with the same vertex set as Γ and where two vertices are adjacent whenever they are at distance k in Γ. Notice that $\Gamma_1(u) = \Gamma(u)$ and $\Gamma_1 = \Gamma$. The k-neighborhood of a vertex u, $N_k(u)$, is the set of vertices which are at distance at most k from u, that is, $N_k(u) := \bigcup_{l=0}^{k} \Gamma_l(u)$. The cardinal of the set $|V \backslash N_k(u)|$ is called the k-excess of the vertex u, and it is denoted by $e_k(u)$. Trivially $e_0(u) = n-1$ and $e_{\varepsilon(u)}(u) = 0$.

Now consider $\boldsymbol{A}$, the adjacency matrix of Γ. The algebra generated by $\boldsymbol{A}$ denoted by $\mathcal{A} = \mathcal{A}(\boldsymbol{A}) := \{p(\boldsymbol{A}),\, p \in \mathbb{R}[x]\}$, is called the adjacency algebra or Bose-Mesner algebra. As usual $\boldsymbol{J}$ stands for the square matrix with all entries equal to 1, and similarly $\boldsymbol{j} \in \mathbb{R}^n$ is the all-1-vector. The spectrum of Γ is the set of different eigenvalues of $\boldsymbol{A}$ together with their multipliticies, and it will be denoted by $\operatorname{sp}\Gamma := \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, with $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and where $m_0 = 1$ since Γ is a connected graph.

2. The predistance polynomials

Given a *mesh* of real numbers, $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ and a set of positive real numbers called *weights*, $g_0, g_1, \ldots, g_d$, verifying $\sum_{i=0}^{d} g_i = 1$, consider the following operation

$$\langle p, q\rangle = \sum_{i=0}^{d} g_i p(\lambda_i) q(\lambda_i). \tag{1}$$

It defines an scalar product in the quotient polynomial algebra $\mathbb{R}[x]/(Z)$, where (Z) is the ideal generated by the polynomial $Z = \prod_{i=0}^{d}(x-\lambda_i)$. A family of polynomials $r_0, r_1, \ldots, r_d$, satisfying $\langle r_k, r_h\rangle = 0$ for any $k \neq h$ and that every polynomial r_k has degree k, is called an orthogonal system. For such a family, it is well known that there exist constants a_i,b_i,c_i, verifying the recurrence condition $xr_i = b_{i-1}r_{i-1} + a_i r_i + c_{i+1}r_{i+1}$, for any $i = 0, 1, \ldots, d$, where $b_{-1} = c_{d+1} = 0$. In particular, we are interested in the orthogonal system associated to (1), called the *canonical orthogonal system*, as it is the only orthogonal system $p_0, p_1, \ldots, p_d$ verifying any of the following equivalent conditions:

(a) $\|p_k\|^2 = p_k(\lambda_0), \quad 0 \leq k \leq d.$

(b) $p_0 + p_1 + \cdots + p_d = \frac{1}{g_0\pi_0} \prod_{h=1}^{d}(x-\lambda_h)$.

(c) $b_k + a_k + c_k = \lambda_0, \quad 0 \le k \le d.$

The symbol π_0 and, in general, π_i is defined as $\pi_i = \prod_{(h\neq i)\, h=0}^{d} |\lambda_i - \lambda_h|$ for every $i = 0, 1, \ldots, d$. For more details about the proofs of these results, we refer the reader to [2].
Moreover, for our purpose we are interested in the explicit expression of $p_d(\lambda_0)$, which can be computed in terms of the eigenvalues as follows:

$$p_d(\lambda_0) = \frac{1}{g_0^2 \pi_0^2} \left(\sum_{i=0}^{d} \frac{1}{g_i \pi_i^2} \right)^{-1}. \tag{2}$$

Now let $\Gamma = (V, E)$ be a connected graph of order n, and let $\boldsymbol{A}$ be its adjacency matrix. From its spectrum $\operatorname{sp}\Gamma$, consider in $\mathbb{R}[x]/(Z)$ the scalar product

$$\langle p, q \rangle = \sum_{i=0}^{d} \frac{m_i}{n} p(\lambda_i) q(\lambda_i). \tag{3}$$

We define as the *predistance polynomials* of Γ the polynomials constituting the orthogonal canonical system associated to this scalar product(3). Observe that the scalar product (3) is the one defined in (1) when the numbers of the mesh are the eigenvalues of $\boldsymbol{A}$ and the weights are the multiplicities divided by n. Therefore, the predistance polynomials verify any of the following equivalent conditions

