

Strict abnormal extremals in nonholonomic and kinematic control systems

MARÍA BARBERO-LIÑÁN*, MIGUEL C. MUÑOZ-LECANDA†

Departamento de Matemática Aplicada IV

Edificio C-3, Campus Norte UPC. C/ Jordi Girona 1. E-08034 Barcelona. Spain

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Abstract

In optimal control problems, there exist different kinds of extremals, that is, curves candidates to be solution: abnormal, normal and strictly abnormal. The key point for this classification is how those extremals depend on the cost function. We focus on control systems such as nonholonomic control mechanical systems and the associated kinematic systems as long as they are equivalent.

With all this in mind, first we study conditions to relate an optimal control problem for the mechanical system with another one for the associated kinematic system. Then, Pontryagin's Maximum Principle will be used to connect the abnormal extremals of both optimal control problems.

An example is given to glimpse what the abnormal solutions for kinematic systems become when they are considered as extremals to the optimal control problem for the corresponding nonholonomic mechanical systems.

Key words: nonholonomic control mechanical systems, kinematic control systems, Pontryagin's Maximum Principle, extremals, abnormality.

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1 Introduction

The problem of shortest paths in subRiemannian geometry has strict abnormal minimizers [11, 12]. That is why the question of the existence of strict abnormal minimizers for optimal control problems associated to mechanical systems is posed here.

**e-mail:* mbarbero@ma4.upc.edu

†*e-mail:* matmcm1@ma4.upc.edu

It will be useful to take advantage of the strict abnormal minimizers in subRiemannian geometry to characterize, at least, the abnormal extremals for some mechanical control systems. In particular, we focus on the nonholonomic ones. They are equivalent to kinematic systems under some assumptions related to the constraint distribution and the distribution spanned by the input vector fields, see for instance [3, 4, 10]. The controls in those mechanical systems are understood as the accelerations, while in the kinematic system the controls are the velocities. The control system in subRiemannian geometry is control-linear as the kinematic systems. From here we connect with the nonholonomic control mechanical system that is the object of study.

Once the equivalence between the mechanical system and the kinematic system is established, we wonder if it is feasible to get a similar connection between optimal control problems associated to the two control systems. If so, the result will be used to try to characterize strict abnormal extremals of optimal control problems for the nonholonomic mechanical systems.

Pontryagin's Maximum Principle [1, 5, 6, 7, 9, 13] defines the different kinds of extremals in optimal control theory: normal, abnormal and strict abnormal. This principle gives necessary conditions to find solutions to optimal control problems. Any curve satisfying these necessary conditions is called extremal. The extremals are abnormal when they only depend on the geometry of the system, in other words, the cost function does not play any role. On the other hand, the cost function takes part in the study of the normal extremals. In order to get a better idea, it is said that Pontryagin's Maximum Principle associates each solution of the optimal control problem with a lift on the cotangent bundle, but this lift of the solution is not necessarily unique. The non-uniqueness makes possible the existence of extremals being normal and abnormal at the same time. Then, a strict abnormal extremal is one that is only abnormal, that is, it only admits one kind of lift.

Moreover, the approach to control mechanical system explained here enlightens how to construct the extended system used in [13] for mechanical systems. In contrast with the work in [5] our modus operandi preserves the second order condition of the extended system, condition also satisfied for the non-extended system.

The paper is organized as follows: In §2 the different definitions and results associated with the optimal control problems for nonholonomic and kinematic systems are described, in particular, the possible equivalence between both problems. After explaining Pontryagin's Maximum Principle in §3, the hamiltonian problems for both control problems are stated in §4 to be able to apply the Principle. It is especially important the definition of extremals for the mechanical case that gives a justification of the study made in [5]. In §4.1 it is showed how to use the strict abnormal minimizers in subRiemannian geometry to characterize the extremals for the corresponding optimal control problem with nonholonomic mechanical system. In the example a strict abnormal minimizer for the time optimal control problem for the mechanical system is obtained.

In the sequel, all the manifolds are real, second countable and \mathcal{C}^∞ . The maps are assumed to be \mathcal{C}^∞ . Sum over repeated indices is understood.

2 Optimal control problem with nonholonomic mechanical systems versus kinematic systems

2.1 Nonholonomic mechanical systems with control

Let (Q, g) be a Riemannian manifold of dimension n and ∇ be the Levi-Civita connection associated to the Riemannian metric g . Let TQ be the tangent bundle with the natural projection $\tau_Q: TQ \rightarrow Q$. Consider $D \subset TQ$, a nonintegrable distribution in Q with rank m and spanned by the *input control vector fields* $\{Y_1, \dots, Y_m\}$.

Let D^\perp be the orthogonal distribution to D according to the metric g . Assume that D^\perp is spanned by $\{Z_1, \dots, Z_{n-m}\}$, a family of vector fields on Q .

It is also possible to consider an external vector field $F \in \mathfrak{X}(Q)$, the set of smooth vector fields on Q . Then, a nonholonomic mechanical system with control is given by $\Sigma = (Q, g, F, D)$. A differentiable curve $\gamma: I \rightarrow Q$ is a *solution of Σ* for certain values of the *control functions* $u^i \in \mathcal{C}^\infty(TQ)$ if it satisfies the conditions

$$\begin{aligned} \nabla_{\dot{\gamma}} \dot{\gamma} &= F \circ \gamma + \sum_{r=1}^{n-m} \lambda^r Z_r \circ \gamma + \sum_{s=1}^m u^s Y_s \circ \gamma, \\ \dot{\gamma} &\in D. \end{aligned} \quad (2.1)$$

where $u: TQ \rightarrow U \subset \mathbb{R}^m$ being U an open set. The Lagrange multipliers λ^j are determined by the condition $\dot{\gamma} \in D$.

The dynamical equations of mechanical systems are second order differential equations in the configuration manifold Q , so they may be rewritten as first order differential equations in TQ using the following vector field along the projection $\pi: TQ \times U \rightarrow TQ$

$$Y = Z_g + F^V + \sum_{r=1}^{n-m} \lambda^r Z_r^V + \sum_{s=1}^m u^s Y_s^V \quad (2.2)$$

where Z_g is the geodesic spray associated with g and Y_s^V is the vertical lift of Y_s , analogously for F^V and Z_r^V . The vector field Y satisfies the second order condition.

