A map characterizing the fuzzy points and columns of a T-indistinguishability operator

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Abstract

A new map (Λ_E) between fuzzy subsets of a universe X endowed with a T-indistinguishability operator E is introduced. The main feature of Λ_E is that it has the columns of E as fixed points, and thus it provides us with a new criterion to decide whether a generator is a column. Two well known maps (ϕ_E and ψ_E) are also reviewed, in order to compare them with Λ_E.

Interesting properties of the fixed points of Λ_E and Λ_E^2 are studied. Among others, the fixed points of Λ_E ($\mathrm{Fix}(\Lambda_E)$) are proved to be the maximal fuzzy points of (X, E) and the fixed points of Λ_E^2 coincide with the Image of Λ_E.

An isometric embedding of X into $\mathrm{Fix}(\Lambda_E)$ is established and studied.

Keywords: fuzzy relation, column of a fuzzy relation, t-norm, T-indistinguishability operator, generator, fuzzy point.

1 Introduction

T-indistinguishability operators (T being a t-norm) are a special kind of fuzzy relations that extent crisp equivalence relations to a fuzzy framework.

They appear under many different names, like fuzzy equivalence relation, fuzzy equality, similarity relation [20], likeness and probabilistic relation among others, depending on the authors or on the chosen t-norm T.

From an structural point of view, it is especially interesting to study the set $H_E \subseteq [0,1]^X$ of all generators or extensional fuzzy subsets of a T-indistinguishability operator E defined on a set X.

The generators are the only fuzzy subsets which are compatible with E, in the same way as the union of equivalence classes are the only crisp subsets compatible with a crisp equivalence relation and they are therefore also called observable fuzzy subsets of the universe. The columns of E are a special kind of generators which are exactly the fuzzy equivalence classes [20], [6], [3].

After this introductory section, Section 2 is devoted to recall some general concepts concerning T-indistinguishability operators. Some results about the maps ϕ_E and ψ_E are reviewed. These maps are key tools to study the structure of H_E, mainly because they allow us to characterize H_E as the set of fixed points of both, ϕ_E and ψ_E. Moreover, for a given fuzzy subset h of X, $\phi_E(h)$ and $\psi_E(h)$ are the smallest generator of E greater or equal than h and the greater generator of E smaller or equal than h and hence its upper and lower approximations in H_E [3]. Actually, H_E can be interpreted as the set of fuzzy subsets of the quotient set X/E (i.e.: $H_E = [0,1]^{X/E}$) and $\phi_E : [0,1]^X \to [0,1]^{X/E}$ is the canonical map. Note that if the indistinguishability operator E is a crisp one, then $\phi_{E|\{0,1\}^X}$ is the crisp canonical map $\pi : X \to X/E$.

A new map is introduced in Section 3 in order to characterize the columns of E. The main results show that fuzzy points can be thought as columns of extensions $(\overline{X}, \overline{E})$ of (X, E) and that the columns of E are the normal fixed points of Λ_E.

In Section 4 the set $\mathrm{Im}(\Lambda_E)$ is characterized as a set of fixed points of Λ_E^2.

Section 5 is devoted to a more detailed study of the fixed points of Λ_E which turn to be the maximal fuzzy points of X. The isometric embedding of X into $\mathrm{Fix}(\Lambda_E)$ is studied.

The paper ends with a section of Concluding Remarks and an example that gives a geometric interpretation of the sets and maps seen in it.

2 Preliminaries

In this section we recall some concepts related to T-indistinguishability operators and some lemmas that will be needed later.

Given a left-continuous t-norm T, its residuation $\hat{T}$ is defined by

$$\hat{T}(x|y) = \sup\{\alpha \in [0,1] \mid T(x,\alpha) \leq y\}$$

for all x, y of $[0,1]$.

It is worth noting that $([0,1], \leq, T)$ is a residuated lattice, and $\hat{T}$ is the corresponding residuation w.r.t. the t-nom T [See, for example [14]]. Further, in a logical context, $\hat{T}$ may be interpreted as the implication $\overrightarrow{T}$ based on the t-norm T.

