

## Impulse Response Recovery of Linear Systems Through Weighted Cumulant Slices

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**Abstract**—Identifiability of the so-called *w-slice* algorithm is proven for ARMA linear systems. Although proofs were developed in the past for the simpler cases of MA and AR models, they were not extendible to general exponential linear systems. The results presented in this paper demonstrate a unique feature of the *w-slice* method, which is unbiasedness and consistency when order is overdetermined, regardless of the IIR or FIR nature of the underlying system, and numerical robustness.

### I. INTRODUCTION

The problem addressed below is the blind estimation of the impulse response of a linear system. It is well known that if the process cannot be assumed to be cyclostationary, only higher-than-second-order statistics (and, in particular, cumulants) of the output process preserve the system phase information [7]. The specific set of cumulant slices and the number of cumulant samples per slice that are needed for identifiability has been considered thoroughly in the past for different algorithms (see [4], for instance). Recently, the authors have proposed a new method that employs linear combinations of slices from (generally) different order cumulants to obtain linear, consistent, low variance estimates [2] and [6]. Developments were built in the special cases of AR and MA models. Although complete and well founded, the rationale followed there did not allow for the extension to the more general ARMA structure. This is specifically the case that we deal with in this paper. The importance of the derivation lies on the fact that the same estimation procedure—i.e., the *w-slice* algorithm—can be applied to obtain consistent estimates of the impulse response of any linear system, regardless of its IIR or FIR nature and with no previous knowledge of the exact model order.

We will present first the problem and a briefing of the results found in [2] and [6] and then the extension of the same principles to the ARMA case, which is the backbone of this correspondence. Finally, we will present some numerical examples illustrating the performance.

To start up with the derivation, first consider a zero-mean, ergodic process  $y(n)$  with finite cumulants. Assume that the process is generated by a causal linear system, whose input/output relation is described by

$$y(n) = \sum_{i=0}^{\infty} h(i)e(n-i). \quad (1)$$

Consider now that the output is corrupted with another process  $v(n)$ :

$$x(n) = y(n) + v(n) \quad (2)$$

fitting the following hypotheses, which will be assumed throughout:

- H<sup>1</sup>. The driving process  $e(n)$  is zero-mean, stationary, i.i.d., non-Gaussian with finite  $k$ th-order cumulants and absolutely summable  $2k$ th-order cumulants.

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H<sup>2</sup>.  $H(z)$  is time invariant, causal, and exponentially stable, that is, all poles are inside the unit circle.

H<sup>3</sup>. The process  $v(n)$  is independent of  $e(n)$ , zero-mean, Gaussian, and of unknown power spectrum.  $v(n)$  is allowed to be non-Gaussian if some  $k$ th-order cumulant cancels, in which case, that  $k$ th order will be used for identification (i.e., if the noise is known to be zero-mean and uniformly distributed—as the quantization noise—then third-order cumulants can be used).

The Barlett–Brillinger–Rosenblatt summation formula [1] relates the  $k$ th-order cumulants of  $x(n)$  to the impulse response of the system:

$$C_{ky}(i_1, i_2, \dots, i_{k-1}) = \gamma_{ke} \sum_{n=0}^{\infty} \prod_{m=0}^{k-1} h(n+i_m) \quad (3)$$

$i_0 = 0, k \geq 2$

where  $\gamma_{ke}$  is the  $k$ th-order cumulant of the excitation process  $e(n)$ . Aiming at the recovery of  $h(n)$  from the  $C_{ky}(\cdot)$  terms, we can express the 1-D slice cumulant of  $k$ th order as the cross correlation

$$C_{ky}(i, i_2, \dots, i_{k-1}) = \sum_{n=-i}^{\infty} h(n+i) \cdot h(n; i_2, \dots, i_{k-1}) \quad (4)$$

where the causal sequence  $h(n; i_2, \dots, i_{k-1})$  is defined as

$$h(n; i_2, \dots, i_{k-1}) = \gamma_{ke} h(n) \prod_{m=2}^{k-1} h(n+i_m). \quad (5)$$

If we use a linear combination of cumulant slices (*w-slice*) it is also possible to obtain a cross correlation of the impulse response and a causal sequence:

$$\begin{aligned} C_w(i) &= w_2 C_{2y}(i) + \sum_{i_2=-N_2}^{M_2} w_3(i_2) C_{3y}(i, i_2) \\ &+ \sum_{i_2=-N_2}^{M_2} \sum_{i_3=-N_3}^{i_2} w_4(i_2, i_3) C_{4y}(i, i_2, i_3) + \dots \\ &= \sum_{n=0}^{\infty} h(n+i) g_w(n) \end{aligned} \quad (6)$$

where  $g_w(n)$  is the causal sequence

$$\begin{aligned} g_w(n) &= w_2 h(n) + \sum_{i_2=-N_2}^{M_2} w_3(i_2) h(n; i_2) \\ &+ \sum_{i_2=-N_2}^{M_2} \sum_{i_3=-N_3}^{i_2} w_4(i_2, i_3) h(n; i_2, i_3) + \dots \end{aligned} \quad (7)$$

