The Manhattan Product of Digraphs *

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Abstract

We give a formal definition of a new product of bipartite digraphs, the Manhattan product, and we study some of its main properties. It is shown that if all the factors of the above product are (directed) cycles, then the digraph obtained is the Manhattan street network. To this respect, it is proved that many properties of these networks, such as high symmetries and the presence of Hamiltonian cycles, are shared by the Manhattan product of some digraphs. Moreover, we prove that the Manhattan product of two Manhattan streets networks is also a Manhattan street network. Also, some necessary conditions for the Manhattan product of two Cayley digraphs to be again a Cayley digraph are given.

1 Introduction

The 2-dimensional Manhattan street network M_2 was introduced simultaneously, in different contexts, by Morillo *et al.* [9] and Maxemchuk [8] as an unidirectional regular mesh structure resembling locally the topology of the avenues and streets of Manhattan (or *l'Eixample* in downtown Barcelona). In fact, M_2 has a natural embedding in the torus and it has been extensively studied in the literature as a model of interconnection networks. For instance, its average distance has been computed by Khasnabish [7] and Chung and Agrawal [3], the generation of routing schemes by Maxemchuk [8]. Moreover, Chung and Agrawal [3] gave its diameter. Varvarigos [10] evaluated again the mean internodal distance and provided a shortest path routing algorism and some Hamiltonian properties.

Recall that a digraph $G = (V, A)$ consists of a set of *vertices* V, together with a set of *arcs* A, which are ordered pairs of vertices, $A \subset V \times V = \{(u, v) : u, v \in V\}$. An arc (u, v) is usually depicted as an arrow with *tail* u (initial vertex) and *head* v (end vertex), that is, $u \to v$. The *indegree* $\delta^-(u)$ (respectively, *outdegree* $\delta^+(u)$) of a vertex u is the number of arcs with tail (respectively, head) u. Then G is δ*-regular* when $\delta^-(u) = \delta^+(u) = \delta$ for every vertex $u \in V$. Given a digraph $G = (V, A)$, its *converse* digraph $\overline{G} = (V, \overline{A})$ is obtained from G by reversing all the orientations of the arcs in A, that is, $(u, v) \in \overline{A}$ if and only if $(v, u) \in A$. The standard definitions and basic results about graphs and digraphs not defined here can be found in [1, 2, 11].

In this paper, we first recall the definition and some of the properties of the Manhattan street network (where the Manhattan product takes its name from). Afterwards we introduce the Manhattan product of (bipartite) digraphs. It is shown that when all the factors

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are (directed) cycles, then the obtained digraph is just the Manhattan street network. Moreover, we prove that the Manhattan product of two Manhattan streets networks is also a Manhattan street network. It is proved that many properties of these networks, such as high symmetries and the presence of Hamiltonian cycles, are shared by the Manhattan product of some digraphs. We also investigate when the Manhattan product of two Cayley digraph is again a Cayley digraph and characterize the corresponding group.

2 Manhattan street networks

In this section, we recall the definition and some basic properties [4, 5] of a class of toroidal directed networks, commonly known as Manhattan street networks.

Given n *even* positive integers $N_1, N_2, \ldots, N_n$, the *n-dimensional Manhattan street network* $M_n = M(N_1, N_2, \ldots, N_n)$ is a digraph with vertex set $V(M_n) = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_n}$. Thus, each of its vertices is represented by an n-vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, with $0 \leq u_i \leq N_i - 1$, $i = 1, 2, \ldots, n$. The arc set $A(M_n)$ is defined by the following adjacencies (here called *i-arcs*):

$$(u_1, \ldots, u_i, \ldots, u_n) \quad \to \quad (u_1, \ldots, u_i + (-1)^{\sum_{j \neq i} u_j}, \ldots, u_n) \qquad (1 \leq i \leq n). \qquad (1)$$

Therefore, M_n is an n-regular digraph on $N = \prod_{i=1}^{n} N_i$ vertices.

The properties of M_n are the following:

- Homomorphism: There exist an homomorphism from M_n to the symmetric digraph of the hypercube Q_n^*, so that M_n is both 2^n-partite and bipartite digraph.
- Vertex-symmetry: The n-dimensional Manhattan street network M_n is a vertex-symmetric digraph.
- Line digraph: For any N_1, N_2, the 2-dim Manhattan street network $M_2(N_1, N_2)$ is a line digraph.
- Diameter: For $N_i > 4$, the diameter of the n-dim Manhattan street network $M_n = M(N_1, N_2, \ldots, N_n)$, $i = 1, 2, \ldots, n$, is

  (*a*) $D(M_n) = \frac{1}{2} \sum_{i=1}^{n} N_i + 1$, if $N_i \equiv 0 \pmod 4$ for any $1 \leq i \leq n$;

  (*b*) $D(M_n) = \frac{1}{2} \sum_{i=1}^{n} N_i$, otherwise.
- Hamiltonicity: The n-dimensional Manhattan street network M_n is Hamiltonian.

