Application of the Forces' Method in Dynamic Systems

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Abstract

We present here some applications of the Forces' method in dynamic systems. In particular, we consider the problem of the approximation of the trajectories of a conservative system of point masses by means of the minimization of the action integral and the computation of planar central configurations.

1 Introduction

The classical N-body problem consists in solving the movement equations of a system of N masses, $m_1, \ldots, m_N$, that interact according to the Newtonian forces $F_i = -\sum_{\substack{j=1 \\ j\neq i}}^{N} m_i m_j \dfrac{x_i - x_j}{|x_i - x_j|^3}$. If the trajectories of the N masses are denoted by $x(t) = (x_1(t), \ldots, x_N(t))$, then the potential and kinetic energies of the system are $V(t) = -\sum_{1\leq i<j\leq N} \dfrac{m_i m_j}{|x_i(t) - x_j(t)|}$ and $K(t) = \dfrac{1}{2}\sum_{i=1}^{N} m_i |v_i(t)|^2$, respectively, where $v_i(t) = x_i'(t)$. The Lagrangian of the system is defined by $L(t) = K(t) - V(t)$, and the action of the system between the instants $t = 0$ and $t = T$ is $A = \displaystyle\int_0^T L(t)\, dt$. According to the fundamentals of Mechanics, the trajectories of the particles between the instants $t = 0$ and $t = T$ must be stationary points for A. Hence, the calculation of these trajectories reduces to a Variational Calculus problem.

We consider here the application of the Forces' method as an alternative technique to tackle directly the problem of the computation of stationary points for the action functional under general conditions. We also study the particular case of the central planar configurations, that can be raised as a problem of static equilibrium.

2 Action minimization

For the numerical estimation of the trajectories of the particles we start by discretizing the interval $[0,T]$ in m subintervals $\Delta t_1,\ldots,\Delta t_m$, $\sum_{i=1}^{m}\Delta t_i = T$. This generates the instants $t_0,\ldots,t_m$, where $t_0=0$, $t_j=\sum_{i=1}^{j}\Delta t_i$, $j=1,\ldots,m$. We consider as the unknowns of the problem the positions of each particle x_i in each instant t_j, that we denote by $x_{i,j}=x_i(t_j)$. We associate to each subinterval Δt_j and to each particle the velocity $v_{i,j}=\dfrac{x_{i,j}-x_{i,j-1}}{\Delta t_j}$, $i=1,\ldots,N$, $j=1,\ldots,m$. Moreover, we assign to each subinterval Δt_k the potential energy value $V_k=-\dfrac{1}{2}\displaystyle\sum_{1\le i<j\le N} m_i m_j\left(\frac{1}{|x_{i,k}-x_{j,k}|}+\frac{1}{|x_{i,k-1}-x_{j,k-1}|}\right)$, $k=1,\ldots,m$. We consider the following quadrature formula for the action integral

$$A\simeq\frac{1}{2}\sum_{k=1}^{m}\left(\sum_{i=1}^{N}m_i|v_{i,k}|^2+V_k\right)\Delta t_k=I(p),$$

where $p=(x_{1,0},\ldots,x_{1,m},\ldots,x_{N,m})$ is the vector that contains the $3N(m+1)$ unknowns corresponding to the discretized problem in the three-dimensional space or the $2N(m+1)$ unknowns in the bi-dimensional case. In these conditions, the solutions of the non-linear system $\dfrac{\partial I}{\partial p}=0$ that satisfy the constrains corresponding to each particular case approximate trajectories stationary for the action.

The system $\dfrac{\partial I}{\partial p}=0$ is highly non-linear and in general it contain a big amount of unknowns. However, we have a robust and efficient algorithm to solve this kind of systems [1, 2]. We consider p as the vector containing the position vectors of a system with $N(m+1)$ virtual particles with masses m_i corresponding to the associated trajectories, and we say that $I(p)$ is their "virtual potential energy". So, the "virtual forces" $F_{i,k}=-\dfrac{\partial I}{\partial x_{i,k}}$, $i=1,\ldots,N$, are given by