The key idea of the *w-slice* method [2], [6] is to choose the weights  $\mathbf{w} = [w_2, w_3(i_2), w_4(i_2, i_3), \dots]$  of the linear combination in such a way that  $C_w(i)$  yields the impulse response  $h(n)$ . According to (6), the *w-slice* can be written in matrix form as

$$C_w = \mathbf{S}_a \mathbf{w} = \mathbf{1} \quad (8)$$

where  $\mathbf{S}_a$  is a  $(P+1)$  rows matrix containing the cumulant samples corresponding to  $(P+1)$  anticausal slices, the weight vector

$\mathbf{w}$  is chosen as the one yielding the  $w$ -slice to be causal, and  $\mathbf{1} = [0, \dots, 0, h(0)]^T$ . Then, as  $\mathbf{w}$  has been computed using the pseudoinverse, it can be used with the causal counterpart of  $\mathbf{S}_a$  in order to estimate  $P + 1$  samples of the impulse response  $\mathbf{h} = [h(0), h(1), \dots, h(P)]^T$  of the system

$$\mathbf{h} = \mathbf{S}_c \mathbf{S}_a^\# \mathbf{1}. \quad (9)$$

We have found that for AR and MA models, it is a sufficient condition that the weights are chosen so that the  $w$ -slice is causal. For MA( $q$ ) models, causality has to be imposed from  $n = -q, \dots, -1$  [2], whereas in AR( $p$ ) models, the minimum range of values is  $n = -p, \dots, -1$  [6]. The right choice of  $P$  and other sufficient conditions to achieve identifiability for ARMA models are derived in Section I-A. In any case, the order of the cumulants can be chosen at will, and second-order statistics are not necessary at all.

## II. ARMA MODELING

Are the approaches found in [2] and [6] valid in the case of ARMA models? Before addressing this question, consider the expression

$$\sum_{i_2=0}^p a(i_2) C_{ky}(i, q - i_2, 0, \dots, 0) = \gamma_{ke} b(q) h(i) \quad (10)$$

$$h(0) = 1$$

which was first derived in [4]. The terms  $a(i)$  are the AR parameters, and  $b(i)$  are the MA parameters. Equation (10) is telling us that there exist a linear combination of  $p + 1$  slices (whose weights are the AR coefficients) that yield the impulse response of the ARMA system. This linear combination is not unique: The convolution of the AR coefficients and the cumulant slices can be repeated up to  $k - 2$  times, always obtaining the impulse response of the system:

$$\begin{aligned} & \sum_{i_2=0}^p \sum_{i_3=0}^p a(i_2) a(i_3) \\ & \cdot C_{ky}(i, q - i_2, q - i_3, 0, \dots, 0) = \gamma_{ke} b^2(q) h(i) \\ & \vdots \\ & \sum_{i_2=0}^p \sum_{i_3=0}^p \cdots \sum_{i_{k-1}=0}^p a(i_2) a(i_3) \cdots a(i_{k-1}) \\ & \cdot C_{ky}(i, q - i_2, q - i_3, \dots, q - i_{k-1}) = \gamma_{ke} b^{k-2}(q) h(i). \end{aligned} \quad (11)$$

We will see in the sequel that the recovery of the impulse response is possible with the same  $w$ -slice approach by imposing mild constraints on the number of slices used

### A. Properties

From (6), it is obvious that the impulse response is found in the  $w$ -slice if  $g_w(n)$  is forced to be the Kronecker delta times a constant. Our interest is to investigate if this goal is possible by forcing a

finite number  $P$  of anticausal samples of the  $w$ -slice to be zero. The impulse response of an ARMA( $p, q$ ) system fits the recursion

$$\sum_{i=0}^p a(i) h(n - i) = \begin{cases} b(n) & 0 \leq n \leq q \\ 0 & \text{elsewhere.} \end{cases} \quad (12)$$