On the other hand, a differentiable curve $\gamma: I \rightarrow Q$ is a *solution of the kinematic system associated to (2.1)*, if there exist $w^i \in \mathcal{C}^\infty(\mathbb{R})$, $i = 1, \dots, m$, such that

$$\dot{\gamma}(t) = \sum_{s=1}^m w^s(t) Y_s(\gamma(t)), \quad (2.3)$$

that is, γ is an integral curve of the vector field $X = \sum_{s=1}^m w^s Y_s$ with $w: \mathbb{R} \rightarrow V \subset \mathbb{R}^m$ being V an open set.

The systems (2.1) and (2.3) are *equivalent* if and only if every solution of (2.1) is also a solution of (2.3) and in the other way round. Notice that, in spite of the equivalence of the systems, a solution to both systems could have different control functions, but the curve on Q is exactly the same.

Remark 2.1. Here, we consider the nonholonomic control system called *fully actuated* because the constraint distribution is exactly the distribution given by the input control vector fields. If the distribution of the input vector fields has rank strictly less than the rank of the constraint distribution, then we have underactuated systems. In this case (2.1) and (2.3) are not equivalent any more, but weak equivalent. See [2, 3, 4, 10] for more details.

Theorem 2.2. [3, 10] *Every fully actuated nonholonomic control system Σ is equivalent to the associated kinematic system.*

2.2 Optimal control

From a control system we define an optimal control problem adding a cost function whose integral must be minimized over solutions of the control system. First, we consider an optimal control problem with a nonholonomic mechanical system. The equivalence of this system with a kinematic system, that is, a control-linear system is known by Theorem 2.2. It should be useful to find a cost function for the kinematic system such that some connection between the optimal solutions to both problems may be established.

Let us point out the importance of this relation between those optimal control problems. To deal with a kinematic system is by far easier than to deal with a mechanical control system, which is either control-affine or nonlinear. Moreover, in [11] the strict abnormal minimizers have been described for the problem of shortest-paths in subRiemannian geometry. Thus it might be expected to characterize abnormal extremals for mechanical control systems using the well-known abnormal minimizers in subRiemannian geometry. The control system in subRiemannian geometry is control-linear, so it can be understood as a kinematic system that comes from a nonholonomic mechanical control system.

Let us consider a cost function $\mathcal{F}: TQ \times U \rightarrow \mathbb{R}$ for the mechanical control system. The optimal control problem for (2.2) is stated as follows.

Problem 2.3. *Given $x_0, x_f \in Q$, find $(\gamma, u): I \rightarrow Q \times U$ such that*

1. γ satisfies the end-point conditions on Q , i.e. $\gamma(t_0) = x_0, \gamma(t_f) = x_f$;
2. $\dot{\gamma}$ is an integral curve of Y , i.e. $\ddot{\gamma}(t) = Y(\dot{\gamma}(t), u(t))$;
3. $(\dot{\gamma}, u)$ gives the minimum of $\int_I \mathcal{F}(\dot{\gamma}(t), u(t)) dt$ among all the curves satisfying 1 and 2.

In optimal control theory, it is common to consider the functional to be minimized as a new coordinate of the system. In this way, all the elements in the optimal control problem are included in a control system, usually called the *extended system* [7, 13]. Nevertheless, the minimization of the functional must be included to the extended system, what turns out to be the minimization of the new coordinate.

In the case of mechanical control systems two new coordinates are added in order to maintain the second order condition of the vector field (2.2). Let $\widehat{Q} = \mathbb{R} \times Q$, then the cost function is considered as a vector field along the projection $\widehat{\pi}: T\widehat{Q} \times U \rightarrow T\widehat{Q}$ with local expression $\mathcal{F}\partial/\partial x^0$. Then (2.1) becomes

$$\widehat{\nabla}_{\dot{\widehat{\gamma}}} \dot{\widehat{\gamma}} = F \circ \widehat{\gamma} + \sum_{r=1}^{n-m} \lambda^r Z_r \circ \widehat{\gamma} + \sum_{s=1}^m u^s Y_s \circ \widehat{\gamma} + \mathcal{F} \circ (\dot{\widehat{\gamma}}, u) \left. \frac{\partial}{\partial x^0} \right|_{\dot{\widehat{\gamma}}}$$

where $\widehat{\gamma}: I \rightarrow \widehat{Q}$ is a differentiable curve and the Levi-Civita connection is extended to \widehat{Q} considering all the new Christoffel symbols equal to zero, and $\pi_2 \circ \dot{\widehat{\gamma}} = \dot{\gamma} \in D$ with the projection $\pi_2: T\widehat{Q} = T\mathbb{R} \times TQ \rightarrow TQ$.

The above second order differential equation admits a first order differential equation given by the vector field

$$\widehat{Y} = v^0 \frac{\partial}{\partial x^0} + \mathcal{F} \frac{\partial}{\partial v^0} + Z_g + F^V + \sum_{r=1}^{n-m} \lambda^r Z_r^V + \sum_{s=1}^m u^s Y_s^V \quad (2.4)$$

along the projection $\widehat{\pi}: T\widehat{Q} \times U \rightarrow T\widehat{Q}$. The differential equations added to (2.2) are

$$\begin{aligned} \dot{x}^0 &= v^0 \\ \dot{v}^0 &= \mathcal{F} \end{aligned}$$

taking into account the extension of the Levi-Civita connection to \widehat{Q} . The value that must be minimized in the optimal control problem is $v^0 = \int_I \mathcal{F} dt$.

Now consider the kinematic system (2.3) with a cost function $\mathcal{G}: Q \times V \rightarrow \mathbb{R}$ such that the problem to be solved is

Problem 2.4. Given $x_0, x_f \in Q$, find $(\gamma, w): I \rightarrow Q \times V$ such that

1. γ satisfies the end-point conditions on Q , i.e. $\gamma(t_0) = x_0$, $\gamma(t_f) = x_f$;
2. γ is an integral curve of $X = \sum_{s=1}^m w^s Y_s$, i.e. $\dot{\gamma}(t) = X(\gamma(t), w(t))$;
3. (γ, w) minimizes $\int_I \mathcal{G}(\gamma(t), w(t)) dt$ among all the curves satisfying 1 and 2.

Remark 2.5. The problems 2.3 and 2.4 are called *fixed time optimal control problems* because the domain of definition of the curves is given. However, the *free time optimal control problems* may also be defined. They consist of having another unknown given by the domain of the definition, that must also be found.

As before, let us extend the control system to the manifold $\widehat{Q} = \mathbb{R} \times Q$ such that we look for integral curves of the vector field

$$\widehat{X} = \mathcal{G} \frac{\partial}{\partial x^0} + \sum_{s=1}^m w^s Y_s \quad (2.5)$$

defined along $\pi_1: \widehat{Q} \times V \rightarrow \widehat{Q}$. The differential equation added to (2.3) is

$$\dot{x}^0 = \mathcal{G}$$

and the value to be minimized is $x^0 = \int_I \mathcal{G} dt$.

