The Geometry of t-Cliques in k-Walk-Regular Graphs *

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Abstract

A graph is walk-regular if the number of cycles of length ℓ rooted at a given vertex is a constant through all the vertices. For a walk-regular graph G with $d+1$ different eigenvalues and spectrally maximum diameter $D=d$, we study the geometry of its d-cliques, that is, the sets of vertices which are mutually at distance d. When these vertices are projected onto an eigenspace of its adjacency matrix, we show that they form a regular tetrahedron and we compute its parameters. Moreover, the results are generalized to the case of k-walk-regular graphs, a family which includes both walk-regular and distance-regular graphs, and their t-cliques or vertices at distance t from each other.

1 Preliminaries

Distance-regular graphs with diameter D can be characterized by the invariance of the number of walks of length $\ell \geq t$ between vertices at a given distance t, $0 \leq t \leq D$ (see e.g. Rowlinson [13] or Fiol[5]). Similarly, walk-regular graphs are characterized by the fact that the number of closed walks of length $\ell \geq 0$ rooted at any given vertex u does not depend on u. Thus, a distance-regular graph is also walk-regular, but the converse, in general, is not true (see e.g. Godsil [10]).

In this paper, we first recall some characterizations and derive some basic results on a walk-regular graph G. Afterwards, this background is used to study the geometry

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of the vertices which are mutually at distance d, where $d+1$ is the number of different eigenvalues in the spectrum of G. More precisely, when the coordinate vectors representing such vertices of G are projected onto the eigenspace of any eigenvalue, we show that the points obtained are the vertices of a regular tetrahedron and we compute their radius (distance from the center to every vertex), edge length and angle formed by the vectors going from the center to each vertex. Then, imposing that such parameters must be nonnegative, some consequences on the eigenvalue multiplicities and the d-clique number, which is the maximum number of vertices at distance d from each other, are derived. Finally, these results are generalized for the so-called k-walk-regular graphs, which were recently introduced by the authors in [3, 7] and their t-cliques, $1 \le t \le k$. These graphs are characterized by the invariance of the number of walks of length $\ell \ge t$ between vertices at a given distance t, $0 \le t \le t$. Then, this family includes both walk-regular ($k=0$) and distance-regular ($k=D$) graphs.

1.1 Background

We begin with some notation and basic results. Throughout this paper, $G=(V,E)$ denotes a simple, connected graph, with order $n=|V|$ and adjacency matrix $\boldsymbol{A}$. The *distance* between two vertices u,v is denoted by $\operatorname{dist}(u,v)$, so that the *eccentricity* of a vertex u is $\operatorname{ecc}(u)=\max_{v\in V}\operatorname{dist}(u,v)$ and the *diameter* of the graph is $D=D(G)=\max_{u\in V}\operatorname{ecc}(u)$. The spectrum of G is

$$\operatorname{sp} G = \operatorname{sp} \boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\},$$

where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ and the superscripts stand for the multiplicities $m_i = m(\lambda_i)$. In particular, note that $m_0 = 1$ (since G is connected) and $m_0 + m_1 + \cdots + m_d = n$. It is well-known that the diameter of G satisfies $D \le d$ (see, for instance, Biggs [1]). Then, a graph with $D = d$ is said to have spectrally maximum diameter. For a given ordering of the vertices, the vector space of linear combinations (with real coefficients) of the vertices of G is identified with $\mathbb{R}^n$, with canonical basis $\{\boldsymbol{e}_u : u \in V\}$. For every $0 \le h \le d$, the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_h = \operatorname{Ker}(\boldsymbol{A} - \lambda_h \boldsymbol{I})$ are given by the (Lagrange interpolation) polynomials of degree d

$$P_h = \frac{1}{\phi_h} \prod_{\substack{i=0 \\ i\ne h}}^{d} (x-\lambda_i) = \frac{(-1)^h}{\pi_h} \prod_{\substack{i=0 \\ i\ne h}}^{d} (x-\lambda_i) \qquad (0 \le h \le d),$$

where $\phi_h = \prod_{i=0, i\ne h}^{d} (\lambda_h - \lambda_i)$ and $\pi_h = |\phi_h|$ are "moment-like" parameters satisfying