Lemma 2.1. *Given a left-continuous t-norm T, we have:*

1. *$\hat{T}(x|y)$ is left continuous and non increasing with respect to the first variable x.*

2. *$\hat{T}(x|y)$ is right continuous and non decreasing with respect to the second variable y.*

Proof. Trivial. □

Lemma 2.2. *Given a left-continuous t-norm T, for any $x, y, z \in [0,1]$ the following relations hold:*

2.2.1. $\hat{T}(1|x) = x$.

2.2.2. $x \leq y \Rightarrow \hat{T}(x|y) = 1$.

2.2.3. $MIN\{\hat{T}(x|y), \hat{T}(y|x)\} = \hat{T}\left(MAX\{x,y\}| \, MIN\{x,y\}\right)$.

2.2.4. $\hat{T}\left(\hat{T}(x|y)|\hat{T}(x|z)\right) \geq \hat{T}(y|z)$

2.2.5. $T(x, \hat{T}(x|y)) \leq y$.

Proof. [3], [19], [1]. 2.2.4 $T(x,y) \leq z$ if, and only if, $x \leq \hat{T}(y|z)$. □

Lemma 2.3. *Given a left-continuous t-norm T, for any $x, y, z \in [0,1]$ the following relation holds:*

$$\hat{T}(x|T(y,z)) \geq T(y, \hat{T}(x|z)).$$

Proof.

$$\hat{T}(x|T(y,z)) = \mathrm{SUP}\{\alpha | T(\alpha, x) \leq T(y,z)).$$

From Lemma 2.2.5

$$T(y, \hat{T}(x|z), x) \leq T(y,z)$$

and the result follows. □

Definition 2.4. *Given a left continuous t-norm T, the biresiduation $\overset{\leftrightarrow}{T}$ of T is defined by*

$$\overset{\leftrightarrow}{T}(x,y) = MIN(\hat{T}(x|y), \hat{T}(y|x)) = T(\hat{T}(x|y), \hat{T}(y|x))$$

$\forall x, y \in [0,1]$.

Example 2.5.

- **Lukasiewicz t-norm:** *If* $T(x,y) = MAX(0, x+y-1)$, *then* $\overset{\leftrightarrow}{T}(x,y) = 1 - |x-y|$.
- **Product t-norm:** *If* $T(x,y) = x \times y$, *then* $\overset{\leftrightarrow}{T}(x,y) = MIN\left(\frac{x}{y}, \frac{y}{x}\right)$.
- **Minimum t-norm:** *If* $T(x,y) = MIN(x,y)$, *then*

$$\overset{\leftrightarrow}{T}(x,y) = \begin{cases} 1 & \text{if } x = y \\ MIN(x,y) & \text{otherwise.} \end{cases}$$

Note. In the sequel T will stand for a left-continuous t-norm.

Definition 2.6. *Given a t-norm T, a T-indistinguishability operator E on a set X is a fuzzy relation on X that satisfies*

1. $E(x,x) = 1 \ \forall x \in X$ *(reflexivity),*
2. $E(x,y) = E(y,x) \ \forall x, y \in X$ *(symmetry),*
3. $T(E(x,y), E(y,z)) \leq E(x,z) \ \forall x, y, z \in X$ *(T-transitivity).*

In [19] it is proved that $\overleftrightarrow{T}$ is a T-indistinguishability operator and that any T-indistinguishability operator can be constructed starting from a family of fuzzy sets.

Lemma 2.7. *Given a fuzzy subset h of a set X, the fuzzy relation E_h defined by*

$$E_h(x,y) = \hat{T}\,(MAX(h(x),h(y)) \mid MIN(h(x),h(y))) = \overleftrightarrow{T}\,((h(x),h(y)))$$

is a T-indistinguishability operator on X.

Theorem 2.8. [19] *Representation Theorem. A fuzzy relation on a set X is a T-indistinguishability operator if and only if there exists a family $\{h_i\}_{i\in I}$ of fuzzy subsets of X such that*

$$E = \underset{i\in I}{INF}\, E_{h_i}.$$

Theorem 2.8 suggests the following definition.

Definition 2.9. *Given a T-indistinguishability operator E on X, a generator of E is a fuzzy set of X that belongs to a generating family of E in the sense of the preceding theorem.*

Next lemma follows immediately.

