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GALERKIN METHOD AND APPROXIMATE TRACKING IN A NON-MINIMUM PHASE BILINEAR SYSTEM

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Abstract. The tracking control of non-minimum phase systems is nowadays an open and challenging field, because a general theory is still not available. This article proposes an indirect control strategy in which a key role is played by the inverse problem that arises and their approximate solutions. These are obtained with the Galerkin method, a standard functional analysis tool. A detailed study of the effect on the output caused by the use of an approximate input is performed. Error bounds are also provided. The technique is motivated through its implementation in basic, DC-to-DC nonlinear power converters that are intended to be used as DC-to-AC voltage sources.

1. **Introduction.** Exact tracking of a known output reference for non-minimum phase systems is developed in [6] and [7] both for the time-invariant and time-varying cases. The method tries to determine a bounded input-state trajectory that achieves a desired output behavior with an inversion-based procedure. If this is possible, a composite control law is used in such a way that its first component produces exact tracking and the second one stabilizes the overall system once linearized about the nominal trajectory. A slightly modified version of this work can be found in [15]. However, plant uncertainties may negatively impact on the output tracking performance in inversion based controllers. [5] contains acceptance bounds on the size of the uncertainties under which is advantageous to use inverse feedforward for linear, time-invariant systems.

Approximate and asymptotic output tracking in sliding modes for certain classes of non-minimum phase and uncertain nonlinear systems is reported in [16] and [17]. The key is in the definition of a proper output reference profile to be followed by the system that avoids unstable internal states. It is known as the stable system center

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design because it is based on the center manifold theory [10], [15]. Application to boost and buck-boost converters [18] and to systems with output delay [19] has already been developed.

Functional analysis is an increasing source of tools for the study of control systems. In the early seventies, the fixed point index theory allowed an elegant mathematical justification for the describing function method [3] (a summary may be found in [15]). Since then, the use of Galerkin approximations has been frequently reported in control literature: in [1], where reduced order model-based calculations are developed for use in on-line, real-time control methodologies, Galerkin expansions are used to compute the full order model. In the framework of optimal control, [11] and [2] use Galerkin approximations to approximately solve the Hamilton-Jacobi-Bellman equations; application to infinite-time and finite-time closed loop composite control of singularly perturbed bilinear systems with respect to performance criterion and to an underwater robot vehicle can be found in [11] and [12], respectively.

Recently, [21] has studied approximate dynamic phasor (i.e., Fourier coefficients over a moving window) models in bilinear dissipative systems with nonlinear lossless. At a certain stage, an infinite system of equations with infinite unknowns is compressed into the subspace spanned by a few harmonics, and functional analysis tools allow existence and convergence results to be proved. Although the problem is not the same, the structure of the paper and the tools used therein are close to our approach.

Consider a bilinear system in which the goal consists of the tracking of a periodic signal by a non-minimum phase state variable. If another state variable is minimum phase, indirect control may be tried, that is, we must force the minimum phase variable to follow a proper signal in such a way that the internal dynamics produces the required tracking result in the output.

The inverse problem that allows the indirect input reference to be obtained may be solved via the Galerkin method. This technique provides an algorithm that finds sets of equations whose solutions are used to build a sequence of approximate solutions of the inverse problem. However, several questions arise naturally when Galerkin expansions are used:

(QA) Do all the Galerkin equations have a solution; that is, does such a sequence really exist?

(QB) Does the Galerkin sequence exhibit any type of convergence to the indirect input reference?

(QC) Which type of response can be expected from the output when approximate inputs are used?

(QD) What can be stated about the influence of intrinsic system restrictions on the signals to be followed when approximate tracking is performed?

(QE) Is it possible to evaluate the input and output errors?

This article addresses a detailed theoretical approach to these problems through the study of the situation that appears in nonlinear power converters. The tracking of signals by the output voltage in basic, nonlinear DC-to-DC power converters is developed in [8]. The problem is proved to be solvable via the indirect control of the output voltage through the input current. The input current reference is a periodic and unstable solution of an Abel type.

In [20], a functional iterative computational scheme is proposed. Only a few iterations (one or two) are needed to obtain the suitable current reference. However, the convergence of the procedure, as well as the frequency and amplitude limitations of the desired AC output voltage reference due to the nature of the procedure and to the physical limitation of the control gains of the system are not studied.

In practical applications, the physical limitations of the devices prevent the improvements coming from the use of higher order harmonics in Galerkin approximations. The lowest possible values of the errors, which equal those resulting from an exact treatment of the problem, are reached with the 2nd approximation [14]. Moreover, a first approximation has been successfully used in the robust tracking control of DC-to-DC nonlinear switched converters [9]. This article also establishes a theoretical base that supports the fact of using such low order approximations.

Preliminary mathematical background is provided in Section 2. Section 3 contains the problem, the main results and a summary of the hypotheses under which these results are established. Question QA is affirmatively answered under sufficient conditions in Section 4. QB is treated in Section 5. Section 6 develops QC and introduces sufficient conditions to guarantee the existence of a periodic output when an approximate current is used in the indirect control; moreover, it studies QD and provides restrictions on the signals to be tracked. In Section 7 the convergence of the output to the desired reference is considered, thus completing the study of QC. The evaluation of input and output errors is performed in Section 8, where question QE is answered. Simulation results are presented in Section 9, while conclusions and proposed further research are contained in Section 10.

2. Preliminaries: mathematical background.

The material in this section has been mainly extracted from [4], [23] and [24]. Most definitions and results given here for Hilbert (H-) spaces, which is the natural framework to our problem, are usually extendable to more general Banach (B-) spaces.

Unless it is not explicitly noted, from now on let X be a real, separable H-space with norm $\|\cdot\|$ and inner product $(\cdot|\cdot)$.

2.1. Galerkin method.

The Galerkin method provides approximations to periodic solutions of differential equations by means of truncated Fourier expansions.

Definition 1. Let $\{X_n\}$ be a sequence of Banach subspaces of X, with $X_n \neq \emptyset$ and $\dim X_n < \infty$, $\forall n$. $\{X_n\}$ is a *Galerkin scheme* in X iff

$$\lim_{n\to\infty} \mathrm{d}_X(u, X_n) = \lim_{n\to\infty} \inf_{v\in X_n} \|u - v\| = 0, \qquad \forall u \in X.$$

Consider the set of square integrable functions in $(0, T)$, denoted $L_2(0, T)$, provided with the usual norm and scalar product. We may find in it the so called *trigonometric system* $\{w_n\}$, $w_n \in L_2$ $\forall n \geq 0$; with $m \geq 1$ and $\omega = 2\pi/T$,

$$w_0 = \frac{1}{\sqrt{T}}, \quad w_{2m-1} = \sqrt{\frac{2}{T}} \cos m\omega t, \quad w_{2m} = \sqrt{\frac{2}{T}} \sin m\omega t. \tag{1}$$

Proposition 1. *(i) $L_2(0, T)$ is a real, separable H-space for which the trigonometric system is a complete orthonormal system.*

(ii) $\{w_n\}$ is a basis of $L_2(0, T)$.

(iii) The sequence $\{X_n\}$, $X_n = span\,\{w_0, \ldots, w_{2n}\}$, is a Galerkin scheme in L_2, each of the X_n being a B-subspace.

(iv) Given $x \in L_2$, the mapping $P_n : L_2 \longrightarrow X_n$ is an orthogonal projection operator into X_n:

$$P_n x = \sum_{j=0}^{2n} (x|w_j) w_j. \tag{2}$$

Let now $F : L_2 \longrightarrow L_2$ be an operator in L_2, and consider the problem

$$Fx = 0. \tag{3}$$

Using $\{w_0, w_1, \ldots\}$ as a basis in L_2, the *Galerkin method* seeks to approximate the solution of (3) replacing $x \in L_2$ by $x_n \in X_n$, $x_n = \sum_{j=0}^{2n} c_{nj} w_j$, and searching for the coefficients $\{c_{nj}\}_j$ that satisfy $P_n F x_n = 0$, equivalent to $(F x_n | w_j) = 0$, $j = 0, \ldots, 2n$, which are known as the *Galerkin equations*.

Remark 1. When a Galerkin approximation is used instead of the exact solution, an error appears due to the fact that, in general, $F x_n \neq 0$. Assuming $P_n F x_n = 0$, the properties of projection operators lead to

$$F x_n = P_n F x_n + (\mathbb{I} - P_n) F x_n = (\mathbb{I} - P_n) F x_n, \tag{4}$$

where $\mathbb{I}$ denotes the identity map.

2.2. **Fixed point index.** The fixed point index can be considered a generalization of the so called *index theory*, which allows to predict the existence of equilibrium points in planar, real systems with few calculations.

Definition 2. Let X be a B-space and $G \subset X$ an open, bounded subset of X, and denote $V(G, X)$ the set of compact[1] mappings $f : \overline{G} \longrightarrow X$ with no fixed points in ∂G. Then, two mappings $f, g \in V(G, X)$ are said to be *homotopically compact* in ∂G iff there exists a mapping H with the following properties:

(P1) $H : \overline{G} \times [0, 1] \longrightarrow X$ is compact;
(P2) $H(x, \lambda) \neq x$, $\forall (x, \lambda) \in \partial G \times [0, 1]$;
(P3) $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ in $\overline{G}$.

In this case, we write $\partial G : f \cong g$. The mapping H is called *compact homotopy* or, simply, *homotopy*.

The system of axioms that define the fixed point index is:

Definition 3. To every $f \in V(G, X)$ let there be assigned an integer $i(f, G)$, called the *fixed point index* of f on G, so that it satisfies the axioms:

(A1) (Normalization). If $f(x) = x_0$, $\forall x \in \overline{G}$ and some fixed $x_0 \in G$, then $i(f, G) = 1$.

(A2) (Kronecker existence principle). If $i(f, G) \neq 0, \exists x \in G$ such that $f(x) = x$.

(A3) (Additivity). We have $i(f, G) = \sum_{j=1}^{n} i(f, G_j)$ whenever $f \in V(G, X)$ and $f \in V(G_j, X)$ $\forall j$, where $\{G_j\}_j$ is a partition of G.

(A4) (Homotopy invariance). If $\partial G : f \cong g$, then $i(f, G) = i(g, G)$.

