UNIFIED FORMALISM FOR NON-AUTONOMOUS MECHANICAL SYSTEMS

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Abstract

We present a unified geometric framework for describing both the Lagrangian and Hamiltonian formalisms of regular and non-regular time-dependent mechanical systems, which is based on the approach of Skinner and Rusk [18]. The dynamical equations of motion and their compatibility and consistency are carefully studied, making clear that all the characteristics of the Lagrangian and the Hamiltonian formalisms are recovered in this formulation. As an example, it is studied a semidiscretization of the nonlinear wave equation proving the applicability of the proposed formalism.

Key words: Lagrangian and Hamiltonian formalisms; autonomous mechanics, symplectic and presymplectic manifolds.

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1 Introduction

In 1983 Skinner and Rusk introduced a representation of the dynamics of an autonomous mechanical system which combines the Lagrangian and Hamiltonian features [18]. The aim of this formulation was to obtain a common framework for both regular and singular dynamics, obtaining simultaneously the Hamiltonian and Lagrangian formulations of the dynamics. Over the

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years, however, Skinner and Rusk's framework was extended in many directions. So, Cantrijn *et al* [2] extended this formalism for explicit time-dependent systems using a jet bundle language. In [6] an extension of this formalism to other kinds of more general time-dependent singular differential equations was given. Cortés *et al* [3] used the Skinner and Rusk formalism to consider vakonomic mechanics and the comparison between the solutions of vakonomic and nonholonomic mechanics. Finally, in [4, 9, 15] the Skinner-Rusk model was developed for classical field theories.

The aim of this paper is to continue the study of the the Skinner-Rusk formalism for time dependent mechanical systems (Section 3), now, carefully studying the dynamical equations of motion and the submanifolds where they are consistent, and showing how the Lagrangian and Hamiltonian descriptions are recovered from this unified framework (Sections 4,5).

The case of field theories was independently developed in [4, 9], and improves the construction given in [2], as it is discussed in Section 7.

As a new application, we analyze the case of semidiscretizations of field theories in Section 6. These methods are designed by numerical schemes that respect physical principles preserved by the continuous systems, specially those described by partial differential equations (PDEs). In this case, there are not only a time dependence (as in ordinary differential equations) but also posses an spatial dependence. Many integration methods, in particular in Hamiltonian dynamics, starts by discretizing the spatial structure (spatial truncation) obtaining a finite dimensional system of ordinary differential equations (ODEs) retaining some physical properties of the original system (see [8]). For simplicity, we restrict ourselves to a particular semidiscretization of the nonlinear wave equation [10, 13] obtaining a unique solution of the dynamics on the secondary constraint submanifold.

All the manifolds are real, second countable and $\mathcal{C}^\infty$. The maps are assumed to be $\mathcal{C}^\infty$. Sum over repeated indices is understood.

2 Non-autonomous Lagrangian and Hamiltonian systems

(See [5, 7, 12, 14, 16] for more details). In the jet bundle description of non-autonomous dynamical systems, the configuration bundle is $\pi\colon E \longrightarrow \mathbb{R}$, where E is a $(n+1)$-dimensional differentiable manifold endowed with local coordinates (t, q^i), and $\mathbb{R}$ has t as a global coordinate. The jet bundle of local sections of π, $J^1\pi$, is the *velocity phase space* of the system, with natural coordinates (t, q^i, v^i), adapted to the bundle $\pi\colon E \longrightarrow \mathbb{R}$, and natural projections are

$$\pi^1\colon J^1\pi \longrightarrow E \quad , \quad \bar{\pi}^1\colon J^1\pi \longrightarrow \mathbb{R}\,.$$

(If $E \equiv \mathbb{R}\times Q$, where Q is a n-dimensional differentiable manifold, then $J^1\pi \simeq \mathbb{R}\times \mathrm{T}Q$).

A Lagrangian density $\mathcal{L} \in \mathit{\Omega}^1(J^1\pi)$ is a $\bar{\pi}^1$-semibasic 1-form on $J^1\pi$, and it is usually written as $\mathcal{L} = L\,\mathrm{d}t$, where $L \in \mathrm{C}^\infty(J^1\pi)$ is the *Lagrangian function* determined by $\mathcal{L}$. Throughout this paper we denote by $\mathrm{d}t$ the volume form in $\mathbb{R}$, and its pull-backs to all the manifolds.

The *Poincaré-Cartan forms* associated with the Lagrangian density $\mathcal{L}$ are defined using the *vertical endomorphism* $\mathcal{V}$ of the bundle $J^1\pi$ (see [5, 17])

$$\Theta_{\mathcal{L}} = i(\mathcal{V})\mathrm{d}\mathcal{L} + \mathcal{L} \in \mathit{\Omega}^1(J^1\pi) \quad ; \quad \Omega_{\mathcal{L}} = -\mathrm{d}\Theta_{\mathcal{L}} \in \mathit{\Omega}^2(J^1\pi)\,.$$

A Lagrangian $\mathcal{L}$ is *regular* if $\Omega_{\mathcal{L}}$ has maximal rank; elsewhere $\mathcal{L}$ is singular. In natural coordinates we have $\mathcal{V} = ({\rm d}q^i - v^i{\rm d}t) \otimes \dfrac{\partial}{\partial v^i} \otimes \dfrac{\partial}{\partial t}$, and

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \frac{\partial L}{\partial v^i}{\rm d}q^i - \left(\frac{\partial L}{\partial v^i}v^i - L\right){\rm d}t \\
\Omega_{\mathcal{L}} &= -\frac{\partial^2 L}{\partial v^j \partial v^i}{\rm d}v^j \wedge {\rm d}q^i - \frac{\partial^2 L}{\partial q^j \partial v^i}{\rm d}q^j \wedge {\rm d}q^i \\
& \quad + \frac{\partial^2 L}{\partial v^j \partial v^i}v^i{\rm d}v^j \wedge {\rm d}t + \left(\frac{\partial^2 L}{\partial q^j \partial v^i}v^i - \frac{\partial L}{\partial q^j} + \frac{\partial^2 L}{\partial t \partial v^j}\right){\rm d}q^j \wedge {\rm d}t \, .
\end{aligned}$$

The regularity condition is equivalent to $det\left(\dfrac{\partial^2 L}{\partial v^i \partial v^j}(\bar{y})\right) \neq 0$, for every $\bar{y} \in J^1\pi$. Geometrically, $\mathcal{L}$ is regular if and only if $(\Omega_{\mathcal{L}}, {\rm d}t)$ is a cosymplectic structure on $J^1\pi$. This means that $\Omega_{\mathcal{L}}$ and ${\rm d}t$ are closed and $\Omega_{\mathcal{L}}^n \wedge {\rm d}t$ is a volume form (see [11]).

The *Lagrangian problem* consists in finding sections $\phi\colon \mathbb{R} \longrightarrow E$ of π, characterized by

$$(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = 0 \quad , \quad \text{for every } X \in \mathfrak{X}(J^1\pi)$$

where $j^1\phi : \mathbb{R} \longrightarrow J^1\pi$ is the 1-jet extension of ϕ. In natural coordinates, if $\phi(t) = (t, \phi^i(t))$, this condition is equivalent to demanding that ϕ satisfies the *Euler-Lagrange equations*

$$\left.\frac{\partial L}{\partial q^i}\right|_{j^1\phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)\Big|_{j^1\phi} = 0 \quad , \quad (\text{for } i = 1, \ldots, n)$$

where $j^1\phi(t) = (t, \phi^i(t), \dot{\phi}^i(t))$. Assuming that these sections are integral curves of vector fields in $J^1\pi$ the corresponding equations for these vector fields are

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0 \quad , \quad i(X_{\mathcal{L}}){\rm d}t = 1 \tag{1}$$

where $X_{\mathcal{L}} \in \mathfrak{X}(J^1\pi)$ is holonomic (recall that a vector field in $J^1\pi$ is said to be holonomic, or also a *second order differential equation* (SODE for simplicity), if its integral curves are holonomic; that is, canonical liftings of sections $\varphi\colon \mathbb{R} \longrightarrow E$). In the regular case, there is a unique solution to these equations. In the singular case the existence of a solution is not assured, except perhaps on some submanifold (or subset) of $J^1\pi$, where the solution is not unique, in general.

