

Algebraic Characterizations of Distance-Regular Graphs *

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Abstract

We survey some old and some new characterizations of distance-regular graphs, which depend on information retrieved from their adjacency matrix. In particular, it is shown that a regular graph with $d + 1$ distinct eigenvalues is distance-regular if and only if a numeric equality, involving only the spectrum of the graph and the numbers of vertices at distance d from each vertex, is satisfied.

Keywords: Distance-regular graph; Adjacency matrix; Spectrum

1 Introduction

Distance-regular graphs were introduced by Biggs in the early seventies (see [?]), by changing a symmetry-type requirement, that of distance-transitivity, to a regularity-type condition concerning the cardinality of some vertex subsets. To be more precise, recall that a graph Γ with diameter D is *distance-transitive* when any two pairs of vertices (u, v) and (x, y) at the same distance $\text{dist}(u, v) = \text{dist}(x, y) \leq D$ are indistinguishable from each other; that is, there is an automorphism of the graph that takes u to x and v to y . Thus, a distance-transitive graph ‘looks the same’ when viewed from each one of such pairs. In particular, for any vertex pair (u, v) and integers $0 \leq i, j \leq D$, the number $p_{ij}(u, v)$ of vertices at distance i from u and at distance j from v only depends on $k := \text{dist}(u, v)$, and we write $p_{ij}(u, v) = p_{ij}^k$ for some constants p_{ij}^k called the *intersection numbers*. Such a condition is precisely the combinatorial property that characterizes (or defines) a *distance-regular* graph,

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and so we will refer to it as characterization **(A)**. In fact, because of the many relations existing between the intersection numbers, we can use a much more economic definition which, for each distance k , considers only the pairs of distances $(i, j) = (k - 1, 1)$, $(k, 1)$, and $(k + 1, 1)$. (As we will show, their corresponding intersection numbers are sufficient to determine the remaining ones; see e.g. [?].) Then, one of the most usual definitions of distance-regularity is the following: **(A')** A graph Γ is distance-regular when, for any two vertices u and v at distance $\text{dist}(u, v) = k$ the numbers c_k , a_k , and b_k of vertices which are adjacent to v , and at distance $k - 1$, k , and $k + 1$, respectively, from u only depend on k . Thus, one intuitive way of looking at distance-regularity is to “hang” the graph from a given vertex and observe the resulting different “layers” in which the vertex set is partitioned; that is, the subsets of vertices at given distances from the root: If vertices in the same layer are “neighbourhood-indistinguishable” from each other, and the whole configuration does not depend on the chosen vertex, the graph is distance-regular.

Since their introduction, distance-regular graphs and their main generalization, the association schemes defined below, have proved to be a key concept in algebraic combinatorics. They have important connections with other branches of mathematics, such as geometry, coding theory, group theory, design theory, as well as with other areas of graph theory. As stated in the preface of the comprehensive textbook of Brouwer, Cohen and Neumaier [?], this is because most finite objects bearing “enough regularity” are closely related to certain distance-regular graphs.

Before proceeding with our exposition, we next introduce some basic notation. Throughout the paper, $\Gamma = (V, E)$ stands for a (simple and finite) connected graph, with vertex set $V = \{u, v, w, \dots\}$ and edge set $E = \{uv, wz, \dots\}$. Adjacency between vertices u and v ($uv \in E$) will be denoted by $u \sim v$. The *eccentricity* of a vertex u is $\text{ecc}(u) := \max_{v \in V} \text{dist}(u, v)$ and the *diameter* of the graph is $D := \max_{u \in V} \text{ecc}(u)$. The set $\Gamma_k(u)$ represents the set of vertices at distance k from vertex u . Thus, the degree of vertex u is $\delta_u := |\Gamma_1(u)| \equiv |\Gamma(u)|$ and our basic definition reads:

(A') A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if, for any two vertices $u, v \in V$ at distance $\text{dist}(u, v) = k$, $0 \leq k \leq D$, the numbers

$$c_k(u, v) := |\Gamma_{k-1}(u) \cap \Gamma(v)|, \quad a_k(u, v) := |\Gamma_k(u) \cap \Gamma(v)|, \quad b_k(u, v) := |\Gamma_{k+1}(u) \cap \Gamma(v)|,$$

do not depend on the chosen vertices u and v , but only on their distance k ; in which case they are denoted by c_k , a_k , and b_k , respectively.

In this work, we aim to survey some other characterizations of distance-regular graphs which are of an algebraic nature. Such characterizations rely mainly on the *adjacency 01-matrix* \mathbf{A} of a graph Γ —with rows and columns indexed by the vertices and $(\mathbf{A})_{uv} = 1$ iff $u \sim v$ —and/or some of its invariants, such as its spectrum

$$\text{sp } \Gamma := \text{sp } \mathbf{A} = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \dots, \lambda_d^{m(\lambda_d)}\}$$

where the eigenvalues λ_i , $0 \leq i \leq d$, are in decreasing order and the superscripts denote multiplicities; or their corresponding eigenspaces

$$\mathcal{E}_i := \text{Ker}(\mathbf{A} - \lambda_i \mathbf{I}) \quad (0 \leq i \leq d).$$

We begin, in the next section, with the more “classical” characterizations, which involve \mathbf{A} and its generalizations in terms of distance, commonly known as the “distance matrices”. In Section 3 we follow our study with the matrices representing the orthogonal projections on the eigenspaces \mathcal{E}_i , the so-called “principal idempotents” of \mathbf{A} . Section 4 is devoted to characterizations which depend, essentially, on the spectrum of \mathbf{A} : the eigenvalues and their multiplicities. In that section we discover that distance-regularity can be thought of as an extremal (numeric) property of the graph; and the corresponding characterizations have the striking particularity of involving only numerical (instead of matricial) identities. Such types of results are extended further in the last section, to cope with characterizations of some particular families of distance-regular graphs, which use only a part of the spectrum.

Throughout the paper we put emphasis on two ideas: The first one is that a characterization involving many identities, like (\mathbf{A}) , has usually its counterpart, in which the number of such conditions has been substantially reduced, like (\mathbf{A}') . In this case, both results are identified with the same letter, the latter with a prime. The second idea is to stress the importance of some families of orthogonal polynomials, the so-called distance and predistance polynomials, in the characterization and study of distance-regularity.

In order to present and relate the above results, we end this introduction by recalling some basic results from algebraic graph theory (for more details, see e.g. [?]):

- (a.1) Since Γ is connected, \mathbf{A} is an irreducible nonnegative matrix. Then, by the Perron-Frobenius theorem, the maximum eigenvalue λ_0 is simple, positive (in fact it coincides with the spectral radius of \mathbf{A}), and has a positive eigenvector $\boldsymbol{\nu}$, say, which is useful to normalize in such a way that $\min_{u \in V} \nu_u = 1$. Moreover, Γ is regular if and only if $\boldsymbol{\nu} = \mathbf{j}$, the all-1 vector (then $\lambda_0 = \delta$, the degree of Γ).
- (a.2) The number of walks of length $l \geq 0$ between vertices u and v is $a_{uv}^l := (\mathbf{A}^l)_{uv}$.
- (a.3) If $\Gamma = (V, E)$ has spectrum $\text{sp} \Gamma = \{\lambda_0^{m(\lambda_0)}, \dots, \lambda_d^{m(\lambda_d)}\}$ then the total number of rooted closed walks of length $l \geq 0$ is $\text{tr} \mathbf{A}^l = \sum_{i=0}^d m(\lambda_i) \lambda_i^l$.
- (a.4) If Γ has $d+1$ distinct eigenvalues, then $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^d\}$ is a basis of the *adjacency* or *Bose-Mesner* algebra $\mathcal{A}(\Gamma)$ of matrices which are polynomials in \mathbf{A} . Moreover, if Γ has diameter D ,

$$\dim \mathcal{A}(\Gamma) = d + 1 \geq D + 1$$

because $\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^D\}$ is a linearly independent set of $\mathcal{A}(\Gamma)$. Hence, the diameter is always less than the number of distinct eigenvalues: $D \leq d$.