3 The Manhattan product and its basic properties

In this section, we present an operation on (bipartite) digraphs which, as a particular case, gives rise to a Manhattan street network. With this aim, let $G_i = (V_i, A_i)$ be n bipartite digraphs with independent sets $V_i = V_{i0} \cup V_{i1}$, $N_i = |V_i|$, $i = 1, 2, \ldots, n$. Let π be the characteristic function of $V_{i1} \subset V_i$ for any i; that is,

$$\pi(u) = \begin{cases} 0 & \text{if } u \in V_{i0}, \\ 1 & \text{if } u \in V_{i1}. \end{cases}$$

Then, the *Manhattan product* $M_n = G_1 \# G_2 \# \cdots \# G_n$ is the digraph with vertex set $V(M_n) = V_1 \times V_2 \times \cdots \times V_n$, and each vertex $(u_1, \ldots, u_i, \ldots, u_n)$ is adjacent to vertices $(u_1, \ldots, v_i, \ldots, u_n)$, $1 \leq i \leq n$, when

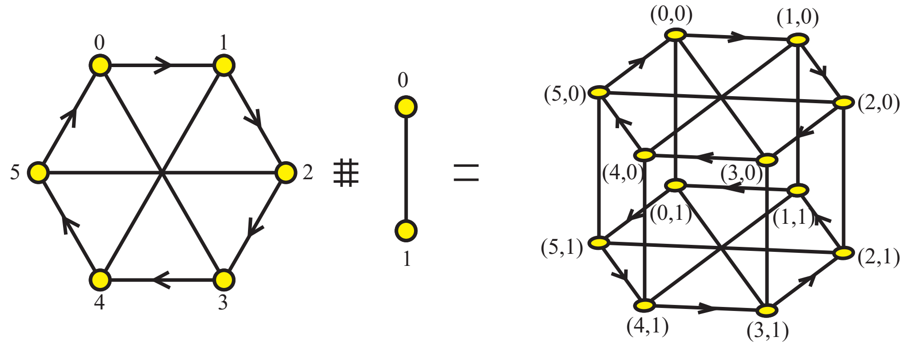

Figure 1: The Manhattan product $Cay(\mathbb{Z}_6, \{1,3\}) \# K_2^*$ (undirected lines stand for pairs of arcs in opposite directions).

- $v_i \in \Gamma^+(u_i)$ if $\sum_{j \neq i} \pi(u_j)$ is even,
- $v_i \in \Gamma^-(u_i)$ if $\sum_{j \neq i} \pi(u_j)$ is odd.

Fig. 1 shows an example of the Manhattan product of the circulant digraph on 6 vertices and steps 1 and 3 (in other words, the Cayley digraph on $\mathbb{Z}_6$ with generating set {1,3}) by the symmetric complete digraph on 2 vertices, K_2^*.

Thus, if every G_i is δ_i-regular, then M_n is a δ-regular digraph, with $\delta = \sum_{i=1}^n \delta_i$, on $N = \prod_{i=1}^n N_i$ vertices.

Some of the basic properties of the Manhattan product, which are a generalization of the properties of the Manhattan street networks given in [4], are presented in the following proposition:

Proposition 3.1. *The Manhattan product $H = G_1 \# G_2 \# \cdots \# G_n$ satisfies the following properties:*

(*a*) *The Manhattan product holds the associative and commutative properties.*

(*b*) *There exists an homomorphism from H to the symmetric digraph of the hypercube Q_n^*. Therefore, H is a bipartite and 2^n-partite digraph.*

(*c*) *For any $n-k$ fixed vertices $x_i \in V_i$, $i = k+1, k+2, \ldots, n$, the subdigraph of H induced by the vertices $(u_1, u_2, \ldots, u_k, x_{k+1}, \ldots, x_n)$ is either the Manhattan product $H_k = G_1 \# G_2 \# \cdots \# G_k$ or its converse $\overline{H}_k$, depending on if $\alpha := \sum_{i=k+1}^n \pi(x_i)$ is even or odd, respectively.*

(*d*) *If each G_i, $i = 1, 2, \ldots, n$, is isomorphic to its converse, then H also is.*

Proof. We only prove the properties (*b*) and (*d*) because the others can be proved similarly as those of the Manhattan street network in [4].

(*b*) The homomorphism from H to Q_n^* is

$$(u_1, u_2, \ldots, u_n) \to (\pi(u_1), \pi(u_2), \ldots, \pi(u_n)),$$

which transform each vertex of H in a binary n-string or as its image vertex in Q_n^*.