By considering the right-most side of (6), we can rewrite (8) as a system matrix with an infinite number of unknowns:

$$\begin{bmatrix} 0 & 0 & \cdots & h(0) & h(1) & \cdots \\ 0 & 0 & \cdots & h(1) & h(2) & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ 0 & h(0) & \cdots & h(P-1) & h(P) & \cdots \\ h(0) & h(1) & \cdots & h(P) & h(P+1) & \cdots \end{bmatrix} \begin{bmatrix} g_w(0) \\ g_w(1) \\ \vdots \\ g_w(P) \\ g_w(P+1) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h(0) \end{bmatrix} \quad (13)$$

or, more compactly

$$\begin{aligned} \mathbf{H}_a \mathbf{g}_w &= (\mathbf{h}_{-P}^{P+1} \quad \mathbf{h}_{-P+1}^{P+1} \quad \cdots \quad \mathbf{h}_0^{P+1} \quad \mathbf{h}_1^{P+1} \quad \cdots) \mathbf{g}_w \\ &= \mathbf{h}_{-P}^{P+1} \end{aligned}$$

where  $\mathbf{h}_Q^{P+1}$  denotes the vector of length  $P + 1$  whose first element is  $h(Q)$ , and the rest of elements are samples of  $h(n)$  with increasing indexes. Among all the columns of this matrix, only the first  $P + 1$  are linearly independent, and they form a basis for the rest of the column vectors. Then, the solution for the  $g_w(\cdot)$  is the addition of the trivial solution  $g_w(n) = \delta(n)$  plus the homogeneous one  $g_w^0(n)$ :

$$(\mathbf{h}_{-P}^{P+1} \quad \mathbf{h}_{-P+1}^{P+1} \quad \cdots \quad \mathbf{h}_0^{P+1} \quad \mathbf{h}_1^{P+1} \quad \cdots) \mathbf{g}_w^0 = \mathbf{0}.$$

If now we try to recover the impulse response by using the same approach as in the MA or AR cases, that is, the linear combination of  $g_w(\cdot)$  with the causal counterpart of the matrix in (13), we obtain the  $(P + 1)$ -long vector containing the estimated impulse response from sample 0 to sample  $P + 1$  ( $\mathbf{h}_0^{P+1}$ ), as shown in (14), which appears at the bottom of the page. The reader should note that the impulse response cannot be recovered if the null subspaces of  $\mathbf{H}_c$  and  $\mathbf{H}_a$  are not the same. This point is shown in the following lemma:

**Lemma 1:** If  $P \geq \max(p, q)$ , and the underlying system is ARMA, then the null subspace of the matrix  $\mathbf{H}_a$  is, in general, not included in the null-space of the matrix  $\mathbf{H}_c$ .

*Proof:* This fact can be easily proven by inspecting (13) and (14) and noticing that the columns of  $\mathbf{H}_c$  are contained in the columns of  $\mathbf{H}_a$ . ■

As the homogeneous solution to (13) brings difficulties, it is interesting to restrict it to be the zero vector. Fortunately, we do not need infinite equations because not all the unknowns  $g_w(n)$  are

$$\mathbf{h}_0^{P+1} = \mathbf{H}_c [\delta(n) + \mathbf{g}_w^0] = \begin{bmatrix} h(0) & h(1) & \cdots & h(P) & h(P+1) & \cdots \\ h(1) & h(2) & \cdots & h(P+1) & h(P+2) & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots \\ h(P-1) & h(P) & \cdots & h(2P-1) & h(2P) & \cdots \\ h(P) & h(P+1) & \cdots & h(2P) & h(2P+1) & \cdots \end{bmatrix} \begin{bmatrix} 1 + g_w^0(0) \\ g_w^0(1) \\ \vdots \\ g_w^0(P) \\ g_w^0(P+1) \\ \vdots \end{bmatrix} \quad (14)$$

independent, as we will see below. Therefore, identifiability will be achieved under two conditions:

- 1) If we can constrain the independence of the  $g_w(n)$  terms to, at most, the first  $P+1$  and then force  $g_w(n) = 0$  for  $n = 1, \dots, P$  there would be no solution for the  $g_w(n)$  lying in the null subspace of  $\mathbf{H}_a$ . In this case, the only solution would be the one given by  $g_w(n) = \delta(n)$ .
- 2) On the other hand, we must allow for the nonhomogeneous solution to (13).

The following lemma deals with point 1.

**Lemma 2:** If  $P - N \geq q$  and  $M + N \geq P$ , then the number of linearly independent variables  $g_w(n)$  is upper bounded by  $P$ .