$$\sum_{h=0}^{d} \frac{(-1)^h}{\pi_h} p(\lambda_h) = 0 \tag{1}$$

for any polynomial p of degree smaller than d (just observe that the coefficient of x^d in both terms of $p(x) = \sum_{h=0}^{d} p(\lambda_h) P_h(x)$ must be zero). In particular, recall that $H = nP_0$ is the Hoffman polynomial, which characterizes the regularity of G by the condition $H(\boldsymbol{A}) = \boldsymbol{J}$, the all-1 matrix (see Hoffman [12]). The matrices $\boldsymbol{E}_h = P_h(\boldsymbol{A})$ corresponding to these orthogonal projections onto $\mathcal{E}_h$ are called the (*principal*) *idempotents* of $\boldsymbol{A}$. Then, the orthogonal decomposition of the unitary vector $\boldsymbol{e}_u$, representing vertex u, is:

$$\boldsymbol{e}_u = \boldsymbol{z}_u^0 + \boldsymbol{z}_u^1 + \cdots + \boldsymbol{z}_u^d, \quad \text{where} \quad \boldsymbol{z}_u^h = P_h(\boldsymbol{A})\boldsymbol{e}_u = \boldsymbol{E}_h\boldsymbol{e}_u \in \mathcal{E}_h.$$

From this decomposition, we define the *u-local multiplicity* of eigenvalue λ_h as

$$m_u(\lambda_h) = \|\boldsymbol{z}_u^h\|^2 = \langle \boldsymbol{E}_h\boldsymbol{e}_u, \boldsymbol{E}_h\boldsymbol{e}_u \rangle = \langle \boldsymbol{E}_h\boldsymbol{e}_u, \boldsymbol{e}_u \rangle = (\boldsymbol{E}_h)_{uu},$$

satisfying $\sum_{h=0}^d m_u(\lambda_h) = 1$ and

$$\sum_{u \in V} m_u(\lambda_h) = \operatorname{tr}(\boldsymbol{E}_h) = m_h (0 \le h \le d), \tag{2}$$

because $\operatorname{sp} \boldsymbol{E}_h = \{0^{n-m_h}, 1^{m_h}\}$ (see Fiol and Garriga [6] for more details).

1.2 Spectral regularity and walk-regularity

We say that G is *spectrally regular* when, for any $h = 0, 1, \ldots, d$, the u-local multiplicity of λ_h does not depend on the vertex u. Then, Eq. (2) implies that each (standard) multiplicity "splits" equitably among the n vertices, giving $m_u(\lambda_h) = \frac{m_h}{n}$.

Let $a_u^{(\ell)} = (\boldsymbol{A}^\ell)_{uu}$ denotes the number of closed walks of length ℓ rooted at vertex u, which can be computed in terms of the local multiplicities as

$$a_u^{(\ell)} = \sum_{h=0}^{d} m_u(\lambda_h)\lambda_h^\ell,$$

see again [6]. When the number $a_u^{(\ell)}$ only depends on ℓ, in which case we write $a_u^{(\ell)} = a^{(\ell)}$, the graph G is called *walk-regular* (a concept introduced by Godsil and McKay in [11]). Notice that, as $a_u^{(2)} = \delta_u$, the degree of vertex u, every walk-regular graph is also regular. Recall also that, if G has $d+1$ distinct eigenvalues, then $\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d\}$ is a basis of the *adjacency* or *Bose-Mesner algebra* $\mathcal{A}(G)$ of matrices which are polynomials in $\boldsymbol{A}$. Therefore, the existence of the set of constants $\mathcal{C} = \{a^{(0)}, a^{(1)}, \ldots, a^{(d)}\}$, such that $a_u^{(\ell)} = a^{(\ell)}$ for every $u \in V$, suffices for assuring walk-regularity.

As it is well known, any distance-regular graph is also walk-regular, but the converse is not true. Actually, as it is pointed out by Godsil [10], there are walk-regular graphs which are neither vertex-transitive nor distance-regular.

In [3] we proved the following result, which characterizes walk-regular graphs:

Proposition 1.1 *A connected graph G is spectrally regular if and only if it is walk-regular.*

1.3 The crossed local multiplicities and the cosines

Here we study some results concerning some parameters of a geometric nature, as the cosines, which will be used to prove our main result in the next section.

