# Iterative approximation of unstable limit cycles for a class of Abel equations

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**Abstract**

This report considers the analytical approximation of unstable limit cycles that may appear in Abel equations written in the normal form. The procedure uses an iterative approach that takes advantage of the contraction mapping theorem. Thus, the obtained sequence exhibits uniform convergence to the target periodic solution. The effectiveness of the technique is illustrated through the approximation of an unstable limit cycle that appears in an Abel equation arising in a tracking control problem that affects an elementary, nonminimum phase, second order bilinear power converter.

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## 1 Introduction

Abel's ordinary differential equations deserve special attention in different areas belonging to the field of nonlinear systems. Below we outline three examples.

First. Bounds on the number of limit cycles in polynomial ODE's such as

$$\dot{x} = A(t)x^3 + B(t)x^2 + C(t)x, \tag{1}$$

which is an Abel equation on the first kind, are obtained in [7]. Recently, new bounds are reported in [1] for the special case $C(t) = 0$. Note that this is a particular case of Pugh's problem [8] which, in turn, is motivated by the still unsolved Hilbert's 16th problem [14].

Second. A series of remarkable second order ODE's of mathematical physics and nonlinear mechanics that include the Emden equation, the Emden-Fowler and the generalized Emden-Fowler equations, the Duffing equation, the generalized Blasius equation and the nonlinear gas pressure diffusion equation, allow admissible functional transformations that reduce them to Abel equations of the second kind [12], i.e.

$$[f_0 + f_1(t)x]\,\dot{x} = g_0(t) + g_1(t)x + g_2(t)x^2. \tag{2}$$

Third. Consider a generic single-input, second order bilinear control system

$$\dot{x} = Ax + \delta(Bx + \gamma)u, \tag{3}$$

with $x, \gamma \in \mathbb{R}^2$, $A, B \in M_2(\mathbb{R})$, and $u$ being a control action that takes real values. It is proved in [10] that the elimination of $u$ in (3) yields a relation between the state vector components $x_1, x_2$ of the form

$$\dot{x}_1 p_1(x_1, x_2) + p_2(x_1, x_2) = \dot{x}_2 q_1(x_1, x_2) + q_2(x_1, x_2),$$

where $p_i, q_i$ are polynomials of degree $i$ in the variables $x_1, x_2$. Hence, when the control action is able to force $x_i = f(t)$, $f(t)$ being a certain command profile, the dynamics of internal state variable $x_j$ are governed by an Abel equation. Alternatively, if the system is nonminimum phase [2] for the output $y = x_i$ and is minimum phase for $y = x_j$, the tracking goal $x_i = f(t)$ may be indirectly achieved by forcing the minimum phase variable $x_j$ to track an appropriate reference $\phi(t)$; this reference has to be chosen in such a way that the system

[2] The internal dynamics are unstable

internal dynamics may yield the desired behavior $x_j = f(t)$. Of course, $\phi(t)$ arises as a solution of an Abel equation, and it is demanded to be bounded and, preferable, periodic when $f(t)$ is periodic. This situation is illustrated in [5] for the nonlinear power converters case.

The work with Abel equations has been long handicapped by the fact that these equations did not admit analytical solution in terms of known functions, except for particular situations [9]. This gap has been apparently overcome with the recent publication [11] of a technique that makes feasible the construction of exact analytical solutions for Abel equations of the second kind written in the so-called *normal form*

$$x\dot{x} - x = G(t), \tag{4}$$

to which types (1) and (2) ODEs are readily transformable [13]. However, the method is not completely friendly in the sense that the expression of the solution involves real roots of a cubic equation that, in turn, contain cosine integral terms.

The Galerkin method is introduced in [6] to find a sequence of periodic approximations uniformly convergent to an unstable, periodic solution of a (4)-type Abel ODE. Moreover, the first approximation is analytically obtainable. Nevertheless, any higher order approximation is to be built from a numerical solution of a nonlinear system of equations, thus resulting in the need for a stable algorithm to compute the Galerkin coefficients.

