On k-Walk-Regular Graphs *

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Abstract

Considering a connected graph G with diameter D, we say that it is *k-walk-regular*, for a given integer k ($0 \leq k \leq D$), if the number of walks of length ℓ between vertices u and v only depends on the distance between them, provided that this distance does not exceed k. Thus, for $k = 0$, this definition coincides with that of walk-regular graph, where the number of cycles of length ℓ rooted at a given vertex is a constant through all the graph. In the other extreme, for $k = D$, we get one of the possible definitions for a graph to be distance-regular. In this paper we show some algebraic characterizations of k-walk-regularity, which are based on the so-called local spectrum and predistance polynomials of G. Moreover, some results concerning some parameters of a geometric nature, such as the cosines, and the spectrum of walk-regular graphs are presented.

1 Introduction

Distance-regular graphs with diameter D can be characterized by the invariance of the number of walks of length $\ell \geq 0$ between vertices at a given distance i, $0 \leq i \leq D$ (see e.g. Rowlinson [14]). Similarly, walk-regular graphs are characterized by the fact that the number of closed walks of length $l \geq 0$ rooted at any given vertex is a constant (see e.g. Godsil [11]). Based on these definitions, in this paper we introduce a generalization (of both distance-regularity and walk-regularity), which we called k-walk-regularity. In particular, we present some algebraic characterizations of k-walk-regular graphs in terms of the so-called local spectrum, which gives information of the graph when it is seen from a vertex, and the predistance polynomials of G. Moreover, some results relating the cosines

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and the maximum-independence number with the spectrum of walk-regular graphs are presented.

We begin with some notation and basic results. Throughout this paper, $G = (V, E)$ denotes a simple, connected graph, with order $n = |V|$ and adjacency matrix $\boldsymbol{A}$. The *distance* between two vertices is denoted by $\operatorname{dist}(u, v)$, so that the *eccentricity* of a vertex is $\operatorname{ecc}(u) = \max_{v \in V} \operatorname{dist}(u, v)$ and the *diameter* of the graph is $D = D(G) = \max_{u \in V} \operatorname{ecc}(u)$. The spectrum of G is denoted by

$$\operatorname{sp} G = \operatorname{sp} \boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\},$$

where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ and the superscripts stand for the multiplicities $m_i = m(\lambda_i)$. In particular, note that $m_0 = 1$ (since G is connected) and $m_0 + m_1 + \cdots + m_d = n$. It is well-known that the diameter of G satisfies $D \leq d$ (see, for instance, [1, 6]). Then, a graph with $D = d$ is said to have spectrally maximum diameter. For a given ordering of the vertices, the vector space of linear combinations (with real coefficients) of the vertices of G is identified with $\mathbb{R}^n$, with canonical basis $\{\boldsymbol{e}_u : u \in V\}$. Let $Z = \prod_{i=0}^{d}(x - \lambda_i)$ be the minimal polynomial of $\boldsymbol{A}$. The vector space $\mathbb{R}_d[x]$ of real polynomials of degree at most d is isomorphic to $\mathbb{R}[x]/(Z)$, and each polynomial $p \in \mathbb{R}_d[x]$ operates on the vector $\boldsymbol{w} \in \mathbb{R}^n$ by $p(\boldsymbol{A})\boldsymbol{w}$. For every $0 \leq k \leq d$, the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}_k = \operatorname{Ker}(\boldsymbol{A} - \lambda_k \boldsymbol{I})$ is given by the polynomial of degree d

$$P_k = \frac{1}{\phi_k} \prod_{\substack{i=0 \\ i \neq k}}^{d} (x - \lambda_i) = \frac{(-1)^k}{\pi_k} \prod_{\substack{i=0 \\ i \neq k}}^{d} (x - \lambda_i),$$

where $\phi_k = \prod_{i=0, i \neq k}^{d} (\lambda_k - \lambda_i)$ and $\pi_k = |\phi_k|$ are "moment-like" parameters satisfying

$$\sum_{\ell=0}^{m} (-1)^\ell \frac{\lambda_\ell^k}{\pi_\ell} = \begin{cases} 0 & \text{if } 0 \leq k < d, \\ 1 & \text{if } k = d, \end{cases} \tag{1}$$

(just express x^ℓ in terms of the basis $\{P_0, P_1, \ldots, P_d\}$ and equate coefficients of degree d). The matrices $\boldsymbol{E}_k = P_k(\boldsymbol{A})$ corresponding to these orthogonal projections are called the (*principal*) *idempotents* of $\boldsymbol{A}$. Then, the orthogonal decomposition of the unitary vector $\boldsymbol{e}_u$, representing vertex u, is:

$$\boldsymbol{e}_u = \boldsymbol{z}_u^0 + \boldsymbol{z}_u^1 + \cdots + \boldsymbol{z}_u^d, \quad \text{where} \quad \boldsymbol{z}_u^k = P_k(\boldsymbol{A})\boldsymbol{e}_u = \boldsymbol{E}_k \boldsymbol{e}_u \in \mathcal{E}_k. \tag{2}$$

In particular, if $\boldsymbol{\nu}$ is an eigenvector of λ_0, then $\boldsymbol{z}_u^0 = \frac{\langle \boldsymbol{e}_u, \boldsymbol{\nu} \rangle}{\|\boldsymbol{\nu}\|^2} \boldsymbol{\nu} = \frac{\nu_u}{\|\boldsymbol{\nu}\|^2} \boldsymbol{\nu}$.