*Proof:* In order to simplify the derivation and with no loss of generality, let us assume we deal only with  $k$ th-order cumulants. Let  $-N$  be the lowest numbered slice of  $k$ th order and  $M$  be the highest numbered slice being taken. Dependency among the  $g_w(n)$  is clearly shown by rewriting (7) in the following way:

$$\begin{aligned} g_w(n) &= \sum_{i=-N}^M w_k(i)h(n; i, 0, \dots, 0) \\ &= \gamma_{k\epsilon} h^{k-2}(n) \sum_{i=-N}^M w_k(i)h(n+i) \end{aligned}$$

or rather, in matrix form,

$$[g_w(0) \ g_w(1) \ \dots \ g_w(P) \ g_w(P+1) \ \dots]^T = \gamma_k \mathbf{D}\mathbf{H}\mathbf{w}$$

where  $\mathbf{D} = \text{diag}[h^{k-2}(0), h^{k-2}(1), \dots, h^{k-2}(P), \dots]$ ,  $\mathbf{w} = [w(M), w(M-1), \dots, w(-N)]^T$  and as given in (15), which appears at the bottom of the page. Clearly, the number of linearly independent variables  $g_w(n)$  is, at most, the rank of  $\mathbf{D}\mathbf{H}$ .  $M$  and  $N$  can be tuned so that the rows rank of the matrix  $\mathbf{H}$  is  $P+1$  by forcing the  $P+2, P+3, \dots$  numbered rows to be linearly dependent of the previous  $P+1$  ones: Consider the ARMA recursion

$$\sum_{i=1}^p a(i)h(n-i) = -h(n) \quad n > q.$$

By looking at the structure of (15), this happens when  $P - N \geq q$ . Then, in order to ensure the rank of  $\mathbf{H}$ , the number of columns must be at least  $P+1$ :  $M + N \geq P$ . Therefore, the rank of  $\mathbf{D}\mathbf{H}$  is, at most,  $P+1$  under the conditions stated, but it may be lower if some samples of the impulse response (the diagonal terms in  $\mathbf{D}$ ) are zero. ■

The conditions to achieve point 2, and hence, the conditions for identifiability are shown in Theorem 1.

**Theorem 1:** The causal part of a  $w$ -slice  $C_w(i)$  contains the impulse response of the underlying ARMA( $p, q$ ) if the first  $P$  samples of the anticausal part are zero, and the following conditions are met:

- 1) The set of slices considered contain the slices  $q-p, q-p+1, \dots, q$ , that is

$$\begin{aligned} M &\geq q \\ N &\geq p - q. \end{aligned}$$

- 2) The number of necessary cumulant samples per slice is  $P - N \geq q$ .

*Proof:* If we are able to force  $P+1$  linearly independent  $g_w(n)$  coefficients, then the homogeneous solution will be the zero vector. This is ensured by Lemma 2. On the other hand, we have to guarantee at least one solution to (8) since  $\mathbf{S}_a$  is of unknown rank. Any of the solutions given by (8) is useful for us, but the solution requiring the least number of slices is the solution  $w_k(i) = a(i)$ . To allow for its existence, the number of slices being used must be at least  $p+1$ :  $M + N \geq p$ ;  $M$  must be at least  $q$ :  $M \geq q$ , and the lag of the right-most element of the first row in  $\mathbf{H}$  should be lower or equal to  $q-p$ :  $q-p \geq -N$ . Finally, condition 2 is extracted from Lemma 2. ■

A similar reasoning can be developed to derive identifiability conditions when different cumulant orders are considered. According to (6), (13) remains unchanged, but (15) has to be enlarged in the number of columns. Consider, for instance, the use of 1-D slices of third- and fourth-order cumulants. Then, we can rewrite (7) as

$$\begin{aligned} g_w(n) &= \sum_{i=-N_3}^{M_3} w_3(i)h(n; i) + \sum_{i=-N_4}^{M_4} w_4(i)h(n; i, 0) \\ &= \gamma_{3\epsilon} h(n) \sum_{i=-N_3}^{M_3} w_3(i)h(n+i) + \gamma_{4\epsilon} h^2(n) \\ &\quad \cdot \sum_{i=-N_4}^{M_4} w_4(i)h(n+i, 0) \end{aligned}$$

and hence, as previously, (15) is modified to

$$\begin{aligned} &\begin{bmatrix} g_w(0) \\ g_w(1) \\ \vdots \\ g_w(P) \\ g_w(P+1) \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} \gamma_3 h(0) \mathbf{h}_{M_3+N_3+1}^{M_3+N_3+1T} & \gamma_4 h^2(0) \mathbf{h}_{M_4+N_4+1}^{M_4+N_4+1T} \\ \gamma_3 h(1) \mathbf{h}_{M_3+1}^{M_3+N_3+1T} & \gamma_4 h^2(1) \mathbf{h}_{M_4+1}^{M_4+N_4+1T} \\ \vdots & \vdots \\ \gamma_3 h(P) \mathbf{h}_{M_3+P}^{M_3+N_3+1T} & \gamma_4 h^2(P) \mathbf{h}_{M_4+P}^{M_4+N_4+1T} \\ \gamma_3 h(P+1) \mathbf{h}_{M_3+P+1}^{M_3+N_3+1T} & \gamma_4 h^2(P+1) \mathbf{h}_{M_4+P+1}^{M_4+N_4+1T} \\ \vdots & \vdots \end{bmatrix} \\ &\quad \cdot \begin{pmatrix} \mathbf{w}_3 \\ \mathbf{w}_4 \end{pmatrix} \\ &= \mathbf{H}\mathbf{w} \end{aligned} \tag{16}$$