This report is devoted to obtaining analytical approximations of unstable limit cycles that may appear in Abel equations written in the normal form (4). The procedure uses a functional iterative approach theoretically supported by the contraction mapping theorem. Hence, the sequence of successive approximations converges uniformly to the searched periodic solution. The method is tested in the solution of a tracking control problem that affects a second order, bilinear, nonminimum phase boost power converter.

The paper is organized as follows. The main result is stated and proved in Sections 2 and 3, respectively. The exemplification of the method in an ideal boost converter is detailed in Section 4, while simulation results are presented in Section 5.

## 2 Main result

Consider the ODE

$$x(1-\dot{x}) = g(t), \tag{5}$$

$g(t)$ being a $T$-periodic, smooth function. It is proved in [5] that

**Theorem 1** *If $g(t) > 0$, then equation (5) has one and only one periodic solution $\phi(t)$, which is positive, hyperbolic and unstable.*

The objective is to compute an analytic sequence of successive approximations of the unstable limit cycle $\phi(t)$ by means of Banach's fixed-point theorem.

Before stating the main result, let us fix some notation. It is well known that $L_2(0,T)$, provided with the scalar product

$$(x|y) = \int_0^T xy, \tag{6}$$

is a real, separable Hilbert space. We may find in $L_2(0,T)$ the trigonometric system $\{w_n\}$, $w_n \in L_2$ $\forall n \geq 0$, with

$$w_0 = \frac{1}{\sqrt{T}},\ w_{2k-1} = \sqrt{\frac{2}{T}}\cos k\omega t,\ w_{2k} = \sqrt{\frac{2}{T}}\sin k\omega t,\ k \geq 1,\ \omega = \frac{2\pi}{T}. \tag{7}$$

The trigonometric system is a complete, orthonormal system in $L_2(0,T)$, thus being a basis in $L_2(0,T)$. Let us now consider the $L_2$ subspaces

$$X_0 = \mathrm{span}\{w_0\}, \qquad \bar{X} = \mathrm{span}\{w_1, w_2, \ldots\}.$$

Any $x \in L_2$ can be uniquely decomposed as

$$x = x_0 + \bar{x}, \qquad x_0 \in X_0, \quad \bar{x} \in \bar{X}. \tag{8}$$

Specifically, for $g \in L_2(0,T)$ we get

$$g = g_0 + \bar{g}, \qquad g_0 \in X_0, \quad \bar{g} \in \bar{X}. \tag{9}$$

Finally, $\bar{X}$ being closed by integration, there exist unique elements $\hat{x}, \hat{g} \in \bar{X}$ such that

$$\dot{\hat{x}} = \bar{x}, \quad \dot{\hat{g}} = \bar{g}. \tag{10}$$

**Assumption A.** Let $g(t)$ be continuous, $T$-periodic, positive and such that

$$g_0\omega > 1 + \sqrt{2\omega\|\bar{g}\|}. \tag{11}$$

The main result reads as follows:

**Theorem 2** *If Assumption A holds, there exist $R \in \mathbb{R}^+$ and a closed subset of $L_2(0,T)$ defined as*

$$M_R = \{\bar{x} \in \bar{X}; \quad \|\bar{x}\| \le R\}, \tag{12}$$

*such that the sequence $\{g_0 + \bar{x}_n\}$, obtained by means of the iterative procedure*

$$\bar{x}_{n+1} = \frac{1}{g_0}\left[\hat{x}_n - \hat{g} - \frac{\bar{x}_n^2 - (\bar{x}_n^2|\omega_0)\omega_0}{2}\right], \quad \bar{x}_0 \in M_R, \tag{13}$$

*converges uniformly to the positive, $T$-periodic solution of*

$$x(1 - \dot{x}) = g(t).$$

## 3 Proof of Theorem 2

The ODE (5) can be written as the fixed point equation

$$x = Ax, \quad x \in L_2(0,T), \tag{14}$$

$A$ being the nonlinear mapping $A : L_2(0,T) \longrightarrow L_2(0,T)$ such that

$$Ax = x\dot{x} + g(t),$$

where, by Assumption A, $g(t)$ is positive and $T$-periodic. The existence and uniqueness of a positive, $T$-periodic solution of (14) is guaranteed by Theorem 1.