This is completed with the following uniqueness principle:

Proposition 2. *(Leray-Schauder). For every mapping $f \in V(G, X)$ and every $V(G, X)$, X being an arbitrary B-space, there is exactly one fixed point index that satisfies axioms (A1)-(A4) of Definition 3.*

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[1]A mapping is compact iff it is continuous and it maps bounded sets into relatively compact sets.

Remark 2. With this tool, the strategy of proving the existence of a fixed point for a certain mapping f consists of relating it by homotopy with a simpler mapping g for which $i(g, G) \neq 0$ happens. Hence, (A4) and (A2) entail the desired result.

Let us finally introduce an alternative way of calculating the fixed point index for $\mathbb{R}^n$ mappings:

Definition 4. Let $f : G \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^1-mapping. The point $\bar{x} \in G$ is a *regular value* of f iff

$$\det \left[\frac{\partial f}{\partial x}(\bar{x}) \right] \neq 0.$$

Definition 5. Let G be an open bounded set in $\mathbb{R}^n$. Then $V_0(G, \mathbb{R}^n)$ denotes the set of all maps f with the following properties:

(i) $f : \overline{G} \longrightarrow \mathbb{R}^n$ is continuous and C^1 in G.

(ii) f has finitely many fixed points, if any, all of which are regular and none of which lies on the boundary ∂G.

Proposition 3. *For every $f \in V_0(G, \mathbb{R}^n)$ let $F(x) = x - f(x)$; then,*

$$i(f, G) = \sum_{j=1}^{m} sgn \left\{ det \left[\frac{\partial F}{\partial x}(x_j) \right] \right\},$$

where $x_1, \dots, x_m$ are all the fixed points of f in G. If f has no fixed points in G, then $i(f, G) = 0$.

2.3. **Weak convergence and Sobolev spaces.** On the one hand, the main use of the weak convergence concept comes down specifically to the following fact: in infinite dimensional B-spaces it is not necessarily true that any bounded sequence contains a convergent subsequence. However, this becomes true in H-spaces when weak convergence is used. On the other hand, Sobolev spaces are of particular interest in our case because weak convergence in the corresponding Sobolev space entails uniform convergence in the set of continuous functions.

Definition 6. Let $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is said to *converge strongly* to $x \in X$ iff

$$\|x_n - x\| \to 0 \quad \text{when} \quad n \to \infty;$$

then we write $x_n \to x$.

(ii) $\{x_n\}$ is said to *converge weakly* to $x \in X$ iff

$$(x_n - x|y) \to 0 \quad \text{when} \quad n \to \infty, \ \forall y \in X;$$

then we write $x_n \rightharpoonup x$.

The most important properties of weak convergence are gathered in the following result:

Proposition 4. *Let $\{x_n\}$ be a bounded sequence in X.*

(i) $\{x_n\}$ has a weakly convergent subsequence.

(ii) If every weakly convergent subsequence of $\{x_n\}$ has weak limit x, then $x_n \rightharpoonup x$.

(iii) If there exist $x \in X$ and a subset $D \subset X$, dense in X and such that $(x_n|y) \to (x|y)$, $\forall y \in D$, then $x_n \rightharpoonup x$.

Definition 7. Let $I \subset \mathbb{R}$. $\mathcal{C}_0^1(I)$ is the set of all real functions $\varphi \in \mathcal{C}^1(I)$ with compact support in I, i.e. such that they take null value everywhere except in a compact subset $K \subset I$ that depends on φ.

Definition 8. The *Sobolev space* $W_2^1(I)$ is defined as

$$W_2^1(I) = \left\{ x \in L_2(I);\ \exists y \in L_2(I),\ (\varphi'|x) = -(\varphi|y), \forall \varphi \in \mathcal{C}_0^1(I) \right\}.$$

y is called the *first generalized derivative* of x, denoted as $y = Dx$.

Remark 3. (i) Let $x \in \mathcal{C}^1(\overline{I})$. Then, the continuous first derivative $x' : \overline{I} \longrightarrow \mathbb{R}$ is also the first generalized derivative of x in $\overline{I}$. This follows immediately from the classical integration by parts formula.

(ii) The generalized derivative of x is unique in $L_2(I)$.

The norm and inner product in W_2^1 are set, respectively, to

$$\|x\|_{W_2^1} = \|x\|_{L_2} + \|Dx\|_{L_2}, \tag{5}$$

$$(x|y)_{W_2^1} = (x|y)_{L_2} + (Dx|Dy)_{L_2}. \tag{6}$$

Outstanding properties of Sobolev spaces are:

Proposition 5. *(i) $H^1(I) = W_2^1(I)$ is a separable H-space with the inner product defined in (6).*

(ii) (Rellich-Kondratjev) The embedding $H^1(I) \subseteq \mathcal{C}(\overline{I})$ is compact; consequently, $x_n \rightharpoonup x$ in H^1 yields $x_n \to x$ in $\mathcal{C}(\overline{I})$.

(iii) Let $\{x_n y_n\}$ be a bounded sequence in H^1 with $x_n \to x$ in $C(\overline{I})$ and $y_n \rightharpoonup y$ in $L_2(I)$. Then, $x_n y_n \rightharpoonup xy$ in $H^1(I)$.

3. **Statement of the problem and preview of the main results.** The boost and buck-boost DC-to-DC bilinear power converters have the following ideal state space model:

$$x' = 1 - u(k + y) \tag{7}$$

$$y' = -\lambda y + ux, \tag{8}$$

where x is proportional to the input current, y is proportional to the output voltage and λ gathers the system parameters. The control variable u takes values in the discrete set $\{0, 1\}$, while $k = 0$ for the boost converter and $k = 1$ for the buck-boost converter. Assuming exact knowledge of the plant parameters and absence of disturbances, the control target focuses on the tracking of periodic references by the state variable y.

As the control gains are fixed, they cannot be tuned attending to each particular reference. Therefore, candidate references must satisfy certain restrictions in order to prevent system saturation [8].

A capital relation between state variables appears after the elimination of u in (7,8), which entails the differential relationship

$$x(1 - x') = (y' + \lambda y)(k + y).$$

Its study [8] reveals both the non-minimum phase character of this system when y is taken as output and the minimum phase feature when the output is x. This stresses the need of indirect control through the state variable x.

Hence, if we succeed in forcing x to follow a T-periodic reference $x = \phi(t)$ satisfying

$$\phi(1 - \phi') = (f' + \lambda f)(k + f), \tag{9}$$

$f(t)$ being a certain T-periodic output reference, the internal dynamics of the system will lead the output to asymptotically track $f(t)$. The reason lies in the fact that

$$(y' + \lambda y)(k + y) = (f' + \lambda f)(k + f) \tag{10}$$

admits $y = f(t)$ as an asymptotically stable solution provided that [8]

$$g = (f' + \lambda f)(k + f) > 0. \tag{11}$$

We also known from [8] that the ODE

$$x(1 - x') = g(t), \tag{12}$$

with $g(t) \in C^\infty$ defined in (11), positive and T-periodic, has a positive, T-periodic, unstable solution $x(t, x_0)$, with $x(0, x_0) = x_0$, denoted $x(t, x_0) = \phi(t)$ from now on. Following the proposal in [8], we look for an analytical approximation of $\phi(t)$ by means of truncated Fourier expansions via the Galerkin method. Hence, questions QA and QB in Section 1 may be particularized as follows:

Does it exit a sequence $\{\phi_n\}$ of solutions of the Galerkin equations associated to (12) that converges to the T-periodic solution of (12)?

Theorems 1 and 2, respectively, provide sufficient conditions to give a positive answer to that question.

Additionally, it is obvious that the use of an n-th Galerkin approximation ϕ_n instead of ϕ will affect the output y, converting it into a y_n that satisfies

$$(y_n' + \lambda y_n)(k + y_n) = \phi_n(1 - \phi_n').$$

According to Subsection 2.1, the right hand side of this equation can be written as

$$\phi_n(1 - \phi_n') = (f' + \lambda f)(k + f) + F\phi_n, \tag{13}$$

which is the equivalent to (9). The error term $F\phi_n$ is given by (4) and is also T-periodic; (10) becomes

$$(y_n' + \lambda y_n)(k + y_n) = g + F\phi_n \tag{14}$$

and, using $G_n(t) = g(t) + F\phi_n(t)$,

$$(y_n' + \lambda y_n)(k + y_n) = G_n. \tag{15}$$

Question QC of Section (1) is now straightforward, and may be splitted in two:

Does it exist asymptotically stable, T-periodic solution for (15), $\forall n \geq 0$, i.e., can we obtain a sequence $\{y_n\}$ with such a feature?

Does the sequence of output responses $\{y_n\}$ converge to the output reference $f(t)$?

The first one is very important, because a negative answer would imply much difficulty or even the impossibility of using this technique. The second one, if true, definitively validates the method from a mathematical viewpoint. These questions have also affirmative answer under appropriate conditions and are given in Theorems 3 and 4, respectively.

Complementary subjects are a study of the restrictions to avoid saturation problems and an evaluation of input and output errors, which correspond to questions QD and QE in Section 1.

As it has been mentioned above, the results exposed in this article are obtained under certain conditions. Some of them are technical, i.e., due to the physical nature of the system under study, and others are fundamental, i.e., sufficient conditions to ensure the fulfillment of mathematical properties. Below follows a list of the hypotheses assumed throughout the article and a brief comment about its origin and use.

Hypothesis H0: The output reference $f(t)$ and the function $g(t)$ are positive, T-periodic and $\mathcal{C}^\infty$. Moreover, in steady state the control action does not saturate:

$$\phi(t) \geq f'(t) + \lambda f(t) > 0$$

Hypothesis H1: Let $g(t)$ be expressed as $g(t) = g_0 + \tilde{g}(t)$, where $g_0 \in \mathbb{R}^+$ is the constant (direct) component of $g(t)$, $\tilde{g}(t)$ is its time-dependent component (with zero mean value) and $\omega = 2\pi T^{-1}$. The following inequalities are fulfilled:

$$g_0\omega > 1, \qquad (g_0\omega - 1)^2 \geq 4\omega \left\| \tilde{g} \right\|.$$

Hypothesis H2: Let $\{G_n\}$ be a sequence obtained by setting $G_n(t) = g(t) + F\phi_n(t)$. Every element of the sequence is positive, i.e., $G_n(t) > 0$, $\forall n \geq 0$.