By Theorem 2.2 we know that (2.1) and (2.3) are equivalent. We are interested in establishing a connection between

$$\left. \begin{array}{l} \dot{x}^0 = v^0 \\ \dot{v}^0 = \mathcal{F} \end{array} \right\}$$

that come from (2.4) and

$$\dot{x}^0 = \mathcal{G}$$

that comes from (2.5).

In some sense, $\mathcal{G} = v^0 = \int \mathcal{F}$, but this equality must be well understood. Observe that \mathcal{G} is a function on $\widehat{Q} \times V$, meanwhile \mathcal{F} is a function on $T\widehat{Q} \times U$. Hence, some simplifications must be considered. Before proceeding with the exact interpretation of $\mathcal{G} = \int \mathcal{F}$, note we also have to check what happens with the minimization conditions when $\mathcal{G} = \int \mathcal{F}$, that is, if the curves minimizing $\int \mathcal{G}$ determine the curves minimizing $\int \mathcal{F}$ and/or in the other way round.

Proposition 2.6. *Let $\mathcal{G}: I \times Q \rightarrow \mathbb{R}$. If $(\dot{\gamma}, u)$ is an optimal curve of a nonholonomic mechanical control system with cost function $\mathcal{F} = \partial \mathcal{G} / \partial t + v^i \partial \mathcal{G} / \partial x^i = \widehat{d}\mathcal{G}: I \times TQ \rightarrow \mathbb{R}$, then there exists $w: I \rightarrow V$ such that (γ, w) is an optimal curve of the kinematic system with cost function \mathcal{G} .*

Proof. If $(\dot{\gamma}, u): I \rightarrow TQ \times U$ is an integral curve of (2.2), then by Theorem 2.2 there exist $w: I \rightarrow V$ such that (γ, w) is an integral curve of (2.3). Thus, it only remains to prove that the optimality condition for \mathcal{F} implies the optimality condition for \mathcal{G} .

As $(\dot{\gamma}, u)$ minimizes $\int \mathcal{F}$, then for any other integral curve $(\tilde{\gamma}, \tilde{u})$ of the vector field (2.2) satisfying the end-point conditions we have

$$\begin{aligned} \mathcal{G}(t, \gamma(t)) - \mathcal{G}(a, \gamma(a)) &= \int_a^t \widehat{d}\mathcal{G}(s, \gamma(s)) = \int_a^t \mathcal{F}(s, \dot{\gamma}(s)) ds < \\ &< \int_a^t \mathcal{F}(s, \dot{\tilde{\gamma}}(s)) ds = \int_a^t \widehat{d}\mathcal{G}(s, \tilde{\gamma}(s)) = \mathcal{G}(t, \tilde{\gamma}(t)) - \mathcal{G}(a, \tilde{\gamma}(a)). \end{aligned}$$

As γ and $\tilde{\gamma}$ satisfy the end-point conditions and none of the cost functions depends on the controls, we have

$$\mathcal{G}(t, \gamma(t)) < \mathcal{G}(t, \tilde{\gamma}(t)),$$

then $\int_I \mathcal{G}(t, \gamma(t)) dt < \int_I \mathcal{G}(t, \tilde{\gamma}(t)) dt$ by the monotony property of the integral. \square

The result just proved holds provided that the cost function for the nonholonomic mechanical system is the total derivative of the cost function for the kinematic system. Observe that both cost functions are independent of the controls.

Remark 2.7. Necessary conditions for a curve to be an optimal solution for a non-holonomic mechanical control system is to be an optimal solution to the optimal control problem for the associated kinematic system.

Remark 2.8. The inverse implication is not necessarily true. If (γ, w) is an optimal curve for the kinematic system, then for any other integral curve $(\tilde{\gamma}, \tilde{w})$ of the kinematic system

$$\int dt \int \mathcal{F}(t, \dot{\gamma}(t)) dt = \int_I \mathcal{G}(t, \gamma(t)) dt < \int_I \mathcal{G}(t, \tilde{\gamma}(t)) dt = \int dt \int \mathcal{F}(t, \dot{\tilde{\gamma}}(t)) dt .$$

The monotony property of the integral is satisfied only in one direction. We should think of conditions such that

$$\left\langle \int_I f < \int_I g \Rightarrow f < g, \quad \text{almost everywhere (a.e.)} \right\rangle$$

In general, we cannot expect better results than a.e., hence we will have optimal curves in a weak sense. For instance, if f and g are both positive or both negative, the implication is satisfied. Moreover, if f and g are continuous functions, then the inequality is satisfied everywhere.

Proposition 2.9. *The time optimal control problem for a nonholonomic mechanical control system is equivalent to the optimal control problem for the associated kinematic systems with $\mathcal{G} = t$.*

Proof. The direct implication is already proved in Proposition 2.6. Let us prove now that the optimal curves for kinematic systems with $\mathcal{G} = t$ are optimal curves for the time optimal problem with nonholonomic mechanical control systems.

If (γ, w) is a minimizer of $\int t dt = t^2/2$ satisfying the kinematic system, then by Theorem 2.2 there exist $u: I \rightarrow U$ such that $(\dot{\gamma}, u)$ is an integral curve of the non-holonomic mechanical control system. For any other integral curve of the kinematic system with the same end-point conditions as γ ,

$$t^2/2 < \tilde{t}^2/2.$$

As t, \tilde{t} are positive numbers, $t < \tilde{t}$. That is $(\dot{\gamma}, u)$ is a minimizer of the time optimal control problem of the statement because of the equivalence of integral curves of (2.2) and (2.3) given by Theorem 2.2 and because of the nature of the cost function. The cost function \mathcal{G} is positive, so we are in one of the cases where the reverse implication of the monotony property of the integral is satisfied. \square

The optimal control problems considered in Proposition 2.9 are free time.

Remark 2.10. Indeed, it is feasible to consider the time-optimal problem for both control systems and they will be equivalent because the time is positive. Thus, to minimize the time or to minimize the time square is exactly the same. Moreover, the curve on the configuration manifolds are related to the same curve on Q since the equations defined by (2.2) and (2.3) also appear in the extended systems (2.4) and (2.5), respectively.

The following corollary links with the fact that some optimal control problems can be understood as time optimal control problems, as for instance happens in the problem of shortest paths in subRiemannian geometry [11].