$$F_{i,k}=m_i\left(v_{i,k}+v_{i,k+1}+\sum_{\substack{j=1\\ j\neq i}}^{N}m_j\frac{\Delta t_k+\Delta t_{k+1}}{2}\,\frac{x_{i,k}-x_{j,k}}{|x_{i,k}-x_{j,k}|^3}\right)$$

when $k=1,\ldots,m-1$, whereas the expressions for the cases $k=0$, $k=m$ are different for closed or open trajectories. Specifically, for open trajectories, $x_{i,0}\neq x_{i,m}$, we have

$$F_{i,0}=m_i\left(v_{i,1}+\sum_{\substack{j=1\\ j\neq i}}^{N}m_j\frac{\Delta t_1}{2}\,\frac{x_{i,0}-x_{j,0}}{|x_{i,0}-x_{j,0}|^3}\right)$$

for $k=0$ and

$$F_{i,m}=m_i\left(v_{i,m}+\sum_{\substack{j=1\\ j\neq i}}^{N}m_j\frac{\Delta t_m}{2}\,\frac{x_{i,m}-x_{j,m}}{|x_{i,m}-x_{j,m}|^3}\right)$$

for $k = m$, and for closed trajectories, $x_{i,0} = x_{i,m}$, the unknown $x_{i,m}$ disappears and we have for $k = 0$

$$F_{i,0} = m_i \left(\frac{x_{i,0} - x_{i,m-1}}{\Delta t_m} + v_{i,1} + \sum_{\substack{j=1 \\ j \neq i}}^{N} m_j \frac{\Delta t_1 + \Delta t_m}{2} \frac{x_{i,0} - x_{j,0}}{|x_{i,0} - x_{j,0}|^3} \right).$$

In general, $F_{i,j}$ is the sum of two components, $F_{i,j}^K$ and $F_{i,j}^V$, corresponding to the kinetic and the potential energies, respectively.

Note that if the initial position and velocity are prescribed for all the particles, then the above expressions reduce to the forward Euler's method.

For the numerical resolution of the non-linear system $\dfrac{\partial I}{\partial p} = 0$ we use the forward Euler scheme $x_{i,j} = x_{i,j} + a\varphi(p)\dfrac{F_{i,j}}{F_{\max}^K}$, where $F_{\max}^K$ is the modulus of the maximum kinetic component of the forces $F_{i,j}$, φ is a function depending on the current position of all the virtual particles and a is a constant coefficient. In the following preliminary examples $\varphi(p)$ is the minimum distance between consecutive virtual particles in a same path, but we are exploring other possible choices.

2.1 Initial and final positions prescribed

In this example we consider the boundary conditions corresponding to prescribe the initial and final positions of all the masses. Specifically, we study a bi-dimensional Big-Bang-Big-Crunch system of $N = 6$ masses with $T = 1$; that is, a system in which 6 masses starts from the same position (the origin) and they go to the same position (the point $(0, 1)$) in a unit time.

The starting positions of the virtual particles have been generated randomly according to a uniform probability density in the square $(0, 1)\times(0, 1)$. The initial trajectory corresponding to each particle was determined by joining the associated points in the order that they were generated (in this example we have used $m = 100$). Fig. 1 shows this starting configuration (left) and also the corresponding final configuration (right). Fig. 2 shows the position of the real masses on the final paths for $t = 0.02$, $t = 0.5$ and $t = 0.98$. As it can be seen, 5 masses form a regular pentagon that grows and then decreases, whereas the 6th particle follows the straight line that joins the origin with the point $(0, 1)$ and it remains in the center of the pentagon.

Taking into account the difficulty of the starting configuration, this example confirms the robustness of our optimization algorithm.