Consider now the *extended momentum phase space* ${\rm T}^*E$, and the *restricted momentum phase space* which is defined by $J^1\pi^* = {\rm T}^*E/\pi^*{\rm T}^*\mathbb{R}$. Local coordinates in these manifolds are (t, q^i, p, p_i) and (t, q^i, p_i), respectively. Then, the following natural projections are

$$\tau^1\colon J^1\pi^* \longrightarrow E \quad , \quad \bar{\tau}^1 = \pi \circ \tau^1\colon J^1\pi^* \longrightarrow \mathbb{R} \quad , \quad \mu\colon {\rm T}^*E \longrightarrow J^1\pi^* \quad , \quad p\colon {\rm T}^*E \longrightarrow \mathbb{R}\, .$$

Let $\Theta \in \mathit{\Omega}^1({\rm T}^*E)$ and $\Omega = -{\rm d}\Theta \in \mathit{\Omega}^2({\rm T}^*E)$ be the canonical forms of ${\rm T}^*E$ whose local expressions are

$$\Theta = p_i{\rm d}q^i + p{\rm d}t \quad , \quad \Omega = {\rm d}q^i \wedge {\rm d}p_i + {\rm d}t \wedge {\rm d}p\, .$$

(In the particular case $E = \mathbb{R} \times Q$, we have ${\rm T}^*E \simeq \mathbb{R} \times \mathbb{R}^* \times {\rm T}^*Q$, and $J^1\pi^* \simeq \mathbb{R} \times {\rm T}^*Q$ and introducing the projections $pr_1\colon {\rm T}^*(\mathbb{R} \times Q) \longrightarrow \mathbb{R} \times \mathbb{R}^*$, $pr_2\colon {\rm T}^*(\mathbb{R} \times Q) \longrightarrow {\rm T}^*Q$, we have $\Theta = pr_1^*\Theta_{\mathbb{R}} + pr_2^*\Theta_Q$ and $\Omega = pr_1^*\Omega_{\mathbb{R}} + pr_2^*\Omega_Q$; where $\Omega_{\mathbb{R}} = -{\rm d}\Theta_{\mathbb{R}} \in \mathit{\Omega}^2(\mathbb{R} \times \mathbb{R}^*)$ and $\Omega_Q = -{\rm d}\Theta_Q \in \mathit{\Omega}^2({\rm T}^*Q)$ denote the natural symplectic forms of $\mathbb{R} \times \mathbb{R}^*$ and ${\rm T}^*Q$).

Being $\Theta_{\mathcal{L}} \in \mathit{\Omega}^1(J^1\pi)$ π^1-semibasic, we have a natural map $\widetilde{\mathcal{FL}}\colon J^1\pi \longrightarrow {\rm T}^*E$, given by

$$\widetilde{\mathcal{FL}}(\bar{y}) = \Theta_{\mathcal{L}}(\bar{y}) \tag{2}$$

which is called the *extended Legendre map* associated to the Lagrangian density $\mathcal{L}$. The *restricted Legendre map* is $\mathcal{FL} = \mu \circ \widetilde{\mathcal{FL}} \colon J^1\pi \longrightarrow J^1\pi^*$. Their local expressions are

$$\widetilde{\mathcal{FL}}^* t = t \quad , \quad \widetilde{\mathcal{FL}}^* q^i = q^i \quad , \quad \widetilde{\mathcal{FL}}^* p_i = \frac{\partial L}{\partial v^i} \quad , \quad \widetilde{\mathcal{FL}}^* p = L - v^i \frac{\partial L}{\partial v^i}$$
$$\mathcal{FL}^* t = t \quad , \quad \mathcal{FL}^* q^i = q^i \quad , \quad \mathcal{FL}^* p_i = \frac{\partial L}{\partial v^i}$$

or, in other words, $\widetilde{\mathcal{FL}}(t, q^i, \dot{q}^i) = (t, q^i, L - v^i \frac{\partial L}{\partial v^i}, \frac{\partial L}{\partial v^i})$ and $\mathcal{FL}(t, q^i, \dot{q}^i) = (t, q^i, \frac{\partial L}{\partial v^i})$. Moreover, we have $\widetilde{\mathcal{FL}}^* \Theta = \Theta_{\mathcal{L}}$, and $\widetilde{\mathcal{FL}}^* \Omega = \Omega_{\mathcal{L}}$.

The Lagrangian $\mathcal{L}$ is regular if, and only if, $\mathcal{FL}$ is a local diffeomorphism. As a particular case, $\mathcal{L}$ is a *hyper-regular* Lagrangian if $\mathcal{FL}$ is a global diffeomorphism.

If $\mathcal{L}$ is a hyper-regular Lagrangian, then $\tilde{\mathcal{P}} = \widetilde{\mathcal{FL}}(J^1\pi)$ is a 1-codimensional, μ-transverse embedded submanifold of T^*E, with natural embedding $\tilde{\jmath}_0 \colon \tilde{\mathcal{P}} \hookrightarrow \mathrm{T}^*E$, which is diffeomorphic to $J^1\pi^*$. This diffeomorphism is the inverse of μ restricted to $\tilde{\mathcal{P}}$, and also coincides with the map $h = \widetilde{\mathcal{FL}} \circ \mathcal{FL}^{-1}$, when it is restricted onto its image (which is just $\tilde{\mathcal{P}}$). This map h is called a *Hamiltonian section*, and is used to construct the *Hamilton-Cartan forms* in $J^1\pi^*$ by making

$$\Theta_h = h^*\Theta \in \mathit{\Omega}^1(J^1\pi^*) \quad , \quad \Omega_h = h^*\Omega \in \mathit{\Omega}^2(J^1\pi^*)\,.$$

Locally, the Hamiltonian section h is specified by $h(t, q^i, p_i) = (t, q^i, -H, p_i)$, where H is the local Hamiltonian function given by $H = p_i(F\mathcal{L}^{-1})^* v^i - (F\mathcal{L}^{-1})^* L$. The local expressions are

$$\Theta_h = p_i \mathrm{d}q^i - H\mathrm{d}t \quad , \quad \Omega_h = \mathrm{d}q^i \wedge \mathrm{d}p_i + \mathrm{d}H \wedge \mathrm{d}t\,.$$

Of course $\mathcal{FL}^*\Theta_h = \Theta_{\mathcal{L}}$, and $\mathcal{FL}^*\Omega_h = \Omega_{\mathcal{L}}$.

The *Hamiltonian problem* consists in finding sections of $\bar{\tau}^1$, $\psi \colon \mathbb{R} \longrightarrow J^1\pi^*$, characterized by

$$\psi^*\, i(X)\Omega_h = 0 \quad , \quad \text{for every } X \in \mathfrak{X}(J^1\pi^*)\ .$$

This condition leads to the *Hamilton equations* which, if $\psi(t) = (t, q^i(t), p_i(t))$, in natural coordinates are

$$\frac{dq^i}{dt} = \left.\frac{\partial H}{\partial p_i}\right|_{\psi} \quad ; \quad \frac{dp_i}{dt} = -\left.\frac{\partial H}{\partial q^i}\right|_{\psi} .$$

Assuming that these sections are integral curves of vector fields $X_h \in \mathfrak{X}(J^1\pi^*)$, the corresponding equations for these vector fields are

$$i(X_h)\Omega_h = 0 \quad , \quad i(X_h)\mathrm{d}t = 1\,.$$

As a final remark, it can be proved that solutions to the Lagrangian and Hamiltonian problems are equivalent, in the sense that they are $\mathcal{FL}$-related; that is,

$$\psi = \mathcal{FL} \circ j^1\phi \quad ; \quad \mathrm{T}\mathcal{FL} \circ X_{\mathcal{L}} = X_h \circ \mathcal{FL}\,. \tag{3}$$

For regular, but not hyper-regular systems, the results are the same, but only locally on open neighbourhoods at every point, instead of $J^1\pi^*$.

A singular Lagrangian $\mathcal{L}$ is *almost-regular* if: $\mathcal{P} = \mathcal{FL}(J^1\pi)$ is a closed submanifold of $J^1\pi^*$ (let $\jmath \colon \mathcal{P} \hookrightarrow J^1\pi^*$ be natural embedding), $\mathcal{FL}$ is a submersion onto its image, and for every $\bar{y} \in J^1\pi$, the fibres $\mathcal{FL}^{-1}(\mathcal{FL}(\bar{y}))$ are connected submanifolds of $J^1\pi$.

If $\mathcal{L}$ is an almost-regular Lagrangian, the submanifold $\mathcal{P}$ of $J^1\pi^*$ is a fibre bundle over E and M. In this case the μ-transverse submanifold $\tilde{\jmath} \colon \tilde{\mathcal{P}} \hookrightarrow \mathrm{T}^*E$ is diffeomorphic to $\mathcal{P}$. This diffeomorphism is denoted by $\tilde{\mu}\colon \tilde{\mathcal{P}} \longrightarrow \mathcal{P}$, and is just the restriction of the projection μ to $\tilde{\mathcal{P}}$. Then, taking the Hamiltonian section $\tilde{h} = \tilde{\jmath} \circ \tilde{\mu}^{-1}$, we define the forms

$$\Theta_h^0 = \tilde{h}^*\Theta \quad ; \quad \Omega_h^0 = \tilde{h}^*\Omega$$

which verify that $\mathcal{FL}_0^*\Theta_h^0 = \Theta_{\mathcal{L}}$ and $\mathcal{FL}_0^*\Omega_h^0 = \Omega_{\mathcal{L}}$ (where $\mathcal{FL}_0$ is the restriction map of $\mathcal{FL}$ onto $\mathcal{P}$). Then, the Hamiltonian problem and the equations of motion are stated as in the hyper-regular case. Now, the existence of a solution to these equations is not assured, except perhaps on some submanifold of $\mathcal{P}$, where the solution is not unique, in general.