Hypothesis H3: Let $\{\phi_n\}$ be a sequence of solutions of the Galerkin equations associated to (12), and let $\{y_n\}$ be the corresponding sequence of solutions of equation (15). For all $n \geq 0$, the system is in the unsaturated zone defined by

$$\phi_n(t) \geq y_n'(t) + \lambda y_n(t) > 0,$$

when it undergoes approximate indirect tracking control.

Remark 4. (i) Hypotheses H0 and H3 are of technical type. In fact, the control gain u takes values in the discrete set $\{0, 1\}$. Hence, any control strategy that yields saturation of u will hardly achieve its target. Notice that the isolation of u in (7) and (8) results in

$$u = \frac{1 - x'}{k + y} = \frac{y' + \lambda y}{x}.$$

The unsaturated regions of the phase plane are calculated by demanding $0 \leq u \leq 1$. Section 4 in [8] contains a detailed study of the characteristic restrictions on switched converters when an exact indirect control is performed, and gives sufficient conditions over f and g to ensure unsaturation of the control action. Section 6 of this article studies the parallel case for an approximate control. H0 is assumed from the very beginning and H3 appears when the convergence of the output responses is considered.

(ii) Hypotheses H1 and H2 are of fundamental type. H1 guarantees the existence of solution for the Galerkin equations associated to the ODE (12), while H2 is mainly used both in the proof of the convergence of the Galerkin sequence of approximate inputs $\{\phi_n\}$ and to ensure the existence of periodic output responses in front of approximate input references.

4. **Solution of the Galerkin equations.** Assume that Hypotheses H0 and H1 are satisfied. Let us take up problem (12), which we now write as

$$Fx = 0, \tag{16}$$

with $F : \mathcal{C}^1_{\text{per}}([0,T]) \subset L_2(0,T) \longrightarrow L_2(0,T)$, where $\mathcal{C}^1_{\text{per}}([0,T])$ stands for the set of continuous, T-periodic functions with continuous first derivative. F is defined as $Fx = x - xx' - g$; therefore, $F\left(\mathcal{C}^1_{\text{per}}([0,T])\right) \subset L_2(0,T)$.

It can be deduced from Section 3 that, under Hypothesis H0, the mapping F has a zero in $\mathcal{C}^1_{\text{per}}([0,T])$ because the solution $\phi(t)$ of (12) is positive, T-periodic and satisfies $\phi - \phi\phi' - g = 0$, the continuity of ϕ and ϕ' being thus guaranteed.

Given the trigonometric system $\{w_n\}$ and an element x_n from the subspace $X_n = \operatorname{span}\{w_0, \ldots, w_{2n}\}$, the Galerkin equations associated to (12) or (16) in X_n are

$$(Fx_n|w_j) = 0, \quad j = 0, \ldots, 2n. \tag{17}$$

Moreover, letting $P_n : L_2(0,T) \longrightarrow X_n$ be a projection operator defined as in (2), its equivalent form

$$P_n F x_n = 0, \tag{18}$$

can also be written, recalling that $P_n x_n = x_n$, $\forall x_n \in X_n$, as

$$x_n - P_n(x_n x_n' + g) = 0. \tag{19}$$

Let us denote by $\widetilde{X}_n$ the B-subspace of the functions with zero mean value: $\widetilde{X}_n = \operatorname{span}\{w_1, \ldots, w_{2n}\}$. Then, the decomposition

$$X_n = X_0 \oplus \widetilde{X}_n \tag{20}$$

allows us to write $x_n = x_{n0} + \tilde{x}_n$, $g = g_0 + \tilde{g}$, x_{n0}, $\tilde{x}_n$, g_0 and $\tilde{g}$ being unique and such that $x_{n0}, g_0 \in X_0$, $\tilde{x}_n \in \widetilde{X}_n$ and $\tilde{g} \in \cup_{n\geq 1}\widetilde{X}_n$. Using these expressions in (19) and observing that $x_n' = \tilde{x}_n'$ and that both $P_n(\tilde{x}_n\tilde{x}_n')$ and $P_n\tilde{g}$ belong to $\widetilde{X}_n$, (19) can be decomposed into

$$\begin{aligned} x_{n0} &= g_0, & (21)\\ \tilde{x}_n &= x_{n0}\tilde{x}_n' + P_n(\tilde{x}_n\tilde{x}_n' + \tilde{g}). & (22) \end{aligned}$$

Notice that (21) and (22) are problems in X_0 and $\widetilde{X}_n$, respectively. Then, for $n = 0$ there is a unique solution $x_0 = x_{00} = g_0$, while for $n \geq 1$ the 0-*th* component is $x_{n0} = g_0$. In this case, equation (22) may be read as the fixed point problem

$$g_0\tilde{x}_n' + P_n(\tilde{x}_n\tilde{x}_n' + \tilde{g}) = \tilde{x}_n. \tag{23}$$

The proof of the existence of a solution of (23) is based on the strategy already mentioned in Remark 2.

Let us set $R > 0$ and define $U_n \subset \widetilde{X}_n$ as $U_n = \{\tilde{x}_n \in \widetilde{X}_n;\ \|\tilde{x}_n'\| < R\}$, where $\|\cdot\|_{L_2} = \|\cdot\|$ from now on. Note that U_n is bounded: as

$$\tilde{x}_n = \sum_{j=1}^{2n} c_{nj} w_j, \quad \frac{\tilde{x}_n'}{\omega} = \sum_{m=1}^{n} k(-c_{n,2m-1} w_{2m} + c_{n,2m} w_{2m-1})$$

with $\omega = 2\pi T^{-1}$,

$$\frac{\|\tilde{x}_n'\|}{\omega} = \sqrt{\sum_{m=1}^{n} m^2 (c_{n,2m-1}^2 + c_{n,2m}^2)} \geq \sqrt{\sum_{j=1}^{2n} c_{nj}^2} = \|\tilde{x}_n\|.$$

This immediately leads to

$$\|\tilde{x}_n\| \leq \frac{R}{\omega}, \qquad \forall \tilde{x}_n \in U_n. \tag{24}$$

With $g(t)$ also fixed, recall the restrictions established in Hypothesis H1:

$$g_0\omega > 1, \qquad (g_0\omega - 1)^2 \geq 4\omega \|\tilde{g}\|. \tag{25}$$

Then, we define the mapping $H_n : \overline{U_n} \times [0,1] \longrightarrow \widetilde{X}_n$ with $H_n(\tilde{x}_n, \lambda) = g_0\tilde{x}_n' + \lambda P_n(\tilde{x}_n\tilde{x}_n' + \tilde{g})$.

Proposition 6. *$H_n(x_n, \lambda)$ verifies that:*

(i) It is compact in $\overline{U_n} \times [0,1]$, $\forall n \geq 1$.

(ii) If (25) are fulfilled, there exists $R > 0$ such that H_n has no fixed points on ∂U_n, $\forall n \geq 1$ and $\forall \lambda \in [0,1]$.

Proof. (i) As H_n has finite dimension and it acts in a closed domain, its continuity ensures its compactness.

(ii) The second statement follows because $\forall n \geq 1$,

$$\Delta_n = \|H_n(\tilde{x}_n, \lambda) - \tilde{x}_n\| \neq 0, \quad \forall \tilde{x}_n \in \partial U_n, \forall \lambda \in [0,1].$$

First note that $\|P_n x\| \leq \|x\|$, $\forall n \geq 0$, $\forall x \in L_2(0,T)$. Therefore, using this relation, the Schwarz inequality, the positivity of g_0 (arising from $g > 0$) and (24), we have that

$$\begin{aligned} \Delta_n &= \|g_0 \tilde{x}'_n + \lambda P_n(\tilde{x}_n \tilde{x}'_n + \tilde{g}) - \tilde{x}_n\| \geq g_0 \|\tilde{x}'_n\| - \lambda \|P_n(\tilde{x}_n \tilde{x}'_n + \tilde{g})\| - \|\tilde{x}_n\| \geq \\ &\geq g_0 \|\tilde{x}'_n\| - \|\tilde{x}_n\| \|\tilde{x}'_n\| - \|\tilde{g}\| - \|\tilde{x}_n\| \geq g_0 R - \frac{R^2}{\omega} - \|\tilde{g}\| - \frac{R}{\omega} = p(R) \end{aligned}$$

The vertex of the inverted parabola $p(R)$ has coordinates

$$(R_v, p(R_v))^\top = \left(\frac{g_0\omega - 1}{2}, \frac{(g_0\omega - 1)^2}{4\omega} - \|\tilde{g}\| \right).$$

Consequently, it is easy to check that the fulfillment of (25) ensures the location of $(R_v, p(R_v))^\top$ in the first quadrant of $\mathbb{R}^2$. It is therefore guaranteed the existence of $R > 0$, $R \in (R_m, R_M)$, with

$$R_{M,m} = \frac{g_0\omega - 1 \pm \sqrt{(g_0\omega - 1)^2 - 4\omega \|\tilde{g}\|}}{2},$$

such that H_n has no fixed points on ∂U_n, $\forall n \geq 1$, $\forall \lambda \in [0,1]$. ☐

Proposition 7. *If (25) are fulfilled, $i(H_n(\tilde{x}_n, 1), U_n) = i(H_n(\tilde{x}_n, 0), U_n) = 1$.*

Proof. When such conditions are satisfied, Proposition 6 ensures that $H_n(\tilde{x}_n, 0)$ and $H_n(\tilde{x}_n, 1)$ are homotopically compact. Therefore, axiom (A4) of Definition 3 guarantees the equality of their fixed point index. It remains to prove that $i(H_n(\tilde{x}_n, 0), U_n) = i(g_0 \tilde{x}'_n, U_n) = 1$. The fixed point problem $g_0 \tilde{x}'_n = \tilde{x}_n$, $\tilde{x}_n \in U_n$, is equivalent to the following problem in $\mathbb{R}^{2n}$: let $R, \omega \in \mathbb{R}^+$ and let $W_n \subset \mathbb{R}^{2n}$ be defined as

$$W_n = \left\{ z_n = (c_{n,l})_{l \leq 2n};\ \sqrt{\sum_{m=1}^{n} m^2 (c_{n,2m-1}^2 + c_{n,2m}^2)} < \frac{R}{w} \right\}.$$