The idempotents of $\boldsymbol{A}$ satisfy the following properties (see e.g. Godsil [11]):

$(a.1)$ $\boldsymbol{E}_k \boldsymbol{E}_h = \begin{cases} \boldsymbol{E}_k & \text{if } k = h, \\ \boldsymbol{0} & \text{otherwise;} \end{cases}$

$(a.2)$ $\boldsymbol{A}\boldsymbol{E}_k = \lambda_k \boldsymbol{E}_k$;

$(a.3)\;\; p(\boldsymbol{A}) = \sum_{k=0}^{d} p(\lambda_k)\boldsymbol{E}_k,\;$ for any polynomial $p \in \mathbb{R}[x]$.

In particular, taking $p = 1$ in $(a.3)$, we have $\boldsymbol{E}_0 + \boldsymbol{E}_1 + \cdots + \boldsymbol{E}_d = \boldsymbol{I}$ (as expected, since the sum of all orthogonal projections gives the original vector). Moreover, taking $p = x^\ell$, each power of $\boldsymbol{A}$ can be expressed as a linear combination of the idempotents $\boldsymbol{E}_k$:

$$\boldsymbol{A}^\ell = \sum_{k=0}^{d} \lambda_k^\ell \boldsymbol{E}_k. \tag{3}$$

2 Spectral regularity and walk-regularity

From the decomposition (2), we define the *u-local multiplicity* of eigenvalue λ_k as

$$m_u(\lambda_k) = \|\boldsymbol{z}_u^k\|^2 = \langle \boldsymbol{E}_k\boldsymbol{e}_u, \boldsymbol{E}_k\boldsymbol{e}_u\rangle = \langle \boldsymbol{E}_k\boldsymbol{e}_u, \boldsymbol{e}_u\rangle = (\boldsymbol{E}_k)_{uu}, \tag{4}$$

(see [7]), satisfying

$$\sum_{k=0}^{d} m_u(\lambda_k) = 1, \quad \sum_{u\in V} m_u(\lambda_k) = m_k \qquad (0 \le k \le d).$$

Indeed, the first equality follows from the unitary character of $\boldsymbol{e}_u$, whereas the second one comes from (4), since

$$\sum_{u\in V} m_u(\lambda_k) = \operatorname{tr}(\boldsymbol{E}_k) = m_k,$$

because $\operatorname{sp} \boldsymbol{E}_k = \{0^{n-m_k}, 1^{m_k}\}$.

We say that G is *spectrally regular* when, for any $k = 0, 1, \ldots, d$, the u-local multiplicity of λ_k does not depend on the vertex u. Then, the above equations imply that (standard) multiplicity "splits" equitably among the n vertices, giving $m_u(\lambda_k) = \frac{m_k}{n}$. In particular, since $m_u(\lambda_0) = \|\boldsymbol{z}_k^0\|^2 = \frac{\nu_u^2}{\|\boldsymbol{\nu}\|^2}$, the spectral regularity implies the regularity of the graph because, in this case, $m_u(\lambda_0) = \frac{1}{n}$ and $\nu_u = \frac{\|\boldsymbol{\nu}\|}{\sqrt{n}}$ for all u.

Let $a_u^{(\ell)} = (\boldsymbol{A}^\ell)_{uu}$ denote the number of closed walks of length ℓ rooted at vertex u. When the number $a_u^{(\ell)}$ only depends on ℓ, in which case we write $a_u^{(\ell)} = a^{(\ell)}$, the graph G is called *walk-regular* (a concept introduced by Godsil and McKay in [12]). Notice that, as $a_u^{(2)} = \delta_u$, the degree of vertex u, every walk-regular graph is also regular. Recall also that, if G has $d+1$ distinct eigenvalues, then $\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d\}$ is a basis of the *adjacency* or *Bose-Mesner algebra* $\mathcal{A}(G)$ of matrices which are polynomials in $\boldsymbol{A}$. Therefore, the existence of the set of constants $\mathcal{C} = \{a^{(0)}, a^{(1)}, \ldots, a^{(d)}\}$ suffices for assuring walk-regularity.

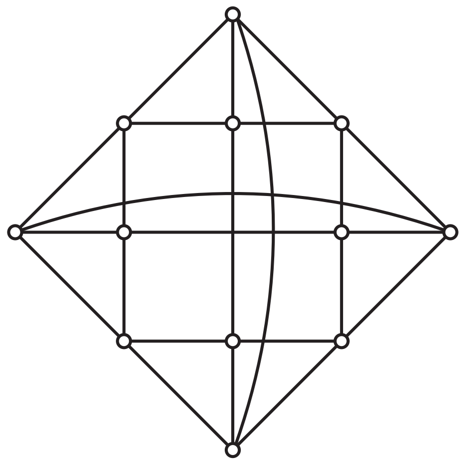

Figure 1: A walk-regular graph which is not distance-regular.

As it is well known, any distance-regular graph is also walk-regular, but the converse is not true. Actually, as it is pointed out by Godsil [11], there are walk-regular graphs which are neither vertex-transitive nor distance-regular. This is the case, for instance, of the graph G depicted in Fig. 1, with spectrum $\operatorname{sp} G = \{4^1, 2^3, 0^3, -2^5\}$ and set of numbers of closed walks $\mathcal{C} = \{1, 0, 4, 4\}$.