Using (8) and considering that

$$\bar{x}\dot{\bar{x}} = \frac{1}{2}\frac{d}{dt}\left(\bar{x}^2\right) \in \bar{X}, \quad \forall \bar{x} \in \bar{X},$$

equation (14) can be splitted into the system

$$x_0 = g_0 \tag{15}$$

$$\bar{x} = x_0\dot{\bar{x}} + \bar{x}\dot{\bar{x}} + \bar{g}, \tag{16}$$

which represent problems defined on $X_0$ and $\bar{X}$, respectively. The objective becomes that of solving the following fixed point equation in $\bar{X}$:

$$\bar{x} = g_0\dot{\bar{x}} + \bar{x}\dot{\bar{x}} + \bar{g}. \tag{17}$$

The equivalent integral problem of (17) can be written as

$$\int_0^t [\bar{x}(s) - \bar{g}(s)]\, dt = g_0 [\bar{x}(t) - \bar{x}(0)] + \frac{1}{2}\left[\bar{x}^2(t) - \bar{x}^2(0)\right],$$

which may also take the form

$$\hat{x}(t) - \hat{g}(t) - [\hat{x}(0) - \hat{g}(0)] = g_0 [\bar{x}(t) - \bar{x}(0)] + \frac{1}{2}\left[\bar{x}^2(t) - \bar{x}^2(0)\right], \tag{18}$$

where $\hat{x}, \hat{g}$ have been introduced in (10). Notice that (18) is not a problem defined on $\bar{X}$ but on $L_2(0,T)$; we may decompose it again making explicit [3] use of the orthogonal projection operator $P_0 : L_2 \longrightarrow X_0$, that acts in the following way:

$$P_0 x = (x|w_0)w_0.$$

Hence, (18) results in

$$\hat{x}(0) - \hat{g}(0) = g_0\bar{x}(0) - \frac{1}{2}\left[P_0\bar{x}^2 - \bar{x}^2(0)\right] \tag{19}$$

$$\hat{x} - \hat{g} = g_0\bar{x} + \frac{1}{2}(\mathbb{I} - P_0)\bar{x}^2. \tag{20}$$

Notice now that, one the one hand, (19) is a problem in $X_0$; on the other hand, (20) is a problem in $\bar{X}$ and any solution of (20) also verifies (19).

Then, let us rewrite (20) in the fixed-point problem form

$$\bar{x} = \frac{1}{g_0}\left[\hat{x} - \hat{g} - \frac{1}{2}(\mathbb{I} - P_0)\bar{x}^2\right]; \tag{21}$$

[3] In fact, it has implicitly been used in the decomposition (15),(16).

aided by the mapping $\bar{A} : \bar{X} \subset L_2(0,T) \longrightarrow L_2(0,T)$ defined as

$$\bar{A}\bar{x} = \frac{1}{g_0}\left[\hat{x} - \hat{g} - \frac{1}{2}(\mathbb{I} - P_0)\bar{x}^2\right],$$

(21) becomes

$$\bar{x} = \bar{A}\bar{x}.$$

**Lemma 3** *If Assumption A holds, then*

$$\alpha = \frac{1}{g_0}\sqrt{\left(g_0 - \frac{1}{\omega}\right)^2 - \frac{2}{\omega}\|\bar{g}\|}$$

*is such that*

$$0 < \alpha \leq 1 - \frac{1}{g_0\omega} < 1.$$

**Proof.** Note that Assumption A leads straightforward to $\alpha > 0$ and

$$0 < 1 - \frac{1}{g_0\omega} < 1.$$

Moreover, again by Assumption A, one gets

$$1 - \alpha = 1 - \sqrt{\left(1 - \frac{1}{g_0\omega}\right)^2 - \frac{2}{g_0^2\omega}\|\bar{g}\|} \geq \frac{1}{g_0\omega} > 0.$$