Consider the set $\mathcal{P}_h(V) = \{\boldsymbol{z}_u^h = \boldsymbol{E}_h\boldsymbol{e}_u : u \in V\}$ of vectors in the m_h-dimensional space $\mathcal{E}_h$. These sets are usually called *eutactic stars* and they have been extensively studied, for instance, see Seidel [14], Rowlinson [13] and Cvetković, Rowlinson and Simić [2]. Then, the spectral regularity of the graph is equivalent to state that, for every $h = 0, 1, \ldots, d$, such vectors define n points (not necessarily different) on the sphere with radius $\sqrt{m_h/n}$. Moreover, for any $h = 1, 2, \ldots, d$, the "center of mass" of the set $\mathcal{P}_h(V)$ is

$$\sum_{u\in V} \boldsymbol{z}_u^h = \boldsymbol{E}_h \sum_{u\in V} \boldsymbol{e}_u = \boldsymbol{E}_h\boldsymbol{j} = \boldsymbol{0}.$$

Fiol, Garriga and Yebra [8] defined the *crossed* (uv-)*local multiplicities* of λ_h, denoted by $m_{uv}(\lambda_h)$, as the uv-entries of the idempotents ($u \neq v$). Now in terms of the orthogonal projection of the canonical vectors $\boldsymbol{e}_u$, the crossed local multiplicities are obtained by the Euclidean scalar products

$$m_{uv}(\lambda_h) := (\boldsymbol{E}_h)_{uv} = \langle \boldsymbol{E}_h\boldsymbol{e}_u, \boldsymbol{e}_v\rangle = \langle \boldsymbol{E}_h\boldsymbol{e}_u, \boldsymbol{E}_h\boldsymbol{e}_v\rangle = \langle \boldsymbol{z}_u^h, \boldsymbol{z}_v^h\rangle \qquad (u, v \in V).$$

The crossed local multiplicities can be used to compute the number of uv-walks of length ℓ as

$$a_{uv}^{(\ell)} = (\boldsymbol{A}^\ell)_{uv} = \sum_{h=0}^{d} m_{uv}(\lambda_h)\lambda_h^\ell. \tag{3}$$

In the following result, we show a characterization for two vertices u, v to be at maximum distance d in a regular graph in terms of the crossed local multiplicities. In particular, it is shown that such multiplicities do not depend on u, v.

Proposition 1.2 *Let $G = (V, E)$ be a regular graph with $d+1$ different eigenvalues. Then, two vertices $u, v \in V$ are at distance d if and only if their crossed uv-local multiplicities are:*

$$m_{uv}(\lambda_h) = \frac{(-1)^h}{n}\frac{\pi_0}{\pi_h} \qquad (0 \leq h \leq d). \tag{4}$$

Proof. Suppose first that $\mathrm{dist}(u, v) = d$. Then, we have:

$$\begin{aligned} m_{uv}(\lambda_h) &= \langle \boldsymbol{z}_u^h, \boldsymbol{z}_v^h\rangle = \langle P_h(\boldsymbol{A})\boldsymbol{e}_u, P_h(\boldsymbol{A})\boldsymbol{e}_v\rangle = \langle P_h(\boldsymbol{A})\boldsymbol{e}_u, \boldsymbol{e}_v\rangle = \frac{(-1)^h}{\pi_h}\langle \boldsymbol{A}^d\boldsymbol{e}_u, \boldsymbol{e}_v\rangle \\ &= \frac{(-1)^h}{\pi_h}\langle \frac{\pi_0}{n}H(\boldsymbol{A})\boldsymbol{e}_u, \boldsymbol{e}_v\rangle = \frac{(-1)^h}{\pi_h}\frac{\pi_0}{n}(H(\boldsymbol{A}))_{uv} = \frac{(-1)^h}{n}\frac{\pi_0}{\pi_h}, \end{aligned}$$

where we have used that the coefficients of x^d in the polynomials P_h and $H = nP_0$ are $\frac{(-1)^h}{\pi_h}$ and $\frac{n}{\pi_0}$, respectively.