Lemma 2.10. *Denoting by H_E the set of generators of E, $h \in H_E$ if and only if $E_h \geq E$.*

The set H_E has been widely studied [2], [14] and its elements have been characterized as the eigenvectors [10], and the generators [9] of E, the fixed points of ϕ_E and ψ_E, the logical states associated to E [18] and their extensional sets [14].

Lemma 2.11. [19] *Given a T-indistinguishability operator E on a set X, and an element $x \in X$, the fuzzy subset h_x of X defined by $h_x(y) = E(x,y)$ $\forall y \in X$ is a generator of E.*

We refer the fuzzy subsets h_x, $x \in X$, as the **columns** of E.

Definition 2.12. *Let $E, \overline{E}$ be two T-indistinguishability operators on X and $\overline{X}$ respectively. $\left(\overline{X}, \overline{E}\right)$ is an* **extension** *of (X,E) if, and only if,* *1.* $X \subseteq \overline{X}$

2. $\overline{E}(x,y) = E(x,y)\ \forall x, y \in X$.

More general,

Definition 2.13. *Let E, F be T-indistinguishability operators on X and Y respectively. A map $\tau : X \to Y$ is an isometric embedding of (X, E) into (Y, F) if, and only if,*

$$E(x,y) = F(\tau(x), \tau(y))\ \forall x, y \in X.$$

As usual, we denote $\leq$ the pointwise order between fuzzy subsets (so, $h \leq h'$ if, and only if, $h(x) \leq h'(x)$, for any $x \in X$). It is a well known fact that $([0,1]^X, \leq)$ is a complete lattice, with meet ($\wedge$) and join ($\vee$) defined in the natural way by $(h \vee h')(x) = \text{SUP}\{h(x), h'(x)\}$, and $(h \wedge h')(x) = \text{INF}\{h(x), h'(x)\}$, for any $x \in X$.

Now let us introduce two maps ($\phi_E, \psi_E : [0,1]^X \to [0,1]^X$) which are key tools in order to study the structure of H_E [2].

The main result concerning these maps is that both, ϕ_E and ψ_E, have H_E as the set of fixed points.

Definition 2.14. *Let E be a T-indistinguishability operator on a set X. The map $\phi_E : [0,1]^X \to [0,1]^X$ is defined by*

$$\phi_E(h)(x) = \underset{y \in X}{SUP}\, T\left(E(x,y), h(y)\right),\ \forall x \in X.$$

Proposition 2.15. [2]. *For all $h, h' \in [0,1]^X$, we have:*

(a) $h \leq h' \Rightarrow \phi_E(h) \leq \phi_E(h')$.

(b) $h \leq \phi_E(h)$.

(c) $\phi_E(h \vee h') = \phi_E(h) \vee \phi_E(h')$.

These properties say that ϕ_E is a fuzzy closure operator.

Proposition 2.16. [2] *Im* $\phi_E = H_E$.

Theorem 2.17. [2] *$h \in H_E$ if, and only if, $\phi_E(h) = h$.*

Proposition 2.18. [2] *For any $h \in [0,1]^X$, $\phi_E(h) = \bigwedge_{h' \in H_E} \{h \leq h'\}$.*

So, $\phi_E(h)$ is the most specific generator that contains h (i.e. $h \leq \phi_E(h)$), and it is the optimal upper bound of h in H_E.

Now, let us study the map ψ_E that sends each fuzzy subset to the greater generator $\psi_E(h)$ contained in h (i.e. $\psi_E(h) \leq h$).

Definition 2.19. *Let E be a T-indistinguishability operator on a set X. The map $\psi_E : [0,1]^X \to [0,1]^X$ is defined by*

$$\psi_E(h)(x) = \underset{y \in X}{INF} \hat{T}\left(E(x,y)|h(y)\right), \quad \forall x \in X.$$

Proposition 2.20. [2] *For all $h, h' \in [0,1]^X$, we have:*

(a) $h \leq h' \Rightarrow \psi_E(h) \leq \psi_E(h')$.

(b) $\phi_E(h) \leq h$.

(c) $\psi_E(h \wedge h') = \psi_E(h) \wedge \psi_E(h')$.

In fact, ψ_E is a fuzzy interior operator.