□

**Lemma 4** *Let $\bar{x}, \hat{x} \in \bar{X}$ be such that $\dot{\hat{x}} = \bar{x}$. Then,*

$$\|\hat{x}\| \leq \frac{1}{\omega}\|\bar{x}\|, \quad \forall \bar{x} \in \bar{X}. \tag{22}$$

**Proof.** Recall that

$$\bar{x} = \sum_{k\geq 1} c_{2k-1}w_{2k-1} + c_{2k}w_{2k}, \qquad \forall \bar{x} \in \bar{X}.$$

Since $\hat{x} \in \bar{X}$ is the only solution of $\dot{\hat{x}} = \bar{x}$, it must be

$$\hat{x} = \sum_{k\geq 1} \frac{1}{k\omega}\left(c_{2k-1}w_{2k} - c_{2k}w_{2k-1}\right), \qquad \text{with} \;\; \omega = \frac{2\pi}{T}.$$

Hence,

$$\|\hat{x}\| = \sqrt{\sum_{k\geq 1} \frac{1}{k^2\omega^2}\left(c_{2k-1}^2 + c_{2k}^2\right)} \leq \frac{1}{\omega}\sqrt{\sum_{k\geq 1}\left(c_{2k-1}^2 + c_{2k}^2\right)},$$

thus yielding (22). □

Set

$$R_- = g_0(1-\alpha) - \frac{1}{\omega}, \tag{23}$$

$$R_a = ag_0 - \frac{1}{\omega}. \tag{24}$$

**Proposition 5** *If Assumption A holds, then,* $\forall a \in (1-\alpha, 1)$ *and* $\forall R \in (R_-, R_a]$*, it follows that:*

*(i)* $M_R$ *is mapped into itself by* $\bar{A}$*, i.e.* $\bar{A} : M_R \subset L_2(0,T) \longrightarrow M_R$*, and*

*(ii)* $\bar{A}$ *is a-contractive on* $M_R$*.*

**Proof.** Let us prove the first statement. In order to have

$$\bar{A}\bar{x} \in M_R, \quad \forall \bar{x} \in M_R,$$

it must be verified that

$$\|\bar{A}\bar{x}\| \leq R, \quad \forall \bar{x} \in M_R.$$

Notice that, using (22),

$$\|\bar{A}\bar{x}\| = \left\| \frac{1}{g_0}\left[\hat{x} - \hat{g} - \frac{1}{2}(\mathbb{I} - P_0)\bar{x}^2\right]\right\| \leq \frac{1}{g_0}\left(\|\hat{x}\| + \|\hat{g}\| + \frac{1}{2}\|\bar{x}^2\|\right) \leq$$
$$\leq \frac{1}{g_0}\left(\frac{1}{\omega}\|\bar{x}\| + \frac{1}{\omega}\|\bar{g}\| + \frac{1}{2}\|\bar{x}\|^2\right) \leq \frac{1}{g_0}\left(\frac{1}{\omega}\|\bar{g}\| + \frac{1}{\omega}R + \frac{1}{2}R^2\right).$$

Therefore, the following inequality must be fulfilled for some $R > 0$:

$$\frac{1}{g_0}\left(\frac{1}{\omega}\|\bar{g}\| + \frac{1}{\omega}R + \frac{1}{2}R^2\right) \leq R.$$

After some manipulations one obtains the equivalent expression

$$p(R) = \frac{1}{2g_0}R^2 - \left(1 - \frac{1}{g_0\omega}\right)R + \frac{1}{g_0\omega}\|\bar{g}\| \leq 0.$$

$p(R)$ is a parabola whose branches point upwards; hence, its negative zone is between the roots $R_\pm$ of $p(R)$:

$$R_\pm = g_0 - \frac{1}{\omega} \pm \sqrt{\left(g_0 - \frac{1}{\omega}\right)^2 - \frac{2}{\omega}\|\bar{g}\|} = g_0(1 \pm \alpha) - \frac{1}{\omega}. \tag{25}$$

As Assumption A and Lemma 3 yield

$$R_+ > R_- = g_0(1-\alpha) - \frac{1}{\omega} \geq 0,$$

it is straightforward that $\bar{A}$ maps $M_R$ into itself $\forall R \in (R_-, R_+] \subset \mathbb{R}^+$.