Conversely, if (4) holds and $\ell < d$, Eq. (3) gives

$$a_{uv}^{(\ell)} = \sum_{h=0}^{d} \frac{(-1)^h}{n} \frac{\pi_0}{\pi_h} \lambda_h^\ell = 0,$$

where we have used (1). Therefore, $\mathrm{dist}(u,v) = d$ as claimed. $\square$

Let $\gamma_{u,v}^h = \gamma(\boldsymbol{z}_u^h, \boldsymbol{z}_v^h)$ denote the angle between the two vectors $\boldsymbol{z}_u^h$, $\boldsymbol{z}_v^h$. In terms of our local multiplicities, the cosines of these angles are:

$$\cos \gamma_{u,v}^h = \frac{\langle \boldsymbol{z}_u^h, \boldsymbol{z}_v^h \rangle}{\|\boldsymbol{z}_u^h\| \, \|\boldsymbol{z}_v^h\|} = \frac{m_{uv}(\lambda_h)}{\sqrt{m_u(\lambda_h) m_v(\lambda_h)}}.$$

These cosines were already considered by Godsil [9, 10] when G is a distance-regular graph.

In particular, if G is spectrally regular, we get:

$$\cos \gamma_{u,v}^h = \frac{(-1)^h}{m_h} \frac{\pi_0}{\pi_h} \qquad (0 \le h \le d), \tag{5}$$

so that $m_h \ge \frac{\pi_0}{\pi_h}$ for any $0 \le h \le d$.

2 The Geometry of d-Cliques in Walk-Regular Graphs

In this section, we assume that the graph $G = (V, E)$, with $\mathrm{sp}\, G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$, is spectrally regular and it has spectrally maximum diameter. Let $U \subset V$ be a subset of r vertices which are at distance d from each other. Let $\mathcal{P}_h(U) = \{\boldsymbol{z}_u^h = \boldsymbol{E}_h \boldsymbol{e}_u : u \in U\}$ be its projection set onto the eigenspace $\mathcal{E}_h = \mathrm{Ker}(\boldsymbol{A} - \lambda_h \boldsymbol{I}) \subset \mathbb{R}^n$ with dimension m_h and let

$$\boldsymbol{c}_r = \frac{1}{r} \sum_{u \in U} \boldsymbol{z}_u^h = \frac{1}{r} \boldsymbol{w}^h \tag{6}$$

be the barycenter of the points in $\mathcal{P}_h(U)$. For each $h = 0, 1, \dots, d$, we are here interested in studying the geometry of the points in $\mathcal{P}_h(U)$. In the next result, we show that such points are the vertices of a tetrahedron whose parameters depend on r, n and the spectrum of G (see Fig. 1 for the case $r = 3$).

Theorem 2.1 *Let G be a walk-regular graph with $\mathrm{sp}\, G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$ and spectrally maximum diameter d. Let $U \subset V$ be a set of r vertices which are mutually at distance d. Then, the set of projected points $\mathcal{P}_h(U) = \{\boldsymbol{z}_u^h : u \in U\}$ are the vertices of a regular tetrahedron with center $\boldsymbol{c}_r$ at distance (from the origin)*

$$S = \sqrt{\tfrac{1}{rn} \left(m_h + (-1)^h (r-1) \tfrac{\pi_0}{\pi_h} \right)}, \tag{7}$$

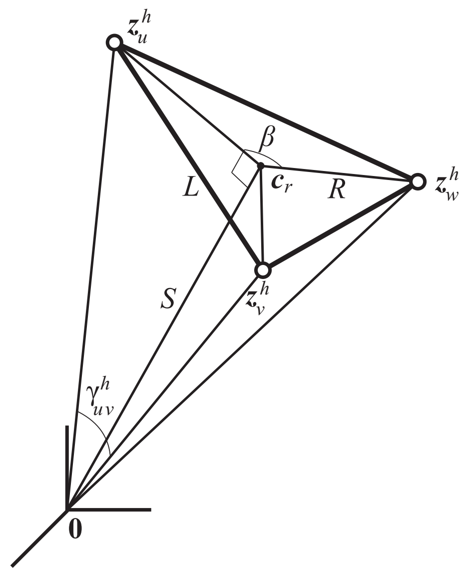

Figure 1: The projection of $r = 3$ vertices of a d-clique.