Proposition 2.21. [2] *Im* $\psi_E = H_E$.

Theorem 2.22. [2]. $h \in H_E$ *if, and only if,* $\psi_E(h) = h$.

Proposition 2.23. [2] *For any* $h \in [0,1]^X$, $\psi_E(h) = \bigvee_{h' \in H_E}\{h' \leq h\}$.

As stated in the Introduction, H_E is the set of fuzzy subsets of the quotient set X/E ($H_E = [0,1]^{X/E}$) and $\phi_E : [0,1]^X \to [0,1]^{X/E}$ is the canonical map.

3 The map Λ_E

In the previous section, generators had been characterized as fixed points of two suitable maps (ϕ_E and ψ_E).

In the present section, we are going to associate a new map (Λ_E) to a given T-indistinguishability operator E, which is also closely related to the structure of E. The main result concerning Λ_E is that it has the columns of E as fixed points.

Definition 3.1. *Let E be a T-indistinguishability operator on a set X. $h \in H_E$ is a fuzzy point of X wrt E if and only if*

$$T\left(h(x_1), h(x_2)\right) \leq E(x_1, x_2),\ \forall x_1, x_2 \in X.$$

P_X will denote the set of fuzzy points of X wrt E.

Next Theorem provides us with a criterion to decide whether a generator is a fuzzy point.

Theorem 3.2. *Let be (X, E) a T-indistinguishability operator. Given $h \in H_E$, these are equivalent statements:*

(a) h is a fuzzy point.

(b) There exists an extension $(\overline{X}, \overline{E})$ of (X, E) such that $h = h_y|_X$, $y \in \overline{X}$ (i.e. $h(x) = \overline{E}(y, x)\ \forall x \in X$).

Proof. b) $\Rightarrow$ a))

$T(h(x_1, x_2) = T(\overline{E}(y, x_1), \overline{E}(y, x_2)) \leq \overline{E}(x_1, x_2) = E(x_1, x_2)$ for all $x_1, x_2 \in X$.

a) $\Rightarrow$ b))

We define a T-indistinguishability operator $\overline{E}$ on the set $\overline{X} = X \cup \{h\}$ as follows:

$$\overline{E}(x_1, x_2) = E(x_1, x_2)\ \forall x_1, x_2 \in X$$
$$\overline{E}(x, h) = \overline{E}(h, x) = h(x)\ \forall x \in X$$
$$\overline{E}(h, h) = 1$$

$\overline{E}$ is reflexive and symmetric and it is an extension of E.

It remains to prove the T-transitivity of $\overline{E}$, i.e. $T(\overline{E}(x, y), \overline{E}(y, z) \leq \overline{E}(x, z)$. There are only four possible cases (non exclusive):

- $x = y, y = z$ or $x = z$ (trivial)
- $x, y, z \in X$ (trivial)
- $y = h$ and $x, z \in X$. In this case, $T(\overline{E}(x, h), \overline{E}(h, z) = T(h(x), h(z)) \leq E(x, z)$.
- $x = h$ and $y, z \in X$. In this case, $T(\overline{E}(h, y), \overline{E}(y, z) = T(h(y), E(y, z) \leq h(z) = \overline{E}(h, z)$, because $h \in H_E$.

□

This theorem characterizes both the columns of (X,E) and the columns of their extensions as exactly the fuzzy points of E. We note $C_E = \{h \in H_E \mid \exists(\overline{X},\overline{E})$ extension of (X,E) and $y \in \overline{X}$ such that $h(x) = \overline{E}(y,x)$, $\forall x \in X\}$.

Of course, $P_X = C_E$ and we will say that a fuzzy point is in C_E when we want to stress the idea that it can be a column of an extension of (X,E).

If h is normal, then $(\overline{X},\overline{E}) = (X,E)$, and $h = h_x$ for some $x \in X$. This particular case is a well known result (see, for example, [14]).

In order to have a characterization of the columns of E, let us introduce the map Λ_E.