(*d*) As the Manhattan product is associative, we only need to deal with the case $H = G_1 \# G_2$. Since, $G_i \cong \overline{G}_i$ by hypothesis, there exist isomorphisms ψ_i, such that

$\Gamma^{\pm}_{G_i}(\psi_i(u_i)) = \psi_i(\Gamma^{\mp}_{G_i}(u_i))$, for all $u_i \in V_i$. As ψ_i is a mapping between stable sets, the parity π in $\overline{G}_i$ can be defined in such a way that $\pi(u_i)$ is even if and only if $\pi(\psi_i(u_i))$ is also even. Then, the mapping Ψ defined in H as

$$\Psi(u_1, u_2) := (\psi_1(u_1), \psi_2(u_2))$$

is the automorphism from H to its converse $\overline{H}$. Indeed, assuming that, for instance, $\pi(u_1), \pi(u_2)$ are even, we have

$$\begin{aligned}
\Psi\big(\Gamma^+_H(u_1, u_2)\big) &= \Psi\big(\Gamma^+_{G_1}(u_1), u_2\big) \cup \Psi\big(u_1, \Gamma^+_{G_2}(u_2)\big) \\
&= \big(\psi_1(\Gamma^+_{G_1}(u_1)), \psi_2(u_2)\big) \cup \big(\psi_1(u_1), \psi_2(\Gamma^+_{G_2}(u_2))\big) \\
&= \big(\Gamma^-_{G_1}(\psi_1(u_1)), \psi_2(u_2)\big) \cup \big(\psi_1(u_1), \Gamma^-_{G_2}(\psi_2(u_2))\big) \\
&= \Gamma^-_H\big(\psi_1(u_1), \psi_2(u_2)\big) \\
&= \Gamma^-_H\big(\Psi(u_1, u_2)\big).
\end{aligned}$$

The other cases, which correspond to other parities of $\pi(u_1)$ and $\pi(u_2)$, can be proved similarly.

□

As an example of a Manhattan product satisfying the property 3.1(e), see again Fig. 1.

4 The Manhattan product and the Manhattan street networks

In this section we show the relationship between the digraphs obtained by the Manhattan product and the Manhattan street networks.

Proposition 4.1. *The Manhattan product of directed cycles with an even order N_i is a Manhattan street network. More precisely, if $G_i = C_{N_i}$, then*

$$C_{N_1} \# C_{N_2} \# \cdots \# C_{N_n} = M(N_1, N_2, \ldots, N_n).$$

Proof. Each cycle C_{N_i} has set of vertices $V_i = \mathbb{Z}_{N_i}$, and adjacencies $\Gamma^+(u_i) = \{u_i + 1 \pmod{N_i}\}$ and $\Gamma^-(u_i) = \{u_i - 1 \pmod{N_i}\}$, such that V_{i0} and V_{i1} are the sets of even and odd vertices, respectively. Thus, the set of vertices in the Manhattan product of directed cycles is $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_n}$ and each vertex $(u_1, \ldots, u_i, \ldots, u_n)$ is adjacent to the vertices $(u_1, \ldots, v_i, \ldots, u_n)$, $1 \le i \le n$, when

- $v_i = u_i + 1$ iff $\sum_{j \ne i} \pi(u_j)$ is even and, hence, $\sum_{j \ne i} u_j$ is also even,
- $v_i = u_i - 1$ iff $\sum_{j \ne i} \pi(u_j)$ is odd and, hence, $\sum_{j \ne i} u_j$ is also odd,

which corresponds to the definition of the Manhattan street network. □

Another expected result of the Manhattan product is the following:

Proposition 4.2. *The Manhattan product of two Manhattan street networks is a Manhattan network. More precisely, if $M^1 = M(N^1_1, N^1_2, \ldots, N^1_{n_1})$ and $M^2 = M(N^2_1, N^2_2, \ldots, N^2_{n_2})$, then*

$$M^1 \# M^2 = M,$$

where $M = M(N^1_1, \ldots, N^1_{n_1}, N^2_1, \ldots, N^2_{n_2})$.

Proof. Both M^1 and M^2 are bipartite digraphs with vertex sets $V^\alpha = \mathbb{Z}_{N_1^\alpha} \times \mathbb{Z}_{N_2^\alpha} \times \cdots \times \mathbb{Z}_{N_{n_\alpha}^\alpha}$, $\alpha = 1, 2$; whereas $M^1 \# M^2$ has vertex set $V = V^1 \times V^2$. Let $V(M)$ be the vertex set of M. Then, we claim that the natural mapping $\Psi : V \to V(M)$, defined by $\Psi(\boldsymbol{u}^1, \boldsymbol{u}^2) = (u_1^1, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2)$ is an isomorphism between the corresponding digraphs. In proving this, let V_0^α and V_1^α be the stable sets of M^α constituted, respectively, by the vertices $\boldsymbol{u}^\alpha = (u_1^\alpha, \ldots, u_{n_\alpha}^\alpha)$ whose sum of components $\sum_{k=1}^{n_\alpha} u_k^\alpha$ is even or odd. With this convention, each vertex $(\boldsymbol{u}^1, \boldsymbol{u}^2)$ of the Manhattan product $M^1 \# M^2$ is adjacent to the vertices $(\boldsymbol{v}^1, \boldsymbol{u}^2)$ and $(\boldsymbol{u}^1, \boldsymbol{v}^2)$ where, for the first ones,

- $\boldsymbol{v}^1 \in \Gamma^+(\boldsymbol{u}^1)$ (in M^1) if $\pi(\boldsymbol{u}^2)$, and hence $\sum_{k=1}^{n_2} u_k^2$, is even;
- $\boldsymbol{v}^1 \in \Gamma^-(\boldsymbol{u}^1)$ (in M^1) if $\pi(\boldsymbol{u}^2)$, and hence $\sum_{k=1}^{n_2} u_k^2$, is odd.