Notice that W_n is open and bounded because the euclidean norm $\|\cdot\|_E$ of its elements is bounded: $\forall z_n \in W_n$,

$$\frac{R}{\omega} > \sqrt{\sum_{k=1}^{n} mk^2 (c_{n,2m-1}^2 + c_{n,2m}^2)} > \sqrt{\sum_{j=1}^{2n} c_{nj}^2} = \|z_n\|_E.$$

Let $f : W_n \longrightarrow \mathbb{R}^{2n}$ be the C^1 mapping

$$f(z_n) = g_0\omega(c_{n2}, -c_{n1}, \ldots, nc_{n,2n}, -nc_{n,2n-1}).$$

The fixed points of f are the solutions of $f(z_n) = z_n$, which can be written as

$$\left\{ \begin{array}{lcl} c_{n,2m-1} &=& k\omega g_0 c_{n,2m} \\ c_{n,2m} &=& -k\omega g_0 c_{n,2m-1}, \end{array} \right. \qquad m = 1, \ldots, n.$$

Then, $\forall j,\ j=1,\dots,2n$,

$$c_{nj} = -j^2\omega^2 g_0^2 c_{nj} \Longrightarrow (1+j^2\omega^2 g_0^2)c_{nj} = 0 \Longrightarrow c_{nj} = 0.$$

Hence, $z_n = 0$ is the only regular fixed point of f:

$$f'(z_n) = \left[\operatorname{diag} \begin{pmatrix} 0 & m\omega g_0 \\ -m\omega g_0 & 0 \end{pmatrix}_{m=1,\dots,n} \right],$$

and, trivially, $\det[f'(0)] = (n!\omega^n g_0^n)^2 \neq 0$. Let now be $F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$, with $F(z_n) = z_n - f(z_n)$. We have that

$$F'(z_n) = \left[\operatorname{diag} \begin{pmatrix} 1 & -k\omega g_0 \\ k\omega g_0 & 1 \end{pmatrix}_{k=1,\dots,n} \right],$$

and, again trivially, $\det[F'(0)] = \prod_{k=1}^{n}(1+k^2w^2g_0^2) > 0$. According to Proposition 3, $i(f, W_n) = 1$, $\forall n \geq 1$, which immediately implies $i(H_n(\tilde{x}_n, 0), U_n) = 1$, $\forall n \geq 1$. □

We are now ready to state and prove the main result of the section.

Theorem 1. *Let us assume that Hypotheses H0 and H1 are satisfied. Then, the Galerkin equations (19) associated with the ODE defined in (12) have solution ϕ_n, $\forall n \geq 0$.*

Proof. For the case $n = 0$, equation (21) leads to $\phi_0 = g_0$. For $n \geq 1$, Proposition 7 and axiom (A2) of Definition 3 ensure the existence of a solution $\tilde{\phi}_n \in U_n$ for the fixed point problem (23). Therefore, $\phi_n = g_0 + \tilde{\phi}_n$ is a solution of the Galerkin equations (19). □

5. **Convergence of the Galerkin approximation.** Theorem 1 in Section 4 ensures that, under Hypotheses H0 and H1, the Galerkin equations (19) associated to the ODE defined in (12) have solution $\forall n \in \mathbb{N}$. In this section we will see that the sequence $\{\phi_n\}$ of solutions of such Galerkin equations exhibits uniform convergence to the periodic solution of (12) by additionally assuming Hypothesis H2. The proof begins with a set of preliminary results and then follows the step-by-step standard procedure.

Preliminary results.

Lemma 1. *Let $p(t), q(t) \in C([0,T])$ be nonzero and T-periodic. Then, the set of functions $V_{p,q} \subset L_2$ defined as*

$$V_{p,q} = \left\{ \ v_{p,q} = p(t)\varphi' + q(t)\varphi,\ \forall \varphi \in \mathcal{C}^1_{\mathrm{per}}([0,T]) \right\},$$

is dense in $L_2(0,T)$.

Proof. We will prove the Lemma by observing that the trigonometric system belongs to $V_{p,q}$, i.e., for every element w_n of the trigonometric system, there exists φ_n in $\mathcal{C}^1_{\mathrm{per}}([0,T])$ such that $p(t)\varphi_n' + q(t)\varphi_n = w_n$. Therefore, let us write this ODE as

$$\varphi_n' = -\frac{q(t)}{p(t)}\varphi_n + \frac{w_n}{p(t)}. \tag{26}$$

The terms in the equation are dominated by the linear part when $t \to \infty$. Even more, the ODE $z' = -p^{-1}(t)q(t)z$ has no T-periodic solutions except $z = 0$: its general solution is

$$z(t) = K \exp\left\{ -\int_0^t \frac{q(s)}{p(s)} ds \right\}$$

and, as $p^{-1}(t)q(t)$ has definite sign, $z(t)$ has also a definite sign and is strictly decreasing or strictly increasing $\forall K \neq 0$. Following [22], we can state that (26) has at least one T-periodic solution. □

Lemma 2. *$\{\phi_n\}$ is such that $\phi_n > 0$, $\forall n \geq n_0$.*

Proof. From Hypothesis H0 it results that $g_0 > 0$. We also know that $\phi_n = g_0 + \tilde{\phi}_n$, $\tilde{\phi}_n \in \widetilde{X}_n$; then, $\phi_n > 0$ at least in an open interval $I \subset (0, T)$. Suppose that a certain ϕ_{n_0} takes negative values. As it is continuous and T-periodic, Rolle's Theorem ensures the existence of a minimum (at $t = \bar{t}$, for example) where it happens $\phi_{n_0}(\bar{t}) < 0$ and $\phi'_{n_0}(\bar{t}) = 0$; therefore, from (13) we find that $G_{n_0}(\bar{t}) = \phi_n(\bar{t})\left[1 - \phi'_{n_0}(\bar{t})\right] = \phi_{n_0}(\bar{t}) < 0$, thus contradicting Hypothesis H2. □

Step 1: A priori estimates.

Assume that $\|\cdot\| = \|\cdot\|_{L_2}$ from now on.

Lemma 3. *Let $\{\phi_n\}$ be a sequence of solutions of the Galerkin equations (17). Then, there exists a constant $R > 0$ such that, $\forall n \in \mathbb{N}$, the following inequalities hold:*
(i) $\|\phi'_n\| < R$,
(ii) $\|\phi_n\| < \|g_0\| + \frac{R}{\omega}$,
(iii) $\|F\phi_n\| < (1 + R + g_0\omega)\frac{R}{\omega} + \|\tilde{g}\|$.

Proof. From decomposition (20) and equation (21) it follows that each solution ϕ_n of the Galerkin equations is such that $\phi_n = g_0 + \tilde{\phi}_n$, with $\tilde{\phi}_n \in U_n$. This last implies $\left\|\tilde{\phi}'_n\right\| < R$, and (24) leads immediately to $\left\|\tilde{\phi}_n\right\| < \frac{R}{\omega}$. Hence, on the one hand,

$$\|\phi'_n\| = \left\|(g_0 + \tilde{\phi}_n)'\right\| = \left\|\tilde{\phi}'_n\right\| < R, \tag{27}$$

$$\|\phi_n\| = \left\|g_0 + \tilde{\phi}_n\right\| \leq \|g_0\| + \left\|\tilde{\phi}_n\right\| < \|g_0\| + \frac{R}{\omega}. \tag{28}$$

On the other hand,

$$\begin{aligned} \|F\phi_n\| &= \|\phi_n - \phi_n\phi'_n - g\| = \left\|\tilde{\phi}_n - g_0\tilde{\phi}'_n - \tilde{\phi}_n\tilde{\phi}'_n - \tilde{g}\right\| \leq \\ &\leq \left\|\tilde{\phi}_n\right\| + g_0\left\|\tilde{\phi}'_n\right\| + \left\|\tilde{\phi}_n\right\|\left\|\tilde{\phi}'_n\right\| + \|\tilde{g}\| < \frac{R}{\omega} + g_0 R + \frac{R^2}{\omega} + \|\tilde{g}\|. \end{aligned}$$

□

Step 2: Existence of weakly convergent subsequences.

Lemma 4. *The error sequence $\{F\phi_n\}$ converges weakly to 0 in $L_2(0, T)$.*

Proof. Lemma 3 (iii) indicates that the sequence $\{F\phi_n\}$ is bounded $\forall n \geq 0$. Moreover, (17) yields $\lim_{n\to\infty}(F\phi_n|w_j) = 0$, $\forall w_j \in \{w_n\}$. As the trigonometric system $\{w_n\}$ is dense in $L_2(0, T)$, Proposition 4 (iii) entails the result. □

Lemma 5. *The sequence* $\{\phi_n\}$ *is such that:*
(i) $\{\phi_n\}$, $\{\phi_n^2\}$ *belong to the Sobolev space* $H^1(0,T)$.
(ii) $\{\phi_n\}$, $\{\phi_n^2\}$ *possess weakly convergent subsequences in* $H^1(0,T)$.

Proof. (i) As ϕ_n is trivially a $C^1(0,T)$ function $\forall n \geq 0$, Remark 3 ensures that its first generalized derivative coincides with the first classical derivative. Moreover, the a priori estimates derived in Lemma 3 (i) and (ii) yield the boundedness of the sequences $\{\phi_n\}$ and $\{\phi_n'\}$ in $L_2(0,T)$ which, in turn, result in $\|\phi_n\|_{H^1}$ also boundend. A similar reasoning yields the parallel result for $\{\phi_n^2\}$.
(ii) It follows directly from Lemma 5 (i) and Proposition 4 (i). □

Lemma 6. *Let* $\left\{\hat{\phi}_n\right\}$ *be a weakly convergent subsequence of* $\{\phi_n\}$, *and let* $\hat{\phi}$ *be its weak limit. Then,*
(i) $\left\{\hat{\phi}_n\right\}$ *converges uniformly to* $\hat{\phi}$ *in* $C([0,T])$.
(ii) $\hat{\phi}_n^2$ *converges weakly to* $\hat{\phi}^2$ *in* $H^1(0,T)$.