radius (*distance from the center to every vertex*)

$$R = \sqrt{\tfrac{r-1}{rn}\left(m_h - (-1)^h \tfrac{\pi_0}{\pi_h}\right)}, \tag{8}$$

and edge length

$$L = \sqrt{\tfrac{2}{n}\left(m_h - (-1)^h \tfrac{\pi_0}{\pi_h}\right)}. \tag{9}$$

Moreover, the angle β *formed by the vectors going from the center* $\boldsymbol{c}_r$ *to each vertex* $\boldsymbol{z}_u^h$ *satisfies* $\cos\beta = \frac{-1}{r-1}$.

Proof. Computing $\|\boldsymbol{w}^h\|^2 = \|\sum_{h=1}^d \boldsymbol{z}_u^h\|^2$, we obtain

$$\begin{aligned}
\|\boldsymbol{w}^h\|^2 &= \sum_{u,v\in U} \langle \boldsymbol{z}_u^h, \boldsymbol{z}_v^h\rangle = \sum_u \|\boldsymbol{z}_u^h\|^2 + \sum_{u\neq v} \langle \boldsymbol{z}_u^h, \boldsymbol{z}_v^h\rangle = \sum_u \frac{m_h}{n} + \sum_{u\neq v} \frac{(-1)^h}{n}\frac{\pi_0}{\pi_h} \\
&= r\frac{m_h}{n} + r(r-1)\frac{(-1)^h}{n}\frac{\pi_0}{\pi_h} = \frac{r}{n}\left(m_h + (-1)^h (r-1)\frac{\pi_0}{\pi_h}\right).
\end{aligned}$$

Then, by (6),

$$S^2 = \|\boldsymbol{c}_r\|^2 = \frac{1}{r^2}\|\boldsymbol{w}^h\|^2 = \frac{1}{rn}\left(m_h + (-1)^h (r-1)\frac{\pi_0}{\pi_h}\right). \tag{10}$$

This yields the value of S in (7).

Moreover, we have

$$\begin{aligned}\langle \boldsymbol{z}_u^h - \boldsymbol{c}_r, \boldsymbol{c}_r\rangle &= \frac{1}{r}\sum_{u,v\in U}\langle \boldsymbol{z}_u^h, \boldsymbol{z}_v^h\rangle - \|\boldsymbol{c}_r\|^2 \\ &= \frac{1}{r}\frac{1}{n}\left(m_h + (-1)^h (r-1)\frac{\pi_0}{\pi_h}\right) - \frac{1}{rn}\left(m_h + (-1)^h (r-1)\frac{\pi_0}{\pi_h}\right) = 0. \qquad (11)\end{aligned}$$

So, the set of points $\mathcal{P}_h(U)$ is in the hyperplane containing the point $\boldsymbol{c}_r$ and orthogonal to the vector from the origin point to $\boldsymbol{c}_r$ (see Fig. 1). Then, from this orthogonality, we have that the square radius is

$$\begin{aligned}R^2 &= \|\boldsymbol{z}_u^h - \boldsymbol{c}_r\|^2 = \|\boldsymbol{z}_u^h\|^2 - \|\boldsymbol{c}_r\|^2 = \frac{m_h}{n} - \frac{1}{rn}\left(m_h + (r-1)(-1)^h\frac{\pi_0}{\pi_h}\right) \\ &= \frac{r-1}{rn}\left(m_h - (-1)^h\frac{\pi_0}{\pi_h}\right). \qquad (12)\end{aligned}$$

Then, the points in $\mathcal{P}_h(U)$ are on the surface of the sphere with center $\boldsymbol{c}_r$ and radius R given by (8).