In the first case,

$$\begin{aligned}(\boldsymbol{v}^1, \boldsymbol{u}^2) &\stackrel{\Psi}{\longrightarrow} (u_1^1, \ldots, u_i^1 + (-1)^{\sum_{j\neq i} u_j^1}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2) \\ &= (u_1^1, \ldots, u_i^1 + (-1)^{\sum_{j\neq i} u_j^1 + \sum_{k=1}^{n_2} u_k^2}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2) \qquad (1 \le i \le n_1).\end{aligned}$$

Analogously, in the second case,

$$\begin{aligned}(\boldsymbol{v}^1, \boldsymbol{u}^2) &\stackrel{\Psi}{\longrightarrow} (u_1^1, \ldots, u_i^1 - (-1)^{\sum_{j\neq i} u_j^1}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2) \\ &= (u_1^1, \ldots, u_i^1 + (-1)^{\sum_{j\neq i} u_j^1 + \sum_{k=1}^{n_2} u_k^2}, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2) \qquad (1 \le i \le n_1).\end{aligned}$$

Altogether, we obtain the vertices adjacent to $\Psi(\boldsymbol{u}^1, \boldsymbol{u}^2) = (u_1^1, \ldots, u_{n_1}^1, u_1^2, \ldots, u_{n_2}^2)$ in M (through all the i-arcs, $1 \le i \le n_1$). The adjacencies through the other i-arcs, $n_1 + 1 \le i \le n_1 + n_2$ come from the vertices $(\boldsymbol{u}^1, \boldsymbol{v}^2)$. $\square$

The result of the above proposition can be seen as a corollary of the proposition 4.1 and the associative property. Indeed,

$$\begin{aligned}M^1 \# M^2 &= M(N_1^1, N_2^1, \ldots, N_{n_1}^1) \# M(N_1^2, N_2^2, \ldots, N_{n_2}^2) \\ &= (C_{N_1}^1 \# C_{N_2}^1 \# \cdots \# C_{N_{n_1}}^1) \# (C_{N_1}^2 \# C_{N_2}^2 \# \cdots \# C_{N_{n_2}}^2) \\ &= C_{N_1}^1 \# C_{N_2}^1 \# \cdots \# C_{N_{n_1}}^1 \# C_{N_1}^2 \# C_{N_2}^2 \# \cdots \# C_{N_{n_2}}^2 \\ &= M(N_1^1, N_2^1, \ldots, N_{n_1}^1, N_1^2, N_2^2, \ldots, N_{n_2}^2) = M.\end{aligned}$$

5 Symmetries

In this section we study the symmetries of the digraphs obtained by the Manhattan product.

Proposition 5.1. *Let G_i be vertex-symmetric digraphs such that they are isomorphic to their converses, $i = 1, 2, \ldots, n$. Then, the Manhattan product $H = G_1 \# G_2 \# \cdots \# G_n$ is vertex-symmetric.*

Proof. As before, let $G_i = (V_i, A_i)$ be digraphs with $V_i = V_{i0} \cup V_{i1}$, $i = 1, 2, \ldots, n$.

First, we show that there exists an automorphism Φ in H, which transforms a vertex $(u_1, u_2, \ldots, u_n)$ into a vertex $(v_1, v_2, \ldots, v_n)$, such that $u_i, v_i \in V_{ij_i}$, for each $i \in \{1, 2, \ldots, n\}$ and some $j_i \in \{0, 1\}$ (that is, both components u_i, v_i are in the same stable set). By hypothesis, there exist automorphisms ϕ_i in G_i, $\Gamma_{G_i}^+(\phi_i(w_i)) = \phi_i(\Gamma_{G_i}^+(w_i))$, for every $w_i \in V_i$, such that $\phi_i(u_i) = v_i$. Then, we define

$$\Phi(w_1, w_2, \ldots, w_n) := (\phi_1(w_1), \phi_2(w_2), \ldots, \phi_n(w_n)).$$

Then, assuming that $\sum_{j\neq i}\pi(w_j)$ is even and, hence, $\sum_{j\neq i}\pi(\phi_j(w_j))$ is also even, we have

$$\begin{aligned}
\Phi\big(\Gamma_H^+(w_1,\dots,w_i,\dots,w_n)\big) &= \Phi\big(w_1,\dots,\Gamma_{G_i}^+(w_i),\dots,w_n\big)\\
&= \big(\phi_1(w_1),\dots,\phi_i(\Gamma_{G_i}^+(w_i)),\dots,\phi_n(w_n)\big)\\
&= \big(\phi_1(w_1),\dots,\Gamma_{G_i}^+(\phi_i(w_i)),\dots,\phi_n(w_n)\big)\\
&= \Gamma_H^+\big(\phi_1(w_1),\dots,\phi_i(w_i),\dots,\phi_n(w_n)\big)\\
&= \Gamma_H^+\big(\Phi(w_1,\dots,w_i,\dots,w_n)\big),
\end{aligned}$$

which proves that Φ is an automorphism. The proof is similar for $\sum_{j\neq i}\pi(w_j)$ odd, by using $\Gamma_{G_i}^-(\phi_i(w_i)) = \phi_i(\Gamma_{G_i}^-(w_i))$.