Proof. (i) Starting from Lemma 5, the result follows from Proposition 5 (ii).
(ii) Immediate from Proposition 5 (iii) after taking into account Lemma 5 (i). □

Step 3: The weak limits are weak solutions of the full equation.

Let us first establish the weak problem associated to the periodic solutions of the ODE (12) in $L_2(0,T)$. Performing a scalar product on both sides of the equation with any function $\varphi \in \mathcal{C}^1_{\text{per}}([0,T])$,

$$(x(1-x')|\varphi) = (g|\varphi) \Longrightarrow (x|\varphi) - \frac{1}{2}((x^2)'|\varphi) = (g|\varphi).$$

Integrating by parts while taking into account the T-periodicity of x and φ yields

$$(x|\varphi) + \frac{1}{2}(x^2|\varphi') = (g|\varphi), \quad \forall \varphi \in \mathcal{C}^1_{\text{per}}([0,T]). \tag{29}$$

Lemma 7. *The classical, positive and* T*-periodic solution* ϕ *of (12) and the weak limit of every weakly convergent subsequence of* $\{\phi_n\}$ *are weak* T*-periodic solutions of (29).*

Proof. The statement is obvious for ϕ. Then, denote $\left\{\hat{\phi}_n\right\}$ a weakly convergent subsequence of $\{\phi_n\}$ with weak limit $\hat{\phi}$ by Lemma 5 (ii). Every element of the subsequence satisfies the ODE (13), now written

$$\hat{\phi}_n(1-\hat{\phi}_n') = g + F\hat{\phi}_n.$$

Therefore,

$$(\hat{\phi}_n(1-\hat{\phi}_n')|\varphi) = (g|\varphi) + (F\hat{\phi}_n|\varphi), \quad \forall \varphi \in \mathcal{C}^1_{\text{per}}([0,T]).$$

The inner product may be expressed as

$$(\hat{\phi}_n|\varphi) - \frac{1}{2}((\hat{\phi}_n^2)'|\varphi) = (g|\varphi) + (F\hat{\phi}_n|\varphi)$$

and, integrating by parts, we easily arrive at

$$(\hat{\phi}_n|\varphi) + \frac{1}{2}(\hat{\phi}_n^2|\varphi') = (g|\varphi) + (F\hat{\phi}_n|\varphi).$$

Moreover, for $n \to \infty$, the weak convergences $F\hat{\phi}_n \rightharpoonup 0$, $\hat{\phi}_n \rightharpoonup \hat{\phi}$ and $\hat{\phi}_n^2 \rightharpoonup \hat{\phi}^2$, ensured by Lemmas 4 and 5 (i) and (ii), lead to

$$(\hat{\phi}|\varphi) + \frac{1}{2}(\hat{\phi}^2|\varphi') = (g|\varphi)$$

. □

Step 4: Uniqueness.

Lemma 8. *Every weakly convergent subsequence of $\{\phi_n\}$ has weak limit ϕ, the classical, positive and T-periodic solution of (12).*

Proof. Consider $\left\{\hat{\phi}_n\right\}$ a subsequence of $\{\phi_n\}$ weakly convergent to a certain $\hat{\phi}$. Lemma 7 ensures that both ϕ and $\hat{\phi}$ satisfy (29). We may then write

$$(\hat{\phi}|\varphi) + \frac{1}{2}(\hat{\phi}^2|\varphi') = (\phi|\varphi) + \frac{1}{2}(\phi^2|\varphi'),$$

which can be re-written as

$$(\hat{\phi} - \phi| \left[\frac{1}{2}(\hat{\phi} + \phi)\varphi' + \varphi)\right]) = 0, \quad \forall \varphi \in \mathcal{C}^1_{\mathrm{per}}([0, T])$$

or, alternatively, $(\hat{\phi} - \phi|v_{p_x,q_x}) = 0$, $\forall v_{p_x,q_x} \in V_{p_x,q_x}$ (see Lemma 1), where now

$$p_x(t) = 2^{-1}\left[\hat{\phi}(t) + \phi(t)\right], \qquad q_x(t) = 1.$$

On the one hand, the positivity of $p_x(t)$ is guaranteed by the fact that ϕ is positive. On the other hand, $\phi_n > 0$, $\forall n \geq 0$, by Hypothesis H2 and $\hat{\phi}$ is, at least, non negative as a consequence of Lemma 2. Therefore, as V_{p_x,q_x} is dense in L_2 by Lemma 1, $\hat{\phi} = \phi$ almost for all t in $[0, T]$. The continuity of both $\hat{\phi}$ and ϕ entails $\hat{\phi}(t) = \phi(t)$, $\forall t \in [0, T]$. □

Theorem 2. *If Hypotheses H0, H1 and H2 are satisfied, the sequence $\{\phi_n\}$ of solutions of the Galerkin equations (17) converges uniformly to the periodic solution ϕ of (12).*

Proof. Lemma 8 guarantees that every weakly convergent subsequence of $\{\phi_n\}$ has a weak limit ϕ. Item (ii) of Proposition 4 leads to $\phi_n \rightharpoonup \phi$ and, finally, Proposition 5 (ii) yields the result. □

6. **System output.** In Sections 4 and 5 it has been shown that Hypotheses H0, H1 and H2 entail the existence of a Galerkin sequence of approximate inputs $\{\phi_n\}$ and its uniform convergence to the T-periodic solution ϕ of (12). In this Section we will try to answer the questions stated in Section 3 about the type of output we can expect when an approximate indirect control, exerted by means of Galerkin approximations, is induced in the system. Specifically, Theorem 3 points out that an indirect control, exerted by means of any $\phi_n(t)$ such that the corresponding $G_n(t)$ is positive, results in a periodic and asymptotically stable output response. Its proof, as well as the proof of its Corollary, are very close to that of Theorem 3.1 and Corollary 3.1 in [8]. Hence, they will be omitted here. Restrictions on the signals to be tracked will also be derived.

Theorem 3. *Assume that Hypotheses H0,H1 and H2 are fulfilled. Consider the ODE (15) as a Cauchy problem with* $y_n(0) = y_{n0}$ *and* G_n *being any function of the sequence* $\{G_n\}$*. Then, equation (15) has one and only one periodic solution in* $\mathbb{R}^+$*, hyperbolic and asymptotically stable.*

Corollary 1. *Assume that Hypotheses H0, H1 and H2 are fulfilled. Then, equation (15) has* $y_n(t) = \bar{y}_n(t)$ *as a single periodic solution in* $(-\infty, -k)$*, hyperbolic and asymptotically stable. Moreover, in the exact problem* ($G_n(t) = g(t)$) *and for* $k = 0$ *it is* $\bar{y}(t) = -f(t)$.

Remark 5. (i) Theorem 3 allows to state that, given a sequence of approximate input references $\{\phi_n\}$, there exists a sequence of output responses $\{y_n\}$ such that, $\forall n \geq 0$, y_n is positive, T-periodic and asymptotically stable.

(ii) Proposition 4.4 in [8] is a particular case of Theorem 3 if we consider $G_n = g$. In this situation, $y(t) = f(t)$ is a solution of the ODE with the same features as the functions y_n.

As seen in Section 3, the basic restriction suffered by our system is due to the fact that its performance is located in a certain region of the phase plane, the so called *unsaturated zone*. This means that two inequalities where we can find inputs and outputs must be satisfied, which will lead to conditions on the output reference. In [8] a study of the situation for an exact input current was performed. Here we will try to establish parallel results for the approximate case.

Consider an ideal steady state where x and y are tracking ϕ_n and y_n, respectively, due to an ideal control action $\bar{u}_n$. System (7,8) can be written as

$$\begin{aligned} \phi_n' &= 1 - \bar{u}_n(k + y_n) \\ y_n' &= -\lambda y_n + \bar{u}_n \phi_n. \end{aligned}$$

The unsaturated region, defined by $0 \leq \bar{u}_n \leq 1$, is

$$0 \leq \frac{1 - \phi_n'}{k + y_n} \leq 1 \quad \text{or, equivalently,} \quad 0 \leq \frac{y_n' + \lambda y_n}{\phi_n} \leq 1.$$

Proposition 8. *If Hypotheses H0, H1 and H2 are verified, the unsaturated region is given by*

$$0 < 1 - \phi_n' \leq k + y_n \quad \textit{or} \quad 0 < y_n' + \lambda y_n \leq \phi_n, \qquad \forall n \geq 0. \tag{30}$$

Proof. Hypothesis H2 ensures that $G_n(t) > 0$, $\forall n \geq 0$, which, in turn, yields $\phi_n > 0$ by Lemma 2. Using (15) it is straightforward to show that $G_n > 0$ also entails $\text{sign}(y_n' + \lambda y_n) = \text{sign}(k + y_n)$. It is then necessary to use the region $0 < y_n' + \lambda y_n \leq \phi_n$. □

A sufficient condition for system (7,8) to lay in the unsaturated zone is:

Proposition 9. *If Hypotheses H0, H1 and H2 are verified and*

$$\inf_{t \in [0,T]} \{G_n(t)\} \geq \|y_n' + \lambda y_n\|_\infty ,$$

the system is in the unsaturated zone defined by Proposition 8.

Proof. Immediate following the proof of Proposition 4.2 in [8]. □

7. **Convergence of the system output.** Recall Remark 5 (i). Uniform convergence of the output sequence $\{y_n\}$ towards the reference profile $f(t)$ will be guaranteed in this Section under the additional assumption of Hypothesis H3. The structure of this Section is very close to that of Section 5. Hence, we will establish some preliminary results and the a priori estimates. As the rest follows identically, we will straightforward state the main result without detailing its proof.

Preliminary results.

Notice that Hypotheses H0, H1 and H2, besides yielding the positivity of y_n, also entail the continuity of y_n, y_n' and y_n''. Then, the Fourier series of both y_n and y_n' are

$$y_n = y_{0n} w_0 + \sum_{m\geq 1} y_{2m-1,n} w_{2m-1} + y_{2m,n} w_{2m}, \tag{31}$$

$$y_n' = \omega \sum_{m\geq 1} m(-y_{2m-1,n} w_{2m} + y_{2m,n} w_{2m-1}), \tag{32}$$

where $\{w_j\}_j$ again stands for the trigonometric system (1).