In our context, we also have the following result:

Proposition 2.1 *A connected graph G is spectrally regular if and only if it is walk-regular.*

Proof. First, note that, by using (3), the number of closed walks $a_u^{(\ell)}$ can be computed in terms of the local multiplicities as

$$a_u^{(\ell)} = (\boldsymbol{A}^\ell)_{uu} = \sum_{k=0}^{d} \lambda_k^\ell (\boldsymbol{E}_k)_{uu} = \sum_{k=0}^{d} m_u(\lambda_k)\lambda_k^\ell. \tag{5}$$

Then, if G is spectrally regular, for any $u \in V$ and $\ell \geq 0$, this gives

$$a_u^{(\ell)} = \frac{1}{n} \sum_{k=0}^{d} m_k \lambda_k^\ell, \tag{6}$$

so that $a_u^{(\ell)}$ is independent of u and G is walk-regular. Conversely, suppose that G is walk-regular and let $a^{(\ell)}$ denote the constant value of $a_u^{(\ell)}$ for every $u \in V$. Then, from

$$a^{(\ell)} = \frac{1}{n} \operatorname{tr}(\boldsymbol{A}^\ell) = \frac{1}{n} \sum_{k=0}^{d} m_k \lambda_k^\ell$$

and (5), we get

$$\sum_{k=0}^{d} \lambda_k^{\ell} \left(\frac{m_k}{n} - m_u(\lambda_k) \right) = 0,$$

for $\ell = 0, 1, \dots, d$. This system has determinant different from 0. So, $\frac{m_k}{n} - m_u(\lambda_k) = 0$ and the graph is spectrally regular. $\square$

Consequently, from now on we will indistinctly say that a graph G is spectrally regular or that it is walk-regular. If this is the case, we can relate the spectrum of G with the number of closed walks $a_u^{(\ell)} = a^{(\ell)}$ in different ways, such as in (6). Also, given the eigenvalues (from which we compute the polynomials P_k) and the set $\mathcal{C} = \{a^{(0)}, a^{(1)}, \dots, a^{(d)}\}$, we can obtain the multiplicities. With this aim, let us introduce the following notation: given a polynomial $p = \sum_{i=0}^{d} \alpha_i x^i$, let $p(\mathcal{C}) = \sum_{i=0}^{d} \alpha_i a^{(i)}$. Note that if the graph is walk-regular, then $(p(\boldsymbol{A}))_{uu} = \sum_{i=0}^{d} \alpha_i (\boldsymbol{A}^i)_{uu} = p(\mathcal{C})$ for any vertex u. Thus, for such a graph,

$$m_k = n\, m_u(\lambda_k) = n(\boldsymbol{E}_k)_{uu} = n(P_k(\boldsymbol{A}))_{uu} = n\, P_k(\mathcal{C}). \tag{7}$$

The predistance polynomials

From the spectrum of a given graph $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$, we consider the following scalar product in $\mathbb{R}_d[x]$:

$$\langle p, q \rangle = \frac{1}{n} \operatorname{tr}(p(\boldsymbol{A}) q(\boldsymbol{A})) = \frac{1}{n} \sum_{k=0}^{d} m_k p(\lambda_k) q(\lambda_k). \tag{8}$$

Then, by using the Gram-Schmidt method and normalizing appropriately, it is immediate to prove the existence and uniqueness of an orthogonal system of polynomials $\{p_k\}_{0 \le k \le d}$ called *predistance polynomials* which, for any $0 \le h, k \le d$, satisfy:

$(b.1)$ $\operatorname{dgr}(p_k) = k$;

$(b.2)$ $\langle p_h, p_k \rangle = 0$ if $h \ne k$;

$(b.3)$ $\|p_k\|^2 = p_k(\lambda_0)$.

In [7, 8] it was shown that such a system is unique and it is also characterized by the any of the two following conditions:

$(c.1)$ $p_0 = 1, \quad a_k + b_k + c_k = \lambda_0$ for $0 \le k \le d$,

where a_k, b_k and c_k are the corresponding coefficients of the three-term recurrence

$$x p_k = b_{k-1} p_{k-1} + a_k p_k + c_{k+1} p_{k+1} \quad (0 \le k \le d),$$

(that is, the Fourier coefficients of $x p_k$ in terms of p_{k-1}, p_k, and p_{k+1}, respectively) initiated with $p_{-1} = 0$ and p_0 any non-zero constant.

$(c.2)$ $H := \sum_{k=0}^{d} p_k = \frac{n}{\pi_0}\prod_{k=1}^{d}(x-\lambda_k) = n\,P_0.$

The reader familiar with the theory of distance-regular graphs will have already noted that the predistance polynomials can be thought as a generalization of the so-called "distance polynomials". Recall that, in a distance-regular graph, such polynomials satisfy

$$p_k(\boldsymbol{A}) = \boldsymbol{A}_k \quad (0 \le k \le d),$$

where $\boldsymbol{A}_k$ stands for the adjacency matrix of the distance-k graph G_k, usually called the *k-th distance matrix* of G (see, for instance, [2]). Also, recall that the polynomial H in $(c.2)$ is the Hoffman polynomial characterizing the regularity of G by the condition $H(\boldsymbol{A}) = \boldsymbol{J}$, the all-1 matrix (see Hoffman [13]).