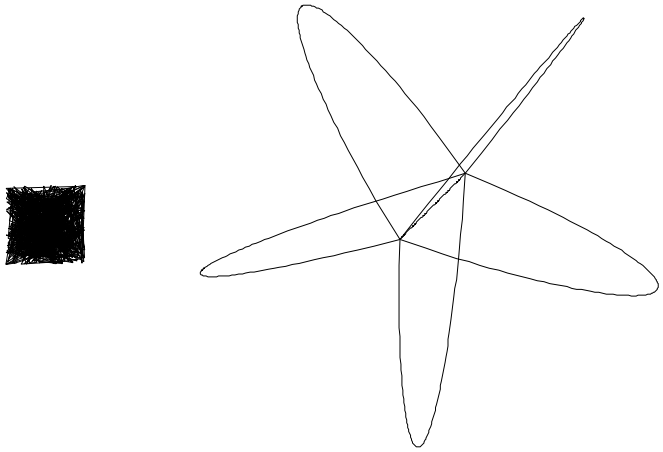

Figure 1: Initial (left) and final (right) configurations for a Big-Bang-Big-Crunch system with $N = 6$ equal masses and $m = 100$ points to define the path of each mass.

2.2 Choreographies

The movement of N equal masses M describe a choreography if all the masses follow periodically the same closed trajectory and the time interval between two consecutive masses is the same. In this case it suffices to fix the period T and minimizing $A = \int_0^{\frac{T}{N}} L(t)\, dt$.

Figures 3, 4 and 5 show different choreografies obtained for $N = 7$, $N = 11$ and $N = 100$ masses.

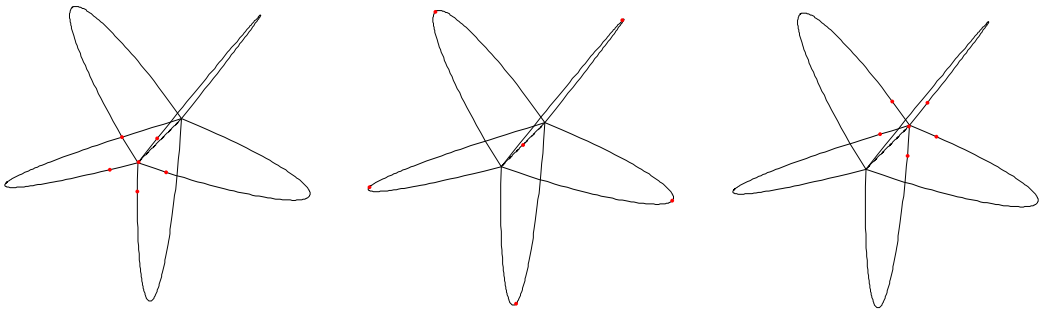

Figure 2: Position of the masses for $t = 0.02$ (left), $t = 0.5$ (center) and $t = 0.98$ (right) according to the paths in Fig. 1 (right).

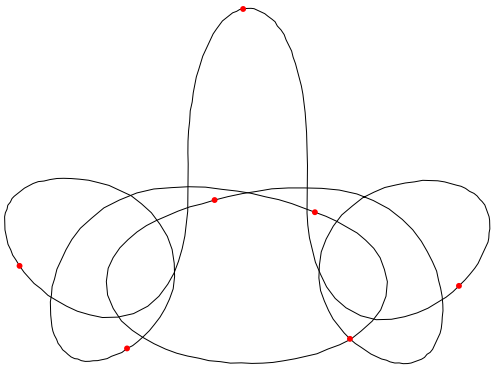

Figure 3: Choreography with $N = 7$ and $m = 350$ points to define the whole curve.

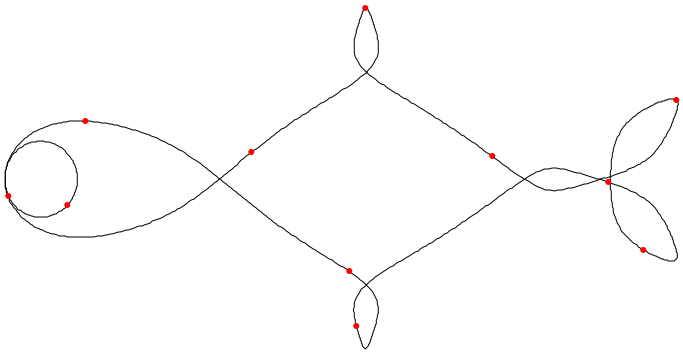

Figure 4: Choreography with $N = 11$ and $m = 1100$ points to define the whole curve.

