

# HIGHER-ORDER CARTAN SYMMETRIES IN $k$ -SYMPLECTIC FIELD THEORY

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## Abstract

For  $k$ -symplectic Hamiltonian field theories, we study infinitesimal transformations generated by certain kinds of vector fields which are not Noether symmetries, but which allow us to obtain conservation laws by means of a suitable generalization of the Noether theorem.

**Key words:** *Symmetries, Conservation laws, Noether theorem, Hamiltonian field theories,  $k$ -symplectic manifolds.*

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## 1 Introduction

The  $k$ -symplectic formalism [1, 4, 6] is the simplest generalization to field theories of the standard symplectic formalism in autonomous Mechanics. It allows us to give a geometric description of certain kinds of field theories: in a local description, those

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theories whose Lagrangian or Hamiltonian functions depend only on the field coordinates and on the partial derivatives of the fields, or on the corresponding moments, but not on the base coordinates. This formalism is based on the polysymplectic formalism developed by Günther [3].

In a previous paper [11] we introduced the notion of *Cartan or Noether symmetry*, and we stated Noether's theorem for Hamiltonian and Lagrangian systems in  $k$ -symplectic field theories. Noether's theorem associates conservation laws to Cartan or Noether symmetries. However, these kinds of symmetries do not exhaust the set of (general) symmetries. As is known, in mechanics there are dynamical symmetries which are not of Noether type, but which also generate conserved quantities (see [5], [8], [9], for some examples). These are the so-called *hidden symmetries*. Different attempts have been made to extend Noether's theorem in order to include these symmetries and the corresponding conserved quantities for mechanical systems (see for instance [12]) and multisymplectic field theories (see [2]).

In this paper we present a generalization of the Noether theorem for  $k$ -symplectic Hamiltonian field theories, which is based in the approach of reference [12] for mechanical systems. This generalization allows us to obtain conservation laws associated to infinitesimal transformations generated by certain kinds of vector fields which are not Noether symmetries.

All manifolds are real, paracompact, connected and  $C^\infty$ . All maps are  $C^\infty$ . Sum over crossed repeated indices is understood.

## 2 $k$ -symplectic Hamiltonian systems

(See [6], [10], [11] for details).

Let  $(T_k^1)^*Q = T^*Q \oplus \dots \oplus T^*Q$  be the *bundle of  $k^1$  covelocities* of an  $n$ -dimensional differentiable manifold  $Q$ , with projection  $\tau^*: (T_k^1)^*Q \rightarrow Q$ . Natural coordinates on  $(T_k^1)^*Q$  are  $(q^i, p_i^A)$ ;  $1 \leq i \leq n$ ,  $1 \leq A \leq k$ .

The *canonical  $k$ -symplectic structure* in  $(T_k^1)^*Q$  is  $(\omega^A, V)$ , where  $V = \ker(\tau^*)$ , and  $\omega^A = (\tau_A^*)^* \omega = -d(\tau_A^*)^* \theta = -d\theta^A$ ;  $\omega = -d\theta$  being the canonical symplectic structure in  $T^*Q$  ( $\theta \in \Omega^1(T^*Q)$  is the *Liouville 1-form*), and  $\tau_A^*: (T_k^1)^*Q \rightarrow T^*Q$  the projection on the  $A^{\text{th}}$ -copy  $T^*Q$  of  $(T_k^1)^*Q$ . Locally

$$\omega^A = -d\theta^A = -d(p_i^A dq^i) = dq^i \wedge dp_i^A.$$

Given a diffeomorphism  $\varphi: Q \rightarrow Q$ , its *canonical prolongation* to  $(T_k^1)^*Q$  is the map  $(T_k^1)^* \varphi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ , which is defined by

$$(T_k^1)^* \varphi(\alpha_{1q}, \dots, \alpha_{kq}) = (T^* \varphi(\alpha_{1q}), \dots, T^* \varphi(\alpha_{kq})), \quad (\alpha_{1q}, \dots, \alpha_{kq}) \in (T_k^1)^*_q Q, \quad q \in Q.$$