In our context, the predistance polynomials allow us to give another characterization of walk-regularity (or spectral regularity), as it is shown in the following result:

Proposition 2.2 *Let G be a connected graph with adjacency matrix $\boldsymbol{A}$ having $d+1$ distinct eigenvalues, and with predistance polynomials $\{p_0, p_1, \ldots, p_d\}$. Then, the two following statements are equivalent:*

- (a) *G is walk-regular.*
- (b) *The matrices $p_k(\boldsymbol{A})$, $1 \le k \le d$, have null diagonals.*

Proof. Assume first that (a) holds: if G is walk-regular, then the diagonal vector of $\boldsymbol{A}^\ell$ is $\mathrm{diag}(\boldsymbol{A}^\ell) = a^{(\ell)}\boldsymbol{j}$. Since $(p_k(\boldsymbol{A}))_{uu} = p_k(\mathcal{C})$ for every vertex u, $\mathrm{diag}(p_k(\boldsymbol{A})) = p_k(\mathcal{C})\boldsymbol{j}$, with $\boldsymbol{j}$ being the all-1 vector. But, for $1 \le k \le d$, we have

$$0 = \langle p_k, p_0\rangle = \frac{1}{n}\,\mathrm{tr}(p_k(\boldsymbol{A})) = p_k(\mathcal{C}),$$

so that $\mathrm{diag}(p_k(\boldsymbol{A})) = \boldsymbol{0}$.

Now suppose that (b) holds. Then, by using the expression

$$x^\ell = \sum_{k=0}^{\ell} \alpha_{\ell k} p_k \tag{9}$$

where $\alpha_{\ell k}$ are the Fourier coefficients of x^ℓ in terms of p_k, we have

$$\mathrm{diag}(\boldsymbol{A}^\ell) = \sum_{k=0}^{\ell} \alpha_{\ell k}\,\mathrm{diag}(p_k(\boldsymbol{A})) = \alpha_{\ell 0}\boldsymbol{j}.$$

Therefore, $a_u^{(\ell)} = \alpha_{\ell 0}$, which is independent of u and the graph is walk-regular. (Notice that $\alpha_{\ell 0} = \frac{\langle x^\ell, 1\rangle}{\|1\|^2} = \frac{1}{n}\sum_{k=0}^{\ell} m_k \lambda_k^\ell$, as expected.) □

Note that property (b) is also satisfied in the case of distance-regularity, as $p_k(\boldsymbol{A}) = \boldsymbol{A}_k$.

3 k-Walk-regular graphs

The above result can be generalized if we consider the following new definition. Let G be a connected graph with diameter D. For a given integer k, $0 \le k \le D$, we say that G is *k-walk-regular* if the number of walks of length ℓ between vertices u, v, that is $a_{uv}^{(\ell)} = (\boldsymbol{A}^\ell)_{uv}$, only depends on the distance between u and v, provided that $\mathrm{dist}(u, v) = i \le k$. If this is the case, we write $a_{uv}^{(\ell)} = a_i^{(\ell)}$. Thus, a 0-walk-regular graph is the same concept as a walk-regular graph. In the other extreme, the distance-regular graphs correspond to the case of D-walk-regular graphs (see e.g. Rowlinson [14]). Note that, obviously, if G is a k-walk-regular graph, then it is also k'-walk-regular for any $k' \le k$. This is consequent with the fact that a distance-regular graph is also walk-regular. To illustrate our new definition, a family of graphs which are 1-walk-regular (but not k-walk-regular for $k > 1$) are the Cartesian products of cycles $C_m \times C_m$ with $m \ge 5$. In fact, notice that all these graphs are vertex- and edge-transitive. For instance, $C_5 \times C_5$ has diameter $D = 2$, number of different eigenvalues $d + 1 = 6$, and sets $\mathcal{C} = \{a_0^{(\ell)}\}_{0\le\ell\le5} = \{1, 0, 4, 0, 36, 4\}$, $\mathcal{W} = \{a_1^{(\ell)}\}_{0\le\ell\le5} = \{0, 1, 0, 9, 1, 100\}$.

As in the case of walk-regularity, the concept of k-walk-regularity can also be seen as the invariance of some entries of the idempotents. By analogy with local multiplicities, which correspond to the diagonal of the matrix, Fiol, Garriga and Yebra [9] called these entries the *crossed* (uv-)*local multiplicities* of λ_k, and they were denoted by $m_{uv}(\lambda_k)$. Now in terms of the orthogonal projection of the canonical vectors $\boldsymbol{e}_u$, the crossed local multiplicities are obtained by the scalar products

$$m_{uv}(\lambda_k) := (\boldsymbol{E}_k)_{uv} = \langle \boldsymbol{E}_k\boldsymbol{e}_u, \boldsymbol{e}_v\rangle = \langle \boldsymbol{E}_k\boldsymbol{e}_u, \boldsymbol{E}_k\boldsymbol{e}_v\rangle = \langle \boldsymbol{z}_u^k, \boldsymbol{z}_v^k\rangle \qquad (u, v \in V).$$

Now, for a given k, $0 \le k \le d$, we say that graph G is *k-spectrally regular* when, for any $h = 0, 1, \ldots, d$, the crossed uv-local multiplicities of λ_h only depend on the distance between u and v, provided that $\mathrm{dist}(u, v) \le k$.