where

$$\mathbf{h}_M^{sT} = [h(M), h(M-1), \dots, h(M-s+1)].$$

Again, at least one solution has to be allowed for the equation  $\mathbf{S}_a \mathbf{w} = \mathbf{1}$ , which may be  $w_k(i) = a(i)$ . Then, by the same reasoning

$$\mathbf{H} = \begin{bmatrix} h(M) & h(M-1) & \dots & h(-N+1) & h(-N) \\ h(M+1) & h(M) & \dots & h(-N+2) & h(-N+1) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ h(M+P) & h(M+P-1) & \dots & h(-N+P+1) & h(-N+P) \\ h(M+P+1) & h(M+P) & \dots & h(-N+P+2) & h(-N+P+1) \\ \vdots & \vdots & \dots & \vdots & \vdots \end{bmatrix} \tag{15}$$

TABLE I  
 AR AND MA ESTIMATES IN MEAN AND STANDARD DEVIATION (50 RUNS OF 2048 SAMPLES)

	Wideband ARMA(3,1)				Narrowband ARMA(3,1)				ARMA(2,2)			
Param.	a(1)	a(2)	a(3)	b(1)	a(1)	a(2)	a(3)	b(1)	a(1)	a(2)	b(1)	b(2)
True	-1.523	0.775	-0.135	-1.5	-1.192	1.073	-0.481	-1.5	-0.8	0.65	0	1.0
w-slice	-1.491 ±0.201	0.754 ±0.133	-0.098 ±0.162	-1.438 ±0.380	-1.201 ±0.088	1.085 ±0.069	-0.470 ±0.165	-1.549 ±0.109	-0.786 ±0.101	0.645 ±0.099	-0.002 ±0.117	1.019 ±0.070
q-slice	-1.379 ±0.269	0.551 ±0.390	-0.039 ±0.163	-1.372 ±0.289	-1.089 ±0.156	0.992 ±0.118	-0.437 ±0.084	-1.243 ±0.349	-0.797 ±0.095	0.656 ±0.077	-0.014 ±0.070	1.002 ±0.082

used in Theorem 1, the following relations must hold:

$$\begin{aligned}
 M_3 &\geq q \\
 M_4 &\geq q \\
 M_3 + N_3 &\geq p \\
 M_4 + N_4 &\geq p \\
 N_3 &\geq p - q \\
 N_4 &\geq p - q.
 \end{aligned}$$

The counterpart for Lemma 2 is discussed now: Matrix  $\mathbf{H}$  has a triangular zero matrix on its upper-right corner, corresponding to the columns containing the fourth-order terms and whose dimensions are set once  $M_4$  and  $N_4$  are tuned. The columns corresponding to the third-order terms also contain a triangular zero matrix on its upper-right corner. Following a reasoning similar to the one in Theorem 1, it is easy to see that the  $P+2, P+3, \dots$  numbered rows are not linearly dependent of the above ones due to the terms  $h(0), h^2(0), h(1), h^2(1), \dots$ , and hence, the rank of (16) is limited by the number of columns  $M_3 + M_4 + N_3 + N_4 + 2$ . In order to limit the number of independent  $g(\cdot)$  terms, the number of columns must be bounded to  $P+1$ :

$$P \geq M_3 + N_3 + M_4 + N_4 + 1.$$

That is, as more cumulants are considered, a larger  $P$  should be used. From the conditions above, it turns out that

$$P \geq 2p + 1.$$

### B. Algorithm

Theorem 1 allows the use of the  $h(n)$  estimation procedure already seen in the MA and AR cases under the constraints previously shown. The resulting set of cumulant slices and lags can be condensed in (8) for 1-D slices of  $k$ th-order cumulants, where  $\mathbf{S}_a$  is the anticausal  $w$ -slice matrix as shown in (16a), which appears at the bottom of the page, with  $P, M$ , and  $N$  chosen according to the values allowed in Section I-A,  $\mathbf{1} = (0, \dots, 0, 1)^T$ , and  $\mathbf{w}$  is the weighting vector  $\mathbf{w} = [w_3(i_1) \dots w_4(i_2) \dots]^T$ .