- (a.5) A graph $\Gamma = (V, E)$ with eigenvalues $\lambda_0 > \lambda_1 > \dots > \lambda_d$ is a regular graph if and only if there exists a polynomial $H \in \mathbb{R}_d[x]$ such that $H(\mathbf{A}) = \mathbf{J}$, the all-1 matrix. This polynomial is unique and it is called the *Hoffman polynomial* [?]. It has zeros at the eigenvalues λ_i , $i \neq 0$, and $H(\lambda_0) = n := |V|$. Thus,

$$H = \frac{n}{\pi_0} \prod_{i=1}^d (x - \lambda_i)$$

where $\pi_0 := \prod_{i=1}^d (\lambda_0 - \lambda_i)$.

2 The distance matrices

A generalization of the adjacency matrix, which is very useful in the study of distance-regular graphs, is the concept of distance- k matrix: Let Γ be a graph with diameter D . For every integer k ($0 \leq k \leq D$), the *distance- k matrix* \mathbf{A}_k of Γ is defined as

$$(\mathbf{A}_k)_{uv} = \begin{cases} 1 & \text{if } \text{dist}(u, v) = k, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\mathbf{A}_0 = \mathbf{I}$, the identity matrix, and $\mathbf{A}_1 = \mathbf{A}$ is the adjacency matrix of the graph. Using these matrices, our first characterization (**A**) now reads:

- (**A**) A graph Γ with diameter D is distance-regular if and only if, for any integers $0 \leq i, j \leq D$, its distance matrices satisfy

$$\mathbf{A}_i \mathbf{A}_j = \sum_{k=0}^D p_{ij}^k \mathbf{A}_k \quad (0 \leq i, j \leq D), \quad (1)$$

for some constants p_{ij}^k (the intersection numbers).

In fact (??) is the main condition (together with $\mathbf{A}_0 := \mathbf{I}$ and $\sum_{k=0}^D \mathbf{A}_k = \mathbf{J}$) required for a set $\{\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_d\}$ of $n \times n$ 01-matrices to constitute a (*symmetric*) *association scheme*; one of the fundamental concepts in combinatorics (see e.g. [?, ?]).

Going back to distance-regularity, notice that, according to the basic definition (**A'**), it suffices to require the existence of the constants $c_k = p_{k-1,1}^k$, $a_k = p_{k,1}^k$ and $b_k = p_{k+1,1}^k$. Consequently (**A**) becomes:

- (**A'**) A graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if, for some constants a_k, b_k, c_k ($0 \leq k \leq D$), $c_0 = b_D = 0$, its distance matrices satisfy the three-term recurrence

$$\mathbf{A}_k \mathbf{A} = b_{k-1} \mathbf{A}_{k-1} + a_k \mathbf{A}_k + c_{k+1} \mathbf{A}_{k+1} \quad (0 \leq k \leq D), \quad (2)$$

where, by convention, $b_{-1} = c_{D+1} = 0$.

The *intersection array* of Γ is then defined as

$$\iota(\Gamma) := \begin{pmatrix} 0 & c_1 & \cdots & c_{D-1} & c_D \\ a_0 & a_1 & \cdots & a_{D-1} & a_D \\ b_0 & b_1 & \cdots & b_{D-1} & 0 \end{pmatrix}.$$

Note that always $a_0 = 0$ and $c_1 = 1$. Moreover, since Γ must be regular of degree δ , say, we have $b_0 = \delta$, and $a_k + b_k + c_k = \delta$ for any $0 \leq k \leq D$. Hence, it is also usual to write the intersection array in its simplified form: $\iota(\Gamma) = \{\delta, b_1, \dots, b_{D-1}; 1, c_2, \dots, c_D\}$.

By applying recurrence (??) starting from $\mathbf{A}_0 = \mathbf{I}$ and $\mathbf{A}_1 = \mathbf{A}$, we see that, in a distance-regular graph, each distance matrix \mathbf{A}_k is a polynomial of degree k in \mathbf{A} ; that is

$$\mathbf{A}_k = p_k(\mathbf{A}) \in \mathcal{A}(\Gamma) \quad (0 \leq k \leq D).$$

(Of course, $p_0 = 1$ and $p_1 = x$.) These polynomials are called the *distance polynomials* and, as we shall see, satisfy some nice properties which facilitate the computation of the different parameters of Γ . To begin with, notice that, since $\mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}_D = \mathbf{J}$, (a.5) implies $p_0 + p_1 + \cdots + p_D = H$, the Hoffman polynomial of Γ . Thus, assuming that Γ has $d + 1$ distinct eigenvalues and using (a.4), we have

$$D \leq d = \text{dgr } H = D.$$

As a conclusion, a distance-regular graph has “spectrally maximum” diameter $D = d$ and, since the $D + 1$ distance matrices are clearly linearly independent, $\{\mathbf{I}, \mathbf{A}, \dots, \mathbf{A}_D\}$ is a basis for the adjacency algebra $\mathcal{A}(\Gamma)$. This important result was first proved by Damerell [?]. Its converse also holds since, in terms of such a basis, every product $\mathbf{A}_i \mathbf{A}_j$ has an expression like (??) asserting the distance-regularity of the graph. This leads us to the following characterization:

(B) A graph Γ with diameter D is distance-regular if and only if $\{\mathbf{I}, \mathbf{A}, \dots, \mathbf{A}_D\}$ is a basis of the adjacency algebra $\mathcal{A}(\Gamma)$.

Moreover, notice that the distance polynomials (distance matrices) are uniquely determined by the constants a_k, b_k, c_k . Thus, as mentioned above, such intersection numbers determine all the others, so justifying definition **(A’)** (for more details, see [?]).

Since the distance polynomials $\{p_k\}_{0 \leq k \leq D}$ satisfy a three-term recurrence, they must constitute an orthogonal system with respect to some (discrete) scalar product. Indeed, with the inner product

$$\langle p, q \rangle_\Gamma := \frac{1}{n} \text{tr}(p(\mathbf{A})q(\mathbf{A})) = \sum_{i=0}^d \frac{m(\lambda_i)}{n} p(\lambda_i)q(\lambda_i) \quad (p, q \in \mathbb{R}_D[x]) \quad (3)$$

(with “normalized” weight function $\rho_i := \frac{1}{n}m(\lambda_i)$, $0 \leq i \leq d$, since $\sum_{i=0}^d \rho_i = 1$) we get

$$\langle p_k, p_l \rangle_\Gamma = \frac{1}{n} \text{tr}(\mathbf{A}_k \mathbf{A}_l) = \begin{cases} n_k & \text{if } k = l, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

where $n_k := |\Gamma_k(u)|$ represents the number of vertices at distance k from any vertex u . This number is independent of u since, using that \mathbf{j} is a λ_0 -eigenvector of \mathbf{A} , we have

$$|\Gamma_k(u)| = (\mathbf{A}_k \mathbf{j})_u = (p_k(\mathbf{A}) \mathbf{j})_u = (p_k(\lambda_0) \mathbf{j})_u = p_k(\lambda_0). \quad (5)$$

Thus, $n_k = p_k(\lambda_0)$ or

$$\|p_k\|_\Gamma^2 = p_k(\lambda_0).$$

(In terms of the intersection numbers, we have $n_k = p_{kk}^0 = b_0 b_1 \cdots b_{k-1} / c_1 c_2 \cdots c_k$.)