At this point, we are ready to give the following result (where "$\circ$" stands for the Schur or Hadamard—componentwise—product of matrices), relating the k-walk-regularity to the k-spectral regularity and the matrices obtained from the predistance polynomials. In the second case, these polynomials give the distance matrices when we look trough a "window" defined by the matrix $\boldsymbol{S}_k = \boldsymbol{A}_0 + \boldsymbol{A}_1 + \cdots + \boldsymbol{A}_k$.

Theorem 3.1 *Let G be a connected graph with adjacency matrix $\boldsymbol{A}$ having $d+1$ distinct eigenvalues, and with predistance polynomials $\{p_0, p_1, \ldots, p_d\}$. Then, for a given integer k, $0 \le k \le D$, the three following statements are equivalent:*

(*a*) *G is k-walk-regular.*

(*b*) *G is k-spectrally regular.*

(*c*) *$\boldsymbol{S}_k \circ p_i(\boldsymbol{A}) = \boldsymbol{S}_k \circ \boldsymbol{A}_i$ for any $0 \le i \le d$.*

Proof. $(a) \Leftrightarrow (b)$: The equivalence between (a) and (b) is proved analogously to the proof of Proposition 2.1. Indeed, from (3), we now have that the number of walks $a_{uv}^{(\ell)}$ can be computed in terms of the crossed uv-local multiplicities as

$$a_{uv}^{(\ell)} = (\boldsymbol{A}^\ell)_{uv} = \sum_{h=0}^{d} m_{uv}(\lambda_h)\lambda_h^\ell. \tag{10}$$

Then, if G is k-spectrally regular, this gives

$$a_{uv}^{(\ell)} = \frac{1}{n}\sum_{k=0}^{d} m_{ih}\lambda_h^\ell, \tag{11}$$

for any $u, v \in V$ such that $\mathrm{dist}(u,v) = i \leq k$, and $\ell \geq 0$. Therefore, $a_{uv}^{(\ell)}$ is independent of u, v, provided that $\mathrm{dist}(u,v) = i \leq k$, and G is k-walk-regular. Conversely, suppose that G is k-walk-regular and consider the set of numbers of (u,v)-walks $\mathcal{W} = \{a_i^{(0)}, a_i^{(1)}, \ldots, a_i^{(d)}\}$, where $i = \mathrm{dist}(u,v) \leq k$. Then, we can obtain the crossed uv-local multiplicities as

$$m_{uv}(\lambda_h) = (\boldsymbol{E}_h)_{uv} = (P_h(\boldsymbol{A}))_{uv} = P_h(\mathcal{W}), \tag{12}$$

which turn out to be independent of u, v and G is k-spectrally regular.

$(a),(b) \Rightarrow (c)$: We want to prove that $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$ if $i \leq k$ and $\boldsymbol{S}_k \circ p_i(\boldsymbol{A}) = \boldsymbol{O}$ otherwise. Then, if G is k-walk-regular, there are constants $a_i^{(\ell)}$, for any $0 \leq i \leq k$ and $\ell \geq 0$ satisfying

$$\boldsymbol{A}^\ell = \sum_{i=0}^{k} a_i^{(\ell)} \boldsymbol{A}_i. \tag{13}$$

where, clearly, $a_i^\ell = 0$ when $\ell < i$. As a matrix equation (writing only the terms with $\ell \leq k$),

$$\begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \boldsymbol{A}^2 \\ \cdot \\ \cdot \\ \boldsymbol{A}^k \end{pmatrix} = \begin{pmatrix} a_0^{(0)} & & & & & \\ a_0^{(1)} & a_1^{(1)} & & & & \\ a_0^{(2)} & a_1^{(2)} & a_2^{(2)} & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ a_0^{(k)} & a_1^{(k)} & \cdot & \cdot & \cdot & a_k^{(k)} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{A} \\ \boldsymbol{A}_2 \\ \cdot \\ \cdot \\ \boldsymbol{A}_k \end{pmatrix}, \tag{14}$$

where the lower triangular matrix $\boldsymbol{T}$, with rows and columns indexed with the integers $0, 1 \ldots, k$, has entries $(\boldsymbol{T})_{\ell i} = a_i^{(\ell)}$. In particular, note that $a_0^{(0)} = a_1^{(1)} = 1$ and $a_0^{(1)} = 0$. Moreover, since $a_k^{(k)} > 0$, such a matrix has an inverse, which is also a lower triangular matrix, and hence each $\boldsymbol{A}_i$ is a polynomial, say q_i, of degree i in $\boldsymbol{A}$. These polynomials are orthogonal with respect to the scalar product (8) since

$$\langle q_i, q_j \rangle = \frac{1}{n}\,\mathrm{tr}(q_i(\boldsymbol{A})q_j(\boldsymbol{A})) = \frac{1}{n}\,\mathrm{tr}(\boldsymbol{A}_i\boldsymbol{A}_j) = 0 \qquad (i \neq j).$$