If  $Z \in \mathfrak{X}(Q)$  has  $h_s: Q \rightarrow Q$  as local 1-parametric group; the *canonical lift* of  $Z$  to  $(T_k^1)^*Q$  is the vector field  $Z^{C*} \in \mathfrak{X}((T_k^1)^*Q)$  whose local 1-parametric group is  $(T_k^1)^*(h_s): (T_k^1)^*Q \rightarrow (T_k^1)^*Q$ . Locally, if  $Z = Z^i \frac{\partial}{\partial q^i}$  then  $Z^{C*} = Z^i \frac{\partial}{\partial q^i} - p_j^A \frac{\partial Z^j}{\partial q^k} \frac{\partial}{\partial p_k^A}$ .

**Definition 1** Let  $T_k^1 M = TM \oplus \dots \oplus TM$  be the bundle of  $k^1$  velocities of a manifold  $M$ . Let us denote by  $\tau: T_k^1 M \rightarrow M$  the canonical projection.

- A  $k$ -vector field on  $M$  is a section  $\mathbf{X}: M \rightarrow T_k^1 M$  of  $\tau$ .  
A  $k$ -vector field  $\mathbf{X}$  defines a family of vector fields  $X_1, \dots, X_k \in \mathfrak{X}(M)$  by  $X_A = \tau_A \circ \mathbf{X}$ , where  $\tau_A: T_k^1 M \rightarrow TM$  is the projection on the  $A^{\text{th}}$ -copy  $TM$  of  $T_k^1 M$ .
- An integral section of  $\mathbf{X}$  at a point  $q \in M$ , is a map  $\psi: U_0 \subset \mathbb{R}^k \rightarrow M$ , with  $0 \in U_0$ , such that  $\psi(0) = q$ ,  $\psi_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) = X_A(\psi(t))$ , for every  $t \in U_0$ ; or what is equivalent,  $\psi$  satisfies that  $\mathbf{X} \circ \psi = \psi^{(1)}$ , where  $\psi^{(1)}$  is the first prolongation of  $\psi$  to  $T_k^1 M$  defined by

$$\begin{aligned} \psi^{(1)}: U_0 \subset \mathbb{R}^k &\longrightarrow T_k^1 M \\ t &\longrightarrow \psi^{(1)}(t) = \left( \psi_*(t) \left( \frac{\partial}{\partial t^1} \Big|_t \right), \dots, \psi_*(t) \left( \frac{\partial}{\partial t^k} \Big|_t \right) \right). \end{aligned}$$

A  $k$ -vector field is integrable if there is an integral section at every point of  $M$ .

The set of  $k$ -vector fields on  $M$  are denoted by  $\mathfrak{X}^k(M)$ .

Now take  $M = (T_k^1)^*Q$ . Let  $H: (T_k^1)^*Q \rightarrow \mathbb{R}$  be a *Hamiltonian function*. The family  $((T_k^1)^*Q, \omega^A, H)$  is a  *$k$ -symplectic Hamiltonian system*. The *Hamilton-de Donder-Weyl (HDW) equations* associated to this system are

$$\frac{\partial H}{\partial q^i} \Big|_{\psi(t)} = - \sum_{A=1}^k \frac{\partial \psi_i^A}{\partial t^A} \Big|_t, \quad \frac{\partial H}{\partial p_i^A} \Big|_{\psi(t)} = \frac{\partial \psi^i}{\partial t^A} \Big|_t, \quad (1)$$

where  $\psi: \mathbb{R}^k \rightarrow (T_k^1)^*Q$ ,  $\psi(t) = (\psi^i(t), \psi_i^A(t))$ , is a solution.