Let us now see that the points in $\mathcal{P}_h(U)$ are mutually at the same distance. Indeed, the square norm

$$\begin{aligned}L^2 &= \|\boldsymbol{z}_u^h - \boldsymbol{z}_v^h\|^2 = \|\boldsymbol{z}_u^h\|^2 + \|\boldsymbol{z}_v^h\|^2 - 2\langle \boldsymbol{z}_u^h, \boldsymbol{z}_v^h\rangle = 2\frac{m_h}{n} - 2\frac{(-1)^h}{n}\frac{\pi_0}{\pi_h} \\ &= \frac{2}{n}\left(m_h - (-1)^h\frac{\pi_0}{\pi_h}\right) \qquad (13)\end{aligned}$$

is independent of the vertices and, if $r \geq 3$, every three points $\boldsymbol{z}_u^h, \boldsymbol{z}_v^h, \boldsymbol{z}_w^h$ determine an equilateral triangle. So, the points in $\mathcal{P}_h(U)$ are the vertices of a regular tetrahedron with center $\boldsymbol{c}_r$, radius R and side L as claimed in (9).

Note that, from (12) and (13), we get that the ratio between the edge length and the radius is $\frac{L}{R} = \sqrt{\frac{2r}{r-1}}$, which is known for a regular tetrahedron with r vertices. Finally, the value of $\cos\beta$, is also a known result and it can be proved in our context. Indeed, let us consider the angle β, which is formed by the vectors $\boldsymbol{z}_u^h - \boldsymbol{c}_r$ and $\boldsymbol{z}_v^h - \boldsymbol{c}_r$. First, by using (12), we get:

$$\langle \boldsymbol{z}_u^h - \boldsymbol{c}_r, \boldsymbol{z}_v^h - \boldsymbol{c}_r\rangle = R^2\cos\beta = \frac{r-1}{rn}\left(m_h - (-1)^h\frac{\pi_0}{\pi_h}\right)\cos\beta.$$

Moreover, by using (11) and (10), we also have

$$\begin{aligned}\langle \boldsymbol{z}_u^h - \boldsymbol{c}_r, \boldsymbol{z}_v^h - \boldsymbol{c}_r\rangle &= \langle \boldsymbol{z}_u^h - \boldsymbol{c}_r, \boldsymbol{z}_v^h\rangle = \frac{(-1)^h}{n}\frac{\pi_0}{\pi_h} - \langle \boldsymbol{c}_r, \boldsymbol{z}_v^h\rangle \\ &= \frac{(-1)^h}{n}\frac{\pi_0}{\pi_h} - \langle \boldsymbol{c}_r, \boldsymbol{z}_v^h - \boldsymbol{c}_r\rangle - \|\boldsymbol{c}_r\|^2 \\ &= \frac{(-1)^h}{n}\frac{\pi_0}{\pi_h} - \frac{1}{rn}\left(m_h + (-1)^h(r-1)\frac{\pi_0}{\pi_h}\right) \\ &= -\frac{1}{rn}\left(m_h - (-1)^h\frac{\pi_0}{\pi_h}\right).\end{aligned}$$

Then, from the above expressions, we obtain $\cos\beta = -\frac{1}{r-1}$, and this completes the proof. □

A straightforward, but interesting, consequence of our theorem is the following result obtained by considering the parity of h (note that, for odd h, the bound of the multiplicity given below is, in general, an improvement of that obtained from (5)):

Corollary 2.2 *Let $G = (V,E)$ be a walk-regular graph with spectrum $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$ with a set $U \subset V$ of r vertices which are at distance d from each other. Then, the eigenvalue multiplicities satisfy the bounds:*

$$m_h \geq \frac{\pi_0}{\pi_h} \quad \text{if } h \text{ is even}, \tag{14}$$

$$m_h \geq (r-1)\frac{\pi_0}{\pi_h} \quad \text{if } h \text{ is odd}. \tag{15}$$

Moreover, equality in (14) is attained if and only if the tetrahedron with vertices $\mathcal{P}_h(U)$ collapses into a point ($L = R = 0$), while equality in (15) is attained if the corresponding tetrahedron is centered at the origin ($S = 0$).