Looking back at (??), we now see that the intersection numbers p_{ij}^k of a distance-regular graph are just the Fourier coefficients of $p_i p_j$ in terms of the basis $\{p_k\}_{0 \leq k \leq D}$:

$$p_{ij}^k = \frac{\langle p_i p_j, p_k \rangle_\Gamma}{\|p_k\|_\Gamma^2} = \frac{1}{p_k(\lambda_0)} \sum_{h=0}^d \frac{m(\lambda_h)}{n} p_i(\lambda_h) p_j(\lambda_h) p_k(\lambda_h) \quad (0 \leq i, j, k \leq d). \quad (6)$$

Moreover, the highest degree polynomial p_d allow us to compute the multiplicities of the eigenvalues of Γ by the formulas:

$$m(\lambda_i) = \frac{\phi_0 p_d(\lambda_0)}{\phi_i p_d(\lambda_i)} \quad (0 \leq i \leq d) \quad (7)$$

where $\phi_i = \prod_{j=0(j \neq i)}^d (\lambda_i - \lambda_j)$ (see Bannai and Ito [?]). The value at λ_0 of the distance polynomial p_d can be computed from the spectrum through the expression

$$p_d(\lambda_0) = n \left(\sum_{i=0}^d \frac{\pi_0^2}{m(\lambda_i) \pi_i^2} \right)^{-1} \quad (8)$$

where the π_i 's are moment-like parameters defined by

$$\pi_i := |\phi_i| = \prod_{j=0, j \neq i}^d |\lambda_i - \lambda_j| \quad (0 \leq i \leq d).$$

These parameters seem to have special relevance to the study of some properties of graphs from their spectra (see [?, ?, ?, ?]).

Alternatively, the multiplicities can also be computed by using all the distance polynomials (see Biggs [?]):

$$m(\lambda_i) = n \left(\sum_{j=0}^d \frac{1}{n_j} p_j(\lambda_i)^2 \right)^{-1} \quad (0 \leq i \leq d) \quad (9)$$

where $n_j = p_j(\lambda_0)$.

As we shall see, sometimes it is useful to consider such an orthogonal system when the graph is not distance-regular, not even regular. Thus, given a graph Γ with spectrum $\text{sp } \Gamma = \{\lambda_0^{m(\lambda_0)}, \dots, \lambda_d^{m(\lambda_d)}\}$, we consider the scalar product $\langle \cdot, \cdot \rangle_\Gamma$ defined as in (??)—with $\|\nu\|^2$ instead of n —and its corresponding orthogonal sequence of polynomials $p_0 (= 1), p_1, \dots, p_d$, which satisfies a three-term recurrence like (??), and is uniquely characterized by any of the following conditions (see [?, ?]):

$$(b.1) \quad \|p_k\|_\Gamma^2 = p_k(\lambda_0);$$

$$(b.2) \quad a_k + b_k + c_k = \lambda_0 \quad (0 \leq k \leq d);$$

$$(b.3) \quad q_d := \sum_{k=0}^d p_k = \frac{\|\boldsymbol{\nu}\|^2}{\pi_0} \prod_{i=1}^d (x - \lambda_i), \quad \text{where } \boldsymbol{\nu} \text{ is the positive eigenvector and} \\ \pi_0 := \prod_{i=1}^d (\lambda_0 - \lambda_i).$$

Such polynomials have been called “proper” in some previous work by the authors but, taking into account their role in the distance-regularity context, it seems more appropriate to call them the *predistance polynomials*. Here we shall use this name. Moreover, note that, when Γ is a regular graph on n vertices, we have $\|\boldsymbol{\nu}\|^2 = \|\boldsymbol{j}\|^2 = n$; hence the above “sum polynomial” q_d coincides with the Hoffman polynomial H in (a.5). In general, the polynomial q_d satisfies $(q_d(\mathbf{A}))_{uv} = \nu_u \nu_v$ for any $u, v \in V$ and so we call it the *Hoffman-like polynomial*.

Besides proving that for every distance-regular graph there exist the distance polynomials, we have shown that the converse result also holds. Indeed, if every distance- k matrix of a graph is a polynomial of degree k in \mathbf{A} , $0 \leq k \leq D$, the above reasonings lead again to $D = d$ and to the orthogonality of these polynomials with respect to the scalar product (??). Consequently, the distance matrices are a basis of $\mathcal{A}(\Gamma)$ and (B) applies. Let us make this result explicit:

- (C) A graph Γ with diameter D is distance-regular if and only if, for any integer k , $0 \leq k \leq D$, the distance- k matrix \mathbf{A}_k is a polynomial of degree k in \mathbf{A} ; that is:

$$\mathbf{A}_k = p_k(\mathbf{A}) \quad (0 \leq k \leq D). \quad (10)$$

As already noticed, the existence of the first two such polynomials, p_0 and p_1 , is always guaranteed since $\mathbf{A}_0 = \mathbf{I}$ and $\mathbf{A}_1 = \mathbf{A}$. In fact, if every vertex $u \in V$ has the maximum possible eccentricity allowed by the spectrum in (a.4) (that is, the number of distinct eigenvalues minus one: $\text{ecc}(u) = d$), the existence of the highest degree distance polynomial suffices:

- (C') A graph Γ with diameter D and $d + 1$ distinct eigenvalues is distance-regular if and only if all its vertices have spectrally maximum eccentricity d ($\Rightarrow D = d$) and the distance matrix \mathbf{A}_d is a polynomial of degree d in \mathbf{A} :

$$\mathbf{A}_d = p_d(\mathbf{A}).$$

This was proved by Garriga, Yebra and the author [?] in the context of “pseudo-distance-regularity”—a generalization of distance-regularity that makes sense even for non-regular graphs.

To give a last characterization of distance-regularity involving the distance matrices, we study now the role of the powers of the adjacency matrix. To this end, note first that, if Γ is distance-regular, the l -th power of \mathbf{A} , $l \geq 0$, can be written in terms of the basis $p_k(\mathbf{A}) = \mathbf{A}_k$, $0 \leq k \leq D(=d)$, as:

$$\mathbf{A}^l = \sum_{h=0}^D \frac{\langle x^l, p_h \rangle}{\|p_h\|^2} \mathbf{A}_h.$$

Thus, the number of walks $a_{uv}^l = (\mathbf{A}^l)_{uv}$ of length l between two vertices u, v at distance k is just the above Fourier coefficient of \mathbf{A}_k :

$$a_{uv}^l = \frac{\langle x^l, p_k \rangle}{\|p_k\|^2} = \frac{1}{p_k(\lambda_0)} \sum_{i=0}^d \frac{m(\lambda_i)}{n} \lambda_i^l p_k(\lambda_i).$$

Hence, this number only depends on $k = \text{dist}(u, v)$, and we write $a_{uv}^l = a_k^l$ and

$$\mathbf{A}^l = \sum_{k=0}^D a_k^l \mathbf{A}_k \quad (l \geq 0). \quad (11)$$

where, clearly, $a_k^l = 0$ when $l < k$.