We denote by  $\mathfrak{X}_H^k((T_k^1)^*Q)$  the set of  $k$ -vector fields on  $(T_k^1)^*Q$  solutions to

$$\sum_{A=1}^k i(X_A) \omega^A = dH.$$

In a local system of canonical coordinates, each  $X_A$  is locally given by  $X_A = (X_A)^i \frac{\partial}{\partial q^i} + (X_A)^B \frac{\partial}{\partial p_i^B}$ , and we obtain that the equation (2) is equivalent to the equations

$$\frac{\partial H}{\partial q^i} = - \sum_{A=1}^k (X_A)^A_i, \quad \frac{\partial H}{\partial p_i^A} = (X_A)^i. \quad (2)$$

If  $\mathbf{X} = (X_1, \dots, X_k)$  is an integrable  $k$ -vector field in  $(T_k^1)^*Q$ , and  $\psi: \mathbb{R}^k \rightarrow (T_k^1)^*Q$  an integral section of  $\mathbf{X}$ , we have that  $\psi(t) = (\psi^i(t), \psi_i^A(t))$  is a solution to the HDW-equations (1) if, and only if,  $\mathbf{X} \in \mathfrak{X}_H^k((T_k^1)^*Q)$ . In fact, if  $\psi(t) = (\psi^i(t), \psi_i^A(t))$  is an integral section of  $\mathbf{X}$ , then

$$\frac{\partial \psi^i}{\partial t^B} = (X_B)^i \quad , \quad \frac{\partial \psi_i^A}{\partial t^B} = (X_B)_i^A . \quad (3)$$

and therefore (2) are the HDW-equations (1).

**Remark 1** We can define the vector bundle morphism

$$\begin{aligned} \omega^\sharp : T_k^1((T_k^1)^*Q) &\rightarrow T^*((T_k^1)^*Q) \\ (v_{p_1}, \dots, v_{p_k}) &\mapsto \sum_{A=1}^k i(v_{p_A}) \omega_p^A , \end{aligned}$$

and we denote with the same symbol its natural extension

$$\begin{aligned} \omega^\sharp : \mathfrak{X}^k((T_k^1)^*Q) &\rightarrow \Omega^1((T_k^1)^*Q) \\ \mathbf{X} = (X_1, \dots, X_k) &\mapsto \sum_{A=1}^k i(X_A) \omega^A . \end{aligned}$$

Then, the solutions to (2) are given by  $\mathbf{X} + \ker \omega^\sharp$ , where  $\mathbf{X}$  is a particular solution.

The equations (1) and (2) are not equivalent because not every solution to the HDW-equations (1) is an integral section of some integrable  $k$ -vector field belonging to  $\mathfrak{X}_H^k((T_k^1)^*Q)$ , unless some additional conditions are required. Thus, we assume the following condition (which holds for a large class of mathematical applications and physical field theories):

**Definition 2** A map  $\psi: \mathbb{R}^k \rightarrow (T_k^1)^*Q$ , solution to the equations (1), is said to be an admissible solution to the HDW-equations for a  $k$ -symplectic Hamiltonian system  $((T_k^1)^*Q, H)$ , if  $\text{Im } \psi$  is an embedded submanifold of  $(T_k^1)^*Q$ .

We say that  $((T_k^1)^*Q, H)$  is an admissible  $k$ -symplectic Hamiltonian system when only admissible solutions to its HDW-equations are considered.

**Proposition 1** Every admissible solution to the HDW-equations (1) is an integral section of an integrable  $k$ -vector field  $\mathbf{X} \in \mathfrak{X}_H^k((T_k^1)^*Q)$ .