(a) $\|p_k\|^2 = p_k(\lambda_0), \quad 0 \le k \le d.$

(b) $p_0 + p_1 + \cdots + p_d = \frac{n}{\pi_0} \prod_{h=1}^{d} (x - \lambda_h).$

(c) $b_k + a_k + c_k = \lambda_0, \quad 0 \le k \le d.$

We note that in the regular case $\frac{n}{\pi_0} \prod_{h=1}^{d} (x - \lambda_h)$ is the Hoffman polynomial H, which verifies $H(\boldsymbol{A}) = \boldsymbol{J}$. Moreover, we have an expression for $p_d(\lambda_0)$ in terms of the eigenvalues of the graph, that is

$$p_d(\lambda_0) = \frac{n}{\pi_0^2} \left(\sum_{i=0}^{d} \frac{1}{m_i \pi_i^2} \right)^{-1}. \tag{4}$$

3. The algebras $\mathcal{A}$ and $\mathcal{D}$

Given a graph Γ, the set $\mathcal{A} = \mathcal{A}(\Gamma) = \{p(\boldsymbol{A}),\, p \in \mathbb{R}[x]\}$ is a vectorial space of dimension $d+1$ and also an algebra with the ordinary product of matrices, known as the Bose-Mesner algebra, and $\{\boldsymbol{I}, \boldsymbol{A}, \ldots, \boldsymbol{A}^d\}$ is a basis of $\mathcal{A}$. Since $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^D$ are linearly independent, we have that $\dim \mathcal{A}(\Gamma) = d+1 \geq D+1$ and, therefore, the diameter is always at most equal to the number of different eigenvalues, or $D \leq d$. Then, a natural question is to enhance the case where the equality is attained, that is $D = d$. In such case, we say that the graph Γ has *spectrally maximum diameter.*

Let $\mathcal{D} = \mathcal{D}(\Gamma)$ be the linear span of the set $\{\boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_D\}$, of which it is a basis or, equivalently, the linear span of $\{\boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_d\}$, where if $D < d$ we take $\boldsymbol{A}_k = 0$ for every $D+1 \leq k \leq d$. It turns out that it forms an algebra with the componentwise Hadamard product of matrices, defined by $(\boldsymbol{X} \circ \boldsymbol{Y})_{uv} = \boldsymbol{X}_{uv}\boldsymbol{Y}_{uv}$. In our context, we will work with the vectorial space $\mathcal{T} = \mathcal{A} + \mathcal{D}$. Note that in the regular case $\boldsymbol{I}$, $\boldsymbol{A}$ and $\boldsymbol{J}$ are matrices in $\mathcal{A} \cap \mathcal{D}$, as $\boldsymbol{A}_0 + \boldsymbol{A}_1 + \cdots + \boldsymbol{A}_D = \boldsymbol{J} = H(\boldsymbol{A}) \in \mathcal{A}$. Thus we have that $\dim \mathcal{T} \leq d + D - 1$. It also holds the equality

$$\boldsymbol{A}_0 + \boldsymbol{A}_1 + \cdots + \boldsymbol{A}_D = \boldsymbol{J} = p_0(\boldsymbol{A}) + p_1(\boldsymbol{A}) + \cdots + p_d(\boldsymbol{A}). \tag{5}$$

For any pair of matrices $\boldsymbol{R}$, $\boldsymbol{S} \in \mathcal{T}$, we obtain

$$\operatorname{tr}(\boldsymbol{R}\boldsymbol{S}) = \sum_u (\boldsymbol{R}\boldsymbol{S})_{uu} = \sum_u \sum_v \boldsymbol{R}_{uv}\boldsymbol{S}_{vu} = \sum_{uv} \boldsymbol{R}_{uv}\boldsymbol{S}_{uv}.$$

Thus we can define an scalar product into $\mathcal{T}$, in two equivalent forms

$$\langle \boldsymbol{R}, \boldsymbol{S} \rangle = \frac{1}{n}\operatorname{tr}(\boldsymbol{R}\boldsymbol{S}) = \frac{1}{n}\sum_{uv} \boldsymbol{R}_{uv}\boldsymbol{S}_{uv} = \frac{1}{n}\sum_{uv} (\boldsymbol{R} \circ \boldsymbol{S})_{uv}. \tag{6}$$

Observe that the factor $1/n$ provides $\|\boldsymbol{I}\| = 1$. Furthermore, we point out that the scalar product (3) in $\mathcal{A}$ can also be expressed as

$$\langle p(\boldsymbol{A}), q(\boldsymbol{A}) \rangle = \frac{1}{n}\operatorname{tr}(p(\boldsymbol{A})q(\boldsymbol{A})) = \frac{1}{n}\sum_{i=0}^{d} m_i p(\lambda_i) q(\lambda_i) = \langle p, q \rangle\,,$$

which is the scalar product (3) in $\mathbb{R}[x]/(Z)$ from which we construct the predistance polynomials.