Conversely, assume that, for a certain graph and any $0 \leq k \leq l \leq D$, there are constants a_k^l satisfying (??). As a matrix equation,

$$\begin{pmatrix} \mathbf{I} \\ \mathbf{A} \\ \mathbf{A}^2 \\ \cdot \\ \cdot \\ \mathbf{A}^D \end{pmatrix} = \begin{pmatrix} a_0^0 & & & & & & \\ a_0^1 & a_1^1 & & & & & \\ a_0^2 & a_1^2 & a_2^2 & & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ a_0^D & a_1^D & \cdot & \cdot & \cdot & a_D^D & \end{pmatrix} \begin{pmatrix} \mathbf{I} \\ \mathbf{A} \\ \mathbf{A}_2 \\ \cdot \\ \cdot \\ \mathbf{A}_D \end{pmatrix} \quad (12)$$

where the lower triangular matrix \mathbf{T} , with rows and columns indexed with the integers $0, 1, \dots, D$, has entries $(\mathbf{T})_{lk} = a_k^l$. In particular, note that $a_0^0 = a_1^1 = 1$ and $a_0^1 = 0$. Moreover, since $a_k^k > 0$, such a matrix has an inverse which is also a lower triangular matrix and hence each \mathbf{A}_k is a polynomial of degree k in \mathbf{A} . Therefore, according to (C), we are dealing with a distance-regular graph. (Of course, the entries of \mathbf{T}^{-1} are the coefficients of the distance polynomials.)

In conclusion, we have justified the following characterization (see e.g. Rowlinson [?]):

- (D) A graph $\Gamma = (V, E)$ is distance-regular if and only if, for each non-negative integer l , the number a_{uv}^l of walks of length l between two vertices $u, v \in V$ only depends on $k = \text{dist}(u, v)$.

Again, we do not need to impose the invariance condition for each value of l . For instance, if Γ is regular we have the following result:

- (D') A regular graph $\Gamma = (V, E)$ with diameter D is distance-regular if and only if there are constants a_k^k and a_k^{k+1} such that, for any two vertices $u, v \in V$ at distance k , we have $a_{uv}^k = a_k^k$ and $a_{uv}^{k+1} = a_k^{k+1}$ for any $0 \leq k \leq D - 1$, and $a_{uv}^D = a_D^D$ for $k = D$.

To illustrate some typical reasonings involving the intersection numbers, let us prove (D') from the usual definition of distance-regularity in (A'): Assume first that Γ is distance-regular. As the result clearly holds for $k = 0$ since $a_{uu}^0 = 1 = a_0^0$ and $a_{uu}^1 = 0 = a_0^1$, we

shall use induction and assume that $a_{uv}^{k-1} = a_{k-1}^{k-1}$ and $a_{uv}^k = a_{k-1}^k$ for any vertices u, v at distance $k-1$. Then, for any vertices u, v at distance k we get:

$$a_{uv}^k = \sum_{w \in \Gamma_{k-1}(u) \cap \Gamma(v)} a_{uw}^{k-1} = a_{k-1}^{k-1} |\Gamma_{k-1}(u) \cap \Gamma(v)| \quad (13)$$

so that $a_k^k = a_{k-1}^{k-1} c_k$. Similarly, using this equality and

$$a_{uv}^{k+1} = \sum_{w \in [\Gamma_{k-1}(u) \cup \Gamma_k(u)] \cap \Gamma(v)} a_{uw}^k = a_{k-1}^k |\Gamma_{k-1}(u) \cap \Gamma(v)| + a_k^k |\Gamma_k(u) \cap \Gamma(v)|, \quad (14)$$

we infer that $a_k^{k+1} = a_{k-1}^k c_k + a_k^{k-1} c_k a_k$.

Conversely, if we suppose that such constants a_k^k and a_{k+1}^k do exist and $\text{dist}(u, v) = k$, from $a_{uv}^k = a_k^k$ and (??) we obtain that

$$c_k(u, v) = |\Gamma_{k-1}(u) \cap \Gamma(v)| = \frac{a_k^k}{a_{k-1}^{k-1}}$$

does not depend on the chosen vertices u and $v \in \Gamma_k(u)$ and so $c_k(u, v) = c_k$. Analogously, from $a_{uv}^{k+1} = a_k^{k+1}$ and (??) we get

$$a_k^{k+1} = a_{k-1}^k \frac{a_k^k}{a_{k-1}^{k-1}} + a_k^k |\Gamma_k(u) \cap \Gamma(v)|$$

where we have used the above value of c_k . Consequently, the value

$$a_k(u, v) = |\Gamma_k(u) \cap \Gamma(v)| = \frac{a_k^{k+1}}{a_k^k} - \frac{a_{k-1}^k}{a_{k-1}^{k-1}}$$

is also independent of the vertices u, v , provided that $\text{dist}(u, v) = k$, and $a_k(u, v) = a_k$. Finally, since Γ is regular, of degree δ say,

$$b_k(u, v) = |\Gamma_{k+1}(u) \cap \Gamma(v)| = \delta - c_k - a_k,$$

shows that b_k is also independent of u, v and, hence, Γ is a distance-regular graph.

3 The idempotent matrices

For each eigenvalue λ_i , $0 \leq i \leq d$, let \mathbf{U}_i be the matrix whose columns form an orthonormal basis of its eigenspace $\mathcal{E}_i := \text{Ker}(\mathbf{A} - \lambda_i \mathbf{I})$. The (*principal*) *idempotents* of \mathbf{A} are the matrices $\mathbf{E}_i := \mathbf{U}_i \mathbf{U}_i^\top$ representing the orthogonal projections onto \mathcal{E}_i . In particular, $\mathbf{E}_0 = \frac{1}{\|\boldsymbol{\nu}\|^2} \boldsymbol{\nu} \boldsymbol{\nu}^\top$, where $\boldsymbol{\nu}$ is the positive (column) λ_0 -eigenvector. Alternatively, in terms of \mathbf{A} ,

$$\mathbf{E}_i = \frac{1}{\phi_i} \prod_{j=0, j \neq i}^d (\mathbf{A} - \lambda_j \mathbf{I}) \quad (0 \leq i \leq d) \quad (15)$$

where $\phi_i = \prod_{j=0, j \neq i}^d (\lambda_i - \lambda_j)$. Accordingly, such matrices satisfy the following properties:

$$(c.1) \quad \mathbf{E}_i \mathbf{E}_j = \begin{cases} \mathbf{E}_i & \text{if } i = j \\ \mathbf{0} & \text{otherwise;} \end{cases}$$

$$(c.2) \quad \mathbf{A} \mathbf{E}_i = \lambda_i \mathbf{E}_i;$$

$$(c.3) \quad p(\mathbf{A}) = \sum_{i=0}^d p(\lambda_i) \mathbf{E}_i, \text{ for any polynomial } p \in \mathbb{R}[x].$$