The second statement of the proposition may be proved as follows. Observe that, using (22), $\forall \bar{x}, \bar{y} \in M_R$,

$$\begin{aligned} \|\bar{A}\bar{x} - \bar{A}\bar{y}\| &= \frac{1}{g_0}\left\|\left[\hat{x} - \hat{g} - \frac{1}{2}(\mathbb{I} - P_0)\bar{x}^2\right] - \left[\hat{y} - \hat{g} - \frac{1}{2}(\mathbb{I} - P_0)\bar{y}^2\right]\right\| \leq \\ \leq \frac{1}{g_0}\left(\|\hat{x} - \hat{y}\| + \frac{1}{2}\|\bar{x}^2 - \bar{y}^2\|\right) &\leq \frac{1}{g_0}\left(\frac{1}{\omega}\|\bar{x} - \bar{y}\| + \frac{1}{2}\|\bar{x} + \bar{y}\|\|\bar{x} - \bar{y}\|\right) \leq \\ &\leq \frac{1}{g_0}\left[\frac{1}{\omega} + \frac{1}{2}\left(\|\bar{x}\| + \|\bar{y}\|\right)\right]\|\bar{x} - \bar{y}\| \leq \frac{1}{g_0}\left(\frac{1}{\omega} + R\right)\|\bar{x} - \bar{y}\| \end{aligned}$$

Assumption A entails $g_0\omega > 1$; hence,

$$a g_0 - \frac{1}{\omega} > 0, \qquad \forall a \in \left(\frac{1}{g_0\omega}, 1\right).$$

Then, $\forall\ a \in I = ((g_0\omega)^{-1}, 1) \subset \mathbb{R}^+$ and $\forall\ R \in \mathbb{R}^+$ such that

$$R \leq R_a = a g_0 - \frac{1}{\omega}, \tag{26}$$

it follows that

$$\|\bar{A}\bar{x} - \bar{A}\bar{y}\| \leq a\|\bar{x} - \bar{y}\|.$$

We must finally check that there exists $a \in I$ such that $R_a \in (R_-, R_+]$. Notice that

$$\begin{aligned} R_a \in (R_-, R_+] &\Longleftrightarrow g_0(1-\alpha) - \frac{1}{\omega} < a g_0 - \frac{1}{\omega} \leq g_0(1+\alpha) - \frac{1}{\omega} \\ &\Longleftrightarrow 1 - \alpha < a \leq 1 + \alpha, \end{aligned}$$

which is indeed true because $\alpha > 0$ and $1 - \alpha \geq (g_0\omega)^{-1}$ from Lemma 3.

Therefore, putting together all the restrictions, we conclude that $\forall\ a \in (1 - \alpha, 1)$ and $\forall\ R \in (R_-, R_a] \subset \mathbb{R}^+$, $M_R$ is mapped into itself by $\bar{A}$ and, at the same time, $\bar{A}$ is $a$-contractive on $M_R$. $\square$

Proposition 2 allows to apply Banach's fixed-point theorem to (21) on the $L_2(0,T)$ subset $M_R$, which yields immediately the statement claimed in Theorem 2.

## 4 Example: output voltage tracking of a sinusoidal reference in a boost converter

The dimensionless model of a boost power converter is

$$\begin{aligned} \dot{z}_1 &= 1 - u z_2, \\ \dot{z}_2 &= -\lambda z_2 + u z_1, \end{aligned} \tag{27}$$

where $z_1$ and $z_2$ stand by the scaled input current and output voltage, respectively, and $\lambda$ is a positive scalar that gathers the converter parameters $R$, $L$ and $C$ (see [5], for example). The control gain $u$ takes values in the discrete set $= \{0, 1\}$.