Lemma 1. *The algebras $\mathcal{A}$ and $\mathbb{R}[x]/(Z)$, with their respective scalar products (6) and (3), are isometric.*

Proof. Just identify both algebras through the isometry $p = p(\boldsymbol{A})$. $\square$

4. The orthogonal projection $\mathcal{D} \to \mathcal{A}$

Throughout all this section we suppose that Γ is a regular graph with spectrally maximum diameter. Consider the euclidean space $\mathcal{T}$ with the scalar product (6) and the orthogonal projection

$$\mathcal{T} \to \mathcal{A} \quad \text{denoted by} \quad \boldsymbol{S} \mapsto \widetilde{\boldsymbol{S}}\,.$$

Using in $\mathcal{A}$ the orthogonal base $p_0, p_1, \ldots, p_d$ of predistance polynomials, this projection can be expressed as

$$\widetilde{\boldsymbol{S}} = \sum_{i=0}^{d} \frac{\langle \boldsymbol{S}, p_i \rangle}{\|p_i\|^2} p_i = \sum_{i=0}^{d} \frac{\langle \boldsymbol{S}, p_i \rangle}{p_i(\lambda_0)} p_i. \tag{7}$$

Now consider the projection of $\boldsymbol{A}_d$

$$\widetilde{\boldsymbol{A}}_d = \sum_{j=0}^{d} \frac{\langle \boldsymbol{A}_d, p_j \rangle}{\|p_j\|^2} p_j = \frac{\langle \boldsymbol{A}_d, p_d \rangle}{\|p_d\|^2} p_d = \frac{\langle \boldsymbol{A}_d, H \rangle}{p_d(\lambda_0)} p_d = \frac{\langle \boldsymbol{A}_d, \boldsymbol{A}_d \rangle}{p_d(\lambda_0)} p_d = \frac{\delta_d}{p_d(\lambda_0)} p_d\,, \tag{8}$$

where $\delta_d = \|\boldsymbol{A}_d\|^2 = \frac{1}{n} \sum_{u,v} (\boldsymbol{A}_d)_{uv} = \frac{1}{n} \sum_{u \in V} |\Gamma_d(u)|$ is the mean degree of Γ_d.

Theorem 2. *For any regular graph Γ, we have $\delta_d \leq p_d(\lambda_0)$, and equality is attained if and only if $\boldsymbol{A}_d = p_d(\boldsymbol{A})$.*

Proof. Consider the equality $\boldsymbol{A}_d = \widetilde{\boldsymbol{A}}_d + \boldsymbol{N}$, on $\boldsymbol{N} \in \mathcal{A}^{\perp}$. Combining both Pitagoras Theorem and Equation (8), we obtain

$$\|\boldsymbol{N}\|^2 = \|\boldsymbol{A}_d\|^2 - \|\widetilde{\boldsymbol{A}}_d\|^2 = \delta_d - \frac{\delta_d^2}{p_d(\lambda_0)} = \delta_d \left(1 - \frac{\delta_d}{p_d(\lambda_0)} \right)\,.$$

This implies the inequality. Moreover, equality is attained if and only if $\boldsymbol{N}$ is zero. $\square$

A similar result was recently proved by Van Dam [6] by using the harmonic mean of the degrees of Γ_d (see also [5]).

We point out that the relation $\delta_d \leq p_d(\lambda_0)$ holds for any graph, not only for graphs with spectrally maximum diameter. In the other cases the inequality is trivial since $0 < p_d(\lambda_0)$.

5. Characterizing distance-regular graphs

A connected graph Γ of diameter D is distance-regular if for any vertex pair (u, v) and integers $0 \leq i, j \leq D$, the numbers $p_{ij}(u, v)$ of vertices at distance i from u and at distance j from v only depends on $k := \partial(u, v)$, and we write $p_{ij}(u, v) = p_{ij}^k$ for some constants p_{ij}^k called the intersection numbers (see e.g. [1]). This definition or characterization can be simplified to some weaker statements.