Lemma 9. *Consider the operator P_0 (see (2)). Then, the sequence $\{y_n\}$ is such that, $\forall n \geq 0$,*
(i) $P_0 y_n' = P_0(y_n y_n') = 0$,
(ii) $P_0 y_n^2 = T^{-\frac{1}{2}} \|y_n\|^2$.

Proof. (i) The first relation, $P_0 y_n' = 0$, is obvious. For the second one, $P_0(y_n y_n') = 2^{-1} P_0(y_n^2)' = 0$, because the derivative operation eliminates the w_0-component.
(ii) Take into account that

$$y_n^2 = \left(\sum_{j\geq 0} y_{jn} w_j\right)^2 = \sum_{j\geq 0} y_{jn}^2 w_j^2 + 2\sum_{i\neq j} y_{in} y_{jn} w_i w_j.$$

The product of two different elements of the trigonometric system is proportional to the product of two different trigonometric functions, which has no component in the w_0 direction. Let us now look at the quadratic terms:

$$\begin{aligned} w_0^2 &= \tfrac{1}{\sqrt{T}} w_0, \\ w_{2m-1}^2 &= \tfrac{2}{T} \cdot \tfrac{1}{2}\left(1 + \cos \tfrac{4\pi m t}{T}\right) = \tfrac{1}{\sqrt{T}} w_0 + \tfrac{1}{\sqrt{2T}} w_{4km-1}, \\ w_{2m}^2 &= \tfrac{2}{T} \cdot \tfrac{1}{2}\left(1 - \cos \tfrac{4\pi m t}{T}\right) = \tfrac{1}{\sqrt{T}} w_0 - \tfrac{1}{\sqrt{2T}} w_{4m-1}. \end{aligned}$$

The result follows immediately. □

A priori estimates.

Lemma 10. *If Hypotheses H0-H3 are verified, the sequence $\{y_n\}$ is such that $\|y_n\|$ and $\|y_n'\|$ are bounded in $L_2(0,T)$, $\forall n \geq 0$.*

Proof. (i) y_n is a periodic and $C^1(0,T)$ function $\forall n \geq 0$. Hence, Remark 3 guarantees that its first generalized derivative coincides with the first classical derivative. As the system is in an unsaturated zone by Hypothesis H3, it follows that $0 < y_n' + \lambda y_n \leq \phi_n$; then,

$$\|\phi_n\| \geq \|y_n' + \lambda y_n\| \geq \|y_n'\| - \lambda \|y_n\|,$$

which yields

$$\|y_n'\| \leq \|\phi_n\| + \lambda \|y_n\|. \tag{33}$$

Thence, as $\|\phi_n\| < \infty$ by (28), it suffices to prove the boundedness of $\|y_n\|$.

Let us now rewrite (15) as $(y_n' + \lambda y_n)(k + y_n) = G_n$. Since $G_n \in L_2(0,T)$, so is the left hand term. Then, projecting over the first element of the trigonometric system we have that

$$P_0(y_n' + \lambda y_n)(k + y_n) = P_0 G_n;$$

using (31) and the properties of P_0, it follows that

$$kP_0 y_n' + P_0 y_n y_n' + \lambda k P_0 y_n + \lambda P_0 y_n^2 = P_0 G_n$$

and, with Lemma 9, we have

$$\lambda k y_{0n} + \lambda T^{-\frac{1}{2}} \|y_n\|^2 = P_0 G_n \leq \|G_n\|. \tag{34}$$

In case that $k = 0$, (34) shows that

$$\|y_n\| \leq \sqrt{\frac{T^{\frac{1}{2}}}{\lambda} \|G_n\|}. \tag{35}$$

For $k = 1$, (34) reads as

$$y_{0n} \leq \frac{\|G_n\|}{\lambda} - \frac{\|y_n\|^2}{\sqrt{T}}. \tag{36}$$

Notice also that, from (32),

$$\|y_n'\|^2 = \omega^2 \sum_{m \geq 1} m^2 \left(y_{2m-1,n}^2 + y_{2m,n}^2\right) = \omega^2(\|y_n^2\| - y_{0n}^2),$$

and this leads to $\|y_n\|^2 \leq y_{0n}^2 + \omega^{-2} \|y_n'\|^2$. Taking into account that $y_{0n} > 0$ (due to the positivity of y_n), we conclude that

$$\|y_n\| \leq y_{0n} + \frac{\|y_n'\|}{\omega}, \qquad \forall n \geq 0. \tag{37}$$

Using (33) and (36) in (37) it follows that

$$\|y_n\| \leq \frac{\|\phi_n\|}{\omega} + \frac{\lambda}{\omega} \|y_n\| + \frac{\|G_n\|}{\lambda} - \frac{\|y_n\|^2}{\sqrt{T}},$$

which becomes

$$\|y_n\|^2 + \sqrt{T}\left(1 - \frac{\lambda}{\omega}\right) \|y_n\| \leq \sqrt{T}\left(\frac{\|\phi_n\|}{\omega} + \frac{\|G_n\|}{\lambda}\right).$$

Then,

$$\|y_n\| \leq \sqrt{\frac{T}{4}\left(1 - \frac{\lambda}{\omega}\right)^2 + T^{\frac{1}{2}}\left(\frac{\|\phi_n\|}{\omega} + \frac{\|G_n\|}{\lambda}\right)} - \frac{\sqrt{T}}{2}\left(1 - \frac{\lambda}{\omega}\right). \tag{38}$$

Since $\|G_n\| = \|g + F\phi_n\| \leq \|g\| + \|F\phi_n\|$, the bounded characters of g and of the sequences $\{\phi_n\}$ and $\{F\phi_n\}$, detailed in (28) and in Lemma 4, ensure $\|y_n\| < \infty$ in both (35) and (38). □

Main result.

Theorem 4. *If Hypotheses H0-H3 are verified, the sequence $\{y_n\}$ of solutions of the ODE (15) converges uniformly to the periodic reference f.*

Proof. The reader is referred to Section 5. □

8. **Error evaluation.** We will evaluate the input and output errors with the L_∞ norm.

8.1. **Input error.** To proceed, let us assume that Hypotheses H0 and H1 are satisfied. Hence, Theorem 1 ensures that the Galerkin equations (19) have solution $\forall n \geq 0$. Denote $e_{nx}(t) = \phi_n(t) - \phi(t)$ the error between an n-th Galerkin approximation and the exact input, with ϕ and ϕ_n satisfying (9) and (13), respectively. Denote also

$$\delta = \frac{\|g\|_\infty}{\inf_{t\in[0,T]}\{g(t)\}}. \tag{39}$$

We know from H0 that g is positive and T-periodic; then, it possesses a non zero, positive infimum which, in turn, guarantees $\delta \in \mathbb{R}^+$.

Theorem 5. *If Hypotheses H0 and H1 are verified, the error e_{nx} satisfies the following inequality:*

$$\|e_{nx}\|_\infty \leq \delta \left\|F\phi_n\right\|_\infty. \tag{40}$$

Proof. From the definition, e_{nx} is continuous, T-periodic and has continuous first derivative, thus exhibiting maximum and minimum values in each closed interval. Then, when we replace ϕ_n by $e_{nx} + \phi$ in (13), we obtain

$$-(\phi + e_{nx})e'_{nx} + (1 - \phi')e_{nx} = F\phi_n.$$

Therefore, at any instant $\bar{t}$ where e_{nx} has an extreme, $e'_{nx}(\bar{t}) = 0$, and the use of (9) yields

$$\frac{\inf\{\phi\}\inf\{F\phi_n\}}{\sup\{g\}} \leq e_{nx} \leq \frac{\sup\{\phi\}\sup\{F\phi_n\}}{\inf\{g\}},$$

where the infimums and supremes are searched on $[0,T]$. With analogous reasoning using and again (9) we arrive at $\inf\{g\} \leq \phi \leq \sup\{g\}$, which leads the previous relation to

$$\frac{\inf\{g\}\inf\{F\phi_n\}}{\sup\{g\}} \leq e_{nx} \leq \frac{\sup\{g\}\sup\{F\phi_n\}}{\inf\{g\}}.$$

Hence, $\|e_{nx}\|_\infty \leq \sup\left\{\delta^{-1}|\inf\{F\phi_n\}|, \delta|\sup\{F\phi_n\}|\right\}$, which entails $\|e_{nx}\|_\infty \leq \delta\left\|F\phi_n\right\|_\infty$, where $\delta \geq 1$ has been used. □

8.2. **Output error.** Recall that Hypotheses H0, H1 and H2, which are assumed in this Subsection, ensure the existence of a sequence $\{y_n\}$ of output responses which are positive and T-periodic $\forall n \geq 0$. Therefore, denote $e_{ny}(t) = y_n(t) - f(t)$ the error between an n-th approximation and the exact output, with y_n satisfying (15).