The final ARMA  $w$ -slice algorithm can be resumed in three steps if upper bounds  $\tilde{p}, \tilde{q}$  of the true ARMA( $p, q$ ) orders are known:

- S1) Computation of the minimum norm weights that yields a causal  $w$ -slice with  $C_w(0) = 1$ :

$$\mathbf{w}_m = \mathbf{S}_a^\# \mathbf{1}. \quad (17)$$

- S2) Estimate the causal part of the impulse response using

$$\mathbf{h}_0 = \mathbf{S}_c \mathbf{w}_m \quad (18)$$

where  $\mathbf{S}_c$  denotes the causal counterpart of  $\mathbf{S}_a$  [2], [6], and  $\mathbf{h}_0 = [h(0), \dots, h(\tilde{p} + \tilde{q})]^T$  is the estimated impulse response.

- S3) Solve for the AR parameters using

$$\begin{bmatrix} h(\tilde{q}) & h(\tilde{q}-1) & \dots & h(\tilde{q}-\tilde{p}+1) \\ h(\tilde{q}+1) & h(\tilde{q}) & \dots & h(\tilde{q}-\tilde{p}+2) \\ \vdots & \vdots & \ddots & \vdots \\ h(\tilde{q}+\tilde{p}-1) & h(\tilde{q}+\tilde{p}-2) & \dots & h(\tilde{q}) \end{bmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{\tilde{p}} \end{pmatrix} = - \begin{bmatrix} h(\tilde{q}+1) \\ h(\tilde{q}+2) \\ \vdots \\ h(\tilde{q}+\tilde{p}) \end{bmatrix}. \quad (19)$$

- S4) Then, the  $b(\cdot)$  coefficients can be estimated by using the AR-compensated impulse response or any other linear or nonlinear method:

$$b(n) = \sum_{k=0}^p a(k)h(n-k). \quad (20)$$

Most of the features seen in [2] and [6] are also encountered in the ARMA case. Some others are as follows:

*Remark 1:* Once the AR parameters have been estimated, as well as  $b(q)$ , the higher order cumulants  $\gamma_{ke}$  of the driving process  $e(n)$  can be computed from (10). Of course, this implies accurate estimation of  $q$ .

*Remark 2:* Unlike other methods, identifiability is guaranteed even if we are only given upper bounds of the true AR and MA orders. In practice, since  $q$  may be unknown,  $N$  should be set to an upper bound of  $p$ , and  $M$  should be set to an upper bound of  $q$ .

*Remark 3:* The matrix in (19) has rank equal to  $p$  (the AR order), provided that  $q$  is greater than or equal to the true MA order and that the ARMA( $p, q$ ) has no pole-zero cancellations (see [3, pp. 244–245]). Hence, we get an estimate for the AR order by analyzing its singular values. The true MA order may be estimated by following the approaches in [5] over the AR-compensated cumulant sequence  $B(m_1, m_2)$ . The comparative test of order determination algorithms in combination with the  $w$ -slice algorithm is beyond the scope of this paper; however, sensitiveness of the  $w$ -slice approach to order overdetermination has been tested in the simulations below.

$$\mathbf{S}_a = \left[ \begin{array}{ccc|ccc|ccc} C_{3y}(-N_3, -P) & \dots & C_{3y}(M_3, -P) & C_{4y}(-N_4, -P) & \dots & C_{4y}(M_4, -P) & \dots & \dots & \dots \\ \vdots & & \vdots & \vdots & & \vdots & & & \\ C_{3y}(-N_3, 0) & \dots & C_{3y}(M_3, 0) & C_{4y}(-N_4, 0) & \dots & C_{4y}(M_4, 0) & \dots & \dots & \dots \end{array} \right] \quad (16a)$$