In particular, taking $p = 1$ in (c.3), we have

$$\mathbf{E}_0 + \mathbf{E}_1 + \cdots + \mathbf{E}_d = \mathbf{I} \quad (16)$$

(as expected, since the sum of all orthogonal projections gives the original vector) and, considering $p = x$, we get the so-called *spectral decomposition theorem*:

$$\sum_{i=0}^d \lambda_i \mathbf{E}_i = \mathbf{A} \quad (17)$$

(see e.g. Godsil [?]). More generally, taking $p = x^l$, each power of \mathbf{A} can be expressed as a linear combination of the idempotents \mathbf{E}_i :

$$\mathbf{A}^l = \sum_{i=0}^d \lambda_i^l \mathbf{E}_i. \quad (18)$$

Consequently, since $\{\mathbf{I}, \mathbf{A}, \dots, \mathbf{A}^d\}$ is a basis of $\mathcal{A}(\Gamma)$, so is the set $\{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_d\}$ —a fact that follows also from (??)—and we can expect to have characterizations of distance-regularity in terms of the entries of the above idempotents. These numbers were called in [?] the *crossed (uv -)local multiplicities* of λ_i , and were denoted by $m_{uv}(\lambda_i)$. Notice that, if \mathbf{z}_{ui} represents the orthogonal projection of the u -canonical vector \mathbf{e}_u on \mathcal{E}_i , that is $\mathbf{z}_{ui} := \mathbf{E}_i \mathbf{e}_u$, the crossed local multiplicities correspond to the scalar products:

$$m_{uv}(\lambda_i) := (\mathbf{E}_i)_{uv} = \langle \mathbf{E}_i \mathbf{e}_u, \mathbf{e}_v \rangle = \langle \mathbf{E}_i \mathbf{e}_u, \mathbf{E}_i \mathbf{e}_v \rangle = \langle \mathbf{z}_{ui}, \mathbf{z}_{vi} \rangle \quad (u, v \in V).$$

For instance, if the graph is regular, then the eigenvector of λ_0 is the all-1 vector \mathbf{j} , and the above gives $m_{uv}(\lambda_0) = \langle \frac{1}{n} \mathbf{j}, \frac{1}{n} \mathbf{j} \rangle = 1/n$ for any $u, v \in V$, and hence $\mathbf{E}_0 = \frac{1}{n} \mathbf{J}$.

Suppose that Γ is a distance-regular graph, so that it has spectrally maximum diameter $D = d$. Then, taking p in (c.3) as the distance-polynomial p_k , $0 \leq k \leq d$, we get

$$\mathbf{A}_k = \sum_{i=0}^d p_k(\lambda_i) \mathbf{E}_i \quad (0 \leq k \leq d)$$

or, in matrix form,

$$\begin{pmatrix} \mathbf{A}_0 \\ \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_d \end{pmatrix} = \begin{pmatrix} p_0(\lambda_0) & p_0(\lambda_1) & \cdots & p_0(\lambda_d) \\ p_1(\lambda_0) & p_1(\lambda_1) & \cdots & p_1(\lambda_d) \\ \vdots & \vdots & & \vdots \\ p_d(\lambda_0) & p_d(\lambda_1) & \cdots & p_d(\lambda_d) \end{pmatrix} \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{E}_1 \\ \vdots \\ \mathbf{E}_d \end{pmatrix}. \quad (19)$$

But, because of the orthogonal property (??) with respect to the scalar product (??), the inverse of the matrix \mathbf{P} , with entries $(\mathbf{P})_{ji} = p_j(\lambda_i)$, $0 \leq j, i \leq d$, is just

$$\mathbf{P}^{-1} = \frac{1}{n} \begin{pmatrix} m(\lambda_0) \frac{p_0(\lambda_0)}{n_0} & m(\lambda_0) \frac{p_1(\lambda_0)}{n_1} & \dots & m(\lambda_0) \frac{p_d(\lambda_0)}{n_d} \\ m(\lambda_1) \frac{p_0(\lambda_1)}{n_0} & m(\lambda_1) \frac{p_1(\lambda_1)}{n_1} & \dots & m(\lambda_1) \frac{p_d(\lambda_1)}{n_d} \\ \vdots & \vdots & \ddots & \vdots \\ m(\lambda_d) \frac{p_0(\lambda_d)}{n_0} & m(\lambda_d) \frac{p_1(\lambda_d)}{n_1} & \dots & m(\lambda_d) \frac{p_d(\lambda_d)}{n_d} \end{pmatrix}$$

where $n_j = p_j(\lambda_0)$. (As a by-product, note that from $(\mathbf{P}^{-1}\mathbf{P})_{ii} = 1$, $0 \leq i \leq d$, we get the formulas for the multiplicities given in (??).) Consequently,

$$\mathbf{E}_l = \sum_{j=0}^d (\mathbf{P}^{-1})_{lj} \mathbf{A}_j = \frac{m(\lambda_l)}{n} \sum_{j=0}^d \frac{p_j(\lambda_l)}{p_j(\lambda_0)} \mathbf{A}_j \quad (0 \leq l \leq d)$$

and, equating the corresponding (u, v) entries, we observe that in a distance-regular graph the crossed uv -local multiplicities only depend on the distance $k = \text{dist}(u, v)$; a result noted e.g. by Godsil in [?]. More precisely, we can write $m_{uv}(\lambda_l) = m_{kl}$, where

$$m_{kl} = \frac{m(\lambda_l) p_k(\lambda_l)}{n p_k(\lambda_0)} \quad (0 \leq k, l \leq d). \quad (20)$$

Again, the converse result also holds because, if

$$\mathbf{E}_l = \sum_{j=0}^D m_{jl} \mathbf{A}_j \quad (0 \leq l \leq d) \quad (21)$$

for some constants m_{jl} , the fact that $\{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_d\}$ is a basis implies that $\{\mathbf{I}, \mathbf{A}, \dots, \mathbf{A}_D\}$ is also a basis, and then **(B)** applies.

All these facts prove the next characterization:

- (E)** A graph Γ with diameter D and eigenvalues $\lambda_0 > \lambda_1 > \dots > \lambda_d$ is a distance-regular graph if and only if, for any given eigenvalue λ_l , $0 \leq l \leq d$, the crossed uv -local multiplicity $m_{uv}(\lambda_l)$ depends only on the distance $\text{dist}(u, v) = k$, $0 \leq k \leq D$.

In fact, in the spirit of our previous results, we can prove the following (see [?]):

- (E')** A regular graph Γ , with eigenvalues $\lambda_0 > \lambda_1 > \dots > \lambda_d$ and diameter D , is distance-regular if and only if there exist constants m_{k1} and m_{kd} such that $m_{uv}(\lambda_1) = m_{k1}$ and $m_{uv}(\lambda_d) = m_{kd}$ for any two vertices $u, v \in V$ with $\text{dist}(u, v) = k$, $0 \leq k \leq D$.

To emphasize the importance of the above constants m_{kl} , we end this section by showing its relation with the so-called ‘‘Krein parameters’’ of a distance-regular graph. First note that the adjacency algebra $\mathcal{A}(\Gamma)$ is also closed under the pointwise product \circ of matrices

(also called *Schur product*), since $\mathbf{A}_i \circ \mathbf{A}_j = \mathbf{O}$ if $i \neq j$ and $\mathbf{A}_i \circ \mathbf{A}_i = \mathbf{A}_i$. Hence, the *Krein parameters* q_{ij}^k are the numbers satisfying

$$\mathbf{E}_i \circ \mathbf{E}_j = \frac{1}{n} \sum_{k=0}^d q_{ij}^k \mathbf{E}_k \quad (0 \leq i, j \leq d).$$

or, using (??),

$$\sum_{h=0}^d m_{hi} m_{hj} \mathbf{A}_h = \frac{1}{n} \sum_{h=0}^d q_{ij}^h \mathbf{E}_h.$$

Then, if we multiply both terms by $\mathbf{E}_k = \sum_{h=0}^d m_{hk} \mathbf{A}_h$ and take traces (this is equivalent to computing the corresponding scalar products), we get

$$\sum_{h=0}^d m_{hi} m_{hj} m_{hk} \operatorname{tr} \mathbf{A}_h^2 = \frac{1}{n} q_{ij}^k \operatorname{tr} \mathbf{E}_k.$$

where $\operatorname{tr} \mathbf{A}_h^2 = \|p_h\|_{\Gamma}^2 = p_h(\lambda_0) = n_h$ and $\operatorname{tr} \mathbf{E}_k = m(\lambda_k)$. Accordingly, using (??),

$$q_{ij}^k = \frac{m(\lambda_i) m(\lambda_j)}{n^2} \sum_{h=0}^d \frac{p_h(\lambda_i) p_h(\lambda_j) p_h(\lambda_k)}{n_h^2} \quad (0 \leq i, j, k \leq d). \quad (22)$$

Like the intersection numbers p_{ij}^k given by (??), the Krein parameters are known to be nonnegative (see e.g. [?, ?]). This fact, known as the *Krein conditions*, together with the integrality of the multiplicities given by (??) or (??), turn to be strong conditions for the existence of a distance-regular graph with a given intersection array.