Theorem 6. *If Hypotheses H0, H1 and H2 are verified, the error e_{ny} is such that*

$$\|e_{ny}\|_\infty \leq \sqrt{\frac{\|F\phi_n\|_\infty}{\lambda}}, \qquad \forall n \geq 0. \tag{41}$$

Proof. Trivially, e_{ny} is continuous, T-periodic and with continuous first derivative $\forall n \geq 0$. Hence, the same process as that in Subsection 8.1 for e_{nx} will be used to compute bounds. When we substitute y_n by $e_{ny} + f$ in (15), we obtain

$$(k + f + e_{ny})e'_{ny} + \lambda e_y^2 + (f' + 2\lambda f + \lambda k)e_{ny} = F\phi_n.$$

At any instant $\bar{t}$ where e_{ny} exhibits an extreme, we will have $\bar{e}'_{ny} = e'_{ny}(\bar{t}) = 0$, and this makes

$$\bar{e}_{ny} = \frac{-\bar{p} \pm \sqrt{\bar{p}^2 + 4\lambda\overline{F\phi_n}}}{2\lambda}, \tag{42}$$

with $\bar{e}_{ny} = e_{ny}(\bar{t})$, $\overline{F\phi_n} = F\phi_n(\bar{t})$ and $\bar{p} = p(\bar{t})$, $p(t)$ being

$$p(t) = f'(t) + 2\lambda f(t) + \lambda k = [f'(t) + \lambda f(t)] + \lambda\,[k + f(t)] \geq 0$$

by Hypothesis H0. The negative option in (42) is incompatible with the fact that we work with $y_n > 0$, because when we consider it, we find that

$$\bar{y}_n - \bar{f} = \frac{-\bar{p} - \sqrt{\bar{p}^2 + 4\lambda\overline{F\phi_n}}}{2\lambda},$$

$\bar{y}_n = y_n(\bar{t})$, $\bar{f} = f(\bar{t})$, which yields

$$\bar{y}_n = -\frac{\bar{f}'}{2\lambda} - \frac{\lambda k + \sqrt{\bar{p}^2 + 4\lambda\overline{F\phi_n}}}{2\lambda},$$

with $\bar{f}' = f'(\bar{t})$. But as we are in an extreme, we may deduct from $e'_{ny}(\bar{t}) = 0$ that $f'(\bar{t}) = y'_n(\bar{t})$. Using (15) and (13) it is possible to find an expression for $y'_n(\bar{t})$ that, once taken to the above equality, results in

$$\bar{y}_n = \frac{\bar{y}_n}{2} - \frac{1}{2\lambda}\cdot\frac{\bar{\phi}_n\left[1 - \bar{\phi}_n\right]}{k + \bar{y}_n} - \frac{\lambda k + \sqrt{\bar{p}^2 + 4\lambda\overline{F\phi_n}}}{2\lambda},$$

with $\bar{\phi}_n = \phi_n(\bar{t})$. Assign

$$q_n = \frac{\bar{\phi}_n\left[1 - \bar{\phi}'_n\right]}{\lambda}, \quad r_n = k + \frac{\sqrt{\bar{p}^2 + 4\lambda\overline{F\phi_n}}}{\lambda}.$$

On the one hand, from (13) and Hypothesis H2 it results that $q_n > 0$, $\forall n \geq 0$; on the other hand, it is also $r_n \geq 0$, $\forall n \geq 0$, because $k \in \{0, 1\}$. Then, the second order equation that gives $\bar{y}_n$ is

$$\bar{y}_n^2 + (k + r)\bar{y}_n + (q_n + kr_n) = 0.$$

The fact that the coefficient of the first order term is positive or null and the independent term is strictly positive prevents the possibility of a positive solution. We must therefore take the positive solution of (42), which leads to

$$\frac{-\bar{p} + \sqrt{\bar{p}^2 - 4\lambda|\overline{F\phi_n}|}}{2\lambda} \leq \bar{e}_{ny} \leq \frac{-\bar{p} + \sqrt{\bar{p}^2 + 4\lambda|\overline{F\phi_n}|}}{2\lambda}. \tag{43}$$

As $a - \sqrt{|b|} \leq \sqrt{a^2 + b} \leq a + \sqrt{|b|}$, $a \geq \sqrt{|b|} \geq 0$, (43) becomes

$$-\sqrt{\lambda|\overline{F\phi_n}|} \leq \lambda\bar{e}_{ny} \leq \sqrt{\lambda|\overline{F\phi_n}|},$$

and the result follows immediately. □

Remark 6. The L_∞ input and output errors norm bounds depend on $\|F\phi_n\|_\infty$. Given a certain ϕ_n, $F\phi_n$ is obtainable from (13):

$$F\phi_n = \phi_n(1 - \phi'_n) - g.$$

Hence, (40) and (41) make sense and can be computed whenever ϕ_n and y_n, respectively, exist.

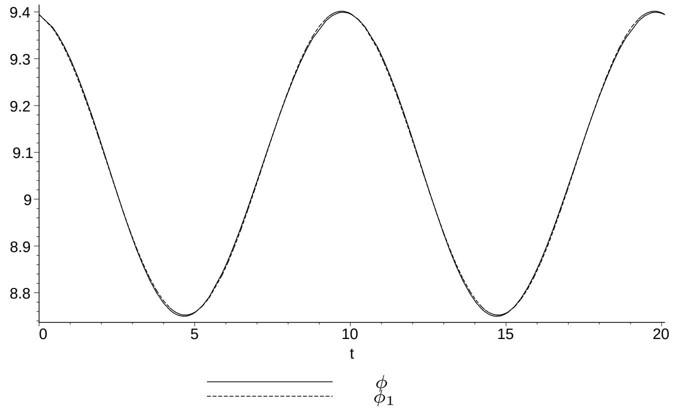

FIGURE 1. The periodic solution $\phi(t)$ of (12) and its first Galerkin approximation $\phi_1(t)$.

9. **Simulation results.** As stated in Section 1, the goal with the DC-to-DC nonlinear power converters is to make them work as DC-to-AC devices. Hence, the technique developed through the article is exemplified with the tracking of a sinusoidal reference profile

$$y_r = f(t) = A + B\sin\omega t$$

with electric network frequency (50 Hz) by a buck-boost converter ($k = 1$). Then, $w = 0.6252$; let also $\lambda = 0.9045$, $A = 2.7$ and $B = 0.3$.

These settings guarantee the fulfillment of Hypothesis H0: it is immediate that both $f(t)$ and $g(t)$ are T-periodic and positive; besides,

$$2.7 = A > \max\left\{B\sqrt{1+\left(\frac{\omega}{\lambda}\right)^2}, B + \frac{A + B\sqrt{1+\left(\frac{\omega}{\lambda}\right)^2}}{A - B\sqrt{1+\left(\frac{\omega}{\lambda}\right)^2}}\right\} = \max\left\{0.36, 1.61\right\},$$

which ensures the presence of the system in the unsaturated zone demanded in H0 (see [8] and [14]). Moreover, they allow the satisfaction of Hypothesis H1, because (see (25)),

$$5.67 = g_0\omega > 1 + 2\sqrt{\omega\,\|\tilde{g}\|} = 4.24.$$

The Galerkin equations (19) have been solved for the cases $n = 1, \ldots, 5$ using the large scale algorithm available with MATLAB 7. This algorithm is a subspace trust region method and is based on an interior-reflective Newton method.

Figure 1 depicts the periodic solution $\phi(t)$ of (12) together with ϕ_1. When ϕ_n, $n = 2, \ldots, 5$, are plotted with ϕ, they are indistinguishable from it. Table 1 indicates the closeness of the approximations to the exact solution providing the absolute and relative errors of $e_{nx} = \phi_n - \phi$, measured with the L_2 and L_∞ norms. Also the errors $F\phi_n$, $n = 1, \ldots, 5$ exhibit a clear tendency to decrease to 0 in Table 2, which contains their L_2 and L_∞ norms.

The verification of Hypothesis H2, which demands the positivity of $G_n(t)$, follows from the fulfillment of the trivial sufficient condition

$$\inf\left\{g\right\} > \left\|F\phi_n\right\|_\infty, \qquad \forall n \geq 0;$$

in our situation,

$$7.26 = \inf\left\{g\right\} > \left\|F\phi_1\right\|_\infty = 0.05,$$

TABLE 1. Absolute and relative errors of the Galerkin approximations measured with the L_2 and the L_∞ norms.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$\|e_{nx}\|_{L_2}$	$1.06\cdot 10^{-2}$	$1.90\cdot 10^{-4}$	$4.21\cdot 10^{-6}$	$1.16\cdot 10^{-7}$	$6.96\cdot 10^{-9}$
$\frac{\|e_{nx}\|_{L_2}}{\|\phi\|_{L_2}}\cdot 100\%$	$3.69\cdot 10^{-4}$	$6.59\cdot 10^{-6}$	$1.47\cdot 10^{-7}$	$4.02\cdot 10^{-9}$	$2.87\cdot 10^{-10}$
$\|e_{nx}\|_{\infty}$	$4.89\cdot 10^{-3}$	$8.80\cdot 10^{-5}$	$1.98\cdot 10^{-6}$	$5.49\cdot 10^{-8}$	$4.30\cdot 10^{-9}$
$\frac{\|e_{nx}\|_{\infty}}{\|\phi\|_{\infty}}\cdot 100\%$	$5.20\cdot 10^{-4}$	$9.36\cdot 10^{-6}$	$2.10\cdot 10^{-7}$	$5.85\cdot 10^{-9}$	$4.57\cdot 10^{-10}$

TABLE 2. L_2 and L_∞ norms of the Galerkin errors $F\phi_n$.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$\|F\phi_n\|_{L_2}$	$1.21\cdot 10^{-1}$	$3.23\cdot 10^{-3}$	$9.56\cdot 10^{-5}$	$3.28\cdot 10^{-6}$	$1.21\cdot 10^{-7}$
$\|F\phi_n\|_{\infty}$	$5.39\cdot 10^{-2}$	$1.45\cdot 10^{-3}$	$4.34\cdot 10^{-5}$	$1.49\cdot 10^{-6}$	$6.80\cdot 10^{-8}$

and Table 2 shows that $\|F\phi_1\|_\infty > \|F\phi_n\|_\infty$, $n=2,\ldots,5$.

The existence of a positive, asymptotically stable periodic output when a Galerkin approximation is used in equation (14) is shown in Figure 2: y_1, corresponding to the use of ϕ_1 in equation (14), is depicted with the reference $f(t)$. Functions y_2 to y_5 cannot be distinguished from f in a plot. Table 3 contains the L_2 and L_∞ norms of the output error $e_{ny} = y_n - f(t)$ in absolute and relative form. Again, the tendency of y_n to f is evident.

The fulfillment of Hypothesis H3 which, in addition to H0, H1 and H2, are sufficient to ensure the convergence of the system output $y_n(t)$ to the output reference profile $f(t)$, is verified with the presence of the approximately controlled systems in the unsaturated zone defined by (see 30)

$$0 \le \bar{u}_n = \frac{1-\phi_n'}{k+y_n} \le 1.$$

The plot of $\bar{u}_n$, $n=1,\ldots,5$, is shown to lie between 0 and 1 in Figure 3.