Moreover, we need an automorphism Ψ, which transforms a vertex $(u_1,\dots,u_i,\dots,u_n)$ into a vertex $(v_1,\dots,v_i,\dots,v_n)$, such that, for $k\neq i$, $u_k, v_k \in V_{kj_k}$ as before, while u_i and v_i belong to different stable sets, for example, $u_i\in V_{i0}$ and $v_i \in V_{i1}$. In this case, the automorphism Ψ is built up in the following way. As each G_i is isomorphic to its converse, there exist automorphisms ψ_k, with $k\neq i$, from G_k to $\overline{G}_k$, $\Gamma_{G_k}^+(\psi_k(w_k)) = \psi_k(\Gamma_{G_k}^-(w_k))$, for every $w_k\in V_k$, such that $\psi_k(u_k)=v_k$; and $\psi_i=\phi_i$ (as in the first case). Then, we define Ψ as

$$\Psi(w_1,\dots,w_i,\dots,w_n) := (\psi_1(w_1),\dots,\psi_i(w_i),\dots,\psi_n(w_n)).$$

Let us now assume that $k=1\neq i$ and that $\sum_{j\neq 1}\pi(w_j)$ is even, so that, $\pi(\phi_i(w_i)) + \sum_{j\neq 1,i}\pi(\psi_j(w_j))$ is odd. Then, we have

$$\begin{aligned}
\Psi\big(\Gamma_H^+(w_1,\dots,w_i,\dots,w_n)\big) &= \Psi\big(\Gamma_{G_1}^+(w_1),\dots,w_i,\dots,w_n\big)\\
&= \big(\psi_1(\Gamma_{G_1}^+(w_1)),\dots,\phi_i(w_i),\dots,\psi_n(w_n)\big)\\
&= \big(\Gamma_{G_1}^-(\psi_1(w_1)),\dots,\phi_i(w_i),\dots,\psi_n(w_n)\big)\\
&= \Gamma_H^+\big(\psi_1(w_1),\dots,\phi_i(w_i),\dots,\psi_n(w_n)\big)\\
&= \Gamma_H^+\big(\Psi(w_1,\dots,w_i,\dots,w_n)\big).
\end{aligned}$$

Thus, Ψ is an automorphism. For the case $\sum_{j\neq 1}\pi(w_j)$ odd, the proof is similar, using $\Gamma_{G_k}^-(\psi_k(w_k)) = \psi_k(\Gamma_{G_k}^+(w_k))$. On the other hand, the case $k=i$ is proved as before, because assuming that $\sum_{j\neq i}\pi(w_j)$ is even, then $\sum_{j\neq i}\pi(\psi_j(w_j))$ is also even. This completes the proof. □

6 Cayley digraphs and the Manhattan product

In this section we investigate when the Manhattan product of Cayley digraphs of is again a Cayley digraph. This generalizes the case studied in [4, 5] of Manhattan street networks, where the factors of the product are directed cycles (see Prop. 4.1), that is, Cayley digraph of the cyclic groups. Because of the associative property of such product (see Prop. 3.1(a)), we only need to study the case of two factors.

Theorem 6.1. *Let $G_1 = \mathrm{Cay}(\Gamma_1,\Delta_1)$ be a bipartite Cayley digraph of the group Γ_1 with generating set $\Delta_1=\{a_1,\dots,a_p\}$ and set of generating relations R_1, such that there exists a group automorphism ψ_1 satisfying $\psi_1(a_i)=a_i^{-1}$, for $i=1,\dots,p$. Let $G_2=\mathrm{Cay}(\Gamma_2,\Delta_2)$ be the bipartite Cayley digraph of the group Γ_2 with generating set $\Delta_2=\{b_1,\dots,b_q\}$ and set of generating relations R_2, such that there exists a group automorphism ψ_2 satisfying $\psi_2(b_j)=b_j^{-1}$, for $j=1,\dots,q$. Then, the Manhattan product $H=G_1 \# G_2$ is the Cayley digraph of the group*

$$\Gamma = \langle \alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q | R'_1, R'_2, (\alpha_i\beta_j)^2 = (\alpha_i\beta_j^{-1})^2 = 1\rangle, \quad i\neq j, \qquad (2)$$

where R'_1 is the same set of generating relations as R_1 changing a_i by α_i (and similarly for R'_2).