3 Computation of central configurations

Without loss of generality, we can assume here $\sum_{i=1}^{N} m_i = N$. Then, the central configurations can be defined, up to rigid motions, as the stationary points for the functional

$$\Omega(x) = \sum_{1 \le i < j \le N} \frac{m_i m_j}{|x_i - x_j|} + \frac{N}{2} \sum_{i=1}^{N} m_i |x_i - x_{\text{cm}}|^2,$$

where $x_{\text{cm}} = \dfrac{1}{N} \sum_{i=1}^{N} m_i x_i$.

In this case we can see $\Omega(x)$ as a virtual potential energy, and the virtual forces $F_i = -\dfrac{\partial \Omega}{\partial x_i}$ are given by $F_i = -\sum\limits_{\substack{j=1 \\ j \neq i}}^{N} m_i m_j \dfrac{x_i - x_j}{|x_i - x_j|^3} + N m_i (x_i - x_{\text{cm}})$. To solve the non-linear system

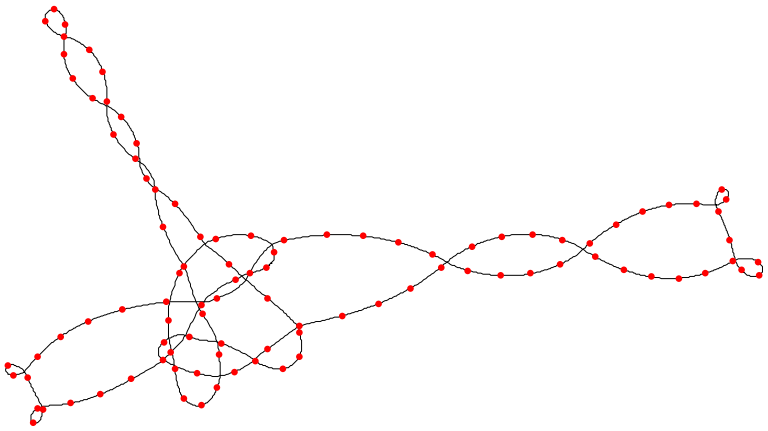

Figure 5: Choreography with $N = 100$ and $m = 2000$ points to define the whole curve.

$\frac{\partial \Omega}{\partial x} = 0$ we have used the forward Euler scheme $x_i = x_i + a \min\limits_{1 \le j < k \le N} \{|x_j - x_k|\} \frac{F_i}{F^V_{\max}}$, where $F^V_{\max} = \max\limits_{1 \le i \le N} \{|F^V_i|\}$, $F^V_i = F_i - N m_i (x_i - x_{\rm cm})$ and a is a constant coefficient.

Figure 6 shows a planar central configuration for $N = 1500$ equal masses.

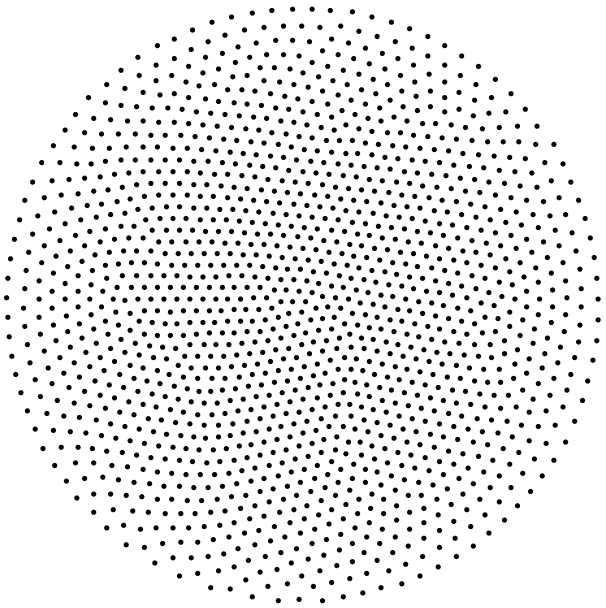

Figure 6: A planar central configuration for $N = 1500$ equal masses.

References

[1] E. Bendito, A. Carmona, A.M. Encinas and J.M. Gesto, Estimation of Fekete Points, *J. Comput. Phys.*, **225** (2007), 2354-2376.

[2] E. Bendito, A. Carmona, A.M. Encinas and J.M. Gesto, Computational Cost of the Fekete Problem, preprint.