(Proof) Let  $\psi: \mathbb{R}^k \rightarrow (T_k^1)^*Q$  be an admissible solution to the HDW-equations (1). By hypothesis,  $\text{Im } \psi$  is a  $k$ -dimensional submanifold of  $(T_k^1)^*Q$ . As  $\psi$  is an embedding, we can define a  $k$ -vector field  $\mathbf{X}|_{\text{Im } \psi}$  (at support on  $\text{Im } \psi$ ), tangent to  $\text{Im } \psi$ , by

$$X_A(\psi(t)) = (\psi)_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) ,$$

which is a solution to (2) on the points of  $\text{Im } \psi$ , since (2) holds on these points as a consequence of (1) and (3). Furthermore,  $\text{Im } \psi$  is a submanifold of  $(T_k^1)^*Q$ ; therefore we can extend this  $k$ -vector field  $\mathbf{X}|_{\text{Im } \psi}$  to an integrable  $k$ -vector field  $\mathbf{X} \in \mathfrak{X}_H^k((T_k^1)^*Q)$  in such a way that this extension is a solution to the equations (2) (note that these equations have solutions everywhere on  $(T_k^1)^*Q$ ), and which obviously has  $\psi$  as an integral section. This extension is made at least locally, and then the global  $k$ -vector field is constructed using partitions of unity. ■

In this way, for admissible  $k$ -symplectic Hamiltonian systems, the field equations (2) are a geometric version of the HDW-equations (1).

### 3 Symmetries and conservation laws

**Definition 3** (Olver [7]) *A conservation law or a conserved quantity of a  $k$ -symplectic Hamiltonian system  $((T_k^1)^*Q, \omega^A, H)$  is a map  $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_k): (T_k^1)^*Q \rightarrow \mathbb{R}^k$  such that the divergence of  $\mathcal{F} \circ \psi = (\mathcal{F}_1 \circ \psi, \dots, \mathcal{F}_k \circ \psi): \mathbb{R}^k \rightarrow \mathbb{R}^k$  is zero for every  $\psi: \mathbb{R}^k \rightarrow (T_k^1)^*Q$  solution to the Hamilton-de Donder-Weyl equations (1); that is,*

$$\sum_{A=1}^k \frac{\partial(\mathcal{F}_A \circ \psi)}{\partial t^A} = 0.$$

For admissible  $k$ -symplectic Hamiltonian systems, conserved quantities can be characterized as follows:

**Proposition 2** *Let  $((T_k^1)^*Q, H)$  be an admissible  $k$ -symplectic Hamiltonian system.*

*A map  $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_k): (T_k^1)^*Q \rightarrow \mathbb{R}^k$  is a conservation law of an admissible  $k$ -symplectic Hamiltonian system if, and only if, for every integrable  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$ , we have that  $\sum_{A=1}^k L(X_A) \mathcal{F}_A = 0$ .*

(Proof) Let  $\mathcal{F} = (\mathcal{F}^1, \dots, \mathcal{F}^k)$  be a conservation law and  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$  an integrable  $k$ -vector field. If  $\psi: \mathbb{R}^k \rightarrow (T_k^1)^*Q$  is an integral section of  $\mathbf{X}$  then:

1. We have that  $\psi$  is a solution to the Hamilton-de Donder-Weyl equation (1).
2. By definition of integral section, we have  $X_A(\psi(t)) = \psi_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right)$ .

Therefore

$$\sum_{A=1}^k L(X_A) \mathcal{F}^A = \sum_{A=1}^k \psi_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right) (\mathcal{F}^A) = \sum_{A=1}^k \frac{\partial(\mathcal{F}^A \circ \psi)}{\partial t^A} \Big|_t = 0.$$

Conversely, let us suppose that every integrable  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k)$  in  $\mathfrak{X}_H^k((T_k^1)^*Q)$  satisfies  $\sum_{A=1}^k L(X_A)\mathcal{F}_A = 0$ , and let  $\psi: \mathbb{R}^k \rightarrow (T_k^1)^*Q$  be an admissible solution to the HDW-equations (1). By Proposition 1 there exists a  $k$ -vector field  $\mathbf{X} \in \mathfrak{X}_H^k((T_k^1)^*Q)$  such that

$$X_A(\psi(t)) = (\psi)_*(t) \left( \frac{\partial}{\partial t^A} \Big|_t \right)$$