Moreover, as $\boldsymbol{A}_i\boldsymbol{j} = q_i(\boldsymbol{A})\boldsymbol{j} = q_i(\lambda_0)\boldsymbol{j}$, the number of vertices at distance i, $0 \le i \le k$, from a given vertex u is a constant through all the graph: $n_i = \Gamma_i(u) = q_i(\lambda_0)$ for every $u \in V$. Thus,

$$\|q_i\|^2 = \frac{1}{n}\operatorname{tr}(q_i^2(\boldsymbol{A})) = \frac{1}{n}\operatorname{tr}(\boldsymbol{A}_i^2) = \frac{1}{n}\operatorname{tr}(\boldsymbol{A}_i) = q_i(\lambda_0)$$

and, therefore, the obtained polynomials are, in fact, the (pre)distance polynomials $q_i = p_i$, $0 \le i \le k$, as claimed. Let us now prove the second part of the statement: if $j > k$, then $p_j(\boldsymbol{A})_{uv} = 0$ provided that $\operatorname{dist}(u,v) \le k$. First, note that, from property $(a.2)$ of the idempotents, we have

$$(p_i(\boldsymbol{A})\boldsymbol{E}_h)_{uu} = p_i(\lambda_h)(\boldsymbol{E}_h)_{uu} = p_i(\lambda_h)m_u(\lambda_h) = p_i(\lambda_h)\frac{m_h}{n} \qquad (15)$$

for any $0 \le h \le d$. But, if $i = \operatorname{dist}(u,v) \le k$, we already know that $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$ and then,

$$(p_i(\boldsymbol{A})\boldsymbol{E}_h)_{uu} = (\boldsymbol{A}_i\boldsymbol{E}_h)_{uu} = \sum_{v\in V}(\boldsymbol{A}_i)_{uv}(\boldsymbol{E}_h)_{uv} = \sum_{v\in\Gamma_i(u)} m_{uv}(\lambda_h) = n_i m_{ih}, \qquad (16)$$

where we have used the invariance of the crossed local multiplicities, $m_{uv}(\lambda_h) = m_{ih}$, and the number of vertices at distance $i(\le k)$ from any given vertex, $n_i = p_i(\lambda_0)$. Equating (15) and (16) we obtain:

$$m_{ih} = \frac{m_h p_i(\lambda_h)}{n p_i(\lambda_0)} \qquad (0 \le i \le k,\ 0 \le h \le d). \qquad (17)$$

Using property $(b.3)$ of the idempotents and these values of the crossed multiplicities, we finally get:

$$\begin{aligned} p_j(\boldsymbol{A})_{uv} &= \sum_{h=0}^{d} p_j(\lambda_h)(\boldsymbol{E}_h)_{uv} = \sum_{h=0}^{d} p_j(\lambda_h)m_{ih} \\ &= \frac{1}{n p_i(\lambda_0)}\sum_{h=0}^{d} m_h p_j(\lambda_h)p_i(\lambda_h) = \frac{1}{p_i(\lambda_0)}\langle p_j, p_i\rangle = 0 \qquad (j > k \ge i). \end{aligned}$$

$(c) \Rightarrow (b)$: Conversely, assume that (c) holds and, for every h, $0 \le h \le d$, consider the expression of $P_h = \sum_{j=0}^{d}\beta_{hj}p_j$, where β_{hj} is the Fourier coefficient of P_h in terms of p_j. Then, if $\operatorname{dist}(u,v) = i \le k$,

$$m_{uv}(\boldsymbol{E}_h)_{uv} = \sum_{j=0}^{d}\beta_{hj}p_j(\boldsymbol{A})_{uv} = \sum_{j=0}^{k}\beta_{hj}(\boldsymbol{A}_j)_{uv} + \sum_{j=k+1}^{d}\beta_{hj}(p_j(\boldsymbol{A}))_{uv} = \beta_{hi}.$$

Consequently, the crossed local multiplicities $m_{uv}(\lambda_h) = \beta_{hi}$ only depend on the distance $\operatorname{dist}(u,v) = i$, and G is k-spectrally regular. (Notice that, $\beta_{hi} = m_{ih} = \frac{\langle P_h, p_i\rangle}{\|p_i\|^2} = \frac{1}{p_i(\lambda_0)n}\sum_{j=0}^{d} m_j P_h(\lambda_j)p_i(\lambda_j) = \frac{m_h p_i(\lambda_h)}{n p_i(\lambda_0)}$, in concordance with (17).) □

4 Spectrum and Diameter

In this last section, we study some results concerning some parameters of a geometric nature, as the cosines and the $(d-1)$-independent number, and the spectrum of walk-regular graphs. In our context, these results are of interest because they apply to all k-regular graphs and, in particular, to distance-regular graphs.

Consider the sets $T_k = \{\boldsymbol{z}_u^k = \boldsymbol{E}_k\boldsymbol{e}_u : u \in V\}$ of vectors in the m_k-dimensional space $\mathrm{Ker}(x-\lambda_k)$. These sets are usually called *eutactic stars* and they have been extensively studied, for instance, see [15, 14, 3]. Then, the spectral regularity of the graph is equivalent to state that, for every $k = 0, 1, \ldots, d$, such vectors define n points (not necessarily different) on the sphere with radius $\sqrt{m_k/n}$. Moreover, for any $k = 1, 2, \ldots, d$, the "center of mass" of the set T_k is

$$\sum_{u\in V} \boldsymbol{z}_u^k = \boldsymbol{E}_k \sum_{u\in V} \boldsymbol{e}_u = \boldsymbol{E}_k\boldsymbol{j} = \boldsymbol{0}.$$