Proof. Since for every $u_1 \in \Gamma_1$ and $i = 1, \ldots, p$

$$\psi_1(u_1 a_i) = \psi(u_1)\psi(a_i) = \psi(u_1) a_i^{-1},$$

then ψ_1 is an (involutive) isomorphism for G_1 to $\overline{G}_1$ preserving colors. The same holds for ψ_2 and G_2. Moreover, since G_1, G_2 are vertex-symmetric, Proposition 5.1 applies and H is also vertex-symmetric.

In fact, we will see that its automorphism group contains a regular subgroup. With this aim, note first that, by using the above automorphisms, we have the following natural way of defining the adjacencies of H (with "colors" denoted by α_i, $1 \le i \le p$, and β_j, $1 \le j \le q$):

$$\begin{array}{lcl} (u_1, u_2) & \stackrel{\alpha_i\text{-arc}}{\longrightarrow} & (u_1, u_2) * \alpha_i = \big(u_1 \psi_1^{\pi(u_2)}(a_i), u_2\big), \\ (u_1, u_2) & \stackrel{\beta_j\text{-arc}}{\longrightarrow} & (u_1, u_2) * \beta_j = \big(u_1, u_2 \psi_2^{\pi(u_1)}(b_j)\big). \end{array}$$

Let us now prove that the mappings ϕ_{1i}, ϕ_{2j}, for $1 \le i \le p$ and $1 \le j \le q$, defined by

$$\begin{aligned} \phi_{1i}(u_1, u_2) &= (a_i u_1, \psi_2(u_2)), \\ \phi_{2j}(u_1, u_2) &= (\psi_1(u_1), b_j u_2), \end{aligned}$$

are all color-preserving isomorphisms of H. Indeed, for all $1 \le i, j \le p$ we have

$$\begin{aligned} \phi_{1i}\big((u_1, u_2) * \alpha_j\big) &= \phi_{1i}\big(u_1 \psi_1^{\pi(u_2)}(a_j), u_2\big) \\ &= \big(a_i u_1 \psi_1^{\pi(u_2)}(a_j), \psi_2(u_2)\big) \\ &= \big(a_i u_1 \psi_1^{\pi(\psi_2(u_2))}(a_j), \psi_2(u_2)\big) \\ &= \big(a_i u_1, \psi_2(u_2)\big) * \alpha_j \\ &= \phi_{1i}(u_1, u_2) * \alpha_j, \end{aligned}$$

where we have used that $\pi(u_2) = \pi(\psi_2(u_2))$ because u_2 can be expressed as the product of the generators b_j and $\pi(b_j) = \pi(\psi_2(b_j)) = \pi(b_j^{-1})$ for all $1 \le j \le q$. Moreover, for all $1 \le i \le p, 1 \le j \le q$, we also have

$$\begin{aligned} \phi_{1i}\big((u_1, u_2) * \beta_j\big) &= \phi_{1i}\big(u_1, u_2 \psi_2^{\pi(u_1)}(b_j)\big) \\ &= \big(a_i u_1, \psi_2(u_2) \psi_2^{\pi(u_1)+1}(b_j)\big) \\ &= \big(a_i u_1, \psi_2(u_2) \psi_2^{\pi(a_j \cdot u_1)}(b_j)\big) \\ &= \big(a_i u_1, \psi_2(u_2)\big) * \beta_j \\ &= \phi_{1i}(u_1, u_2) * \beta_j, \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \phi_{2i}\big((u_1, u_2) * \alpha_j\big) &= \phi_{2i}(u_1, u_2) * \alpha_j, \quad 1 \le i \le q, 1 \le j \le p \\ \phi_{2i}\big((u_1, u_2) * \beta_j\big) &= \phi_{2i}(u_1, u_2) * \beta_j, \quad 1 \le i, j \le q. \end{aligned}$$

To see that the permutation group $\Gamma = \langle \phi_{1i}, \phi_{2j} | 1 \le i \le p, 1 \le j \le q \rangle$ acts transitively on $\Gamma_1 \times \Gamma_2$, that is, the vertex set of H, it is enough to show that any vertex (u_1, u_2) can be mapped into vertex (e_1, e_2) (where e_1 and e_2 stand for the identity elements of Γ_1 and Γ_2, respectively) since, as it was mentioned above, H is vertex-symmetric.

To this end, as Δ_1 is a generating set, u_1^{-1} can be expressed in the form, say, $u_1^{-1} = a_{i_1}a_{i_2}\cdots a_{i_r}$. Then,

$$\begin{aligned}\phi_{1i_1}\phi_{1i_2}\cdots\phi_{1i_r}(u_1,u_2) &= \big(a_{i_1}a_{i_2}\cdots a_{i_r},\psi_2^r(u_2)\big)\\ &= \big(e_1,\psi_2^r(u_2)\big)\\ &= (e_1,v_2),\end{aligned}$$

where $v_2 = u_2^{(-1)^r}$ is either u_2 or u_2^{-1} according to the parity of r. In any case, as Δ_2 is also a generating set, the inverse of this element can be written as, say, $v_2^{-1} = b_{j_1}b_{j_2}\cdots b_{j_s}$. Then,

$$\begin{aligned}\phi_{2j_1}\phi_{2j_2}\cdots\phi_{2j_s}(e_1,v_2) &= \big(\psi_1^s(e_1),e_2\big)\\ &= (e_1,e_2),\end{aligned}$$

as claimed.