4 The spectrum

Of course, it would be nice to have characterizations of distance-regularity involving only the spectrum. The first question is: *Can we see from the spectrum of a graph whether it is distance-regular?* In this context, it has been known for a long time that the answer is ‘yes’ when $D \leq 2$ and ‘not’ if $D \geq 4$. Indeed, a graph with diameter $D = 2$ is strongly regular iff it is regular (a property that can be identified from the spectrum) and has three distinct eigenvalues ($d = 2$). On the other hand, the ‘Hoffman graph’ [?], with diameter $D = 4$, is cospectral with the (2-antipodal) distance-regular graph Q_4 —the 4-cube—but it is not distance-regular. Thus, the only undecided case was $D = 3$, but Haemers [?] gave also a negative answer constructing many Hoffman-like counterexamples for this diameter (see also [?]). Thus, in general the spectrum is not sufficient to ensure distance-regularity and, if we want to go further, we must require the graph to satisfy some additional conditions. In this direction, Van Dam and Haemers [?] showed that, in the case $D = 3$, such a condition could be the specification of the number $n_d(u) = |\Gamma_d(u)|$ of vertices at “extremal distance” $D = d$ of any given vertex $u \in V$. Then, Garriga and the author [?] solved the general case, characterizing distance-regular graphs as those regular graphs whose number of vertices at distance d from each vertex is what it should be; that is, the value of $p_d(\lambda_0)$ given in (??).

To make explicit these and some other related results, we first introduce a local version of the predistance polynomials and enunciate a key result involving them: Namely, an upper bound for their value at λ_0 and the characterization of the case when the bound is attained (for more details, see [?, ?]). To construct such polynomials we use the diagonal entries of the above idempotents, that is the crossed uv -local multiplicities when $u = v$. In this case $m_{uu}(\lambda_i) = (\mathbf{E}_i)_{uu} = \|\mathbf{z}_{ui}\|^2 \geq 0$, denoted also by $m_u(\lambda_i)$, is referred to as the *u-local multiplicity of λ_i* . (In particular, from $\mathbf{E}_0 = (1/\|\boldsymbol{\nu}\|^2)\boldsymbol{\nu}\boldsymbol{\nu}^\top$ we see that $m_u(\lambda_0) = \nu_u^2/\|\boldsymbol{\nu}\|^2$.) In [?] it was noted that when the graph is “seen” from vertex u , the u -local multiplicities play a role similar to that of the standard multiplicities, so justifying the name. Indeed, by (??) note that, for each vertex u , the u -local multiplicities of all the eigenvalues add up to 1; whereas the multiplicity of each eigenvalue λ_i is the sum, extended to all vertices, of its local multiplicities since $\text{tr } \mathbf{E}_i = \text{tr}(\mathbf{U}_i\mathbf{U}_i^\top) = m(\lambda_i)$. Moreover, by (??) the number of closed walks of length l going through vertex u , a_{uu}^l , can be computed in a similar way as the whole number of such rooted walks in Γ is computed by using the “global” multiplicities—see (a.3). The u -local multiplicities are precisely the squares of the “angles” at u , introduced by Cvetković as the cosines $\cos \beta_{ui}$, $0 \leq i \leq d$, with β_{ui} being the angle between \mathbf{e}_u and the eigenspace \mathcal{E}_i . For a number of applications of these parameters, see for instance the recent book of Cvetković, Rowlinson, and Simić [?].

Using the local multiplicities as the values of the weight function, we can now define the *(u-)local scalar product*:

$$\langle p, q \rangle_u := (p(\mathbf{A})q(\mathbf{A}))_{uu} = \sum_{i=0}^d m_u(\lambda_i) p(\lambda_i) q(\lambda_i) \quad (23)$$

with normalized weight function $\rho_i := m_u(\lambda_i)$, $0 \leq i \leq d$, since $\sum_{i=0}^d \rho_i = 1$. Notice that the scalar product in (??) is simply the average, over all vertices, of the local scalar products:

$$\langle p, q \rangle_\Gamma = \frac{1}{n} \sum_{u \in V} \langle p, q \rangle_u. \quad (24)$$

Associated to this product, we define a new orthogonal sequence of polynomials $\{p_k^u\}_{0 \leq k \leq d_u}$ (where d_u is the number of eigenvalues $\lambda_i \neq \lambda_0$ such that $m_u(\lambda_i) \neq 0$) with $\text{dgr } p_k^u = k$, called the *(u-local) predistance polynomials*, satisfying the same properties as the predistance polynomials. For instance,

$$\langle p_k^u, p_l^u \rangle_u = \delta_{kl} p_k^u(\lambda_0) \quad \text{and} \quad \|p_k^u\|_u^2 = p_k^u(\lambda_0). \quad (25)$$

Before presenting the main property of these polynomials, we need to introduce a little more notation. Let $N_k(u)$ be the set of vertices that are at distance not greater than k from u , the so-called *k-neighbourhood* of u . For any vertex subset U , let $\boldsymbol{\rho}U$ be the *characteristic vector* of U ; that is $\boldsymbol{\rho}U := \sum_{u \in U} \mathbf{e}_u$. So, $\boldsymbol{\rho}N_k(u)$ is just the u column (or row) of the sum matrix $\mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}_k$, with square (Euclidean) norm $\|\boldsymbol{\rho}N_k(u)\|^2 = s_k(u) := |N_k(u)|$. Let us also consider the sums of successive polynomials p_k^u , denoted by $q_k^u := \sum_{h=0}^k p_h^u$, and satisfying

$$q_k^u(\lambda_0) = \sum_{h=0}^k p_h^u(\lambda_0) = \sum_{h=0}^k \|p_h^u\|_u^2 = \|q_k^u\|_u^2. \quad (26)$$

In [?] Garriga and the author proved the following result: Let u be a fixed vertex of a regular graph Γ . Then, for any polynomial $q \in \mathbb{R}_k[x]$,

$$\frac{q(\lambda_0)}{\|q\|_u} \leq \|\rho N_k(u)\| \quad (27)$$

and equality holds if and only if

$$\frac{1}{\|q\|_u} q(\mathbf{A})\mathbf{e}_u = \frac{1}{\|\rho N_k(u)\|} \rho N_k(u). \quad (28)$$

Moreover, if this is the case, q is any multiple of q_k^u , say $q = q_k^u$ (equality understood in the quotient ring $\mathbb{R}[x]/I$ where I is the ideal generated by the polynomial with zeroes the $d_u + 1$ eigenvalues with non-null u -local multiplicity). Hence, using (??), Eqs. (??) and (??) become:

$$q_k^u(\lambda_0) = s_k(u) \quad \text{and} \quad q_k^u(\mathbf{A})\mathbf{e}_u = \rho N_k(u). \quad (29)$$