(A) *A graph Γ with diameter D is distance-regular if and only if, for any two vertices u, $v \in V$ at a distance k, there exist the numbers $p_{k+1,1}^k$, p_{k1}^k, $p_{k-1,1}^k$, for any $0 \leq k \leq D$.*

This characterization can also be translated in terms of k-distance matrices terms as follows:

(B) *A graph Γ with diameter D is distance-regular if and only if, for any two vertices u, $v \in V$ at a distance k, if there exist numbers b_k, a_k, c_k (the proper intersection numbers) such that*

$$\boldsymbol{A}\boldsymbol{A}_k = b_{k-1}\boldsymbol{A}_{k-1} + a_k\boldsymbol{A}_k + c_{k+1}\boldsymbol{A}_{k+1} \quad (0 \leq k \leq D), \tag{9}$$

where $b_{-1} = c_{D+1} = 0$. Besides for any $0 \leq k \leq D$ the sum $b_k + a_k + c_k$ is equal to the degree of the graph, and $b_k = p_{k+1,1}^k$, $a_k = p_{k1}^k$, $c_k = p_{k-1,1}^k$.

By iteratively applying (9) we obtain that there exist polynomials p_k, with degree equal to their subindex, such that $\boldsymbol{A}_k = p_k(\boldsymbol{A})$ for every $0 \leq k \leq D$. In this case we say that the distance-matrices are polynomial. In particular it holds $(xp_D - b_{D-1}p_{D-1} - a_Dp_D)(\boldsymbol{A}) = 0$, that is, there is a polynomial of degree $D + 1$ that annihilates $\boldsymbol{A}$. Therefore $d + 1 \leq D + 1$, implies $D = d$ and the distance-regular graphs have spectrally maximum diameter. Conversely, suppose that, in a regular connected graph Γ, the distance-matrices $\boldsymbol{A}_0, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_d$, with $\boldsymbol{A}_d \neq 0$, are polynomial: $\boldsymbol{A}_k = p_k(\boldsymbol{A})$. This implies that the graphs Γ_k are regular of degree $p_k(\lambda_0)$. Then, from

$$\langle p_h, p_k \rangle = \langle p_h(\boldsymbol{A}), p_k(\boldsymbol{A}) \rangle = \langle \boldsymbol{A}_h, \boldsymbol{A}_k \rangle = 0 \quad \text{for } h \neq k,$$
$$\|p_h\|^2 = \|\boldsymbol{A}_h\|^2 = \frac{1}{n} n p_k(\lambda_0) = p_k(\lambda_0) \quad \text{for } h = k,$$

we obtain that the p_k's are the predistance polynomials. Their recurrent relation turns out into the equalities (9) and the graph is distance-regular. Therefore the distance-regularity of a graph can also be characterized in the following way:

(C) *A graph Γ is distance-regular if and only if the k-distance matrices of the graph are polynomials, for every $0 \leq k \leq D$.*

(These polynomials are just the predistance polynomials which, in the case of having distance-regularity, are simply called the *distance polynomials*).

As before, the number of conditions in (**C**) can reduced drastically, as it was shown in [3].

(D) *A graph Γ is distance-regular if and only if Γ is regular, has spectrally maximum diameter and the matrix $\boldsymbol{A}_D$ is polynomial.*

Here we will give a shorter proof of (**D**). Let us see that, within the above conditions on Γ, if $\boldsymbol{A}_d \in \mathcal{A}$ then $\boldsymbol{A}_k \in \mathcal{A}$ for every $0 \leq k \leq d-1$. If $\boldsymbol{A}_d = q(\boldsymbol{A})$ it is clear that q has degree d, and also that Γ_d is a regular graph with degree $\delta_d = q(\lambda_0)$. Let us prove that $q = p_d$. Indeed, $\|q\|^2 = \langle \boldsymbol{A}_d, \boldsymbol{A}_d \rangle = \langle \boldsymbol{A}_d, \boldsymbol{J} \rangle = \delta_d = q(\lambda_0)$. Moreover, for every $r \in \mathbb{R}_{d-1}[x]$, we have $\langle q, r \rangle = \langle \boldsymbol{A}_d, r(\boldsymbol{A}) \rangle = 0$. Therefore, $q = p_d$. From the equality (5) and the recurrence satisfied for the predistance polynomials, we get:

$$\begin{aligned} \boldsymbol{A}_0 + \boldsymbol{A}_1 + \cdots + \boldsymbol{A}_{d-1} &= p_0(\boldsymbol{A}) + p_1(\boldsymbol{A}) + \cdots + p_{d-1}(\boldsymbol{A}) \qquad (10) \\ \boldsymbol{A}\boldsymbol{A}_d &= b_{d-1} p_{d-1}(\boldsymbol{A}) + a_d \boldsymbol{A}_d \qquad (11) \end{aligned}$$