3 Unified formalism

We define the *extended jet-momentum bundle* $\mathcal{W}$ and the *restricted jet-momentum bundle* $\mathcal{W}_r$

$$\mathcal{W} = J^1\pi \times_E \mathrm{T}^*E \quad , \quad \mathcal{W}_r = J^1\pi \times_E J^1\pi^*$$

with natural coordinates (t, q^i, v^i, p, p_i) and (t, q^i, v^i, p_i), respectively. Natural submersions are

$$\begin{array}{c} \rho_1\colon \mathcal{W} \longrightarrow J^1\pi \ , \ \rho_2\colon \mathcal{W} \longrightarrow \mathrm{T}^*E \ , \ \rho_{_E}\colon \mathcal{W} \longrightarrow E \ , \ \rho_{_\mathbb{R}}\colon \mathcal{W} \longrightarrow \mathbb{R} \\ \rho_1^r\colon \mathcal{W}_r \longrightarrow J^1\pi \ , \ \rho_2^r\colon \mathcal{W}_r \longrightarrow J^1\pi^* \ , \ \rho_{_E}^r\colon \mathcal{W}_r \longrightarrow E \ , \ \rho_{_\mathbb{R}}^r\colon \mathcal{W}_r \longrightarrow \mathbb{R} \, , \end{array} \tag{4}$$

with $\pi^1 \circ \rho_1 = \tau^1 \circ \mu \circ \rho_2 = \rho_{_E}$. For $\bar{y} \in J^1\pi$, $\mathbf{p} \in \mathrm{T}^*E$, and $[\mathbf{p}] = \mu(\mathbf{p}) \in J^1\pi^*$, there is also the natural projection

$$\begin{array}{cccc} \mu_{\mathcal{W}} & : & \mathcal{W} & \longrightarrow & \mathcal{W}_r \\ & & (\bar{y}, \mathbf{p}) & \mapsto & (\bar{y}, [\mathbf{p}]) \end{array}$$

The bundle $\mathcal{W}$ is endowed with the following canonical structures:

Definition 1 *1. The* coupling 1-form *in* $\mathcal{W}$ *is the* $\rho_{_\mathbb{R}}$*-semibasic* 1*-form* $\hat{\mathcal{C}} \in \mathit{\Omega}^1(\mathcal{W})$ *defined as follows: for every* $w = (j^1\phi(t), \alpha) \in \mathcal{W}$ *(that is,* $\alpha \in T^*_{\rho_E(w)}E$*) and* $V \in \mathrm{T}_w\mathcal{W}$*, then*

$$\hat{\mathcal{C}}(V) = \alpha(\mathrm{T}_w(\phi \circ \rho_{_\mathbb{R}})V) \, .$$

2. The canonical 1-form $\Theta_{\mathcal{W}} \in \mathit{\Omega}^1(\mathcal{W})$ *is the* $\rho_{_E}$*-semibasic form defined by* $\Theta_{\mathcal{W}} = \rho_2^*\Theta$*.*
The canonical 2-form *is* $\Omega_{\mathcal{W}} = -\mathrm{d}\Theta_{\mathcal{W}} = \rho_2^*\Omega \in \mathit{\Omega}^2(\mathcal{W})$*.*

Being $\hat{\mathcal{C}}$ a $\rho_{_\mathbb{R}}$-semibasic form, there is $\hat{C} \in \mathrm{C}^\infty(\mathcal{W})$ such that $\hat{\mathcal{C}} = \hat{C}\mathrm{d}t$. Note also that $\Omega_{\mathcal{W}}$ is degenerate, its kernel being the ρ_2-vertical vectors; then $(\mathcal{W}, \Omega_{\mathcal{W}})$ is a presymplectic manifold.

The local expressions for $\Theta_{\mathcal{W}}$, $\Omega_{\mathcal{W}}$, and $\hat{\mathcal{C}}$ are

$$\Theta_{\mathcal{W}} = p_i\mathrm{d}q^i + p\mathrm{d}t \quad , \quad \Omega_{\mathcal{W}} = -\mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}p \wedge \mathrm{d}t \quad , \quad \hat{\mathcal{C}} = (p + p_iv^i)\mathrm{d}t \, .$$

Given a Lagrangian density $\mathcal{L} \in \mathit{\Omega}^1(J^1\pi)$, we denote $\hat{\mathcal{L}} = \rho_1^*\mathcal{L} \in \mathit{\Omega}^1(\mathcal{W})$, and we can write $\hat{\mathcal{L}} = \hat{L}\mathrm{d}t$, with $\hat{L} = \rho_1^*L \in \mathrm{C}^\infty(\mathcal{W})$. We define a *Hamiltonian submanifold*

$$\mathcal{W}_0 = \{w \in \mathcal{W} \mid \hat{\mathcal{L}}(w) = \hat{\mathcal{C}}(w)\} \, .$$

So, $\mathcal{W}_0$ is the submanifold of $\mathcal{W}$ defined by the regular constraint function $\hat{C} - \hat{L} = 0$, which is globally defined in $\mathcal{W}$ using the dynamical data and the geometry. In local coordinates it is

$$\hat{C} - \hat{L} = p + p_iv^i - \hat{L}(t, q^j, v^j) = 0 \, .$$

The natural embedding is $\jmath_0 \colon \mathcal{W}_0 \hookrightarrow \mathcal{W}$. We have the projections (submersions), see diagram (5):

$$\rho_1^0 \colon \mathcal{W}_0 \longrightarrow J^1\pi \ , \ \rho_2^0 \colon \mathcal{W}_0 \longrightarrow \mathrm{T}^*E \ , \ \rho_E^0 \colon \mathcal{W}_0 \longrightarrow E \ , \ \rho_{\mathbb{R}}^0 \colon \mathcal{W}_0 \longrightarrow \mathbb{R}$$

which are the restrictions to $\mathcal{W}_0$ of the projections (4), and

$$\hat{\rho}_2^0 = \mu \circ \rho_2^0 \colon \mathcal{W}_0 \longrightarrow J^1\pi^* \, .$$

Local coordinates in $\mathcal{W}_0$ are (t, q^i, v^i, p_i), and we have that

$$\begin{array}{lcl} \rho_1^0(t,q^i,v^i,p_i) = (t,q^i,v^i) & , & \jmath_0(t,q^i,v^i,p_i) = (t,q^i,v^i,L - v^i p_i, p_i) \\ \hat{\rho}_2^0(t,q^i,v^i,p_i) = (t,q^i,p_i) & , & \rho_2^0(t,q^i,v^i,p_i) = (t,q^i,L - v^i p_i, p_i) \, . \end{array}$$

Proposition 1 *$\mathcal{W}_0$ is a 1-codimensional $\mu_{\mathcal{W}}$-transverse submanifold of $\mathcal{W}$, diffeomorphic to $\mathcal{W}_r$.*

(*Proof*) For every $(\bar{y}, \mathbf{p}) \in \mathcal{W}_0$, we have $L(\bar{y}) \equiv \hat{L}(\bar{y}, \mathbf{p}) = \hat{C}(\bar{y}, \mathbf{p})$, and

$$(\mu_{\mathcal{W}} \circ \jmath_0)(\bar{y}, \mathbf{p}) = \mu_{\mathcal{W}}(\bar{y}, \mathbf{p}) = (\bar{y}, \mu(\mathbf{p})) \ .$$

First, $\mu_{\mathcal{W}} \circ \jmath_0$ is injective: let $(\bar{y}_1, \mathbf{p}_1), (\bar{y}_2, \mathbf{p}_2) \in \mathcal{W}_0$, then we have

$$(\mu_{\mathcal{W}} \circ \jmath_0)(\bar{y}_1, \mathbf{p}_1) = (\mu_{\mathcal{W}} \circ \jmath_0)(\bar{y}_2, \mathbf{p}_2) \ \Rightarrow \ (\bar{y}_1, \mu(\mathbf{p}_1)) = (\bar{y}_2, \mu(\mathbf{p}_2)) \ \Rightarrow \ \bar{y}_1 = \bar{y}_2 \ , \ \mu(\mathbf{p}_1) = \mu(\mathbf{p}_2)$$

hence $L(\bar{y}_1) = L(\bar{y}_2) = \hat{C}(\bar{y}_1, \mathbf{p}_1) = \hat{C}(\bar{y}_2, \mathbf{p}_2)$. In a local chart, the third equality gives

$$p(\mathbf{p}_1) + p_i(\mathbf{p}_1) v^i(\bar{y}_1) = p(\mathbf{p}_2) + p_i(\mathbf{p}_2) v^i(\bar{y}_2)$$

but $\mu(\mathbf{p}_1) = \mu(\mathbf{p}_2)$ implies $p_i(\mathbf{p}_1) = p_i([\mathbf{p}_1]) = p_i([\mathbf{p}_2]) = p_i(\mathbf{p}_2)$; then $p(\mathbf{p}_1) = p(\mathbf{p}_2)$, and $\mathbf{p}_1 = \mathbf{p}_2$.