Thus, since  $\sum_{A=1}^k L(X_A)\mathcal{F}_A = 0$ , from the above identity we obtain that

$$\sum_{A=1}^k \frac{\partial(\mathcal{F}_A \circ \psi)}{\partial t^A} \Big|_t = 0.$$

■

**Definition 4** Let  $((T_k^1)^*Q, \omega^A, H)$  be a  $k$ -symplectic Hamiltonian system.

1. A symmetry is a diffeomorphism  $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$  such that for every solution  $\psi$  to the HDW equations (1), we have that  $\Phi \circ \psi$  is also a solution.

If  $\Phi = (T_k^1)^*\varphi$  for some  $\varphi: Q \rightarrow Q$ , the symmetry  $\Phi$  is said to be natural.

2. An infinitesimal symmetry is a vector field  $Y \in \mathfrak{X}((T_k^1)^*Q)$  whose local flows are local symmetries.

If  $Y = Z^{C^*}$  for  $Z \in \mathfrak{X}(Q)$ , the infinitesimal symmetry  $Y$  is said to be natural.

**Proposition 3** Let  $((T_k^1)^*Q, \omega^A, H)$  be an admissible  $k$ -symplectic Hamiltonian system. A diffeomorphism  $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$  is a symmetry if, and only if, for every integrable  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$  we have that  $\Phi_*\mathbf{X} = (\Phi_*X_1, \dots, \Phi_*X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$ , it is integrable, and its integral sections are  $\Phi \circ \psi$ , for any integral section  $\psi$  of  $\mathbf{X}$ .

(Proof) Let  $\Phi: (T_k^1)^*Q \rightarrow (T_k^1)^*Q$  be a diffeomorphism and  $\mathbf{X} = (X_1, \dots, X_k)$  an integrable  $k$ -vector field in  $\mathfrak{X}_H^k((T_k^1)^*Q)$ . Then every integral section  $\psi$  of  $\mathbf{X}$  is a solution to the HDW-equations (1) and satisfy  $X_A(\psi(t)) = \psi_*(t)(\partial/\partial t^A)$ . From this we have  $\Phi_*(\psi(t))X_A(\psi(t)) = (\Phi \circ \psi)_*(\partial/\partial t^A)$ , thus  $\Phi \circ \psi$  is an integral section of  $\Phi_*\mathbf{X}$ , and so  $\Phi_*\mathbf{X}$  is integrable.

Now, since  $\Phi$  is a symmetry, then  $\Phi \circ \psi$  is a solution to the HDW-equations (1) and as it is an integral section of  $\Phi_*\mathbf{X}$  we deduce that  $\Phi_*\mathbf{X} \in \mathfrak{X}_H^k((T_k^1)^*Q)$ .

Let  $\psi$  be an admissible solution to the HDW-equations (1), then by Proposition 1, there exists  $\mathbf{X} \in \mathfrak{X}_H^k((T_k^1)^*Q)$  such that  $\psi$  is an integral section of  $\mathbf{X}$ . Then  $\Phi \circ \psi$  is an integral section of  $\Phi_*\mathbf{X} \in \mathfrak{X}_H^k((T_k^1)^*Q)$ , and thus  $\Phi \circ \psi$  is a solution to the HDW-equations (1). ■

As a consequence of this, if  $\Phi$  is a symmetry we have that  $\Phi_*\mathbf{X} - \mathbf{X} \in \ker \omega^\sharp$ .

