

AN APPLICATION OF FUNCTIONAL ANALYSIS IN A PREDATOR - PREY SYSTEM

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ABSTRACT. In this paper I want to show some advantages of the Geometric Iteration to a possible study of a predator-prey system. The Geometric Iteration presents characteristics such as continuity, derivability and integrability which could allow to analyze when a predator can still reach it is prey even when the prey has a initial advantage.

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1. PRELIMINARIES

I can start assuming a predator-prey system whose motivation was the model established in [4]. We can suppose that two animals make the same number of jumps or strides per minute, but one of them makes bigger jumps than the other. Also, both animals are encouraged or discouraged. The lenght of the initial jump of the predator is b and its factor of encouragement is q , while the lenght of the initial jump of the prey is a and its factor of encouragement is p , but both of them have periods of run m and l respectively. We will imagine a planet in which the predators and the preys move by jumps of incremental or decremental longitude using the Geometric Iteration, which will be defined below. We could consider $a, b \in \mathbb{R}_+ = \mathbb{R}^+ \cup \{0\}, m, l \in \mathbb{Z}_+ = \mathbb{Z}^+ \cup \{0\}, p, q \in \mathbb{R}$, but only we will discuss some special cases. In some cases, we assume that the prey has k steps ahead of the predator as initial advantage; in other cases, it is said that both animals are persistent all times no matter if our prey has an initial advantage. We will work at the second case and we will give the following definition to determine one kind of displacement, velocity or aceleration of the creatures.

Definition 1.1. *Given $a \in \mathbb{R}_+, p \in \mathbb{R}, |a|+|p| \neq 0$, define the first geometric of p on a by $G_{(p,1)}(a) = a^p$. Recursively, for $n \geq 2$, define the n th geometric of p on a by*

$$(1) \quad G_{(p,n)}(a) = a^{p^n} G_{(p,n-1)}(a)$$

The next lemma give us another way to obtain recursively the n th geometric of p on a .

Lemma 1.2. $\forall n \geq 2, a \in \mathbb{R}_+, p \in \mathbb{R}$.

$$(2) \quad G_{(p,n)}(a) = a^p (G_{(p,n-1)}(a))^p$$

Proof. It is follow by induction on n . \square

Definition 1.3. Given $G_{(p,n)}(a)$ as in 1.1, let $m \in \mathbb{Z}^+$ define the m -geometric iteration of p, n on a by

$$(3) \quad G_{(p,n)}^m = G_{(p,n)} \circ G_{(p,n)}^{m-1}$$

where $G_{(p,n)}^0 = G_{(p,n)}$.

2. PROPERTIES OF THE GEOMETRIC ITERATION

In this section we establish some analytic and algebraic properties for Geometric Iteration

Lemma 2.1. Let be $a \in \mathbb{R}_+$, $p \in \mathbb{R}$, $l \in \mathbb{Z}_+$ be fixed, $|a|+|p| \neq 0$. Then $\forall n \in \mathbb{N}$, we have $G_{(1,n)}^l(a) = a^{n^{l+1}}$

$$(4) \quad G_{(p,n)}^l(a) = a^{\left(\frac{p(1-p^n)}{1-p}\right)^{l+1}}, p \neq 1.$$

Proof. Using 1.1 and induction on n . \square

Theorem 2.2. If $G_{(p,n)}^l(a)$ is as in 2.1, $|p| < 1$, then

$$(5) \quad G_{(p,n)}^l(a) \rightarrow a^{\left(\frac{p}{1-p}\right)^{l+1}} \text{ whenever } n \rightarrow \infty.$$

Proof. By 2.1, we can write $G_{(p,n)}^l(a) = a^{\left(\frac{p(1-p^n)}{1-p}\right)^{l+1}}$, now we can see clearly that $\lim_{n \rightarrow \infty} G_{(p,n)}^l(a) = a^{\left(\frac{p}{1-p}\right)^{l+1}}$. \square

Definition 2.3. $G_p^l(a) := a^{\left(\frac{p}{1-p}\right)^{l+1}}$ is called the convergence of the l -geometric iteration of p on a .

Theorem 2.4. $G_{(p,n)}^l$ and G_p^l are automorphisms defined onto \mathbb{R}_+ with the product.

Proof. Let be $H_1 := G_{(p,n)}^l$, $H_2 := G_p^l$, clearly $H_1(ab) = H_1(a)H_1(b)$, $H_2(ab) = H_2(a)H_2(b)$, $H_1(\mathbb{R}^+) = \mathbb{R}^+$, $H_2(\mathbb{R}^+) = \mathbb{R}^+$, $H_1(a) = H_1(b) \Leftrightarrow a = b$, $H_2(a) = H_2(b) \Leftrightarrow a = b$, finally $H_1^{-1}(a) = H_1(a^{-1})$ and $H_2^{-1}(a) = H_2(a^{-1})$. \square

Theorem 2.5. Let be $G_{(p,n)}^l$ and G_p^l as in 2.2, $K_1 = \{G_{(p,n)}^l : p \in \mathbb{R}, l \in \mathbb{Z}_+\}$, $K_2 = \{G_p^l : p \in \mathbb{R}, l \in \mathbb{Z}_+\}$, then (K_1, \circ) and (K_2, \circ) have the structure of an abelian group, where \circ is the composition of functions.

Proof. Take $e_1 = G_{(1,n)}^l \in K_1$, $e_2 = G_{\left(\frac{1}{2}, n\right)}^l \in K_2$, 2.1 and 2.3. \square

Now we will see the main theorem, which involve analytical and topological structures. I omit the proof because it is very long, but is completely made in [1].