Let $\gamma_{u,v}^k = \gamma(\boldsymbol{z}_u^k, \boldsymbol{z}_v^k)$ denote the angle between two vectors $\boldsymbol{z}_u^k$, $\boldsymbol{z}_v^k$. Note that, since $\boldsymbol{z}_u^0 = (1/n)\boldsymbol{j}$, we always have $\gamma_{u,v}^0 = 0$. In terms of our local multiplicities, the cosines of these angles are:

$$\cos\gamma_{u,v}^k = \frac{\langle \boldsymbol{z}_u^k, \boldsymbol{z}_v^k\rangle}{\|\boldsymbol{z}_u^k\|\|\boldsymbol{z}_v^k\|} = \frac{m_{uv}(\lambda_k)}{\sqrt{m_u(\lambda_k)m_v(\lambda_k)}}. \tag{18}$$

These cosines were already considered by Godsil [10, 11] when G is a distance-regular graph. He referred to them as the *uv-cosines*, denoted by $w_{uv} = w_{uv}(\lambda_k)$. For such a graph, and assuming that $\mathrm{dist}(u,v) = \ell$, they satisfy the formula (see [5]):

$$w_\ell(\lambda_k) = \frac{p_\ell(\lambda_k)}{p_\ell(\lambda_0)} \quad (0 \le k \le d).$$

For spectrally-regular graphs, we have the following result:

Proposition 4.1 *Let $G = (V,E)$ be a spectrally regular graph with $d+1$ eigenvalues. Then, two vertices $u, v \in V$ are at (spectrally maximum) distance d if and only if*

$$\cos\gamma_{u,v}^k = \frac{(-1)^k}{m_k}\frac{\pi_0}{\pi_k} \qquad (0 \le k \le d). \tag{19}$$

Proof. For $0 \le k \le d$, we compute the following:

$$\left.\begin{array}{l} \langle \boldsymbol{z}_u^k, \boldsymbol{z}_v^k\rangle = \langle P_k(\boldsymbol{A})\boldsymbol{e}_u, P_k(\boldsymbol{A})\boldsymbol{e}_v\rangle = \langle P_k(\boldsymbol{A})\boldsymbol{e}_u, \boldsymbol{e}_v\rangle = \frac{(-1)^k}{\pi_k}\langle \boldsymbol{A}^d\boldsymbol{e}_u, \boldsymbol{e}_v\rangle \\ \langle \boldsymbol{z}_u^k, \boldsymbol{z}_v^k\rangle = \|\boldsymbol{z}_u^k\|\|\boldsymbol{z}_v^k\|\cos\gamma_{u,v}^k \end{array}\right\} \Rightarrow$$

$$\cos\gamma_{u,v}^k = \frac{(-1)^k}{\pi_k}\frac{n}{m_k}\langle \boldsymbol{A}^d\boldsymbol{e}_u, \boldsymbol{e}_v\rangle.$$

For $k=0$, we have $1=\frac{n}{\pi_0}\langle \boldsymbol{A}^d\boldsymbol{e}_u,\boldsymbol{e}_v\rangle$. Then,

$$\cos\gamma_{u,v}^k=\frac{(-1)^k}{m_k}\frac{\pi_0}{\pi_k}\qquad (0\le k\le d).$$

Reciprocally, let $Q_1(\boldsymbol{A}),Q_2(\boldsymbol{A}),\ldots,Q_d(\boldsymbol{A})$ be the polynomials of degree $d-1$ defined by $(\boldsymbol{A}-\lambda_0)Q_k(\boldsymbol{A})=P_k(\boldsymbol{A})$, namely,

$$Q_k(\boldsymbol{A})=\frac{(-1)^k}{\pi_k}\prod_{\substack{i=1\\ i\neq k}}^{d}(\boldsymbol{A}-\lambda_i).$$

The set $\{Q_1(\boldsymbol{A}),Q_2(\boldsymbol{A}),\ldots,Q_d(\boldsymbol{A})\}$ is a base in $\mathbb{R}_{d-1}$ and, for $1\le k\le d$, each element $Q_k(\boldsymbol{A})$ holds:

$$\begin{aligned}
&Q_k(\lambda_0)=\frac{(-1)^k}{\lambda_0-\lambda_k}\frac{\pi_0}{\pi_k};\\
&Q_k(\lambda_i)=0\qquad \text{for } 1\le i\le d,\ i\neq k;\\
&Q_k(\lambda_k)=\frac{(-1)^k}{\lambda_k-\lambda_0}.
\end{aligned}$$

Then, for $1\le k\le d$, we get:

$$\begin{aligned}
\langle Q_k(\boldsymbol{A})\boldsymbol{e}_u,\boldsymbol{e}_v\rangle &= \langle Q_k(\lambda_0)\boldsymbol{z}_u^0+Q_k(\lambda_k)\boldsymbol{z}_u^k,\boldsymbol{e}_v\rangle=Q_k(\lambda_0)\langle\boldsymbol{z}_u^0,\boldsymbol{z}_v^0\rangle+Q_k(\lambda_k)\langle\boldsymbol{z}_u^k,\boldsymbol{z}_v^k\rangle\\
&= \frac{(-1)^k}{\lambda_0-\lambda_k}\frac{\pi_0}{\pi_k}\frac{1}{n}+\frac{1}{\lambda_k-\lambda_0}\frac{m_k}{n}\frac{(-1)^k}{m_k}\frac{\pi_0}{\pi_k}=0.
\end{aligned}$$

So, $\langle\boldsymbol{A}^i\boldsymbol{e}_u,\boldsymbol{e}_v\rangle=0$ for $i\le d-1$ and the vertices u,v are at distance d. □

Notice that, according to (18), the above condition (19) can also be given in terms of the crossed local multiplicities as:

$$m_{uv}(\lambda_k)=\frac{(-1)^k}{n}\frac{\pi_0}{\pi_k}\qquad (0\le k\le d).$$

As a consequence, by using (10) and (1), we have that, under the hypotheses of Proposition 4.1, the number of walks of length d between vertices u,v at maximum distance d is $a_{uv}^{(d)}=\pi_0/n$ (independent of such vertices, as if the graph were distance-regular).