Thus, the group Γ is a regular subgroup of the automorphism group of H and the Manhattan product is a Cayley digraph of Γ with generators $\alpha_i \equiv \phi_{1i}$ and $\beta_j \equiv \phi_{2j}$. Regarding the structure of Γ, let us check only one of the defining relations in (2), as the others can be proved similarly.

$$\begin{aligned}(\phi_{1i}\phi_{2j})^2(u_1,u_2) &= \phi_{1i}\phi_{2j}\phi_{i1}\phi_{2j}(u_1,u_2)\\ &= \phi_{1i}\phi_{2j}\phi_{1i}\big(\psi_1(u_1),b_ju_2\big)\\ &= \phi_{1i}\phi_{2j}\big(a_i\psi_1(u_1),b_j^{-1}\psi_2(u_2)\big)\\ &= \phi_{1i}\big(a_i^{-1}\psi_1^2(u_1),\psi_2(u_2)\big)\\ &= \big(\psi_1^2(u_1),\psi_2^2(u_2)\big)\\ &= (u_1,u_2).\end{aligned}$$

□

This result can be compared with the well-known following one [11]: If G_1 and G_2 are, respectively, Cayley digraphs of the groups $\Gamma_1 = \langle a_1,\ldots,a_p|R_1\rangle$ and $\Gamma_2 = \langle b_1,\ldots,b_q|R_2\rangle$, then its direct product $G_1\square G_2$ is the Cayley digraph of the group

$$\Gamma = \Gamma_1\times\Gamma_2 = \langle\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q|R_1',R_2',\alpha_i\beta_j = \beta_j\alpha_i\rangle.$$

As an example of direct product of Cayley digraphs, see Fig. 2, to be compared with Fig. 1.

7 An alternative definition

When each of the factors G_i of the Manhattan product has a polarity, that is, there exists an involutive automorphism from G_i to $\overline{G}_i$, we can give the following alternative definition.

Proposition 7.1. *Let ψ_i be an involutive automorphism from G_i to $\overline{G}_i$, for $i = 1,2,\ldots,n$. Then, the Manhattan product $H = G_1 \# G_2 \# \ldots \# G_n$ is the digraph with vertex set $V(M_n) = \mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}\times\cdots\times\mathbb{Z}_{N_n}$ and the following adjacencies ($i = 1,2,\ldots,n$):*

$$(u_1,u_2,\ldots,u_i,\ldots,u_n) \quad\rightsquigarrow\quad (\psi_1(u_1),\psi_2(u_2),\ldots,v_i,\ldots,\psi_n(u_n)),$$

where $v_i\in\Gamma^+(u_i)$.

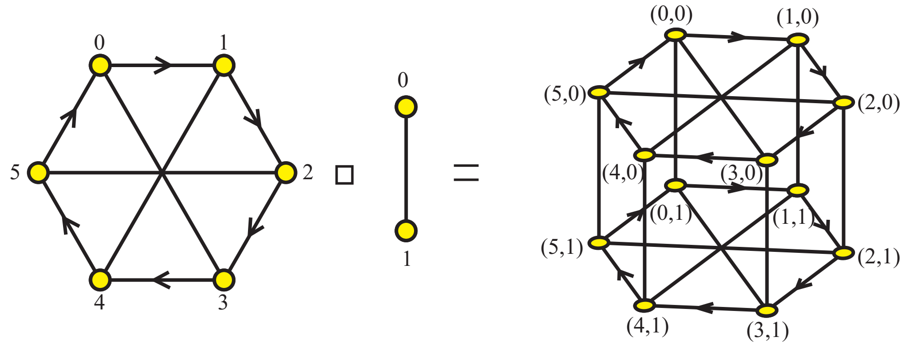

Figure 2: The direct product $Cay(\mathbb{Z}_6, \{1,3\})\Box K_2^*$ (undirected lines stand for pairs of arcs in opposite directions).

Proof. For the sake of simplicity, we respectively write the adjacencies of the first definition and the alternative one as ($i = 1, 2, \ldots, n$):

$$(u_1, \ldots, u_i, \ldots, u_n) \;\to\; \big(u_1, \ldots, \Gamma^{(-1)^{\sum_{j\neq i}\pi(u_j)}}(u_i), \ldots, u_n\big), \tag{3}$$

$$(u_1, \ldots, u_i, \ldots, u_n) \;\rightsquigarrow\; \big(\psi_1(u_1), \ldots, \Gamma^+(u_i), \ldots, \psi_n(u_n)\big), \tag{4}$$

where $\Gamma^{+1} \equiv \Gamma^+$ and $\Gamma^{-1} \equiv \Gamma^-$.