Theorem 2.6. *Let $l \in \mathbb{Z}^+$, $p \in \mathbb{R}$, $\forall n$, $G_{(p,n)}^l : A \rightarrow B$, where $A, B \subset \mathbb{R}^+$ are compact sets, $G_{(p,n)}^l$, $\frac{dG_{(p,n)}^l(x)}{dx}$, $\int_0^x G_{(p,n)}^l(t)dt$, are sequences of homeomorphisms onto compact subsets. If $|p| < 1$ then, $G_{(p,n)}^l(x)$, $\frac{dG_{(p,n)}^l(x)}{dx}$, $\int_0^x G_{(p,n)}^l dt$ converge weakly, strongly and uniformly to $G_p^l(x)$, $\left(\frac{p}{1-p}\right)^l \frac{G_p^l(x)}{x}$, $\frac{(1-p)^l}{p^l+(1-p)^l} x G_p^l(x)$ respectively. Furthermore $G_p^l(x)$, $\left(\frac{p}{1-p}\right)^l \frac{G_p^l(x)}{x}$, $\frac{(1-p)^l}{p^l+(1-p)^l} x G_p^l(x)$ are homeomorphisms onto compact subsets. These homeomorphisms are uniformly and completely continuous.*

This theorem show that $G_{(p,n)}^l$ could be defined in zero, but is not continuous in zero.

Corollary 2.7. *If $-1 < p < 1$ then the following statements hold*

- (i) $\lim_{l \rightarrow \infty} G_p^l(x) = 1$, $-1 < p < \frac{1}{2}$,
- (ii) $\lim_{l \rightarrow \infty} G_p^l(x) = x$, $p = \frac{1}{2}$ and
- (iii) $\lim_{l \rightarrow \infty} G_p^l(x) = \infty$, $p > \frac{1}{2}$.

3. ONE APPLICATION OF THE GEOMETRIC ITERATION

Thanks to Theorem 2.6 is possible to apply the geometric iteration in a model predator-prey where the *catchup* between predator and prey is studied instead of the populational behavior of the species. Suppose that we have two animals that take the same number of jumps per minute and their displacement is given by the Geometric Iteration. If the prey jump has lenght a , it is factor of encouregment is given by p , and those of the predator have lenght b , it is factor of encouregment is gived by q , but both of them have periods of run l and m respectively. Let us assume that the prey starts k steps ahead of the predator. After n steps the distance between the two is

$$(6) \quad G_{(p,n+k)}^l(a) - G_{(q,n)}^m(b).$$

Let us suppose that both animals run all the time, now the distance between the two is

$$(7) \quad G_p^l(a) - G_q^m(b).$$

In both cases, we have to study the necessary conditions for the prey to run away from the predator. Derivation and integration are necessities to determine their speeds and other applications.

Note that if $a = 3, b = 2, p = \frac{1}{4}, q = \frac{1}{5}, l = m = 0$, on the graphs, we observe that the prey has run away from it is predator, because for all $n \in \mathbb{N}$, $G_p^l(a) - G_q^m(b) > 0$. It means that the prey and predator starts from zero without advantage. No matter if the predator and prey run all the time, the predator will never catchup our prey, because the predator was unattentive when the hunt started.

4. REMARK

We can observe that if $p = \frac{\alpha}{\beta}$, whenever $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{Z}^*$, then $G_{(p,n)}(f(x))$ converges uniformly to $(f(x))^{\frac{\alpha}{\beta}}$, but if $\alpha = \beta - 1$ then $G_{(p,n)}(f(x))$ converges uniformly to $(f(x))^\alpha$. The functions $G_{(p,n)}(f(x))$ stick onto $(f(x))^{\frac{p}{1-p}}$, peculiarly

$G_{(\frac{1}{2},n)}(f(x))$ stick on $f(x)$. The fixed points of $G_{(p,n)}(f(x))$ are obtained when $(f(x))^{\frac{p}{1-p}} = x$. We can see that if $f(x) = x$, then $G_{(p,n)}(f(x))$ has as fixed points to 0 and 1. For $G_{(f(x),n)}(a) = a^{f(x)\frac{1-(f(x))^n}{1-f(x)}}$, whenever $|f(x)| < 1$ we can see that $G_{f(x)}(a) = a^{\frac{f(x)}{1-f(x)}}$. In the previous theorems we defined mappings on \mathbb{R}_+ , but there is no problem if are they defined on \mathbb{C} and take $a = re^{i\theta}$. We can apply the Geometric iteration in other fields, for example financial problems and market behavior. There are some open questions such as:

The differential equation $\frac{dG}{dt} = a^{p^t}G$, with initial condition $G_1 = a^p$ has an unique solution. What is the solution?

The family of curves $G = ((1-p)a^p t + c)^{\frac{1}{1-p}}$, satisfies $\frac{dG}{dt} = a^p G^p$. In what points $G = ((1-p)a^p t + c)^{\frac{1}{1-p}}$ is solution to $\frac{dG}{dt} = a^{p^t}G$?

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REFERENCES

- [1] P. B. Acosta-Humanez. *Homomorphisms in Discrete Dynamical Systems*, M.A Dissertation, Sergio Arboleda University, Bogota 2003.
- [2] P. B. Acosta-Humanez. La exponencial iterada en un sistema depredador-presa, *Memorias Vialtopo 22*, Pontificia Universidad Javeriana, Bogota 2002.
- [3] G. Bachman and L. Narici, *Functional Analysis*, Dover, New York, 1966.
- [4] R. L. Devaney, K. Josic, M. Moreno Rocha and others. Playing catchup with iterated exponentials. Department of Mathematics and Statistics, Boston University, *Bob Devaney Home Page*, 2:1–6, 2002.
- [5] R. L. Devaney and M. Moreno Rocha, A semilinear Model for exponential Dynamics and Topology. *Topology Proceedings*, 26 (2001).