As a direct consequence of Proposition 4.1, we have the following result:

Corollary 4.2 *The eigenvalue multiplicities of a (connected) spectrally regular graph with spectrally maximum diameter satisfy*

$$m_k\ge\frac{\pi_0}{\pi_k}\qquad (0\le k\le d).$$

Let $\alpha \equiv \alpha(G) \equiv \alpha_{d-1}(G)$ be the $(d-1)$-independence number of G, that is, the maximum number of vertices which are at distance d from each other. Note that, for a graph G, the property of having spectrally maximum diameter is equivalent to have $\alpha \geq 2$.

Proposition 4.3 *The $(d-1)$-independence number of a spectrally regular graph G satisfies the bound*

$$\alpha \leq 1 + \min_{\substack{1\leq k\leq d \\ k \text{ odd}}} \left\{ m_k \frac{\pi_k}{\pi_0} \right\}.$$

Proof. Let $u_1, u_2, \ldots, u_r$ be vertices at distance d from each other. Consider their spectral decompositions:

$$\begin{aligned}
\boldsymbol{e}_{u_1} &= \boldsymbol{z}_{u_1}^0 + \boldsymbol{z}_{u_1}^1 + \cdots + \boldsymbol{z}_{u_1}^d \\
\boldsymbol{e}_{u_2} &= \boldsymbol{z}_{u_2}^0 + \boldsymbol{z}_{u_2}^1 + \cdots + \boldsymbol{z}_{u_2}^d \\
&\vdots \\
\boldsymbol{e}_{u_r} &= \boldsymbol{z}_{u_r}^0 + \boldsymbol{z}_{u_r}^1 + \cdots + \boldsymbol{z}_{u_r}^d.
\end{aligned}$$

Then, $\boldsymbol{e}_{u_1} + \boldsymbol{e}_{u_2} + \cdots + \boldsymbol{e}_{u_r} = \boldsymbol{w}^0 + \boldsymbol{w}^1 + \cdots + \boldsymbol{w}^d$, where $\boldsymbol{w}^k = \boldsymbol{z}_{u_1}^k + \boldsymbol{z}_{u_2}^k + \cdots + \boldsymbol{z}_{u_r}^k \in \operatorname{Ker}(x - \lambda_k)$. Computing $\|\boldsymbol{w}^k\|^2$, we obtain

$$\begin{aligned}
\|\boldsymbol{w}^k\|^2 &= \sum_{i,j=1}^{r} \langle \boldsymbol{z}_{u_i}^k, \boldsymbol{z}_{u_j}^k \rangle = \sum_{i=1}^{r} \|\boldsymbol{z}_{u_i}^k\|^2 + \sum_{\substack{i,j=1 \\ i\neq j}}^{r} \langle \boldsymbol{z}_{u_i}^k, \boldsymbol{z}_{u_j}^k \rangle \\
&= \sum_{i=1}^{r} \frac{m_k}{n} + \sum_{\substack{i,j=1 \\ i\neq j}}^{r} \frac{m_k}{n} \frac{m_k}{n} \frac{(-1)^k}{m_k} \frac{\pi_0}{\pi_k} = r \frac{m_k}{n} + r(r-1) \frac{(-1)^k}{n} \frac{\pi_0}{\pi_k} \\
&= \frac{r}{n} \left(m_k + (-1)^k (r-1) \frac{\pi_0}{\pi_k} \right).
\end{aligned}$$

For k even, the inequality $\|\boldsymbol{w}^k\|^2 \geq 0$ is irrelevant. However, for k odd, it imposes that $m_k \geq (r-1) \frac{\pi_0}{\pi_k}$ or, equivalently, $r \leq 1 + m_k \frac{\pi_k}{\pi_0}$. □

Then, from the above results we get:

Corollary 4.4 *Let G be a (connected) spectrally regular graph with spectrum* $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$*, spectrally maximum diameter d, and $(d-1)$-independence number α. Then, the eigenvalue multiplicities satisfy the bounds*

$$\begin{aligned}
m_k &\geq \frac{\pi_0}{\pi_k} \quad \text{if } k \text{ is even,} \\
m_k &\geq (\alpha - 1)\frac{\pi_0}{\pi_k} \quad \text{if } k \text{ is odd.}
\end{aligned}$$

A result to be compared with the following characterization of antipodal distance-regular graphs given in [4].

Proposition 4.5 *A distance-regular graph G with spectrum* $\mathrm{sp}\, G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$ *is r-antipodal ($r \geq 2$) if and only if its eigenvalue multiplicities satisfy:*

$$\begin{aligned} m_k &= \frac{\pi_0}{\pi_k} \quad (k \text{ even}), \\ m_k &= (r-1)\frac{\pi_0}{\pi_k} \quad (k \text{ odd}). \end{aligned}$$

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