Corollary 2.11. *For a nonholonomic mechanical control system, an optimal control problem equivalent to a time optimal control problem admits an equivalent time optimal control problem for the associated kinematic system.*

The proof of this corollary is obtained from Proposition 2.9 and Remark 2.10.

3 Pontryagin's Maximum Principle

Pontryagin's Maximum Principle has been widely discussed and used in Optimal Control Theory since the second half of the 20th century [5, 6, 7, 9, 13].

Let Q be a smooth n -dimensional manifold and $U \subset \mathbb{R}^m$ a bounded subset. Let X be a vector field along the projection $\pi: Q \times U \rightarrow Q$. If (x^i) are local coordinates on Q , the local expression of the vector field is $X = f^i \partial / \partial x^i$ where f^i are functions defined on an open set of $Q \times U$. Given $\mathcal{F}: Q \times U \rightarrow \mathbb{R}$, consider the functional

$$\mathcal{S}[\gamma, u] = \int_I \mathcal{F}(\gamma, u) dt$$

defined on curves (γ, u) with a compact interval as domain.

To be able to state the Maximum Principle we need a hamiltonian formalism. Now, we define the equivalent extended optimal control problem on $\widehat{Q} = \mathbb{R} \times Q$ with the projection $\widehat{\pi}: \widehat{Q} \times U \rightarrow \widehat{Q}$.

Let \widehat{X} be the vector field along the projection $\widehat{\pi}: \widehat{Q} \times U \rightarrow \widehat{Q}$ given by:

$$\widehat{X}(x^0, x, u) = \mathcal{F}(x, u) \partial / \partial x^0|_{(x^0, x, u)} + X(x, u),$$

where x^0 is the natural coordinate on \mathbb{R} .

Problem 3.1. (Extended Optimal Control Problem, $\widehat{\text{OCP}}$) Given $Q, U, X, \mathcal{F}, I, x_0, x_f$. Find $(\widehat{\gamma}, u): I \rightarrow \widehat{Q} \times U$ such that

1. $\widehat{\gamma}$ satisfies the end-point conditions: $\widehat{\gamma}(t_0) = (0, x_0), \widehat{\gamma}(t_f) = x_f$;
2. $\dot{\widehat{\gamma}}(t) = \widehat{X}(\widehat{\gamma}(t), u(t))$ almost everywhere $t \in I$;

3. $\gamma^0(t_f)$ is minimum over all curves satisfying 1 and 2.

The key point for considering the extended optimal control problem is that the functional to be minimized is the coordinate x^0 in \mathbb{R} . This is really useful in the proof of Pontryagin's Maximum Principle and in a first characterization of the abnormal extremals since the direction of decreasing of the functional is easily identified.

From \widehat{OCP} , we state a hamiltonian problem that will lead to Pontryagin's Maximum Principle.

Let $T^*\widehat{Q}$ be the cotangent bundle with its natural symplectic structure denoted by ω . For each $u \in U$, $H^u: T^*\widehat{Q} \rightarrow \mathbb{R}$ is the hamiltonian function defined by

$$H^u(\widehat{x}, \widehat{p}) = H(\widehat{x}, \widehat{p}, u) = \langle \widehat{p}, \widehat{X}(\widehat{x}, u) \rangle = p_0 \mathcal{F}(x, u) + \sum_{i=1}^m p_i f^i(x, u).$$

The tuple $(T^*\widehat{Q}, \omega, H^u)$ is a hamiltonian system. The hamiltonian vector field associated with H is a vector field along the projection $\widehat{\pi}_1: T^*\widehat{Q} \times U \rightarrow T^*\widehat{Q}$ given by \widehat{X}^{T^*} , the cotangent lift of \widehat{X} [5].

Problem 3.2. (Hamiltonian Problem, HP)

Given \widehat{OCP} , find $(\widehat{\sigma}, u): I \rightarrow T^*\widehat{Q} \times U$ such that

1. if $\widehat{\gamma} = \pi_{\widehat{Q}} \circ \widehat{\sigma}$, $\gamma = \widehat{\pi}_2 \circ \widehat{\gamma}$ where $\widehat{\pi}_2: \widehat{Q} \rightarrow Q$, then $\widehat{\gamma}(t_0) = (0, x_0)$ and $\gamma(t_f) = x_f$;
2. $\dot{\widehat{\sigma}}(t) = \widehat{X}^{T^*}(\widehat{\sigma}(t), u(t))$ almost everywhere $t \in I$.

Locally the curve $(\widehat{\sigma}, u)$ satisfies Hamilton's equations of the system $(T^*\widehat{Q}, \omega, H^u)$,

$$\begin{aligned} \dot{x}^0 &= \frac{\partial H^u}{\partial p_0} = \mathcal{F} & \dot{p}_0 &= -\frac{\partial H^u}{\partial x^0} = 0 \\ \dot{x}^i &= \frac{\partial H^u}{\partial p_i} = f^i & \dot{p}_i &= -\frac{\partial H^u}{\partial x^i} = -p_0 \frac{\partial \mathcal{F}}{\partial x^i} - p_j \frac{\partial f^j}{\partial x^i}. \end{aligned}$$

Note that there is no initial condition for $\widehat{p} = (p_0, p_1, \dots, p_n)$ in HP , hence it is not a Cauchy initial value problem. This initial condition is chosen so that the necessary conditions of Pontryagin's Maximum Principle are satisfied, in fact, the proof of Theorem 3.3 consists of finding a suitable initial condition [1, 7, 9, 13].

Theorem 3.3. (Pontryagin's Maximum Principle, PMP)

Let $(\widehat{\gamma}, u): I \rightarrow \widehat{Q} \times U$ be a solution of the extended optimal control problem. Then there exists $(\widehat{\sigma}, u): I \rightarrow T^*\widehat{Q} \times U$, with fiber momenta coordinates $\widehat{\lambda}(t) \in T_{\widehat{\gamma}(t)}^* \widehat{Q}$ such that:

1. $(\widehat{\sigma}, u)$ is a solution of the Hamiltonian Problem;

2. $\widehat{\gamma} = \pi_{\widehat{Q}} \circ \widehat{\sigma}$;
3. (a) $H(\widehat{\sigma}(t), u(t)) = \max_{\widetilde{u} \in U} H(\widehat{\sigma}(t), \widetilde{u})$ almost everywhere;
 (b) $\max_{\widetilde{u} \in U} H(\widehat{\sigma}(t), \widetilde{u}) = \text{constant}$ everywhere;
 (c) $(\lambda_0, \lambda(t)) \neq 0$ for each $t \in I$.