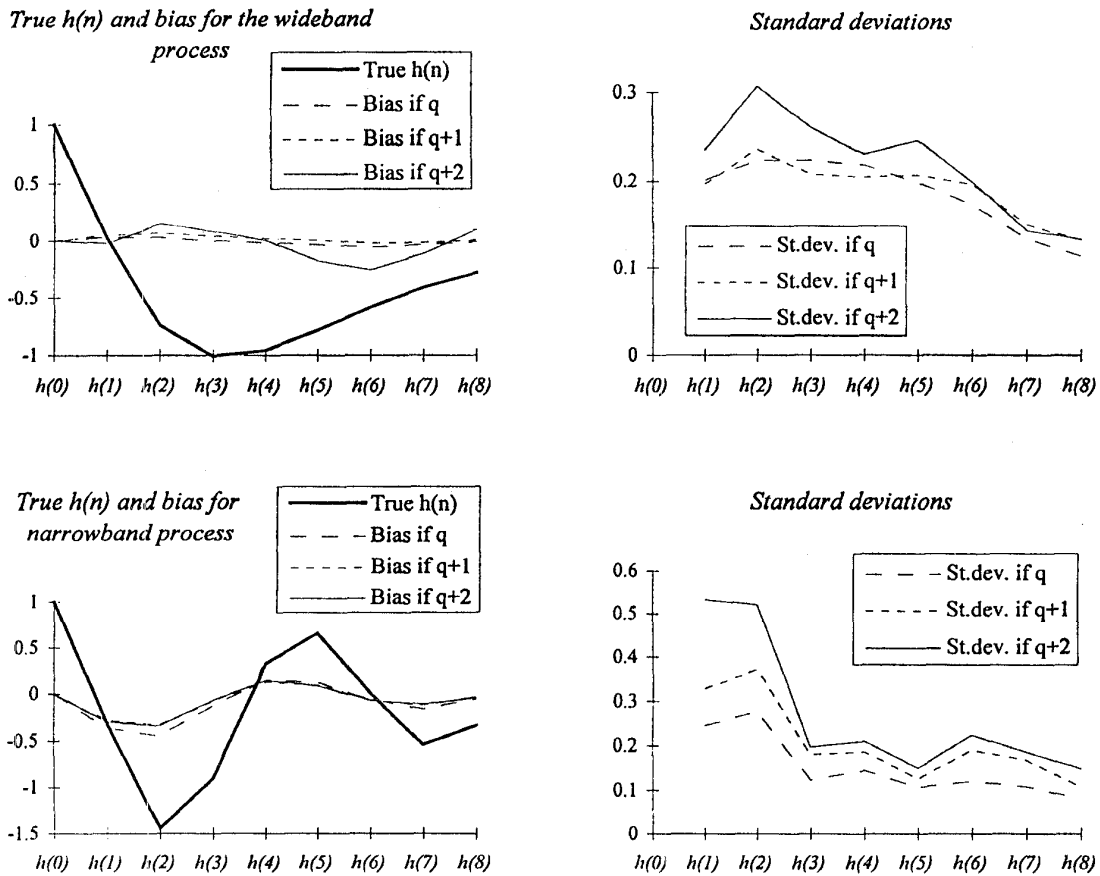


Fig. 1. Bias and standard deviation of the estimated impulse response of both the wideband and narrowband nonminimum phase ARMA process computed for the  $w$ -slice, for different (over)estimations of the true MA order.

*Remark 4:* In practice, theoretical cumulants are substituted by the sample estimates, which are known to be consistent and asymptotically Gaussian. For a large number of signal samples  $N$ , the covariance of the estimated sample cumulants is inversely proportional to  $N$ . Since the  $w$ -slice is a linear transformation, a similar behavior is expected for the estimated linear impulse response, as well as for the AR parameters. In [6], one can find a simulation in which the evolution of the variance of the estimated AR parameters is plotted versus  $N$ . Reasonable  $1/N$  dependence was found for  $N > 1000$  on that AR(2) process.

### III. NUMERICAL EXAMPLES

In the first example, three different models have been tested by performing 50 independent Monte Carlo runs. Third-order statistics have been used, and the minimum set of slices to guarantee identifiability in each method. Each record contains 2048 noiseless samples of i.i.d., zero-mean, exponentially distributed samples that have been filtered through two different bandpass ARMA(3, 1) models containing an allpass term as follows:

$$H(z) = \frac{B(z)}{A(z)} = \frac{1 - bz^{-1}}{(1 - b^{-1}z^{-1})(1 - 2r \cos \theta z^{-1} + r^2 z^{-2})} \quad (21)$$

Wideband process:  $b = 1.5$   $r = 0.45$   $\theta = 0.3142$  rad  
 Narrowband process:  $b = 1.5$   $r = 0.85$   $\theta = 1.2566$  rad

and a third ARMA(2, 2) model proposed in [8] that contains zeros on the unit circle:

$$H(z) = \frac{B(z)}{A(z)} = \frac{1 + z^{-2}}{1 - 0.8z^{-1} + 0.65z^{-2}} \quad (22)$$

Results assuming the true order known are shown in Table I for the  $w$ -slice and for the  $q$ -slice algorithm [8]. As can be seen, the  $w$ -slice approach exhibits similar performance in variance to the  $q$ -slice method but somewhat lower bias.

In the second example, we have tested the robustness of the method to order overdetermination. The  $q$ -slice algorithm is very sensitive to this situation and becomes inconsistent. Both the wideband and the narrowband ARMA(3, 1) process have been tested, using 2048 samples of an exponentially distributed process in every one of the 50 Monte Carlo runs. We have assumed an AR order of 5 and values for the estimated MA order ( $\hat{q}$ ) of 1 (true order), 2, and 3. The set of third-order cumulant slices used is  $M = 5$ ,  $N = 5 - \hat{q}$  in all cases, and the value of  $P$  has been set to  $\hat{q} + N$ . The number of the estimated samples of the causal impulse response (the number of rows in  $S_c$ ) has been seven in all cases. Results are shown in Fig. 1, which depicts a similar behavior in bias and standard deviation for the estimated impulse response. This is not an unexpected result since the method is still consistent, and we are just using more cumulant slices.