Second, $\mu_{\mathcal{W}} \circ \jmath_0$ is onto, then, if $(\bar{y}, [\mathbf{p}]) \in \mathcal{W}_r$, there exists $(\bar{y}, \mathbf{q}) \in \jmath_0(\mathcal{W}_0)$ such that $[\mathbf{q}] = [\mathbf{p}]$. In fact, it suffices to take $[\mathbf{q}]$ such that, in a local chart of $J^1\pi \times_E \mathrm{T}^*E = \mathcal{W}$

$$p_i(\mathbf{q}) = p_i([\mathbf{p}]) \ , \ p(\mathbf{q}) = L(\bar{y}) - p_i([\mathbf{p}]) v^i(\bar{y}) \, .$$

Finally, since $\mathcal{W}_0$ is defined by the constraint function $\hat{C} - \hat{L}$ and, as $\ker \mu_{\mathcal{W}*} = \left\{ \frac{\partial}{\partial p} \right\}$ locally and $\dfrac{\partial}{\partial p}(\hat{C} - \hat{L}) = 1$, then $\mathcal{W}_0$ is $\mu_{\mathcal{W}}$-transversal. ∎

As a consequence of this result, the submanifold $\mathcal{W}_0$ induces a section $\hat{h} \colon \mathcal{W}_r \longrightarrow \mathcal{W}$ of the projection $\mu_{\mathcal{W}}$. Locally, $\hat{h}$ is specified by giving the local *Hamiltonian function* $\hat{H} = -\hat{L} + p_i v^i$; that is, $\hat{h}(t, q^i, v^i, p_i) = (t, q^i, v^i, -\hat{H}, p_i)$. In this sense, $\hat{h}$ is a *Hamiltonian section* of $\mu_{\mathcal{W}}$.

So we have the following diagram

$$\begin{tikzcd}
 & J^1\pi & \\
\mathcal{W}_0 \arrow[ur, "\rho_1^0"] \arrow[r, "\jmath_0"] \arrow[dr, "\rho_2^0"] \arrow[ddr, "\hat{\rho}_2^0"'] & \mathcal{W} \arrow[u, "\rho_1"] \arrow[r, "\mu_{\mathcal{W}}"] \arrow[d, "\rho_2"] & \mathcal{W}_r \arrow[ul, "\rho_1^r"'] \arrow[dl, "\rho_2 \circ \hat{h}"'] \arrow[ddl, "\rho_2^r"] \\
 & \mathrm{T}^*E \arrow[d, "\mu"] & \\
 & J^1\pi^* &
\end{tikzcd} \tag{5}$$

Remark: Observe that, from the Hamiltonian $\mu_{\mathcal{W}}$-section $\hat{h}\colon \mathcal{W}_r \longrightarrow \mathcal{W}$ in the extended unified formalism, we can recover the Hamiltonian μ-section $\tilde{h} = \tilde{\jmath} \circ \tilde{\mu}^{-1}\colon \mathcal{P} \longrightarrow \mathrm{T}^*E$ in the standard Hamiltonian formalism assuming that $\mathcal{L}$ is almost-regular. In fact, given $[\mathbf{p}] \in J^1\pi^*$, the section $\hat{h}$ maps every point $(\bar{y}, [\mathbf{p}]) \in (\rho_2^r)^{-1}([\mathbf{p}])$ into $\rho_2^{-1}[\rho_2(\hat{h}(\bar{y}, [\mathbf{p}]))]$. Now, the crucial point is the projectability of the local function $\hat{H}$ by ρ_2. However, $\dfrac{\partial}{\partial v^i}$ being a local basis for $\ker \rho_{2*}$, $\hat{H}$ is ρ_2-projectable if, and only if, $p_i = \dfrac{\partial L}{\partial v^i}$, and this condition is fulfilled when $[\mathbf{p}] \in \mathcal{P} = \mathrm{Im}\,\mathcal{FL} \subset J^1\pi^*$, which implies that $\rho_2[\hat{h}((\rho_2^r)^{-1}([\mathbf{p}]))] \in \tilde{\mathcal{P}} = \mathrm{Im}\,\widetilde{\mathcal{FL}} \subset \mathrm{T}^*E$. Then, the Hamiltonian section $\tilde{h}$ is defined as

$$\tilde{h}([\mathbf{p}]) = (\rho_2 \circ \hat{h})[(\rho_2^r)^{-1}(\jmath([\mathbf{p}]))] = (\tilde{\jmath} \circ \tilde{\mu}^{-1})([\mathbf{p}]) \ , \ \text{for every } [\mathbf{p}] \in \mathcal{P}.$$

So we have the diagram

$$\begin{tikzcd}
\tilde{\mathcal{P}} \arrow[r, "\tilde{\jmath}"] \arrow[d, shift left, "\tilde{\mu}"] & T^*E \arrow[d, "\mu"] & \mathcal{W} \arrow[l, "\rho_2"'] \\
\mathcal{P} \arrow[u, shift left, "\tilde{\mu}^{-1}"] \arrow[ur, "\tilde{h}"] \arrow[r, "\jmath"'] & J^1\pi^* & \mathcal{W}_r \arrow[l, "\rho_2^r"] \arrow[u, "\hat{h}"']
\end{tikzcd}$$

For (hyper) regular systems this diagram is the same with $\mathcal{P} = \mathrm{Im}\,\mathcal{FL} = J^1\pi^*$.

Finally, we can define the forms

$$\Theta_0 = \jmath_0^*\Theta_{\mathcal{W}} = \rho_2^{0*}\Theta \in \mathit{\Omega}^1(\mathcal{W}_0) \quad , \quad \Omega_0 = \jmath_0^*\Omega_{\mathcal{W}} = \rho_2^{0*}\Omega \in \mathit{\Omega}^2(\mathcal{W}_0)$$

with local expressions

$$\Theta_0 = (L - p_i v^i)\mathrm{d}t + p_i \mathrm{d}q^i \quad , \quad \Omega_0 = \mathrm{d}(p_i v^i - L) \wedge \mathrm{d}t - \mathrm{d}p_i \wedge \mathrm{d}q^i \tag{6}$$

and we have the presymplectic Hamiltonian systems $(\mathcal{W}_0, \Omega_0)$ and $(\mathcal{W}_r, \Omega_r)$, with $\Omega_r = \hat{h}^*\Omega_0$.

4 The dynamical equations for sections

Now we establish the dynamical problem for the system $(\mathcal{W}_0, \Omega_0)$ which, as a consequence of the diffeomorphism stated in Proposition 1, is equivalent to making it for the system $(\mathcal{W}_r, \Omega_r)$.

The *Lagrange-Hamiltonian problem* associated with the system $(\mathcal{W}_0, \Omega_0)$ consists in finding sections of $\rho_{\mathbb{R}}^0$, $\psi_0\colon \mathbb{R} \longrightarrow \mathcal{W}_0$, which are characterized by the condition

$$\psi_0^*\, i(Y_0)\Omega_0 = 0 \quad , \quad \text{for every } Y_0 \in \mathfrak{X}(\mathcal{W}_0) \ . \tag{7}$$

This equation gives different kinds of information, depending on the type of the vector fields Y_0 involved. In particular, using $\hat{\rho}_2^0$-vertical vector fields, denoted by $\mathfrak{X}^{\mathrm{V}(\hat{\rho}_2^0)}(\mathcal{W}_0)$, we have:

Lemma 1 *If $Y_0 \in \mathfrak{X}^{\mathrm{V}(\hat{\rho}_2^0)}(\mathcal{W}_0)$, then $i(Y_0)\Omega_0$ is $\rho_{\mathbb{R}}^0$-semibasic.*

(*Proof*) A simple calculation in coordinates leads to this result. In fact, taking $\left\{\frac{\partial}{\partial v^i}\right\}$ as a local basis for the $\hat{\rho}_2^0$-vertical vector fields, and bearing in mind (6) we obtain the $\rho_{\mathbb{R}}^0$-semibasic forms

$$i\left(\frac{\partial}{\partial v^i}\right)\Omega_0 = \left(p_i - \frac{\partial L}{\partial v^i}\right)\mathrm{d}t \ .$$

■

As an immediate consequence, when $Y_0 \in \mathfrak{X}^{\mathrm{V}(\hat{\rho}_2^0)}(\mathcal{W}_0)$, condition (7) does not depend on the derivatives of ψ_0: it is a pointwise (algebraic) condition. We can define the submanifold

$$\mathcal{W}_1 = \{(\bar{y}, \mathbf{p}) \in \mathcal{W}_0 \mid i(V_0)(\Omega_0)_{(\bar{y},\mathbf{p})} = 0, \text{ for every } V_0 \in \mathrm{V}_{(\bar{y},\mathbf{p})}(\hat{\rho}_2^0)\}$$

where $\mathrm{V}(\hat{\rho}_2^0)$ denotes the $\hat{\rho}_2^0$-vertical vectors. $\mathcal{W}_1$ is called the *first constraint submanifold* of the Hamiltonian pre-multisymplectic system $(\mathcal{W}_0, \Omega_0)$, as every section ψ_0 solution to (7) must take values in $\mathcal{W}_1$. We denote by $\jmath_1 \colon \mathcal{W}_1 \hookrightarrow \mathcal{W}_0$ the natural embedding.