If the domain of definition of the curves is not given, that is, free optimal control problems, see Remark 2.5, then Pontryagin's Maximum Principle provides us the same necessary conditions as Theorem 3.3, but it also guarantees that the maximum of the Hamiltonian is zero everywhere.

Remark 3.4. As a consequence of conditions (3.a) and (3.b) the Hamiltonian along the optimal curve with its corresponding momenta is constant almost everywhere $t \in I$, and in particular it is zero in free time optimal control problems. This will be used in §4.1.

As we said previously, the proof of Theorem 3.3 consists of choosing the initial condition for the fibers of the cotangent bundle in a suitable way. In fact, it is chosen such that

$$\begin{aligned} \langle \widehat{\sigma}(t_f), \widehat{v}(t_f) \rangle &\leq 0 \\ \langle \widehat{\sigma}(t_f), (-1, \mathbf{0}) \rangle &\geq 0 \end{aligned} \quad (3.6)$$

where $\widehat{v}(t_f)$ are the perturbation vectors given by

$$\widehat{v}(t_f) = \widehat{X}(\widehat{\gamma}(t_f), u_{t_f}) - \widehat{X}(\widehat{\gamma}(t_f), u(t_f)) \quad (3.7)$$

obtained from a determined variation of the control with value $u_{t_f} \in U$, see [1, 7, 9, 13], and $(-1, \mathbf{0})$ is the direction of decreasing in the functional. Both vectors are in $T_{\widehat{\gamma}(t_f)}\widehat{Q}$. Note that the initial condition is, indeed, final since it is taken at final time.

Observe that Maximum Principle guarantees the existence of a covector along the optimal curve, but it does not say anything about the uniqueness of the covector. Actually, this covector may not be unique. Depending on the covector we associate with the optimal curves, different kinds of curves are defined.

Definition 3.5. A curve $(\widehat{\gamma}, u): I \rightarrow \widehat{Q} \times U$ for \widehat{OCP} is

1. an **extremal** if there exist $\widehat{\sigma}: I \rightarrow T^*\widehat{Q}$ such that $\widehat{\gamma} = \pi_{\widehat{Q}} \circ \widehat{\sigma}$ and $(\widehat{\sigma}, u)$ satisfies the necessary conditions of PMP;
2. a **normal extremal** if it is an extremal with $\lambda_0 = -1$;
3. an **abnormal extremal** if it is an extremal with $\lambda_0 = 0$;
4. a **strictly abnormal extremal** if it is not a normal extremal, but it is abnormal.

For the abnormal extremals, $\lambda_0 = 0$, the cost function disappears from the hamiltonian function. Then, it is said that the abnormal extremals only depend on the geometry of the control system. In contrast with the normal and strict abnormal extremals where the cost function plays a role. In the case of strict abnormality, the cost function is used to prove that the extremal is not normal.

4 Hamiltonian problems for nonholonomic mechanical systems versus kinematic systems

In order to make profit of the optimal control problems defined in §2.2, let us associate them with a hamiltonian problem in the sense of Pontryagin's Maximum Principle given in §3.

For the extended mechanical system \widehat{Y} given in (2.4) we have the hamiltonian function $H_m: T^*T\widehat{Q} \times U \rightarrow \mathbb{R}$ defined by

$$(\widehat{\Lambda}, u) \longmapsto \langle \widehat{\Lambda}, v^0 \frac{\partial}{\partial x^0} + \mathcal{F} \frac{\partial}{\partial v^0} + Z_g + F^V + \sum_{r=1}^{n-m} \lambda^r Z_r^V + \sum_{s=1}^m u^s Y_s^V \rangle .$$

The Lagrange multipliers λ^j are fixed because they are chosen in such a way that the part of the geodesic spray that is not in the distribution D is deleted. Another way to consider the Lagrange multipliers is modifying the connection, see [8].

For simplicity, we consider the system with null connection and without external forces. Then the Lagrange multipliers are zero and the local expression of the hamiltonian function is

$$H_m = p_0 v^0 + q_0 \mathcal{F} + p_i v^i + \sum_{s=1}^m q_i u^s Y_s^i ,$$

with Hamilton's equations

$$\begin{aligned} \dot{x}^0 &= v^0 & \dot{p}_0 &= 0 \\ \dot{x}^i &= v^i & \dot{p}_i &= -q_0 \frac{\partial \mathcal{F}}{\partial x^i} - q_j u^s \frac{\partial Y_s^j}{\partial x^i} \\ \dot{v}^0 &= \mathcal{F} & \dot{q}_0 &= -p_0 \\ \dot{v}^i &= u^s Y_s^i & \dot{q}_i &= -p_i \end{aligned} \quad (4.8)$$

where p_i are the momenta of the states and q_i are the corresponding momenta to the velocities.

Observe that the Hamiltonian is autonomous. Pontryagin's Maximum Principle for this problem tells us that the elementary perturbation vector at time t for $u_1 \in U$ along an optimal curve is given by $\widehat{Y}(\widehat{\gamma}(t), u_1) - \widehat{Y}(\widehat{\gamma}(t), u(t))$, see (3.7),

$$\widehat{v}_m(t) = \sum_{i=l}^m (u_1^s - u^s(t)) Y_l^V + (\mathcal{F}(\widehat{\gamma}(t), u_1) - \mathcal{F}(\widehat{\gamma}(t), u(t))) \frac{\partial}{\partial v^0} \Big|_{\widehat{\gamma}(t)} . \quad (4.9)$$

The covector $\widehat{\Lambda}$ associated to the optimal curve through Pontryagin's Maximum Principle satisfies a separating condition analogous to (3.6)

$$\begin{aligned}\langle \widehat{\Lambda}(t), \widehat{v}_m(t) \rangle &= \langle \widehat{q}(t), \widehat{v}_m(t) \rangle \leq 0 \\ \langle \widehat{\Lambda}(t), (0, \mathbf{0}, -1, \mathbf{0}) \rangle &= -q_0(t) \geq 0.\end{aligned}$$

The vectors $\widehat{v}_m(t)$ and $(0, \mathbf{0}, -1, \mathbf{0})$ are in $T_{\widehat{\gamma}(t)}T\widehat{Q}$. Here we do not use the vector $(-1, \mathbf{0})$, but $(0, \mathbf{0}, -1, \mathbf{0})$, the direction of decreasing in the functional $\int_I \mathcal{F}$. Remember that the value to be minimized is v^0 .