Proof. If h is even, Theorem 2.1 gives:

$$S = \sqrt{\tfrac{1}{rn}\left(m_h + (r-1)\tfrac{\pi_0}{\pi_h}\right)}, \quad R = \sqrt{\tfrac{r-1}{rn}\left(m_h - \tfrac{\pi_0}{\pi_h}\right)} \geq 0, \quad L = \sqrt{\tfrac{2}{n}\left(m_h - \tfrac{\pi_0}{\pi_h}\right)} \geq 0,$$

whence any of the two last inequalities yields (14), and equalities are attained ($R = L = 0$) if and only if $m_h = \frac{\pi_0}{\pi_h}$. In this case, the tetrahedron collapses into a single point $\boldsymbol{c}_r$ at the minimum possible distance (from the origin)

$$S = \sqrt{\tfrac{1}{n}\tfrac{\pi_0}{\pi_h}} = \sqrt{\tfrac{m_h}{n}} = \|\boldsymbol{z}_u^h\|,$$

for any u, as expected.

However, if h is odd, Theorem 2.1 gives:

$$S = \sqrt{\tfrac{1}{rn}\left(m_h - (r-1)\tfrac{\pi_0}{\pi_h}\right)} \geq 0, \quad R = \sqrt{\tfrac{r-1}{rn}\left(m_h + \tfrac{\pi_0}{\pi_h}\right)}, \quad L = \sqrt{\tfrac{2}{n}\left(m_h + \tfrac{\pi_0}{\pi_h}\right)},$$

whence the first inequality yields (15), and equality is attained ($S = 0$) if and only if $m_h = (r-1)\frac{\pi_0}{\pi_h}$. Then, in this case, the tetrahedron is centered at the origin and have the minimum possible dimensions

$$R = \sqrt{\tfrac{r-1}{n}\tfrac{\pi_0}{\pi_h}} = \sqrt{\tfrac{m_h}{n}} = \|\boldsymbol{z}_u^h\|,$$

for any u, as expected, and

$$L = \sqrt{\tfrac{2r}{n}\tfrac{\pi_0}{\pi_h}} = \sqrt{\tfrac{m_h}{n}}\sqrt{\tfrac{2r}{r-1}} = R\sqrt{\tfrac{2r}{r-1}}.$$

□

In fact, the above two extreme cases are attained for antipodal distance-regular graphs because of the following characterization given by Fiol in [4]:

Proposition 2.3 *A distance-regular graph G with spectrum* $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$ *is r-antipodal ($r \geq 2$) if and only if its eigenvalue multiplicities satisfy:*

$$\begin{aligned} m_h &= \frac{\pi_0}{\pi_h} \quad (h \text{ even}), \\ m_h &= (r-1)\frac{\pi_0}{\pi_h} \quad (h \text{ odd}). \end{aligned}$$

Let $\omega_d \equiv \omega_d(G)$ be the *d-clique number* of G, that is, the maximum number of vertices which are at distance d from each other. Note that, for a graph G, the property of having spectrally maximum diameter is equivalent to have $\omega_d \geq 2$. Notice also that from (15) the d-clique number of a spectrally regular graph G satisfy the bound

$$\omega_d \leq 1 + \min_{\substack{1 \leq h \leq d \\ h \text{ odd}}} \left\{ m_h \tfrac{\pi_h}{\pi_0} \right\}, \tag{16}$$

a result proved in [3]. Moreover, since in a m_h-dimensional space the maximum number of points mutually at a given distance is $m_h + 1$, by Theorem 2.1, we have

$$\omega_d \leq 1 + \min_{1 \leq h \leq d} \left\{ m_h \,\middle|\, m_h \neq \tfrac{\pi_0}{\pi_h} \text{ if } h \text{ is even} \right\}. \tag{17}$$

3 The Geometry of t-Cliques in k-Walk-Regular Graphs

The above results can be easily extended to the case of k-walk-regular graphs and t-cliques. A connected graph G with diameter D is said to be k-walk-regular for some integer k, $0 \leq k \leq D$, if the number of walks of length ℓ between vertices u and v only depends on $t = \operatorname{dist}(u,v)$, provided that $t \leq k$. In particular, 0-walk-regular graphs are walk-regular, whereas D-walk-regular graphs are distance-regular. Some properties and characterizations of k-walk-regular graphs were given by the authors in [3], where it was also proved that a graph G is k-walk-regular if and only if, for any $h = 0, 1, \ldots, d$, its crossed uv-local multiplicities of λ_h only depend on $t = \operatorname{dist}(u,v)$, for $t \leq k$. More precisely,