As it has been already commented in Section 1, the elimination of the control action $u$ in (27) yields a differential relation between state variables that does not depend on $u$:

$$z_1(1 - \dot{z}_1) = z_2(\dot{z}_2 + \lambda z_2). \tag{28}$$

Let the control goal be the tracking of a positive, sinusoidal output reference profile $z_{2d}(t) = A + B \sin \omega t$, with period $T = 2\pi\omega^{-1}$. It is proved in [5] that the system is nonminimum phase when the output is set to $y = z_2$. Alternatively, it is minimum phase in case that the output is changed to $y = z_1$: note that in this case the internal dynamics equation is

$$z_2(\dot{z}_2 + \lambda z_2) = z_{1d}(1 - \dot{z}_{1d}),$$

and the change of variable $z = 2^{-1} z_2^2$ yields the linear ODE

$$\dot{z} + 2\lambda z = z_{1d}(1 - \dot{z}_{1d}); \tag{29}$$

the claim follows because of the positivity of $\lambda$.

Hence, the tracking objective may be achieved by means of an indirect control: the state variable $z_1$ may be forced to follow a reference $z_{1d}(t)$ that satisfies an Abel equation of the second kind yet in normal form:

$$z_{1d}(1 - \dot{z}_{1d}) = g(t), \tag{30}$$

where

$$g(t) = [1 + z_{2d}(t)]\,[\dot{z}_{2d}(t) + \lambda z_{2d}(t)] = \lambda\left(A^2 + \frac{B^2}{2}\right) + \\ + AB\omega\cos\omega t + 2AB\lambda\sin\omega t - \frac{B^2\lambda}{2}\cos 2\omega t + \frac{B^2\omega}{2}\sin 2\omega t.$$

For the control method to be physically realizable, two essential conditions must be fulfilled. On the one hand, the current reference $z_{1d}$ must be bounded and, preferable, periodic. On the other hand, the system performance has to lay in a region of the phase plane where the control action does not saturate (recall that $u \in \{0, 1\}$. Both demands have a direct translation on the reference candidate parameters. Following [5] and [10], it is sufficient for (30) to have a single, positive, $T$-periodic solution $z_{1d}(t) = \phi(t)$ and, at the same time, ensure the presence of the system in the non saturation zone during the steady state when $(z_1, z_2) = (z_{1d}(t), z_{2d}(t))$, that:

$$A > B\sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2} > 0, \tag{31}$$

$$A \geq B + \frac{A + B\sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2}}{A - B\sqrt{1 + \left(\frac{\omega}{\lambda}\right)^2}}. \tag{32}$$

According to Theorem 1, such a solution will be unstable. Furthermore, Assumption A is verified iff

$$\lambda\omega\left(A^2 + \frac{B^2}{2}\right) > 1 + \left\{B^2\pi\omega\left[(4A^2 + B^2)\omega^2 + (16A^2 + B^2)\lambda^2\right]\right\}^{\frac{1}{4}}. \tag{33}$$

In case that (31), (32) and (33) are fulfilled, the unstable limit cycle may be approximated using the iterative approach established in Theorem 2.

Note that if one assumes fixed values for $A$, $B$ and $\lambda$, with $A > B > 0$, the admissible values for the frequency $\omega$ are to be found in the region of the first quadrant defined by the intersection of a degree 4 polynomial in $\omega$ with a straight line orthogonal to the $\omega$-axis:

$$\omega \leq \lambda \sqrt{\frac{(A-B)^2 - (A+B)}{B(A-B+1)}}, \tag{34}$$

$$\left[(2A^2+B^2)\lambda\omega - 2\right]^4 > 16B^2\pi\omega\left[(4A^2+B^2)\omega^2 + (16A^2+B^2)\lambda^2\right]. \tag{35}$$