An analogous separating condition must be satisfied for the vector $(-1, \mathbf{0}, -1, \mathbf{0})$ in order not to contradict the hypothesis of optimality in Theorem 3.3, see [1, 7, 9, 13] for the details of that contradiction. But if $(-1, \mathbf{0}, 0, \mathbf{0})$, the direction of decreasing in x^0 , is in the same half-space as the perturbation vectors, we do not necessarily arrive at a contradiction because, in general, a decreasing in x^0 does not imply a decreasing in v^0 .

Thus in the mechanical case the momenta must separate all the perturbation vectors from the vectors $(0, \mathbf{0}, -1, \mathbf{0})$ and $(-1, \mathbf{0}, -1, \mathbf{0})$, what implies the nonpositiveness of q_0 . Taking into account Hamilton's equations (4.8), p_0 is constant and normalizing can be consider to be 0, -1 or 1 , then the different possibilities for the momenta are:

1. $p_0 = 0$ and $q_0 = 0$. Here the cost function does not take part in the computations. Note that in this case $(-1, \mathbf{0}, 0, \mathbf{0})$ is in the separating hyperplane defined by the kernel of the momenta.
2. $p_0 = 0$ and $q_0 = -1$. Then the cost function appears in the computations. As in previous item, $(-1, \mathbf{0}, 0, \mathbf{0})$ is in the separating hyperplane.
3. $p_0 = -1$ and $q_0 = t + A$. The separating conditions will be satisfied depending on the value of the final time and the constant A . It is necessary that $A < 0$ and $t_f \leq -A$. In this case, $(-1, \mathbf{0}, 0, \mathbf{0})$ is also separated from the perturbation vectors.
4. $p_0 = 1$ and $q_0 = -t + A$. The separating conditions will be satisfied depending on the value of the final time and the constant A . It is necessary that $A > 0$ and $t_0 \geq A + 1$. In this case, $(-1, \mathbf{0}, 0, \mathbf{0})$ is contained in the half-space where the perturbation vectors are. Thus it could be associated with a perturbation vector, depending on the directions that are covered by the perturbations of the controls.

To sum up, the last two previous cases cause more difficulty to chose the initial condition for the momenta and the final time if a free optimal control problem is being considered. Pontryagin's Maximum Principle guarantees the existence of a momenta, but without determining it. Hence, we can chose the momenta that appear in the

cases 1 and 2. Under that restriction, q_0 is a constant that plays a similar role that the constant in Definition 3.5. Moreover, our mechanical Hamiltonian turns out to be the Hamiltonian considered in [5] to apply Pontryagin's Maximum Principle for affine connection control systems. Thus, the framework described here guarantees that the second order condition is satisfied in the approach given in [5] because it corresponds with our case $p_0 = 0$.

In the extended problem for the mechanical system we have added two new coordinates, thus two new covectors have appeared. If we look at Definition 3.5, it is not clear how to define the extremals in this case. What we have to remember is that the abnormal extremals are characterized only using the geometry of the system before extending it, that is, the cost function does not play any role in the computation of abnormal extremals. For the mechanical Hamiltonian H_m , this will happen if and only if p_0 and q_0 vanish simultaneously. Otherwise, the extremals are normal.

Definition 4.1. A curve $(\hat{\gamma}, u): I \rightarrow T\hat{Q} \times U$ for the optimal control problem 2.3 is

1. a **normal extremal** if it is an extremal with either p_0 being a nonzero constant or $q_0 = -1$, in the latter $p_0 = 0$;
2. an **abnormal extremal** if it is an extremal with $p_0 = q_0 = 0$;

For the kinematic system, the hamiltonian function is

$$\begin{aligned} H_k: T^*\hat{Q} \times V &\longrightarrow \mathbb{R} \\ (\hat{a}, w) &\longmapsto \langle \hat{a}, \mathcal{G} \frac{\partial}{\partial x^0} + \sum_{s=1}^m w^s Y_s \rangle \end{aligned} \quad (4.10)$$

with local expression

$$H_k = a_0 \mathcal{G} + \sum_{l=1}^m a_l w^l Y_l^i,$$

and Hamilton's equations are given by

$$\begin{aligned} \dot{x}^0 &= \mathcal{G} & \dot{a}_0 &= 0 \\ \dot{x}^i &= w^s Y_s^i & \dot{a}_i &= -a_0 \frac{\partial \mathcal{G}}{\partial x^i} - a_j w^s \frac{\partial Y_s^j}{\partial x^i} \end{aligned} \quad (4.11)$$

The elementary perturbation vector along the optimal curve at t for $w_1 \in V$ is

$$\hat{v}_k(t) = \sum_{s=1}^m (w_1^s - w^s(t)) Y_s + (\mathcal{G}(\hat{\gamma}(t), w_1) - \mathcal{G}(\hat{\gamma}(t), w(t))) \frac{\partial}{\partial x^0} \Big|_{\hat{\gamma}(t)} \quad (4.12)$$

according to (3.7). The covector \hat{a} defined along the optimal curve that comes from Pontryagin's Maximum Principle satisfies a separating condition analogous to (3.6)

$$\begin{aligned} \langle \hat{a}(t), \hat{v}_k(t) \rangle &\leq 0 \\ \langle \hat{a}(t), (-1, \mathbf{0}) \rangle = -a_0 &\geq 0 \end{aligned}$$

where $\widehat{v}_k(t)$ and $(-1, \mathbf{0})$ are in $T_{\widehat{\gamma}(t)}\widehat{Q}$. Here the definitions of extremals is exactly the same as in Definition 3.5 because there is only one more momentum as happens in §3.

Thus we have two different hamiltonian problems, one defined in $T^*T\widehat{Q} \times U$ and the other one defined in $T^*\widehat{Q} \times V$. We wonder if there is any way to relate the momenta of both problems that not only satisfy Hamilton's equations, but also the necessary conditions of Pontryagin's Maximum Principle. Using the Tulczyjew diffeomorphism $\phi_{\widehat{Q}}$ defined in [14] there is a natural way to go from $T^*T\widehat{Q}$ to $T^*\widehat{Q}$ with local expression,

$$\begin{array}{ccccc} T^*(T\widehat{Q}) & \xrightarrow{\phi_{\widehat{Q}}} & T(T^*\widehat{Q}) & \xrightarrow{\tau_{T^*\widehat{Q}}} & T^*\widehat{Q} \\ (x, v, p, q) & \longmapsto & (x, q, v, p) & \longmapsto & (x, q) \end{array} \quad (4.13)$$

and it is also possible to go in the other way round as follows

$$\begin{array}{ccc} & T(T^*\widehat{Q}) & \xrightarrow{\phi_{\widehat{Q}}^{-1}} & T^*(T\widehat{Q}) \\ & (x, q, \dot{x}, \dot{q}) & & (x, \dot{x}, \dot{q}, q) \\ & \nearrow & & \\ T^*\widehat{Q} & & & \\ (x, q) & & & \\ \uparrow & & & \\ I & & & \end{array} \quad (4.14)$$

where all the coordinates are function of t and (x, q, \dot{x}, \dot{q}) is the canonical lift of $(x(t), q(t))$ to the tangent bundle.