Locally, $\mathcal{W}_1$ is defined in $\mathcal{W}_0$ by the constraints $p_i = \dfrac{\partial L}{\partial v^i}$. Moreover:

Proposition 2 *$\mathcal{W}_1$ is the graph of $\widetilde{\mathcal{FL}}$; that is, $\mathcal{W}_1 = \{(\bar{y}, \widetilde{\mathcal{FL}}(\bar{y})) \in \mathcal{W} \mid \bar{y} \in J^1\pi\}$.*

(*Proof*) Consider $\bar{y} \in J^1\pi$, let $\phi \colon \mathbb{R} \longrightarrow E$ be a representative of $\bar{y}$, and $\mathbf{p} = \widetilde{\mathcal{FL}}(\bar{y})$. For every $U \in \mathrm{T}_{\bar{\pi}^1(\bar{y})}\mathbb{R}$, consider $V = \mathrm{T}_{\bar{\pi}^1(\bar{y})}\phi(U)$ and its canonical lifting $\bar{V} = \mathrm{T}_{\bar{\pi}^1(\bar{y})}j^1\phi(U)$. From the definition of the extended Legendre map (2) we have $(\mathrm{T}_{\bar{y}}\pi^1)^*(\widetilde{\mathcal{FL}}(\bar{y})) = (\Theta_{\mathcal{L}})_{\bar{y}}$, then

$$i(\bar{V})[(\mathrm{T}_{\bar{y}}\pi^1)^*(\widetilde{\mathcal{FL}}(\bar{y}))] = i(\bar{V})(\Theta_{\mathcal{L}})_{\bar{y}}\,.$$

Furthermore, as $\mathbf{p} = \widetilde{\mathcal{FL}}(\bar{y})$, we also have that

$$\begin{aligned} i(\bar{V})[(\mathrm{T}_{\bar{y}}\pi^1)^*(\widetilde{\mathcal{FL}}(\bar{y}))] &= i(\mathrm{T}_{\bar{\pi}^1(\bar{y})}j^1\phi(U))[(\mathrm{T}_{\bar{y}}\pi^1)^*\mathbf{p}] = i((\mathrm{T}_{\bar{y}}\pi^1)_*(\mathrm{T}_{\bar{\pi}^1(\bar{y})}j^1\phi(U)))\mathbf{p} \\ &= i(\mathrm{T}_{\bar{\pi}^1(\bar{y})}\phi(U))\mathbf{p} = i(V)\mathbf{p}\,. \end{aligned}$$

Therefore we obtain

$$i(U)(\phi^*\mathbf{p}) = i(U)[(j^1\phi)^*(\Theta_{\mathcal{L}})_{\bar{y}}]$$

and bearing in mind the definition of the coupling form $\mathcal{C}$, this condition becomes

$$i(U)(\hat{\mathcal{C}}(\bar{y}, \mathbf{p})) = i(U)[(j^1\phi)^*\Theta_{\mathcal{L}})_{\bar{y}}]\,.$$

Since it holds for every $U \in \mathrm{T}_{\bar{\pi}^1(\bar{y})}\mathbb{R}$, we conclude that $\hat{\mathcal{C}}(\bar{y}, \mathbf{p}) = [(j^1\phi)^*\Theta_{\mathcal{L}}]_{\bar{y}}$, or equivalently, $\hat{\mathcal{C}}(\bar{y}, \mathbf{p}) = \hat{L}(\bar{y}, \mathbf{p})$, where we have made use of the fact that $\Theta_{\mathcal{L}}$ is the sum of the Lagrangian density $\mathcal{L}$ and a contact form $i(\mathcal{V})\mathrm{d}\mathcal{L}$ (vanishing by pull-back of lifted sections). This is the condition defining $\mathcal{W}_0$, and thus we have proved that $(\bar{y}, \widetilde{\mathcal{FL}}(\bar{y})) \in \mathcal{W}_0$, for every $\bar{y} \in J^1\pi$; that is, graph $\widetilde{\mathcal{FL}} \subset \mathcal{W}_0$. Furthermore, graph $\widetilde{\mathcal{FL}}$ and $\mathcal{W}_1$ are defined as subsets of $\mathcal{W}_0$ by the same local conditions: $p_i - \dfrac{\partial L}{\partial v^i} = 0$. So we conclude that graph $\widetilde{\mathcal{FL}} = \mathcal{W}_1$. ■

As $\mathcal{W}_1$ is the graph of $\widetilde{\mathcal{FL}}$, it is diffeomorphic to $J^1\pi$. Every section $\psi_0 \colon \mathbb{R} \longrightarrow \mathcal{W}_0$ is of the form $\psi_0 = (\psi_{\mathcal{L}}, \psi_{\mathcal{H}})$, with $\psi_{\mathcal{L}} = \rho_1^0 \circ \psi_0 \colon \mathbb{R} \longrightarrow J^1\pi$, and if ψ_0 takes values in $\mathcal{W}_1$ then $\psi_{\mathcal{H}} = \widetilde{\mathcal{FL}} \circ \psi_{\mathcal{L}} \colon \mathbb{R} \longrightarrow \mathrm{T}^*E$. In this way every constraint, differential equation, etc. in the unified formalism can be translated to the Lagrangian or the Hamiltonian formalisms by restriction to the first or the second factors of the product bundle.

However, as was pointed out before, the geometric condition (7) in $\mathcal{W}_0$, which can be solved only for sections $\psi_0 \colon \mathbb{R} \longrightarrow \mathcal{W}_1 \subset \mathcal{W}_0$, is stronger than the Lagrangian condition $\psi_{\mathcal{L}}^*\, i(Z)\Omega_{\mathcal{L}} = 0$, (for every $Z \in \mathfrak{X}(J^1\pi)$) in $J^1\pi$, which can be translated to $\mathcal{W}_1$ by the natural diffeomorphism between them. The reason is that, as ρ_1^0 is a submersion, and $\mathcal{W}_1$ is a ρ_1^0-transversal submanifold of $\mathcal{W}_0$ (as a consequence of Proposition 2), we have the splitting $\jmath_1^*\mathrm{T}\mathcal{W}_0 = \mathrm{T}\mathcal{W}_1 \oplus_{\mathcal{W}_1} \jmath_1^*\mathrm{V}(\rho_1^0)$, $\jmath_1 \colon \mathcal{W}_1 \hookrightarrow \mathcal{W}_0$ being the natural embedding. Therefore the additional information comes from the ρ_1^0-vertical vectors, and is just the holonomic condition. In fact:

Theorem 1 *Let $\psi_0 \colon \mathbb{R} \longrightarrow \mathcal{W}_0$ be a section fulfilling equation (7), $\psi_0 = (\psi_{\mathcal{L}}, \psi_{\mathcal{H}}) = (\psi_{\mathcal{L}}, \widetilde{\mathcal{FL}} \circ \psi_{\mathcal{L}})$, where $\psi_{\mathcal{L}} = \rho_1^0 \circ \psi_0$. Then:*

1. *$\psi_{\mathcal{L}}$ is the canonical lift of the projected section $\phi = \rho_E^0 \circ \psi_0 \colon \mathbb{R} \longrightarrow E$ (that is, $\psi_{\mathcal{L}}$ is a holonomic section).*
2. *The section $\psi_{\mathcal{L}} = j^1\phi$ is a solution to the Lagrangian problem, and the section $\mu \circ \psi_{\mathcal{H}} = \mu \circ \widetilde{\mathcal{FL}} \circ \psi_{\mathcal{L}} = \mathcal{FL} \circ j^1\phi$ is a solution to the Hamiltonian problem.*

Conversely, for every section $\phi \colon \mathbb{R} \longrightarrow E$ such that $j^1\phi$ is a solution to the Lagrangian problem (and hence $\mathcal{FL} \circ j^1\phi$ is a solution to the Hamiltonian problem) we have that the section $\psi_0 = (j^1\phi, \widetilde{\mathcal{FL}} \circ j^1\phi)$, is a solution to (7).