Then, if $\partial(u,v) > d-1$ we have $(p_{d-1}(\boldsymbol{A}))_{uv} = 0$, since p_{d-1} has degree $d-1$. If $\partial(u,v) = d-1$, the equality (10) implies that $(p_{d-1}(\boldsymbol{A}))_{uv} = 1$. Otherwise the equality (11) implies $(p_{d-1}(\boldsymbol{A}))_{uv} = 0$. Therefore, $p_{d-1}(\boldsymbol{A}) = \boldsymbol{A}_{d-1}$. Suppose now that $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$ for $d \geq i \geq k+1$. Then, we have the equalities:

$$\begin{aligned} \boldsymbol{A}_0 + \boldsymbol{A}_1 + \cdots + \boldsymbol{A}_{k} &= p_0(\boldsymbol{A}) + p_1(\boldsymbol{A}) + \cdots + p_{k}(\boldsymbol{A}) \qquad (12) \\ \boldsymbol{A}\boldsymbol{A}_{k+1} &= b_k p_k(\boldsymbol{A}) + a_{k+1} \boldsymbol{A}_{k+1} + c_{k+2} \boldsymbol{A}_{k+2} \qquad (13) \end{aligned}$$

As before, from the degree of p_k we can deduce that, if $\partial(u,v) > k$ then $(p_k(\boldsymbol{A}))_{uv} = 0$. If $\partial(u,v) = k$, using (12) we have $(p_k(\boldsymbol{A}))_{uv} = 1$. Otherwise the equality (13) yields $(p_k(\boldsymbol{A}))_{uv} = 0$. Thus, we get $p_k(\boldsymbol{A}) = \boldsymbol{A}_k$ which, by recurrence, proves the result in (**D**).

Now observe that in a distance-regular graph, the $(d-1)$-excess of any vertex is the mean degree of $\boldsymbol{A}_d$, which can be calculated as $p_d(\lambda_0)$, and recall Equation (4). As a consequence of the projection method introduced in Section 4, we present a simple proof of the Spectral Excess Theorem, first given in [4]. Another simple approach was used by Van Dam [6] to prove the same result.

Theorem 3. *A regular connected graph Γ with $d+1$ different eigenvalues is distance-regular if and only if the mean $(d-1)$-excess of the vertices of the graph is*

$$\frac{n}{\pi_0^2}\left(\sum_{i=0}^{d}\frac{1}{m_i\pi_i^2}\right)^{-1}. \tag{14}$$

Proof. First suppose that Γ is distance-regular, then $\boldsymbol{A}_d = p_d(\boldsymbol{A})$ and Γ_d is regular of degree $p_d(\lambda_0)$. From Equation (4) the condition yields. Conversely, if the mean degree of Γ_d is not null, the graph Γ has spectrally maximum diameter, as $D = d$. Thus $\delta_d = p_d(\lambda_0)$ and by Theorem 2 it holds that $\boldsymbol{A}_d$ is polynomial. Finally, the characterization (**D**) establishes the distance-regularity of the graph. □

6. Acknowledgments

The authors wish to thank the support by the Ministry of Science and Technology (Spain) with the European Regional Development Fund under projects MTM2005-08990-C02-01 and TEC2005-03575.

References

[1] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance Regular Graphs, Springer-Verlag, Berlin 1989.

[2] M. Cámara, J. Fàbrega, M.A. Fiol, E. Garriga, Some families of orthogonal polynomials of a discrete variable and their applications to graphs and codes, UPC-prints, http://hdl.handle.net/2117/736.

[3] M.A. Fiol, E. Garriga, J.L.A. Yebra, Locally Pseudo-distance-regular graphs, J. Combin. Theory Ser. B 68, 1996, 179205.

[4] M.A. Fiol, E. Garriga, From local adjacency polynomials to locally pseudo-distance-regular graphs, J. Combin. Theory Ser. B 71, 1997, 162-183.

[5] M.A. Fiol, Algebraic characterizations of distance-regular graphs, Discrete Math. 246, 2002, 111-129.

[6] E.R. van Dam, The spectral excess theorem for distance-regular graphs: a global (over)view, Electron. J. Combin. 15 (2008), R129.