From here, we could think that knowing the momenta for the mechanical system the covector for the kinematic system is given by the momenta of the velocities. But this is not true in general because the momenta for the kinematic system we are looking for must also satisfy the other necessary conditions of Pontryagin's Maximum Principle. Moreover, both hamiltonian functions are not exactly the same as shows (4.8) and (4.11).

In the sequel the cost functions considered for both problems are either equal to 1, that is time optimal problems, or the cost function given in Proposition 2.6.

Proposition 4.2. *Let $(\widehat{\Lambda}, \dot{\widehat{\gamma}}): I \rightarrow T^*(T\widehat{Q})$ be a covector curve along an optimal solution for the nonholonomic mechanical system, Problem 2.3. If there exists a $t_1 \in I$ such that $\langle \widehat{q}(t_1), \widehat{v}_k(t_1) \rangle \leq 0$ for every elementary perturbation vector of the kinematic system, then $\widehat{q}(t_1)$ is the initial condition for the covector to solve the Hamilton's equation of the kinematic system, being $\widehat{\gamma}$ an extremal for the kinematic Pontryagin's Maximum Principle.*

Proof. As an optimal solution to the nonholonomic mechanical system is given, by Proposition 2.6 and Remark 2.10 there exist controls such that the same curve on \widehat{Q} is an optimal solution to the kinematic system. Thus, we can apply Pontryagin's Maximum Principle that assures the existence of kinematic momenta. But if for

some $t_1 \in I$, we have $\langle \widehat{q}(t_1), \widehat{v}_k(t_1) \rangle \leq 0$, this $\widehat{q}(t_1)$ determines the initial condition for the momenta to integrate Hamilton's equations such that all the necessary conditions of kinematic Pontryagin's Maximum Principle are satisfied. The sign of the above inequality remains invariant because of a property of the integral curves of the complete lift and the cotangent lift of a vector field on \widehat{Q} [1]. \square

Corollary 4.3. *The abnormal optimal curves for nonholonomic mechanical system with covectors satisfying the hypothesis in the above proposition determine abnormal optimal curves for the kinematic system.*

Proof. The momenta of abnormal extremals for nonholonomic mechanical system are $p_0 = q_0 = 0$. If the hypothesis in the previous proposition are satisfied, then the initial condition for the momenta of the kinematic system are $\widehat{q}(t_0)$, that is, $a_0(t_0) = 0$. As a_0 is constant because of Hamilton's equations (4.11), the abnormal solutions for the mechanical case determine abnormal solutions for the kinematic case using Proposition 2.6 and 4.2. \square

Remark 4.4. There is an analogous result for the normal solutions as long as the momentum for p_0 is taken to be equal to 0, that is, if we consider the case of normal solutions for mechanical systems with $p_0 = 0$ and q_0 to be a nonzero negative constant.

Remark 4.5. Observe that the extremals for the kinematic system are extremals for the mechanical control system. But from the kinematic momenta is not necessarily possible to find the mechanical momenta, as the example in §4.1 shows.

4.1 Example

For instance, it can be proved that the example of strict abnormal minimizer given in [11] understood as a solution to a nonholonomic control mechanical system is a strict abnormal minimizer.

Let $Q = \mathbb{R}^3$ with local coordinates (x, y, z) . We consider the distribution given by

$$D = \ker \omega = \ker(x^2 dy - (1-x) dz) = \text{span}\{\partial/\partial x, (1-x)\partial/\partial y + x^2\partial/\partial z\} = \text{span}\{X, Y\}.$$

Consider the Riemannian metric on Q , $g = dx \otimes dx + \psi(x)(dy \otimes dy + dz \otimes dz)$, where $\psi(x) = ((1-x)^2 + x^4)^{-1}$. Observe that X and Y are a g -orthonormal basis of sections of Q .

The hamiltonian function for the time optimal control problem for the kinematic system associated to D is

$$H_k(\widehat{a}, w_1, w_2) = a_0 + a_1 w_1 + a_2 w_2 (1-x) + a_3 x^2 w_2.$$

The curve $(\gamma, w): [0, 1] \rightarrow Q \times V$, $t \mapsto (0, t, 0, 0, 1)$ satisfying the initial conditions $\gamma(0) = (0, 0, 0)$ and $\gamma(1) = (0, 1, 0)$ is a local strict abnormal minimizer for the

time-optimal problem. It is impossible to find momenta with $a_0 = -1$ verifying all the necessary conditions of Pontryagin's Maximum Principle. Let us check it, the corresponding Hamilton equations for abnormality and normality are

$$\begin{aligned}\dot{x}_0 &= 1 & \dot{a}_0 &= 0 \\ \dot{x} &= w_1 & \dot{a}_1 &= a_2 w_2 - 2x w_2 a_3 \\ \dot{y} &= w_2(1-x) & \dot{a}_2 &= 0 \\ \dot{z} &= x^2 w_2 & \dot{a}_3 &= 0\end{aligned}$$

Assume that the control set is open, then the maximization of the Hamiltonian over the controls has as necessary conditions that $\partial H_k / \partial w_1 = a_1 = 0$, $\partial H_k / \partial w_2 = a_2(1-x) + a_3 x^2 = 0$. Along the curve γ , we have $a_1 = 0$ and $a_2 = 0$. The abnormal momenta are $\hat{a}: [0, 1] \rightarrow T^*\widehat{Q}$, $t \mapsto (0, 0, 0, a_3)$ along $\hat{\gamma}(t)$ with a_3 being a nonzero constant. Observe that $H_k(\hat{a}(t), w_1, w_2) = 0$ for all $t \in [0, 1]$.

For the normal case, $a_0 = -1$ and the necessary conditions for the maximization of the Hamiltonian over the controls are the same along γ : $a_1 = 0$, $a_2 = 0$. But then $H_k(\hat{a}(t), w_1, w_2) = -1 \neq 0$ for all $t \in [0, 1]$ contradicting a necessary condition of Pontryagin's Maximum Principle, see Remark 3.4. Thus, as mentioned, γ is a strict abnormal extremal. The local optimality is proved in [11].