**Proposition 4** *Let  $((T_k^1)^*Q, \omega^A, H)$  be an admissible  $k$ -symplectic Hamiltonian system. If  $Y \in \mathfrak{X}((T_k^1)^*Q)$  is an infinitesimal symmetry, then for every integrable  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$  we have that  $[Y, \mathbf{X}] = ([Y, X_1], \dots, [Y, X_k]) \in \ker \omega^\sharp$ .*

(Proof) As  $Y$  is an infinitesimal symmetry, denoting by  $F_t$  the local 1-parameter groups of diffeomorphisms generated by  $Y$ , we have that  $F_{t*}\mathbf{X} - \mathbf{X} \in \ker \omega^\sharp$ . Then, if  $\{\mathbf{Z}^1, \dots, \mathbf{Z}^r\} = \{(Z_1^1, \dots, Z_k^1), \dots, (Z_1^r, \dots, Z_k^r)\}$  is a local basis of  $\ker \omega^\sharp$ , we have that  $F_{t*}\mathbf{X} - \mathbf{X} = g_\alpha \mathbf{Z}^\alpha$ ,  $\alpha = 1, \dots, r$ , with  $g_\alpha: \mathbb{R} \times (T_k^1)^*Q \rightarrow \mathbb{R}$  (they are functions that depend on  $t$ ); that is

$$F_{t*}\mathbf{X} - \mathbf{X} = (F_{t*}X_1 - X_1, \dots, F_{t*}X_k - X_k) = (g_\alpha Z_1^\alpha, \dots, g_\alpha Z_k^\alpha) = g_\alpha \mathbf{Z}^\alpha.$$

Therefore

$$\begin{aligned} [Y, \mathbf{X}] &= \mathbf{L}(Y)\mathbf{X} = (\mathbf{L}(Y)X_1, \dots, \mathbf{L}(Y)X_k) = \left( \lim_{t \rightarrow 0} \frac{F_{t*}X_1 - X_1}{t}, \dots, \lim_{t \rightarrow 0} \frac{F_{t*}X_k - X_k}{t} \right) \\ &= \left( \lim_{t \rightarrow 0} \frac{g_\alpha}{t} Z_1^\alpha, \dots, \lim_{t \rightarrow 0} \frac{g_\alpha}{t} Z_k^\alpha \right) = (f_\alpha Z_1^\alpha, \dots, f_\alpha Z_k^\alpha) = f_\alpha \mathbf{Z}^\alpha \in \ker \omega^\sharp, \end{aligned}$$

where  $f_\alpha: (T_k^1)^*Q \rightarrow \mathbb{R}$ . ■

**Proposition 5** *Let  $((T_k^1)^*Q, \omega^A, H)$  be an admissible  $k$ -symplectic Hamiltonian system. If  $Y \in \mathfrak{X}((T_k^1)^*Q)$  is an infinitesimal symmetry, then for every  $\mathbf{Z} \in \ker \omega^\sharp$ , we have that  $[Y, \mathbf{Z}] \in \ker \omega^\sharp$ .*

(Proof) For every  $\mathbf{Z} \in \ker \omega^\sharp$ , there exist integrable  $k$ -vector fields  $\mathbf{X}, \mathbf{X}' \in \mathfrak{X}_H^k((T_k^1)^*Q)$  such that  $\mathbf{X}' - \mathbf{X} = \mathbf{Z}$ ; therefore  $[Y, \mathbf{Z}] = [Y, \mathbf{X}'] - [Y, \mathbf{X}] \in \ker \omega^\sharp$ , since  $[Y, \mathbf{X}'], [Y, \mathbf{X}] \in \ker \omega^\sharp$ , by Proposition 4. ■

## 4 Higher-order Cartan symmetries. Noether's theorem

Noether's theorem allows us to associate conservation laws to certain kinds of symmetries: the so-called *infinitesimal Cartan* or *Noether symmetries*, which are vector fields  $Y \in \mathfrak{X}((T_k^1)^*Q)$  such that: (i)  $\mathbf{L}(Y)\omega^A = 0$ , and (ii)  $\mathbf{L}(Y)H = 0$  (see [11]).

Now we introduce new kinds of generators of conservation laws which are not of this type (we restrict ourselves to the infinitesimal case).