(*Proof*) 1. Taking $\left\{ \frac{\partial}{\partial p_i} \right\}$ as a local basis for the ρ_1^0-vertical vector fields:

$$i\left(\frac{\partial}{\partial p_i}\right)\Omega_0 = v^i \mathrm{d}t - \mathrm{d}q^i$$

so that for a section ψ_0 we have

$$0 = \psi_0^* \left[i\left(\frac{\partial}{\partial p_i}\right)\Omega_0 \right] = \left(v^i - \frac{\partial q^i}{\partial t} \right) \mathrm{d}t$$

and thus the holonomy condition appears naturally within the unified formalism. So we have that $\psi_0 = \left(t, q^i, \dfrac{dq^i}{dt}, \dfrac{\partial L}{\partial v^i} \right)$, since ψ_0 takes values in $\mathcal{W}_1$, and hence it is of the form $\psi_0 = (j^1\phi, \widetilde{\mathcal{FL}} \circ j^1\phi)$, for $\phi = (t, q^i) = \rho_E^0 \circ \psi_0$.

2. Consider the diagram

$$\begin{tikzcd}
 & & \mathcal{W} \arrow[dll, "\rho_1"'] \arrow[dr, "\rho_2"] & & & \\
 & & \mathcal{W}_0 \arrow[u, "\jmath_0"] \arrow[r, "\rho_2^0"] \arrow[dll, "\rho_1^0"] & \mathrm{T}^*E & & \\
J^1\pi \arrow[drr, "\pi^1"] & & \mathcal{W}_1 \arrow[u, "\jmath_1"] \arrow[ll, "\rho_1^1"'] \arrow[r, "\rho_2^1"] \arrow[d, "\rho_E^1"] & J^1\pi^* \arrow[dl, "\tau^1"] & & \mathrm{T}^*E \\
 & & E & & & \\
 & & \mathbb{R} \arrow[u, "\phi"] \arrow[uu, bend right, "\psi_1"'] \arrow[uuu, bend right, "\psi_0"'] \arrow[uull, "\psi_{\mathcal{L}} = j^1\phi"] \arrow[uurrr, "\psi_{\mathcal{H}} = \widetilde{\mathcal{FL}} \circ j^1\phi"'] & & &
\end{tikzcd}$$

Since sections $\psi_0 \colon \mathbb{R} \longrightarrow \mathcal{W}_0$ solution to (7) take values in $\mathcal{W}_1$, we can identify them with sections $\psi_1 \colon \mathbb{R} \longrightarrow \mathcal{W}_1$. These sections ψ_1 verify, in particular, that $\psi_1^* \, i(Y_1)\Omega_1 = 0$ holds for every $Y_1 \in \mathfrak{X}(\mathcal{W}_1)$. Obviously $\psi_0 = \jmath_1 \circ \psi_1$. Moreover, as $\mathcal{W}_1$ is the graph of $\widetilde{\mathcal{FL}}$, denoting by $\rho_1^1 = \rho_1^0 \circ \jmath_1 \colon \mathcal{W}_1 \longrightarrow J^1\pi$ the diffeomorphism which identifies $\mathcal{W}_1$ with $J^1\pi$, if we define $\Omega_1 = \jmath_1^*\Omega_0$, we have that $\Omega_1 = \rho_1^{1*}\Omega_{\mathcal{L}}$. In fact; as $(\rho_1^1)^{-1}(\bar{y}) = (\bar{y}, \widetilde{\mathcal{FL}}(\bar{y}))$, for every $\bar{y} \in J^1\pi$, then $(\rho_2^0 \circ \jmath_1 \circ (\rho_1^1)^{-1})(\bar{y}) = \widetilde{\mathcal{FL}}(\bar{y}) \in \mathrm{T}^*E$, and hence

$$\Omega_{\mathcal{L}} = (\rho_2^0 \circ \jmath_1 \circ (\rho_1^1)^{-1})^*\Omega = [((\rho_1^1)^{-1})^* \circ \jmath_1^* \circ \rho_2^{0*}]\Omega = [((\rho_1^1)^{-1})^* \circ \jmath_1^*]\Omega_0 = ((\rho_1^1)^{-1})^*\Omega_1 \,.$$

Now, let $X \in \mathfrak{X}(J^1\pi)$. We have

$$\begin{aligned} (j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} &= (\rho_1^0 \circ \psi_0)^*\, i(X)\Omega_{\mathcal{L}} = (\rho_1^0 \circ \jmath_1 \circ \psi_1)^*\, i(X)\Omega_{\mathcal{L}} \\ &= (\rho_1^1 \circ \psi_1)^*\, i(X)\Omega_{\mathcal{L}} = \psi_1^*\, i((\rho_1^1)_*^{-1} X)(\rho_1^{1*}\Omega_{\mathcal{L}}) = \psi_1^*\, i(Y_1)\Omega_1 \\ &= \psi_1^*\, i(Y_1)(\jmath_1^*\Omega_0) = (\psi_1^* \circ \jmath_1^*)\, i(Y_0)\Omega_0 = \psi_0^*\, i(Y_0)\Omega_0 \end{aligned} \tag{8}$$

where $Y_0 \in \mathfrak{X}(\mathcal{W}_0)$ is such that $Y_0 = \jmath_{1*}Y_1$. But as $\psi_0^*\, i(Y_0)\Omega_0 = 0$, for every $Y_0 \in \mathfrak{X}(\mathcal{W}_0)$, then we conclude that $(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = 0$, for every $X \in \mathfrak{X}(J^1\pi)$.

Conversely, let $j^1\phi\colon \mathbb{R} \longrightarrow J^1\pi$ such that $(j^1\phi)^*\, i(X)\Omega_{\mathcal{L}} = 0$, for every $X \in \mathfrak{X}(J^1\pi)$, and define $\psi_0\colon \mathbb{R} \longrightarrow \mathcal{W}_0$ as $\psi_0 = (j^1\phi, \widetilde{\mathcal{FL}} \circ j^1\phi)$ (observe that ψ_0 takes its values in $\mathcal{W}_1$). Taking into account that, on the points of $\mathcal{W}_1$, every $Y_0 \in \mathfrak{X}(\mathcal{W}_0)$ splits into $Y_0 = Y_0^1 + Y_0^2$, with $Y_0^1 \in \mathfrak{X}(\mathcal{W}_0)$ tangent to $\mathcal{W}_1$, and $Y_0^2 \in \mathfrak{X}^{\mathrm{V}(\rho_1^0)}(\mathcal{W}_0)$, we have that

$$\psi_0^*\, i(Y_0)\Omega_0 = \psi_0^*\, i(Y_0^1)\Omega_0 + \psi_0^*\, i(Y_0^2)\Omega_0 = 0$$

since for Y_0^1 the same reasoning as in (8) leads to

$$\psi_0^*\, i(Y_0^1)\Omega_0 = (j^1\phi)^*\, i(X_0^1)\Omega_{\mathcal{L}} = 0$$

(where $X_0^1 = (\rho_1^1)_* Y_0^1$), and for Y_0^2, following the same reasoning as in (8), a local calculus gives

$$\psi_0^*\, i(Y_0^2)\Omega_0 = (j^1\phi)^* \left[\left(f_i(x) \left(v_\alpha^A - \frac{\partial q^i}{\partial x^\alpha} \right) \right) \mathrm{d}t \right] = 0$$

since $j^1\phi$ is a holonomic section and $Y_0^2 = f_i \dfrac{\partial}{\partial p_i}$. The result for the sections $\psi_{\mathcal{H}} = \widetilde{\mathcal{FL}} \circ j^1\phi$ is a direct consequence of the first equivalence relations (3). ■

Remark: The results in this section can also be recovered in coordinates taking an arbitrary local vector field $Y_0 = f\dfrac{\partial}{\partial t} + f^i \dfrac{\partial}{\partial q^i} + g^i \dfrac{\partial}{\partial v^i} + h_i \dfrac{\partial}{\partial p_i} \in \mathfrak{X}(\mathcal{W}_0)$, then

$$\begin{aligned} i(Y_0)\Omega_0 &= -f \left(p_i \mathrm{d}v^i + v^i \mathrm{d}p_i - \frac{\partial L}{\partial q^i}\mathrm{d}q^i - \frac{\partial L}{\partial v^i}\mathrm{d}v^i \right) \\ &\quad - f^i \left(\frac{\partial L}{\partial q^i}\mathrm{d}t + \mathrm{d}p_i \right) + g^i \left(p_i - \frac{\partial L}{\partial v^i} \right) \mathrm{d}t + h_i (v^i \mathrm{d}t - \mathrm{d}q^i) \end{aligned}$$

and, for a section ψ_0 fulfilling (7),

$$0 = \psi_0^*\, i(Y_0)\Omega_0 = \left[f^i \left(\frac{dp_i}{dt} - \frac{\partial L}{\partial q^i} \right) + g^i \left(p_i - \frac{\partial L}{\partial v^i} \right) + h_i \left(v^i - \frac{dq^i}{dt} \right) \right] \mathrm{d}t \tag{9}$$

reproduces the holonomy condition, the restricted Legendre map (that is, the definition of the momenta), and the Euler-Lagrange equations. The coefficient of the component f vanishes as a consequence of the last equations.