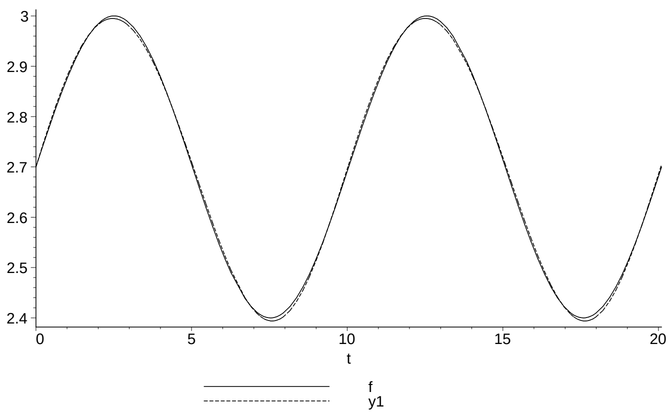

FIGURE 2. The reference $f(t)$ and its approximation $y_1(t)$, obtained using the first Galerkin approximation $\phi_1(t)$ of $\phi(t)$ in (14).

TABLE 3. Absolute and relative errors of the output measured with the L_2 and L_∞ norms.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$\|e_{ny}\|_{L_2}$	$1.55 \cdot 10^{-2}$	$3.45 \cdot 10^{-4}$	$8.50 \cdot 10^{-6}$	$3.51 \cdot 10^{-7}$	$1.33 \cdot 10^{-8}$
$\frac{\|e_{ny}\|_{L_2}}{\|f\|_{L_2}} \cdot 100$ %	$1.81 \cdot 10^{-3}$	$4.02 \cdot 10^{-5}$	$9.90 \cdot 10^{-7}$	$4.09 \cdot 10^{-8}$	$1.54 \cdot 10^{-9}$
$\|e_{ny}\|_\infty$	$7.95 \cdot 10^{-3}$	$1.73 \cdot 10^{-4}$	$4.21 \cdot 10^{-6}$	$1.29 \cdot 10^{-7}$	$5.56 \cdot 10^{-9}$
$\frac{\|e_{ny}\|_\infty}{\|f\|_\infty} \cdot 100$ %	$2.65 \cdot 10^{-3}$	$5.77 \cdot 10^{-5}$	$1.40 \cdot 10^{-6}$	$4.31 \cdot 10^{-8}$	$1.85 \cdot 10^{-9}$

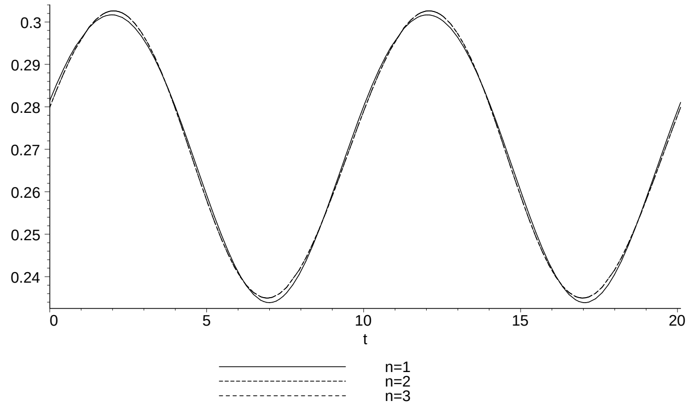

FIGURE 3. Detail of the ideal control functions $\bar{u}_n$, $n = 1, \ldots, 5$, laying in the unsaturated zone: $0 \leq \bar{u}_n \leq 1$.

The satisfaction of the input and output error bounds determined in equations (40) and (41), respectively, can be observed in Table 4 by checking the positivity of the differences

$$E_{nx} = \delta \left\| F\phi_n \right\|_\infty - \left\| e_{nx} \right\|_\infty , \qquad E_{ny} = \sqrt{\frac{\left\| F\phi_n \right\|_\infty}{\lambda}} - \left\| e_{ny} \right\|_\infty .$$

Take into account that δ, defined in (39), reaches the value $\delta = 1.5155$.

TABLE 4. Fulfillment of the input and output error bounds.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
E_{nx}	$7.68 \cdot 10^{-2}$	$2.12 \cdot 10^{-3}$	$6.38 \cdot 10^{-5}$	$2.20 \cdot 10^{-6}$	$9.88 \cdot 10^{-8}$
E_{ny}	$2.36 \cdot 10^{-1}$	$3.99 \cdot 10^{-2}$	$6.91 \cdot 10^{-3}$	$1.28 \cdot 10^{-3}$	$2.43 \cdot 10^{-4}$

Finally, notice that the method developed in [20] cannot be applied here, at least in low dimension, because it yields an apparently divergent sequence of approximations of ϕ.

10. **Conclusions.** The tracking problem in a certain class of non-minimum phase, bilinear systems is solvable via an inversion procedure that uses Galerkin expansions. Leray-Schauder fixed point index theory has been used to prove the existence

of a sequence of approximate solutions for the internal dynamics equation. This sequence shows uniform convergence to the exact input reference, and an error bound has been derived.

The system output exhibits a periodic and asymptotically stable behavior when indirect control using the sequence of Galerkin approximations is performed. In turn, the sequence of periodic outputs converges uniformly to the original target function under a reasonable hypothesis. Error bounds have also been obtained.

These results lay on three hypotheses stated in Section 3. Notice that Hypothesis H2 can be weakened by assuming the property to be fulfilled not $\forall n \geq 0$, but from a certain $n \geq n_0$.

A further research may consider the robustness of the technique. Galerkin expansions have already been used in the design of robust controllers for infinite dimensional systems [13]. In our case, it is well known that indirect control schemes use to be very sensitive to disturbances and parameter uncertainties. However, a first step has been done in [9], where robust tracking control for boost and buck-boost converters is achieved through an adaptive control that incorporates on-line updating of the disturbed parameter via a first order Galerkin approximation of the indirect reference. The scheme shows reasonably rapid speed of identification and good simulation results.

REFERENCES

[1] H. Banks, R. C. del Rosario, and R. C. Smith, *Reduced-order model feedback control design: Numerical implementation in a thin shell model*, IEEE Trans. Automat. Contr., **45** (2000), no. 7, 1312–1324.

[2] R. W. Beard, G. N. Saridis, and J. T. Wen, *Galerkin approximations of the generalized Hamilton-Jacobi-Bellman equations*, Automatica, **33** (1997), no. 12, 2159–2177.

[3] A. Bergen and R. Franks, *Justification of the describing function method*, SIAM J. Contr., **9** (1971), no. 4, 568–589.

[4] H. Brézis, Análisis Funcional, Alianza Editorial, Madrid, 1984.

[5] S. Devasia, *Robust inversion-based feedforward controllers for output tracking under plant uncertainty* in *Proc. Amer. Contr. Conf.*, Chicago, Illinois, (2000), 497–502.

[6] S. Devasia, D. Chen and B.Paden, *Nonlinear inversion-based output tracking*, IEEE Trans. Automat. Contr., **41** (1996), no. 7, 930–942.

[7] S. Devasia and B. Paden, *Stable inversion for nonminimum-phase time-varying systems*, IEEE Trans. Automat. Contr., **43** (1998), no. 2, 283–288.

[8] E. Fossas and J. M. Olm, *Asymptotic tracking in dc-to-dc nonlinear power converters*, Discrete Continuous Dynam. Systems Series B, **2** (2002), no. 2, 295–307.

[9] E. Fossas, J. Olm, and A. Zinober, *Robust tracking control of DC-to-DC nonlinear power converters*, in *Proc. IEEE Int. Conf. Decis. Contr.*, Atlantis, Paradise Island, Bahamas, (2004), 5291-5296.

[10] A. Isidori, Nonlinear Control Systems, Springer-Verlag, Berlin, 1989.

[11] Y. Kim, B. Kim, and M. Lim, *Composite control for singularly perturbed bilinear systems via successive Galerkin approximation*, IEE Proc. Contr. Theory Appl., **150** (2003), no. 5, 483–488.

[12] T. McLain and R. W. Beard, *Successive Galerkin approximations to the nonlinear optimal control of an underwater robot vehicle*, in *Proc. IEEE Int. Conf. Robotics and Automation*, Leuven, Belgium, (1998), 762–767.

[13] K. Morris, *Robustness of controllers designed using Galerkin type approximations*, in *Proc. IEEE Int. Conf. Decis. Contr.*, Tampa, FL, (1989), 2679–2684.

[14] J. Olm, *Asymptotic tracking with DC-to-DC bilinear power converters*, Ph.D. dissertation, Tech. Univ. of Catalonia, Barcelona, 2004. http://www.tdx.cesca.es.

[15] S. Sastry, Nonlinear Systems. Analysis, Stability and Control, Springer, New York, 1999.

[16] I. Shkolnikov and Y. Shtessel, *Tracking in a class of nonminimum-phase systems with nonlinear internal dynamics via sliding mode control using method of system center*, Automatica, **38** (2000), 837–842.

[17] ——, *Tracking controller design for a class of nonminimum phase systems via the method of system center*, IEEE Trans. Automat. Contr., **46** (2001), no. 10, 1639–1642.

[18] Y. Shtessel, A. Zinober, and I. Shkolnikov, *Sliding Mode Control of Boost and Buck-Boost Power Converters Using Method of Stable System Centre*, Automatica, **39** (2003), 1061-1067.

[19] ——, *Sliding Mode Control for Nonlinear SSystems with Output Delay Via Method of Stable System Centre*, Journal of Dynamic Systems, Measurement and Control, **25** (2003), no. 2, 253–257.

[20] H. Sira-Ramírez, *DC-to-AC power conversion on a 'boost' converter*, Int. J. Robust Nonlinear Control, **11** (2001), 589–600.

[21] G. Tadmor, *On approximate phasor models in dissipative bilinear systems*, IEEE Trans. Circuits Syst. I, **49** (2002), no. 8, 1167–1179.

[22] F. Verhulst, Nonlinear Differential Equations and Dynamical Systems, Springer-Verlag, Berlin, 1990.

[23] E. Zeidler, Nonlinear Functional Analysis and its Applications - Part I: Fixed-Point Theorems, Springer-Verlag, New York, 1993.

[24] ——, Nonlinear Functional Analysis and its Applications - Part IIA: Linear Monotone Operators, Springer-Verlag, New York, 1990.

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