According to the metric, the Christoffel symbols that do not vanish are

$$\begin{aligned}\Gamma_{22}^1 &= \Gamma_{33}^1 &= \frac{1-x-2x^3}{((1-x)^2+x^4)^2} \\ \Gamma_{12}^2 &= -\Gamma_{21}^1 &= \frac{1-x-2x^3}{(1-x)^2+x^4} \\ \Gamma_{13}^3 &= -\Gamma_{31}^1 &= \frac{1-x-2x^3}{(1-x)^2+x^4}\end{aligned}$$

where 1 stands for coordinate x and so on. Observe that the connection associated to the metric does not have zero torsion.

Having this in mind, the hamiltonian function for the mechanical system is

$$H_m(\widehat{\Lambda}, u_1, u_2) = p_0 v_0 + q_0 + p_1 v_1 + p_2 v_2 + p_3 v_3 + q_1(-\Gamma_{22}^1 v_2^2 - \Gamma_{33}^1 v_3^2 + u_1) + q_2 u_2(1-x) + q_3 x^2 u_2.$$

Hamilton's equations are

$$\begin{aligned}\dot{x}_0 &= v_0 & \dot{p}_0 &= 0 \\ \dot{x} &= v_1 & \dot{p}_1 &= \frac{\partial \Gamma_{22}^1}{\partial x} q_1 v_2^2 + \frac{\partial \Gamma_{33}^1}{\partial x} q_1 v_3^2 + q_2 u_2 - 2x u_2 q_3 \\ \dot{y} &= v_2 & \dot{p}_2 &= 0 \\ \dot{z} &= v_3 & \dot{p}_3 &= 0 \\ \dot{v}_0 &= 1 & \dot{q}_0 &= -p_0 \\ \dot{v}_1 &= -\Gamma_{22}^1 v_2^2 - \Gamma_{33}^1 v_3^2 + u_1 & \dot{q}_1 &= -p_1 \\ \dot{v}_2 &= u_2(1-x) & \dot{q}_2 &= -p_2 + 2q_1 \Gamma_{22}^1 v_2 \\ \dot{v}_3 &= x^2 u_2 & \dot{q}_3 &= -p_3 + 2v_3 \Gamma_{33}^1 q_1\end{aligned}$$

The strict abnormal minimizer for the kinematic system becomes the extremal $\hat{\gamma}(t) = (t, 0, t, 0, 1, 0, 1, 0)$ for the mechanical system. Substituting into the first column of Hamilton's equations along $\hat{\gamma}$ we have $u_1 = 1$ and $u_2 = 1$.

Remark 4.6. The control are different for the equivalent control systems, as was mentioned in §2.1.

Necessary conditions for the maximization of the Hamiltonian H_m over the controls along the extremal are $q_1 = 0$ and $q_2 = 0$. From the second column in Hamilton's equations we have

$$\dot{p}_1 = 0, \quad p_1 = 0, \quad p_2 = 0, \quad \dot{q}_3 = -p_3$$

where p_3 is constant. These are valid for abnormality and normality because of the considered cost function.

The abnormal momenta, $p_0 = q_0 = 0$, is $\widehat{\Lambda}(t) = (0, 0, 0, p_3, 0, 0, 0, -p_3t + A)$ with p_3 and A being constants, that cannot vanish simultaneously. If now we evaluate the Hamiltonian, $H_m(\widehat{\Lambda}(t), u_1, u_2) = 0$. Thus, the abnormal minimizer for the kinematic system is an abnormal extremal in the mechanical case.

Let us try to find the normal momenta, that is, either $q_0 = -1$ or $p_0 = -1$. The different cases are:

1. $p_0 = -1$ then by Hamilton's equations $q_0(t) = t + B$ with a constant B ;
2. $p_0 = 0$, then $q_0 = -1$.

Thus, either $\widehat{\Lambda}_1(t) = (-1, 0, 0, p_3, t+B, 0, 0, -p_3t+A)$ or $\widehat{\Lambda}_2(t) = (0, 0, 0, p_3, -1, 0, 0, -p_3t+A)$ along $\widehat{\gamma}$. If we evaluate the Hamiltonian H_m at these covectors,

$$\begin{aligned} H_m(\widehat{\Lambda}_1(t), u_1, u_2) &= -1 + t + B, \\ H_m(\widehat{\Lambda}_2(t), u_1, u_2) &= -1 \end{aligned}$$

None of the previous values are zero almost everywhere on $[0, 1]$. Thus, the strict abnormal minimizer for the kinematic system is not a normal extremal for the mechanical case. Therefore, we have a strict abnormal extremal for the nonholonomic mechanical system.

As for the elementary perturbation vectors (4.9), (4.12) along the extremals considered, we have

$$\begin{aligned} \widehat{v}_k(t) &= \tilde{w}_1 \frac{\partial}{\partial x} + (\tilde{w}_2 - 1) \frac{\partial}{\partial y} \\ \widehat{v}_m(t) &= (\tilde{u}_1 - 1) \frac{\partial}{\partial v_1} + (\tilde{u}_2 - 1) \frac{\partial}{\partial v_2}. \end{aligned}$$

For the momenta found, the conditions (3.6) are

$$\begin{aligned} \langle (0, 0, 0, a_3), \widehat{v}_k(t) \rangle &= 0 \leq 0 \\ \langle (0, 0, 0, p_3, 0, 0, 0, -p_3t + A), \widehat{v}_m(t) \rangle &= 0 \leq 0. \end{aligned}$$

Observe that the kinematic momenta and the mechanical momenta are related through (4.13), (4.14). From a kinematic system we recover the mechanical momenta at every time $t \in I$ when $p_3 = 0$ and $A = a_3$. But the way to understand the relation is:

given a time t_1 , the initial condition for the kinematic momenta is $-p_3t_1 + A$. After integrating Hamilton's equations, the momenta do not necessarily satisfy the relation at every time because mechanical and kinematic Hamilton's equations are different, although this relation is satisfied at time t_1 . The same happens in the other way round from the kinematic momenta to the mechanical momenta. Thus it is highlighted the fact that the mapping defined using Tulczyjew's diffeomorphism does not establish a one-to-one relation between the momenta of both Hamilton's equations for every time.

Remark 4.7. Due to Proposition 2.9 and Remark 2.10, the strict abnormal extremal found for the mechanical case is also a local strict abnormal minimizer for the time optimal control problem for the control system given by D .

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