Summarizing, the equation (7) gives different kinds of information, depending on the verticality of the vector fields Y_0 involved. In particular, we obtain equations of three different classes:

1. Algebraic (not differential) equations, in coordinates $p_i = \dfrac{\partial L}{\partial v^i}$, which determine a subset $\mathcal{W}_1$ of $\mathcal{W}_0$, where the sections solution must take their values. These can be called *primary Hamiltonian constraints*, and in fact they generate, by $\hat{\rho}_2^0$ projection, the primary constraints of the Hamiltonian formalism for singular Lagrangians, i.e., the image of the Legendre transformation, $\mathcal{FL}(J^1\pi) \subset J^1\pi^*$.

2. The holonomic differential equations, in coordinates $v^i = \dfrac{dq^i}{dt}$, forcing the sections solution ψ_0 to be lifting of π-sections. This property reflects the fact that the geometric condition in the unified formalism is stronger than the usual one in the Lagrangian formalism.

3. The classical Euler-Lagrange equations, in coordinates

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial^2 L}{\partial v^j \partial v^i}\frac{d^2 q^j}{dt^2} + \frac{\partial^2 L}{\partial q^j \partial v^i}\frac{dq^j}{dt} + \frac{\partial^2 L}{\partial t \partial v^i} = \frac{\partial L}{\partial q^i} \tag{10}$$

which are obtained from $\dfrac{dp_i}{dt} = \dfrac{\partial L}{\partial q^i}$, using the previous equations.

5 The dynamical equations for vector fields

Proposition 3 *The problem of finding sections solutions to (7) is equivalent to finding the integral curves of a vector field* $X_0 \in \mathfrak{X}(\mathcal{W}_0)$*, which is tangent to* $\mathcal{W}_1$ *and satisfies that*

$$i(X_0)\Omega_0 = 0 \quad , \quad i(X_0)\mathrm{d}t = 1\,. \tag{11}$$

(*Proof*) In a natural chart in $\mathcal{W}_0$, the local expression of a vector field $X_0 \in \mathfrak{X}(\mathcal{W}_0)$ is

$$X_0 = f\frac{\partial}{\partial t} + F^i\frac{\partial}{\partial q^i} + G^i\frac{\partial}{\partial v^i} + H_i\frac{\partial}{\partial p_i}\,.$$

Then, the second equation (11) leads to $f = 1$, and the first gives

$$\text{coefficients in } \mathrm{d}p_i: \quad F^i = v^i \tag{12}$$

$$\text{coefficients in } \mathrm{d}v^i: \quad p_i = \frac{\partial L}{\partial v^i} \tag{13}$$

$$\text{coefficients in } \mathrm{d}q^i: \quad H_i = \frac{\partial L}{\partial q^i} \tag{14}$$

$$\text{coefficients in } \mathrm{d}t \;: \quad -F^i\frac{\partial L}{\partial q^i} + G^i\left(p_i - \frac{\partial L}{\partial v^i}\right) + H_i v^i = 0\,. \tag{15}$$

Now, if $\psi_0 = (t, q^i(t), v^i(t), p_i(t))$ is an integral curve of X_0, we have that $F^i = \dfrac{dq^i}{dt}$, $G^i = \dfrac{dv^i}{dt}$, $H_i = \dfrac{dp_i}{dt}$, and then (see equation (9)):

- Equations (12) are the holonomy condition.
- The algebraic equations (13) are the compatibility conditions defining $\mathcal{W}_1$.
- Using (12) and (13), equations (14) are the Euler-Lagrange equations (10).
- Taking into account (12) and (14), equation (15) holds identically.

Observe that the condition that X_0 (if it exists) must be tangent to $\mathcal{W}_1$ holds also identically from the above equations, since

$$0 = X_0\left(p_i - \frac{\partial L}{\partial v^i}\right) = -\frac{\partial^2 L}{\partial v^i \partial v^j}G^i - \frac{\partial^2 L}{\partial t \partial v^j} - \frac{\partial^2 L}{\partial q^i \partial v^j}v^i + \frac{\partial L}{\partial q^j} \qquad (\text{on } \mathcal{W}_1)$$

are the Euler-Lagrange equations again. Observe that, if L is a regular Lagrangian, these equations allow us to determine the functions $G^i = \dfrac{dv^i}{dt}$. If L is singular, then a constraint algorithm must be used in order to obtain a final constraint submanifold $\mathcal{W}_f$ (if it exists) where consistent solutions exist, that is, X_0 must be tangent to $\mathcal{W}_f$ (see [2] and Section 6 for details). ∎

Now, the equivalence of the unified formalism with the Lagrangian and Hamiltonian formalisms can be recovered as follows, where $\mathfrak{X}_{\mathcal{W}_1}(\mathcal{W}_0)$ is the set of vector fields on $\mathcal{W}_0$ with support in $\mathcal{W}_1$.

Theorem 2 *Let X_0 be a vector field in $\mathcal{W}_0$ which is the solution to the equations (11). Then the vector field $X_{\mathcal{L}} \in \mathfrak{X}(J^1\pi)$, defined by $X_{\mathcal{L}} \circ \rho_1^0 = \mathrm{T}\rho_1^0 \circ X_0$, is a holonomic vector field solution to the equations (1).*

Conversely, every holonomic vector field solution to the equations (1) can be recovered in this way from a vector field $X_0 \in \mathfrak{X}_{\mathcal{W}_1}(\mathcal{W}_0)$.

(*Proof*) Let X_0 be a vector field on $\mathcal{W}_0$, which is a solution to (11). As sections $\psi_0 \colon \mathbb{R} \longrightarrow \mathcal{W}_0$ solution to the geometric equation (7) must take value in $\mathcal{W}_1$, then X_0 can be identified with a vector field $X_1 \colon \mathcal{W}_1 \longrightarrow \mathrm{T}\mathcal{W}_1$ (i.e., $\mathrm{T}\jmath_1 \circ X_1 = X_0|_{\mathcal{W}_1}$), and hence there exists $X_{\mathcal{L}} \colon J^1\pi \longrightarrow \mathrm{T}(J^1\pi)$ such that $X_1 = \mathrm{T}(\rho_1^1)^{-1} \circ X_{\mathcal{L}} \in \mathfrak{X}(\mathcal{W}_1)$. Therefore, as a consequence of the item 1 in Theorem 1, for every section ψ_0 solution to (7), there exists $X_{\mathcal{L}}^0 \in \mathfrak{X}(j^1\phi(\mathbb{R}))$ such that $\mathrm{T}\jmath_\phi \circ X_{\mathcal{L}}^0 = X_{\mathcal{L}}|_{j^1\phi(\mathbb{R})}$, where $\jmath_\phi \colon j^1\phi(\mathbb{R}) \longrightarrow E$ is the natural embedding. So, $X_{\mathcal{L}}$ is $\bar{\pi}^1$-transversal and holonomic. Then, bearing in mind that $\jmath_1^*\Omega_0 = \rho_1^{1*}\Omega_{\mathcal{L}}$, we have

$$\jmath_1^*\, i(X_0)\Omega_0 = i(X_1)(\jmath_1^*\Omega_0) = i(X_1)(\rho_1^{1*}\Omega_{\mathcal{L}}) = \rho_1^{1*}\, i(X_{\mathcal{L}})\Omega_{\mathcal{L}}$$

then $i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$ because $i(X_0)\Omega_0 = 0$. A similar reasoning leads us to prove that, if $i(X_0)\mathrm{d}t = 1$, then $i(X_{\mathcal{L}})\mathrm{d}t = 1$.

Conversely, given a holonomic vector field $X_{\mathcal{L}}$, from $i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$, and taking into account the above chain of equalities, we obtain that $i(X_0)\Omega_0 \in [\mathfrak{X}(\mathcal{W}_1)]^0$ (the annihilator of $\mathfrak{X}(\mathcal{W}_1)$). Moreover, $X_{\mathcal{L}}$ being holonomic, X_0 is holonomic, and then the extra condition $i(Y_0)\, i(X_0)\Omega_0 = 0$ is also fulfilled for every $Y_0 \in \mathfrak{X}^{\mathrm{V}(\rho_1^0)}(\mathcal{W}_0)$. Thus, remembering that $\jmath_1^*\mathrm{T}\mathcal{W}_0 = \mathrm{T}\mathcal{W}_1 \oplus_{\mathcal{W}_1} \jmath_1^*\mathrm{V}(\rho_1^0)$, we conclude that $i(X_0)\Omega_0 = 0$. To prove that if $i(X_{\mathcal{L}})\mathrm{d}t = 1$, then $i(X_0)\mathrm{d}t = 1$ is trivial. ∎

Finally, the Hamiltonian formalism is recovered using the second equivalence relations (3). The proof for the almost-regular case follows in a straightforward way.