**Definition 5** Let  $((T_k^1)^*Q, \omega^A, H)$  be a  $k$ -symplectic Hamiltonian system. A vector field  $Y \in \mathfrak{X}((T_k^1)^*Q)$  is said to be an infinitesimal Cartan or Noether symmetry of order  $n$  if:

1.  $Y$  is an infinitesimal symmetry.
2.  $L^n(Y)\omega^A := \overbrace{L(Y) \dots L(Y)}^n \omega^A = 0$ , but  $L^m(Y)\omega^A \neq 0$ , for  $m < n$ .
3.  $L(Y)H = 0$ .

In the particular case that  $Y = Z^{C^*}$  for some  $Z \in \mathfrak{X}(Q)$ , the infinitesimal Cartan (Noether) symmetry of order  $n$  is said to be natural.

For  $n = 1$  we recover the definition of infinitesimal Cartan (Noether) symmetry. Observe that infinitesimal Cartan symmetries of order  $n > 1$  are not infinitesimal Cartan symmetries.

**Proposition 6** If  $Y \in \mathfrak{X}((T_k^1)^*Q)$  is a infinitesimal Cartan symmetry of order  $n$  of a  $k$ -symplectic Hamiltonian system, then the forms  $L^{n-1}(Y)i(Y)\omega^A \in \Omega^1((T_k^1)^*Q)$  are closed.

(Proof) From the definition 5, we obtain

$$0 = L^n(Y)\omega^A = L^{n-1}(Y)L(Y)\omega^A = L^{n-1}(Y)d i(Y)\omega^A = dL^{n-1}(Y)i(Y)\omega^A .$$

■

**Proposition 7** Let  $Y \in \mathfrak{X}((T_k^1)^*Q)$  be an infinitesimal Cartan symmetry of order  $n$  of a  $k$ -symplectic Hamiltonian system  $((T_k^1)^*Q, \omega^A, H)$ . Then, for every  $p \in (T_k^1)^*Q$ , there is an open neighbourhood  $U_p \ni p$ , such that:

1. There exist  $g^A \in C^\infty(U_p)$ , which are unique up to constant functions, such that

$$L^{n-1}(Y)i(Y)\omega^A = dg^A, \quad (\text{on } U_p) . \quad (4)$$

2. There exist  $\xi^A \in C^\infty(U_p)$ , verifying that  $L^n(Y)\theta^A = d\xi^A$ , on  $U_p$ ; and then

$$g^A = i(Y)\theta^A - \xi^A, \quad (\text{up to a constant function, on } U_p) \quad (5)$$



(Proof)

1. It is an immediate consequence of Proposition 6 and the Poincaré Lemma.
2. We have that

$$dL^n(Y)\theta^A = L^n(Y)d\theta^A = -L^n(Y)\omega^A = 0$$

and hence  $L^n(Y)\theta^A$  are closed forms. Therefore, by the Poincaré Lemma, there exist  $\xi^A \in C^\infty(U_p)$ , verifying that  $L^n(Y)\theta^A = d\xi^A$ , on  $U_p$ . Furthermore, as (4) holds in  $U_p$ , we obtain that

$$\begin{aligned} d\xi^A &= L^n(Y)\theta^A = L^{n-1}(Y)L(Y)\theta^A = L^{n-1}(Y)\{di(Y)\theta^A + i(Y)d\theta^A\} \\ &= dL^{n-1}(Y)i(Y)\theta^A - L^{n-1}(Y)i(Y)\omega^A = d\{i(Y)\theta^A - g^A\} \end{aligned}$$

and thus (5) holds. ■

Finally, Noether's theorem can be generalized for these higher-order Cartan symmetries and admissible  $k$ -symplectic Hamiltonian systems as follows:

**Theorem 1** (Noether): *If  $Y \in \mathfrak{X}((T_k^1)^*Q)$  is an infinitesimal Cartan symmetry of order  $n$  of an admissible  $k$ -symplectic Hamiltonian system  $((T_k^1)^*Q, \omega^A, H)$ , then*

$$g = (g^1, \dots, g^k) = (i(Y)\theta^1 - \xi^1, \dots, i(Y)\theta^k - \xi^k)$$

*is a conserved quantity; that is, for every integrable  $k$ -vector field  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$ , we have that  $\sum_{A=1}^k L(X_A)g^A = 0$  (on  $U_p$ ).*

(Proof) If  $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}_H^k((T_k^1)^*Q)$ , taking (4) into account we have

$$\sum_{A=1}^k L(X_A)g^A = \sum_{A=1}^k i(X_A)dg^A = \sum_{A=1}^k i(X_A)L^{n-1}(Y)i(Y)\omega^A.$$

Then, if  $n = 2$ , we have

$$\begin{aligned} \sum_{A=1}^k L(X_A)g^A &= \sum_{A=1}^k i(X_A)L(Y)i(Y)\omega^A = \sum_{A=1}^k \{L(Y)i(X_A) - i([Y, X_A])\}i(Y)\omega^A \\ &= \sum_{A=1}^k \{-L(Y)i(Y)i(X_A) + i(Y)i([Y, X_A])\}\omega^A \\ &= -L(Y)i(Y)dH = -L^2(Y)H = 0, \end{aligned}$$

since as  $Y$  is an infinitesimal Cartan symmetry of order  $n$ , it is a symmetry; then  $L(Y)H = 0$  and, by Proposition 4,  $[Y, \mathbf{X}] \in \ker \omega^\sharp$ , and hence  $\sum_{A=1}^k i([Y, X_A])\omega^A = 0$ .

If  $n = 3$ , by an analogous reasoning, we obtain

$$\begin{aligned} \sum_{A=1}^k L(X_A)g^A &= \sum_{A=1}^k i(X_A)L^2(Y)i(Y)\omega^A = \sum_{A=1}^k i(X_A)L(Y)L(Y)i(Y)\omega^A \\ &= \sum_{A=1}^k \{L(Y)i(X_A) - i([Y, X_A])\}L(Y)i(Y)\omega^A \\ &= \sum_{A=1}^k \{L(Y)i(X_A)L(Y) - i([Y, X_A])L(Y)\}i(Y)\omega^A \\ &= \sum_{A=1}^k \{L^2(Y)i(X_A) - 2L(Y)i([Y, X_A]) + i([Y, [Y, X_A]])\}i(Y)\omega^A \\ &= \sum_{A=1}^k L^2(Y)i(X_A)\omega^A = L^2(Y)H = 0, \end{aligned}$$

since, by Proposition 4,  $[Y, \mathbf{X}] \in \ker \omega^\sharp$  and, by Proposition 5,  $\sum_{A=1}^k i([Y, [Y, X_A]])\omega^A = 0$ .

For  $n > 3$ , we arrive at the same result by repeating the above procedure  $n - 2$  times. Thus, taking into account Proposition 2, we have proved that  $g = (g^1, \dots, g^k)$  is a conservation law. ■

**Remark:**  $k$ -symplectic Lagrangian systems can be defined in  $T_k^1Q = TQ \oplus \dots \oplus TQ$ , starting from a Lagrangian function  $L \in C^\infty(T_k^1Q)$ , and using the canonical structures of this  $k$ -tangent bundle for defining a family of  $k$  Lagrangian forms  $\omega_L^A \in \Omega^2(T_k^1Q)$ , and the Energy Lagrangian function  $E_L \in C^\infty(T_k^1Q)$  (see [6], [11]). Then, if the Lagrangian is regular,  $(T_k^1Q, \omega_L^A, E_L)$  is a  $k$ -symplectic Hamiltonian system with Hamiltonian function  $E_L$ , and all the definitions and results in Sections 3 and 4 are applied to this case.

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