$$m_{uv}(\lambda_h) = \frac{m_h}{n} \frac{p_t(\lambda_h)}{p_t(\lambda_0)}, \tag{18}$$

where $\{p_h\}_{0 \leq h \leq d}$ are the so-called predistance polynomials, which are orthogonal with respect to the scalar product

$$\langle p, q \rangle = \frac{1}{n} \operatorname{tr}(p(\boldsymbol{A}) q(\boldsymbol{A})) = \frac{1}{n} \sum_{h=0}^{d} m_h p(\lambda_h) q(\lambda_h)$$

and they satisfy $\operatorname{dgr}(p_h) = h$ and the normalization condition $\|p_h\|^2 = p_h(\lambda_0)$ for any h, $0 \leq h \leq d$. (This makes sense as always $p_h(\lambda_0) > 0$.) For more details about such polynomials see Fiol and Garriga [6].

In this context, the invariance of the crossed local multiplicities allows us to study the geometry of t-cliques, which are the sets of vertices mutually at distance t, for $1 \le t \le k$. Note that a 1-clique is simply a clique (or complete subgraph).

Theorem 3.1 *Let G be a k-walk-regular graph with $\mathrm{sp}\, G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$. Let $U \subset V$ be a set of r vertices which are mutually at distance t, for some $1 \le t \le k$. Then, the set of projected points $\mathcal{P}_h(U) = \{\boldsymbol{z}_u^h : u \in U\}$ are the vertices of a regular tetrahedron with center $\boldsymbol{c}_r$ at distance S from the origin, radius R and edge length L given by:*

$$\begin{aligned} S &= \sqrt{\tfrac{m_h}{rn}\left(1+(r-1)\tfrac{p_t(\lambda_h)}{p_t(\lambda_0)}\right)}, \\ R &= \sqrt{\tfrac{r-1}{r}\,\tfrac{m_h}{n}\left(1-\tfrac{p_t(\lambda_h)}{p_t(\lambda_0)}\right)}, \\ L &= \sqrt{\tfrac{2m_h}{n}\left(1-\tfrac{p_t(\lambda_h)}{p_t(\lambda_0)}\right)}. \end{aligned}$$

The proof of this result is as in Theorem 2.1 by using (18) instead of (4).

Note that now the tetrahedron collapses into a point if and only if $p_t(\lambda_h) = p_t(\lambda_0)$ whereas it is centered at the origin if and only if $p_t(\lambda_h) = -\frac{p_t(\lambda_0)}{r-1} = p_t(\lambda_0)\cos\beta$.

With respect to the t-clique number ω_t, that is, the maximum number of vertices which are mutually at distance t, we find analogous results to (16) and (17):

$$\omega_t \le 1 + \min_{1\le h\le d}\left\{\tfrac{p_t(\lambda_0)}{|p_t(\lambda_h)|} \mid p_t(\lambda_h) < 0\right\}, \tag{19}$$

$$\omega_t \le 1 + \min_{1\le h\le d}\{m_h \mid p_t(\lambda_h) \ne p_t(\lambda_0)\}. \tag{20}$$

Notice that in both cases the conditions on the values of $p_t(\lambda_h)$ and $p_t(\lambda_0)$ assure that the projection is a proper tetrahedron. In particular, if $t = 1$, we have $p_1 = x$ and Eqs. (19) and (20) give the following upper bounds for the clique number of a k-walk-regular graph $(k \ge 1)$:

$$\begin{aligned} \omega_1 &\le 1 + \min_{1\le h\le d}\left\{\tfrac{\lambda_0}{|\lambda_h|}\right\} = 1 - \tfrac{\lambda_0}{\lambda_d}, \\ \omega_1 &\le 1 + \min_{1\le h\le d}\{m_h\}. \end{aligned}$$

The first bound was proved by Delsarte for distance-regular graphs and, more generally, by Hoffman for regular graphs (see, for instance, Godsil [10]). Both bounds are attained, for example, if $G = K_{r,r,\overset{m}{\dots},r}$, the m-partite complete graph, with spectrum $\mathrm{sp}\, K_{r,r,\overset{m}{\dots},r} = \left\{(m-1)r, 0^{m(r-1)}, -r^{m-1}\right\}$ and clique number $\omega_1 = m$.

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