6 An example: spatial semidiscretization of the nonlinear wave equation

Consider the nonlinear wave equation given by

$$u_{tt} = \frac{d}{dx}\left(\frac{\partial \sigma}{\partial u_x}(t, u_x)\right) - \frac{\partial g}{\partial u}(t, u) \tag{16}$$

where $u : U \subset \mathbb{R}^2 \to \mathbb{R}$, $u(t, x)$ and σ and g are smooth functions and we impose periodic boundary conditions $u(t, x) = u(t, x + K)$, $K > 0$. Different choices of the functions σ and g idealize one-dimensional models for fluids and materials.

Equation (16) corresponds to the Euler-Lagrange equation derived extremizing the action functional

$$u \mapsto \int_0^T \int_0^K \left(\frac{1}{2}u_t^2 - \sigma(t, u_x) - g(t, u)\right) dt\, dx\,,$$

where we will assume in the sequel the regularity condition $\dfrac{\partial^2 \sigma}{\partial u_x^2} \neq 0$.

One basic idea towards a geometric discretization [8, 10, 13] of this type of equations is first to introduce an spatial truncation, that reduce the PDE (16) to a system of ODEs preserving many of its geometrical properties. Hence, we replace the x-derivative in the Lagrangian by a simple difference (for simplicity, we will work with a uniform grid of $N+1$ points, $h = K/N$) as follows:

$$L(t, u_i, (u_i)_t) = \sum_{i=0}^{N-1} \left[\frac{1}{2} \left(\frac{(u_i)_t + (u_{i+1})_t}{2} \right)^2 - \sigma \left(t, \frac{u_{i+1} - u_i}{h} \right) - g \left(t, \frac{u_{i+1} + u_i}{2} \right) \right],$$

In a more convenient notation, we are working with the Lagrangian function $L : \mathbb{R} \times T\mathbb{R}^{N+1} \longrightarrow \mathbb{R}$:

$$L(t, q^i, v^i) = \sum_{i=0}^{N-1} \left[\frac{1}{2} \left(v^{i+1/2} \right)^2 - \sigma \left(t, w^i \right) - g \left(t, q^{i+1/2} \right) \right].$$

where $w^i = \dfrac{q^{i+1} - q^i}{h}$ and $Q^{i+1/2} = \dfrac{Q^{i+1} + Q^i}{2}$, $i = 0 \ldots, N-1$, $Q = q, v$. Now, following the notation in previous sections, we find that

$$\Theta_0 = (L(t, q^i, v^i) - p_i v^i)\, dt + p_i\, dq^i \ , 0 \leq i \leq N.$$

Consider now a vector field

$$X_0 = f \frac{\partial}{\partial t} + F^i \frac{\partial}{\partial q^i} + G^i \frac{\partial}{\partial v^i} + H_i \frac{\partial}{\partial p_i},$$

satisfying the equations:

$$i_{X_0} \Omega_0 = 0, \qquad i_{X_0} dt = 1.$$

It is easy to deduce that:

$$\begin{cases} f = 1 \\ F^i = v^i, \\ H_0 = \dfrac{1}{h} \dfrac{\partial \sigma}{\partial u_x} \left(t, w^0 \right) - \dfrac{1}{2} \dfrac{\partial g}{\partial u} \left(t, q^{0+1/2} \right) \\ H_i = \dfrac{1}{h} \left(\dfrac{\partial \sigma}{\partial u_x}(t, w^i) - \dfrac{\partial \sigma}{\partial u_x}(t, w^{i-1}) \right) - \dfrac{1}{2} \left(\dfrac{\partial g}{\partial u}(t, q^{i+1/2}) + \dfrac{\partial g}{\partial u}(t, q^{i-1/2}) \right), \quad 1 \leq i \leq N-1 \\ H_N = -\dfrac{1}{h} \dfrac{\partial \sigma}{\partial u_x} \left(t, w^{N-1} \right) - \dfrac{1}{2} \dfrac{\partial g}{\partial u} \left(t, q^{N-1/2} \right) \end{cases}$$

and the constraints defining $\mathcal{W}_1$:

$$p_0 = \frac{1}{2} v^{0+1/2}, \qquad p_i = \frac{1}{2} (v^{i-1/2} + v^{i/2}), \qquad p_N = \frac{1}{2} v^{N-1/2}.$$

Since X_0 must be tangent to $\mathcal{W}_1$ then we obtain the additional conditions

$$\begin{aligned} 0 &= X_0(p_0 - \frac{1}{2} v^{0+1/2}) = -\frac{G^0 + G^1}{4} + \frac{1}{h} \frac{\partial \sigma}{\partial u_x} \left(t, w^0 \right) - \frac{1}{2} \frac{\partial g}{\partial u} \left(t, q^{0+1/2} \right) \\ 0 &= X_0(p_i - \frac{1}{2}(v^{i-1/2} + v^{i/2})) = -\frac{G^{i-1} + 2G^i + G^{i+1}}{4} + \frac{1}{h} \left(\frac{\partial \sigma}{\partial u_x}(t, w^i) - \frac{\partial \sigma}{\partial u_x}(t, w^{i-1}) \right) \\ & \quad -\frac{1}{2} \left(\frac{\partial g}{\partial u}(t, q^{i+1/2}) + \frac{\partial g}{\partial u}(t, q^{i-1/2}) \right), \quad 1 \leq i \leq N-1 \\ 0 &= X_0(p_N - \frac{1}{2} v^{N-1/2}) = -\frac{G^{N-1} + G^N}{4} - \frac{1}{h} \frac{\partial \sigma}{\partial u_x} \left(t, w^{N-1} \right) - \frac{1}{2} \frac{\partial g}{\partial u} \left(t, q^{N-1/2} \right) \end{aligned}$$

From these last equations we obtain $G^0, G^1, \cdots, G^{N-1}$ in terms of G^N and the additional constraint

$$\sum_{i=0}^{N-1} (-1)^i \frac{\partial\sigma}{\partial u_x}\left(t, w^i\right) = 0$$

which determines the new constraint submanifold, $\mathcal{W}_2$. Again, the condition of tangency of X_0 to $\mathcal{W}_2$ gives us a new constraint:

$$\sum_{i=0}^{N-1} (-1)^i \left[\frac{\partial^2\sigma}{\partial u_x \partial t}\left(t, w^i\right) + \frac{v^{i+1} - v^i}{h} \frac{\partial^2\sigma}{\partial u_x^2}\left(t, w^i\right)\right] = 0.$$

determining the constraint submanifold, $\mathcal{W}_3$. From it, we obtain that

$$\begin{aligned}\sum_{i=0}^{N-1} (-1)^i \left[\frac{\partial^3\sigma}{\partial u_x \partial^2 t}\left(t, w^i\right)\right. &+ \frac{v^{i+1} - v^i}{h} \frac{\partial^3\sigma}{\partial u_x^2 \partial t}\left(t, w^i\right) \\ &\left. + \left(\frac{v^{i+1} - v^i}{h}\right)^2 \frac{\partial^3\sigma}{\partial u_x^3}\left(t, w^i\right) + \frac{G^{i+1} - G^i}{h} \frac{\partial^2\sigma}{\partial u_x^2}\left(t, w^i\right)\right] = 0\end{aligned}$$

which uniquely determines the remaining coefficient G^N form the regularity condition $\dfrac{\partial^2\sigma}{\partial u_x^2} \neq 0$.

7 Conclusion and outlook

Following the Skinner-Rusk model for autonomous mechanical systems, we have presented a generalized framework for describing both Lagrangian and Hamiltonian time dependent mechanical systems.

The key tool of this construction is the coupling form which is defined using the natural geometric structure of the manifold $\mathcal{W} = J^1\pi \times_E \mathrm{T}^*E$. This function allows us to define in a natural way a submanifold $\mathcal{W}_0$ of $\mathcal{W}$, which is diffeomorphic to $\mathcal{W}_r = J^1\pi \times_E J^1\pi^*$, the true space of physical variables. Then, the compatibility of the dynamical equations stated in $\mathcal{W}_0$ gives a new submanifold $\mathcal{W}_1$ which is identified with the graph of the Legendre map $\widetilde{\mathcal{FL}}$, where all the characteristic features of the Lagrangian and Hamiltonian formalisms of time-dependent regular and singular non-autonomous systems are recovered.

This unified formalism constitutes an alternative but equivalent approach to that given by Cantrijn *et al* in [2]. The essential difference is that, in this work, the dynamical equations are established directly in $\mathcal{W} = J^1\pi \times_E \mathrm{T}^*E$. These equations are compatible in a 1-codimensional submanifold of $\mathcal{W}$, but the dynamical solution is undetermined, even in the regular case. In order to overcome this trouble, the authors are forced to introduce a new constraint, in such a way that the resulting submanifold is the graph of the Legendre map. As a consequence, they are unable to define intrinsically the submanifold of physical states $\mathcal{W}_0$. In our model, the introduction of the coupling form gets round all the above problems.

The Skinner-Rusk unified formalism which is developed here has been used to give a new geometric framework for time-dependent optimal control problems in [1], where some interesting examples are analyzed. Following the above example the developed formalism could be applied to optimal control problems in partial differential equations where the spatial semidiscretization is used to solve.

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