The isomorphism from the first definition to the alternative one is:

$$\Phi(u_1, \ldots, u_i, \ldots, u_n) = \Big(\psi_1^{\sum_{j\neq 1}\pi(u_j)}(u_1), \ldots, \psi_i^{\sum_{j\neq i}\pi(u_j)}(u_i), \ldots, \psi_n^{\sum_{j\neq n}\pi(u_j)}(u_n)\Big).$$

Indeed, let us see that this mapping preserves the adjacencies. First, by (3), we have

$$\Phi\big(\Gamma^+(u_1, \ldots, u_i, \ldots, u_n)\big) =$$
$$\Big(\psi_1^{\sum_{j\neq 1}\pi(u_j)+1}(u_1), \ldots, \psi_i^{\sum_{j\neq 1}\pi(u_j)}\big(\Gamma^{(-1)^{\sum_{j\neq i}\pi(u_j)}}(u_i)\big), \ldots, \psi_n^{\sum_{j\neq 1}\pi(u_j)+1}(u_1)\Big). \tag{5}$$

Whereas, by (4), we have

$$\Gamma^+\big(\Phi(u_1, \ldots, u_i, \ldots, u_n)\big) =$$
$$\Big(\psi_1^{\sum_{j\neq 1}\pi(u_j)+1}(u_1), \ldots, \Gamma^+\big(\psi_i^{\sum_{j\neq i}\pi(u_j)}(u_i)\big), \ldots, \psi_n^{\sum_{j\neq n}\pi(u_j)+1}(u_n)\Big). \tag{6}$$

To check that the i-th entry in (5) and (6) represents the same set, we distinguish two cases:

- If $\sum_{j\neq i}\pi(u_j) = \alpha$ is an even number, then $\psi_i^\alpha = Id$ (as ψ_i is involutive) and $Id\big(\Gamma^+(u_i)\big) = \Gamma^+\big(Id(u_i)\big)$.
- If $\sum_{j\neq i}\pi(u_j) = \beta$ is an odd number, then $\psi_i^\beta = \psi_i$ and $\psi_i\big(\Gamma^-(u_i)\big) = \Gamma^+\big(\psi_i(u_i)\big)$ (as ψ_i is an automorphism from G_i to $\overline{G_i}$) .

□

In the case of the Manhattan street network M_n, $G_i = C_i$ (Prop. 4.1). Then, a simple way of choosing the involutive automorphisms is $\psi_i(u_i) = -u_i \bmod N_i$ (in fact, it is readily checked that any isomorphism from C_i to $\overline{C_i}$ is involutive). That gives the following definition of M_n [4, 5]: The Manhattan street network $M_n = M_n(M_1, \ldots, M_n)$ is the digraph with vertex set $\mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_n}$ and the adjacencies

$$(u_1, \ldots, u_i, \ldots, u_n) \;\rightsquigarrow\; (-u_1, \ldots, u_i + 1, \ldots, -u_n) \qquad (1 \leq i \leq n).$$

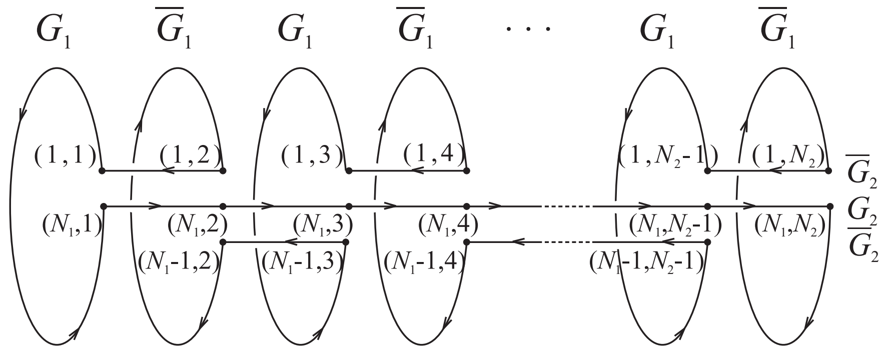

Figure 3: A Hamiltonian cycle in the Manhattan product $G_1 \# G_2$.

8 Hamiltonian Cycles

Next we give a result on the Hamiltonicity of the Manhattan product of two digraphs with Hamiltonian paths, as a generalization of a theorem in [4, 5] about the Hamiltonicity of the Manhattan street network.

Theorem 8.1. *If G_1 and G_2 have both a Hamiltonian path, then their Manhattan product $H = G_1 \# G_2$ is Hamiltonian.*

Proof. We use the same idea as in the proof of Theorem 5.1 in [4], which allows to construct a Hamiltonian cycle in H, from the Hamiltonian paths in G_1 and G_2, say $1 \to 2 \to \cdots \to N_1$ and $1' \to 2' \to \cdots \to N_2$ respectively. With this aim, we appropriately joint N_2 Hamiltonian paths (some of them without an arc) of N_2 subdigraphs isomorphic to G_1 or $\overline{G}_1$ (see Prop. 3.1(c)). Such paths are joined by using three copies of the Hamiltonian path (two of them with alternative arc removed) of subdigraphs isomorphic to G_2 or $\overline{G}_2$. See the selfexplanatory Fig. 3. □

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