## IV. CONCLUSIONS

We have presented a proof for the consistency of the  $w$ -slice algorithm in the estimation of the impulse response of an ARMA linear system. Its extension to the ARMA case is an important feature of the method, which allows impulse response recovery even if the FIR or IIR nature is unknown. Simulations have shown good performance and robustness to order overdetermination.

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## Complex Linear-Quadratic Systems for Detection and Array Processing

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**Abstract**—Linear-quadratic (LQ) filters for detection and estimation are widely used in the real case. We investigate their extension in the complex case, which introduces various new questions. In particular, we calculate the optimum LQ array receiver in a non-Gaussian environment by using the deflection criterion and evaluate some of its performance.

## I. INTRODUCTION

Linear-quadratic (LQ) systems are widely used in many areas of signal processing and especially in detection problems. As an example, the optimum receiver for the detection of a normal signal in a normal noise is an LQ system. Most of the results known for LQ systems are established in the *real* case. However, these assumptions are too restrictive for various problems and especially in narrowband array processing. Even if the physical signals received by the sensors

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are real, there is a great advantage in the narrowband case to work with the complex representation, for example, by using the analytic signal (see [1, p. 229]). This especially allows definition of the complex steering vector characterizing the geometrical structure of the problem.

The most general form of a complex LQ filter calculating an output  $y$  in terms of a vector input  $\mathbf{x}$  is

$$y(\mathbf{x}) = c + \mathbf{x}^H \mathbf{h}_1 + \mathbf{x}^T \mathbf{h}_2 + \mathbf{x}^H \mathbf{M}_1 \mathbf{x} + \mathbf{x}^H \mathbf{M}_2 \mathbf{x}^* + \mathbf{x}^T \mathbf{M}_3 \mathbf{x} \quad (1)$$

where

- $c$  is a constant;
- $\mathbf{h}_1$  is a complex vector;
- $\mathbf{h}_2$  is a complex vector;
- $\mathbf{M}_i$  are three complex matrices.

In this equation,  $\mathbf{x}^H$  means transposition and complex conjugation. Note that because of the symmetry of the last two quadratic terms, there is no loss of generality in assuming that the matrices  $\mathbf{M}_2$  and  $\mathbf{M}_3$  are symmetric. The introduction of complex signals and systems yields significant changes in statistical signal processing problems. The main purpose of this correspondence is to calculate the vectors and matrices appearing (1) in such a way that  $y$  satisfies some optimality criterion introduced in the next section.

## II. STATEMENT OF THE PROBLEM

The basic detection problem consists of deciding between two simple hypotheses  $H_0$  and  $H_1$  from an observation vector  $\mathbf{x}$ . When the probability distributions of  $\mathbf{x}$  under  $H_0$  and  $H_1$  are known, the optimum procedure consists in comparing the likelihood ratio (LR) to a threshold. Our basic assumption is that we are not in this situation and that our knowledge concerning the statistical properties of  $\mathbf{x}$  is much lower. If, for instance, this knowledge is limited to second-order properties of  $\mathbf{x}$  under  $H_0$  and  $H_1$ , which means that only the mean values and the covariance matrices are known, it is possible to calculate the linear filter that maximizes the output signal to noise ratio, and, in the case of nonrandom signals, this leads to the famous matched filter used in many areas of statistical signal processing (see [1, p. 555]). The output signal-to-noise ratio is also called the *deflection* and can be defined for any filter  $y(\mathbf{x})$  by the expression

$$D(y) \triangleq \frac{|E_1(y) - E_0(y)|^2}{V_0(y)} \quad (2)$$

where  $E_0$  and  $E_1$  are expectations under  $H_0$  and  $H_1$ , respectively, and  $V_0$  is the variance under  $H_0$ . This deflection was introduced long time ago and has been used under various assumptions, especially in the context of array processing [2]–[5]. Even if it has been essentially used in the linear case, there is no reason to limit its use to linear systems. Therefore, if the moments up to the order 4 are known, it is possible to calculate the deflection of (1) and to find the system giving its maximum value. This work extends to the complex case results obtained in [6] for the real case.

We shall outline only the principles of the method used in order to maximize the deflection of systems like (1). The first point is to note that the deflection is invariant under affine transformation, and then it is appropriate to use this property to work with LQ systems having an output with zero mean value under  $H_0$ . This is realized by subtracting the mean value. By assuming that the input vector  $\mathbf{x}$