**Remark 6** *Besides the Galerkin method [6],[16], alternative attempts to solve equation (30) may be found in the literature. Among them one finds [3], which addresses the exact tracking of non-causal references in nonminimum phase, time-varying systems. Essentially, the authors find a bounded solution for the unstable internal dynamics with an iteration procedure that involves a backwards time integration in each step. Nevertheless, this procedure yields high sensitivity to external disturbances. Hence, alternative approximation methods that are able to incorporate robust control strategies may offer a performance improvement.*

*In [2], a bounded solution of (30) is approximated in the following way: first of all, the expression of the equilibrium point that may correspond to the regulation case is obtained. Afterwards, the constant output is replaced by the actual time-varying reference. As the approximate solution depends on the plant parameters, disturbance compensation is feasible. However, a severe compromise between the reference and the system parameters is necessary in order to maintain the tracking error in acceptable bounds.*

*Finally, [15] solves the ODE using the flatness [4] property of the system to design a functional iterative procedure that yields successive approximations of the minimum phase state variable reference. Hence, on-line dynamic compensation of perturbations should be achieved by means of an appropriate disturbance identification scheme. However, the power of the method is limited by the fact that no convergence study of the approximation sequence is provided.*

*The improvement represented by the iterative approach presented in this report is particularly increased in its application to nonminimum phase nonlinear power converters. This is due to the fact that, the successive approximations being analytical and explicitly dependent on the system parameters, on-line rejection of disturbances by means of dynamic compensation is also feasible.*

**Remark 7** *When the system is indirectly controlled with successive approximations of the current reference $\phi(t)$, i.e. with elements of $\{z_{1n}\}$, one gets a sequence of outputs $\{z_{2n}\}$ obtained from (29). Such a sequence converges uniformly to the output command profile $f(t)$ [6].*

## 5 Simulation results

The technique has been tested in a boost converter with $V_g = 50V$, $L = 0.018H$, $C = 0.00022F$ and $R = 10\Omega$. The output voltage reference profile has been set to:

$$v_C = 135 + 15\sin(2\pi\nu\tau),$$

with $\nu = 50Hz$. The corresponding values in normalized variables are $\lambda = 0.9045$ and

$$z_{2d}(t) = 2.7 + 0.3\sin\omega t,$$

where $\omega = 0.6252$. These settings guarantee the fulfillment of (31), (32) and (33). Furthermore, assuming that the values of $A$, $B$ and $\lambda$ are fixed, (34) and (35) are satisfied $\forall\ \omega \in (0, 0.02) \cup (0.44, 1.48)$. This corresponds to $\nu \in (0, 1.60Hz) \cup (35.19Hz, 118.37Hz)$.

The iterative procedure introduced in Theorem 2 to approximate the unstable limit cycle of (30) has been run with initial condition $\bar{z}_{10} = 0$. The expression of the first and second approximations are:

$$z_{10} = \lambda\left(A^2 + \frac{B^2}{2}\right), \tag{36}$$

$$z_{11} = z_{10} + \frac{8\lambda AB\cos\omega t - 4AB\omega\sin\omega t + B^2\omega\cos 2\omega t + B^2\lambda\sin 2\omega t}{2\lambda\omega(2A^2 + B^2)}. \tag{37}$$

Figure 1 depicts the $T$-periodic solution of (5) $\phi(t)$ and the approximations $z_{10}$, $z_{11}$, $z_{12}$ and $z_{13}$. It can be noticed that they converge to $\phi(t)$, $z_{13}$ being undistinguishable from $\phi(t)$. This fact may be better observed in Table 1, which contains the $L_2$ and $L_\infty$ absolute and relative errors of the approximations $z_{11}$ to $z_{15}$. Although the convergence is faster when using Galerkin approximations (see Table 1 in [6]), it has already been pointed out in Section 1 that high order Galerkin coefficients are not easily obtainable.

Finally, the average continuous controls $\bar{u}_n$,

$$\bar{u}_n = \frac{1 - \dot{z}_{1n}}{z_{2n}},$$

that maintain the tracking situation $(z_1, z_2) = (z_{1n}, z_{2n})$ in the steady state are shown to lay in the non saturation zone $0 < \bar{u}_n < 1$, $n = 1, \ldots, 4$, in Figure 2.