Let us now use the above results to obtain a new characterization of distance-regularity. We already know that, in a distance-regular graph Γ , the number of vertices at distance not greater than k from any given vertex u is a constant since

$$s_k(u) = \sum_{h=0}^k n_k(u) = \sum_{h=0}^k p_k(\lambda_0) = q_k(\lambda_0).$$

In order to show that the converse also holds, let Γ be a regular graph with predistance polynomials $\{p_k\}_{0 \leq k \leq d}$, and consider, for some fixed k , the sum polynomial $q_k := \sum_{h=0}^k p_k$ which also satisfies $q_k(\lambda_0) = \|q_k\|_\Gamma^2$. Then, by (??), we have $q_k(\lambda_0)/\|q_k\|_u \leq \|\rho N_k(u)\|$, or

$$\frac{\|q_k\|_u^2}{q_k(\lambda_0)^2} \geq \frac{1}{\|\rho N_k(u)\|^2} = \frac{1}{s_k(u)} \quad (u \in V). \quad (30)$$

Then, by adding over all vertices we get

$$\sum_{u \in V} \frac{1}{s_k(u)} \leq \frac{1}{q_k(\lambda_0)^2} \sum_{u \in V} \|q_k\|_u^2 = \frac{n}{q_k(\lambda_0)^2} \|q_k\|_\Gamma^2 = \frac{n}{q_k(\lambda_0)}$$

where we have used the relationship (??) between the scalar products involved. Thus, we conclude that $q_k(\lambda_0)$ never exceeds the harmonic mean of the numbers $s_k(u)$:

$$q_k(\lambda_0) \leq \frac{n}{\sum_{u \in V} \frac{1}{s_k(u)}}. \quad (31)$$

What is more, equality can only hold if and only if all inequalities in (??) are also equalities and, hence, $q_k = \alpha_u q_k^u$ for every vertex $u \in V$ and some constants α_u . Let us see that all these constants are equal to 1. Let u, v be two adjacent vertices and assume $k \geq 1$. Using the second equality in (??) we have that $(q_k^u(\mathbf{A}))_{uv} = (q_k^v(\mathbf{A}))_{vu} = 1$ and, therefore,

$\frac{1}{\alpha_u}(q_k(\mathbf{A}))_{uv} = \frac{1}{\alpha_v}(q_k(\mathbf{A}))_{vu} = 1$. Hence $\alpha_u = \alpha_v$ and, since Γ is supposed to be connected, $q_k = \alpha q_k^u$ for some constant α and any vertex u . Moreover, using these equalities and (??),

$$\frac{n}{\alpha}q_k(\lambda_0) = \sum_{u \in V} q_k^u(\lambda_0) = \sum_{u \in V} \|q_k^u\|_u^2 = \frac{1}{\alpha^2} \sum_{u \in V} \|q_k\|_u^2 = \frac{n}{\alpha^2} \|q_k\|_{\Gamma}^2 = \frac{n}{\alpha^2} q_k(\lambda_0)$$

whence $\alpha = 1$ and $q_k = q_k^u$ for any $u \in V$. Consequently, by (??), $q_k(\mathbf{A})e_u = \rho N_k(u)$ for every vertex $u \in V$. In matrix form,

$$q_k(\mathbf{A}) = \mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}_k.$$

Then, if we assume that Γ has $d+1$ eigenvalues and the above holds for any $1 \leq k \leq d$ (the case $k=0$ being trivial since $q_0 = p_0 = 1$), we have that $p_k(\mathbf{A}) = q_k(\mathbf{A}) - q_{k-1}(\mathbf{A}) = \mathbf{A}_k$ for any $1 \leq k \leq d$ and, by (C), Γ is a distance-regular graph. We have just proved our next characterization.

(F) A regular graph Γ with n vertices and predistance polynomials $\{p_k\}_{0 \leq k \leq d}$ is distance-regular if and only if

$$q_k(\lambda_0) = \frac{n}{\sum_{u \in V} \frac{1}{s_k(u)}} \quad (0 \leq k \leq d) \quad (32)$$

where $q_k = p_0 + \cdots + p_k$.

By using the spectrum of the Laplacian matrix of Γ , an equivalent result was proved by Hajaj in [?], where the sequence of the above harmonic means, for $k = 0, 1, \dots, d$, is referred to as the “harmonic mean of the growth” of Γ .

Alternatively, considering the “base vertices” one by one, we may give a characterization which does not use the sum polynomials q_k or the harmonic means of the $s_k(u)$ ’s:

(G) A graph $\Gamma = (V, E)$ with predistance polynomials $\{p_k\}_{0 \leq k \leq d}$ is distance-regular if and only if the number of vertices at distance k from every vertex $u \in V$ is

$$n_k(u) = p_k(\lambda_0) \quad (0 \leq k \leq d). \quad (33)$$

Indeed, if (??) holds we have $q_k(\lambda_0) = p_0(\lambda_0) + \cdots + p_k(\lambda_0) = n_0(u) + \cdots + n_k(u) = s_k(u)$ for every vertex u , and (??) trivially applies. Notice also that, in this case, we do not need to assume the regularity of the graph, since it is guaranteed by considering $k=1$ in (??): $\delta_u = n_1(u) = p_1(\lambda_0)$ for any $u \in V$ (whence $p_1(\lambda_0) = \lambda_0$).

But, once more, not all the conditions in (??) or (??) are necessary to ensure distance-regularity. In fact, if the graph is regular (which guarantees the case $k=1$ since then $p_1 = x$), only the case $k=d-1$ matters. This is because, if (??) is satisfied for such a k , we infer that $q_{d-1}(\mathbf{A}) = \sum_{h=0}^{d-1} \mathbf{A}_h$ and so, $p_d(\mathbf{A}) = H(\mathbf{A}) - q_{d-1}(\mathbf{A}) = \mathbf{J} - \mathbf{A}_{d-1} = \mathbf{A}_d$, where H is the Hoffman polynomial defined in (a.5). Thus, from (C’), we can state the following:

(**F'**) A regular graph Γ with n vertices and spectrum $\text{sp}\Gamma = \{\lambda_0^{m(\lambda_0)}, \dots, \lambda_d^{m(\lambda_d)}\}$ is distance-regular if and only if

$$\frac{\sum_{u \in V} \frac{n}{n-n_d(u)}}{\sum_{u \in V} \frac{n_d(u)}{n-n_d(u)}} = \sum_{i=0}^d \frac{\pi_0^2}{m(\lambda_i)\pi_i^2}. \quad (34)$$

In the above we have used that, by property (b.3), $q_{d-1}(\lambda_0) = q_d(\lambda_0) - p_d(\lambda_0) = n - p_d(\lambda_0)$, and the value of $p_d(\lambda_0)$ given in (??). Alternatively, considering again the vertices one by one, we obtain the following result:

(**G'**) A regular graph $\Gamma = (V, E)$ with n vertices and spectrum $\text{sp}\Gamma = \{\lambda_0^{m(\lambda_0)}, \dots, \lambda_d^{m(\lambda_d)}\}$ is distance-regular if and only if the number of vertices at (spectrally maximum) distance d from each vertex $u \in V$ is

$$n_d(u) = n \left(\sum_{i=0}^d \frac{\pi_0^2}{m(\lambda_i)\pi_i^2} \right)^{-1}. \quad (35)$$

As already mentioned, this was proved by Garriga and the author in [?], generalizing some previous results of Haemers and Van Dam [?] (the case $d = 3$), and Garriga, Yebra and the author [?] (the case $n_d(u) = 1$). Finally, notice that, since $\mathbf{A}_k = p_k(\mathbf{A})$ implies $n_k(u) = p_k(\lambda_0)$ for every $u \in V$ —see (??)—both characterizations (**C'**) and (**G'**) are closely related.