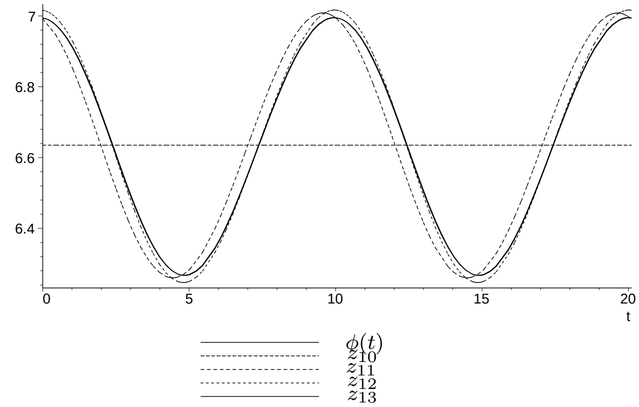


Fig. 1. The $T$-periodic solution of (5), i.e $\phi(t)$, and the approximated solutions $z_{10}$, $z_{11}$, $z_{12}$ and $z_{13}$.

Table 1
Absolute and relative errors of several input current approximations measured with the $L_2$ and the $L_\infty$ norms.

| | $z_{11}$ | $z_{12}$ | $z_{13}$ | $z_{14}$ | $z_{15}$ |
|---|---|---|---|---|---|
| $\|e_{z_{1n}}\|_{L_2}$ | $1.97 \cdot 10^{-1}$ | $4.75 \cdot 10^{-2}$ | $1.13 \cdot 10^{-2}$ | $2.69 \cdot 10^{-3}$ | $6.32 \cdot 10^{-4}$ |
| $\frac{\|e_{z_{1n}}\|_{L_2}}{\|\phi\|_{L_2}} \cdot 100\%$ | $9.34 \cdot 10^{-1}$ | $2.26 \cdot 10^{-1}$ | $5.38 \cdot 10^{-2}$ | $1.28 \cdot 10^{-2}$ | $3.00 \cdot 10^{-3}$ |
| $\|e_{z_{1n}}\|_\infty$ | $9.32 \cdot 10^{-2}$ | $2.19 \cdot 10^{-2}$ | $6.04 \cdot 10^{-3}$ | $1.24 \cdot 10^{-3}$ | $3.48 \cdot 10^{-4}$ |
| $\frac{\|e_{z_{1n}}\|_\infty}{\|\phi\|_\infty} \cdot 100\%$ | $1.33 \cdot 10^{0}$ | $3.12 \cdot 10^{-1}$ | $8.62 \cdot 10^{-2}$ | $1.77 \cdot 10^{-2}$ | $4.97 \cdot 10^{-3}$ |

Notice that $\bar{u}_4$ can be hardly distinguished from $\bar{u}_3$; $\bar{u}_5$ is also undistinguishable from both of them in a plot. The output responses $z_{2n}$ may be obtained from (29) (recall also Remark 7) .

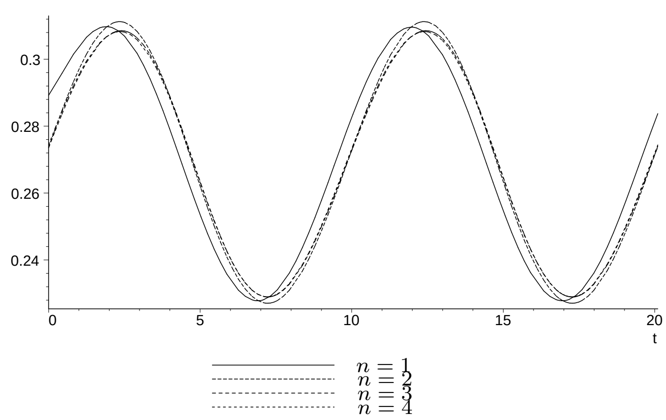


Fig. 2. Detail of the ideal control functions $\bar{u}_n$, $n = 1, \ldots, 4$ laying in the unsaturated zone.

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