5 Characterizing some families

In this last section we shall give some “spectral” characterizations which apply to certain specific families of distance-regular graphs. Now the main results will be labeled with lower case letters, to emphasize their more restricted framework. We begin with the above case of 2-antipodal graphs—that is, $n_d(u) = 1$ —where, in fact, only the distinct eigenvalues matter, as their multiplicities can be deduced from them by the formulae

$$m_i = \frac{\pi_0}{\pi_i} \quad (0 \leq i \leq d).$$

Furthermore, in this case we do not need to require regularity since it is implied by condition (??); see [?]. Then characterization (**G'**) becomes:

(**a**) A graph Γ with eigenvalues $\lambda_0 > \lambda_1 > \dots > \lambda_d$ is a 2-antipodal distance-regular graph if and only if, for each vertex u of Γ ,

$$n_d(u) = n \left(\sum_{i=0}^d \frac{\pi_0}{\pi_i} \right)^{-1} = 1. \quad (36)$$

The graphs satisfying the second equality of (??), that is $\sum_{i=0}^d(\pi_0/\pi_i) = n$, are called *boundary graphs* since they satisfy an extremal property that arises from a bound for the diameter of a graph in terms of its distinct eigenvalues. Namely, it was proved in [?] that, if Γ is regular,

$$\sum_{i=0}^d \frac{\pi_0}{\pi_i} > n \quad \Rightarrow \quad D \leq d - 1.$$

From the result in (??), some other spectral characterizations have been given for special classes of distance-regular graphs. Thus, a generalization of (a) for r -antipodal distance-regular graphs was proved in [?], and reads as follows:

- (b) A regular graph Γ , with eigenvalues $\lambda_0 > \lambda_1 > \dots > \lambda_d$, is an r -antipodal distance-regular graph if and only if the distance graph Γ_d (that is, the graph whose adjacency matrix is \mathbf{A}_d) is constituted by disjoint copies of the complete graph K_r , with r satisfying an expression in terms of n and the distinct eigenvalues. Namely,

$$r = 2n \left(\sum_{i=0}^d \frac{\pi_0}{\pi_i} \right)^{-1}. \quad (37)$$

Note that the case $r = 2$ corresponds to (??). In other words, we can restate (b) in the following way (see [?]):

- (b) A regular graph Γ as above is an r -antipodal distance-regular graph iff the distance- d graph Γ_d is constituted by $\frac{1}{2} \left(\sum_{i=0}^d \frac{\pi_0}{\pi_i} \right)$ disjoint copies of the complete graph K_r .

Recall that a δ -regular graph Γ on n vertices is called $(n, \delta; a, c)$ -strongly regular if every pair of adjacent (respectively nonadjacent) vertices have a (respectively c) common neighbours. Thus, if connected, a strongly regular graph Γ is the same as a distance-regular graph with diameter two. Otherwise, it is known that Γ is constituted by several copies of K_r .

Grouping the ideas of distance-regularity and strong-regularity, Fiol [?] proposed the following definition: A graph Γ with diameter $D = d$ is called $(n, \delta; a, c)$ -strongly distance-regular if Γ is distance-regular and its distance- d graph Γ_d is strongly regular with the indicated parameters. Some known examples of such graphs are the connected strongly regular graphs, with distance- d graph $\Gamma_d = \bar{\Gamma}$ (the complement of Γ), and the r -antipodal distance-regular graphs with $\Gamma_d = mK_r$ (m disjoint copies of K_r) so that they are $(n, \delta; r - 1, 0)$ -strongly distance-regular graphs. Hence, some spectral conditions for a regular graph to be strongly distance-regular have been already given above. In particular, notice that 2-antipodal distance-regular graphs characterized in (a) correspond to the case $a = c = 0$. In this context, the more general case $a = c$ was dealt with in [?], where one can find the following result:

- (c) Let Γ be a regular graph on n vertices, with distinct eigenvalues $\lambda_0 > \lambda_1 > \dots > \lambda_d$, $d > 1$. Then Γ is $(n, \delta; c, c)$ -strongly distance-regular if and only if, the number of

vertices at distance d from every vertex u of G is:

$$n_d(u) = \frac{n(n-1)}{\left(\sum_{i=1}^d \frac{\pi_0}{\pi_i}\right)^2 + n - 1}. \quad (38)$$

Moreover, in such a case, the above parameters are $\delta = n_d := n_d(u)$, $c = n_d(n_d - 1)/(n - 1)$, and the multiplicity of eigenvalue λ_i is

$$m_i = \frac{\pi_0}{\pi_i} \sqrt{\frac{(n-1)n_d}{n-n_d}} = (n-1) \frac{\frac{1}{\pi_i}}{\sum_{j=1}^d \frac{1}{\pi_j}} \quad (1 \leq i \leq d). \quad (39)$$

The case $d = 3$ of **(c)** was proved by Van Dam [?] using the Laplacian matrix of Γ and Haemers' method of eigenvalue interlacing [?]. In this case, he also offered examples of graphs satisfying the result. Namely, the odd graph O_4 defined below (4-regular, $n = 35$, $n_3 = 18$), and the generalized hexagons $GH(q, q)$, with q a prime power, ($(q+1)$ -regular, $n = 2(q+1)(q^4 + q^2 + 1)$, $n_3 = q^5$); for a detailed description of these graphs, see e.g. [?]-[?],[?].

On the other hand notice that, for the case $n_d(u) = 1$, the results **(a)** and **(c)** are equivalent since, in both cases, Γ must be a 2-antipodal distance-regular graph.

Finally, for general values of a and c , and $d = 3$, the following characterization has been recently given in [?].

- (d)** A regular graph Γ , with n vertices and distinct eigenvalues $\lambda_0 > \lambda_1 > \lambda_2 > \lambda_3$, is strongly distance-regular if and only if $\lambda_2 = -1$, and, for every vertex u of Γ ,

$$n_3(u) = \frac{(n - \lambda_0 - 1)[\pi_0/(\lambda_0 + 1) - n(\lambda_0 + \lambda_1\lambda_3)]}{\pi_0 - n(\lambda_0 + \lambda_1\lambda_3)}. \quad (40)$$

If this is the case, a and c satisfy also some expressions in terms of the eigenvalues.

Although we are not aware, up to now, of any generalization of the above theorem for $d > 3$, we do not in fact know of any example of a strongly distance-regular graph with diameter greater than three (apart from the r -antipodal ones). This suggests we end with the following conjecture: A (connected) regular graph Γ , with n vertices and distinct eigenvalues $\lambda_0 > \lambda_1 > \dots > \lambda_d$, is strongly distance-regular if and only if one of the three following conditions holds.

1. Γ is strongly regular ($d = 2$);
2. $d = 3$, $\lambda_2 = -1$, and Γ_3 is k -regular with degree k satisfying (??);
3. Γ is an antipodal distance-regular graph (that is, Γ_d is constituted by disjoint copies of K_r with r satisfying (??)).

Thus this result, if true, would imply that any strongly distance-regular graph is antipodal or has diameter at most three.

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