Definition 3.3. *Let E be an T-indistinguishability operator an a set X. The map $\Lambda_E : [0,1]^X \to [0,1]^X$ is defined by*

$$\Lambda_E(h)(x) = \underset{y\in X}{INF}\hat{T}\left(h(y)|E(y,x)\right), \quad \forall x \in X.$$

It is easy to check that, in a crisp setting, Λ_E acts simply by intersecting equivalence classes: $\Lambda(h) = \bigcap_{x\in h} h_x$ where h_x (the column of x) is in this case $\overline{x}$ (the cluster or equivalence class of x with respect to E).

So that in a crisp framework only three different situations may occur, namely:

- $h \neq \emptyset$ and there exists $x \in X$ such that $h \subseteq h_x$. In this case, $\Lambda_E(h) = h_x$. ($\Lambda_E(h)$ is the intersection of exactly one equivalence class h_x).
- $\Lambda_E(h) = \emptyset$ in any other situation with $h \neq \emptyset$ ($\Lambda_E(h)$ is then the intersection of two or more equivalence classes).
- $\Lambda_E(\emptyset) = X$ (Note that $\emptyset \subseteq h_x$ for all $x \in X$).

In other words, if a crisp subset A of X is contained in exactly one equivalence class $\overline{x}$ of E, then $\Lambda_E(A) = \overline{x}$. If A intersects more than an equivalence class of E, then $\Lambda_E(A) = \emptyset$ and $\Lambda_E(\emptyset) = X$.

This summarizes the situation in the crisp case. However, not such a trivial discussion can give us understanding enough in the fuzzy case, mainly due to two reasons. First, there exist columns h_y having their centers or prototypical elements y outside X, (as it states Theorem 3.2). And second, the map Λ_E^2 (which in the crisp case is a trivial one, fixing the columns and sending X to $\emptyset$ and $\emptyset$ to X) plays here an important role as will be seen in the next Section.

Some general properties concerning Λ_E are:

Proposition 3.4. *Given $h_1, h_2 \in [0,1]^X$, we have:*

(a) $\Lambda_E(h_1) \geq \Lambda_E(h_2)$ *if* $h_1 \leq h_2$

(b) $\Lambda_E(h_1 \vee h_2) = \Lambda_E(h_1) \wedge \Lambda_E(h_2)$

(c) $\Lambda_E(h_1 \wedge h_2) \geq \Lambda_E(h_1) \vee \Lambda_E(h_2)$.

Proof. Trivial. □

Proposition 3.5. *Let be $h \in [0,1]^X$ and $\alpha \in [0,1]$*

(a) $\Lambda_E\left(T(\alpha, h)\right) = \hat{T}(\alpha|\Lambda_E(h))$

(b) $\Lambda_E(\hat{T}(\alpha|h)) \geq T(\alpha, \Lambda_E(h))$.

Proof. (a) $\Lambda_E(T(\alpha,h))(x) = \mathrm{INF}_{y\in X}\, \hat{T}\left(T(\alpha, h(y))|E(y,x)\right) =$

$$\hat{T}\left(\alpha|\operatorname*{INF}_{y\in X} \hat{T}(h(y)|E(y,x))\right) = \hat{T}(\alpha|\Lambda_E(h)x)), \text{ for all } x \in X.$$

(b) $\Lambda_E(\hat{T}(\alpha|h))(x) = \mathrm{INF}_{y\in X}\, \hat{T}\left(\hat{T}(\alpha|h(y))|E(y,x)\right) \geq$

$$T\left(\alpha, \operatorname*{INF}_{y\in X} \hat{T}(h(y)|E(y,x))\right) = \hat{T}(\alpha|\Lambda_E(h)x)), \text{ for all } x \in X.$$

□

The following two theorems establish the relation between $\mathrm{Fix}(\Lambda_E)$ (the set of fixed points of Λ_E) and the columns of E.

Theorem 3.6. *Fix* $(\Lambda_E) \subseteq C_E = P_X$.

Proof. Let $h \in [0,1]^X$ be a fixed point of Λ_E, i.e., $\Lambda_E(h) = h$.

Being $\Lambda_E(h)(x) = \mathrm{INF}_{y\in X}\, \hat{T}(h(y)|E(x,y))$, then $\Lambda_E(h) = h$ implies that $\hat{T}(h(y)|E(x,y)) \geq h(x)$ for all $y \in X$, and also that $T(h(x), h(y)) \leq E(x,y)$ for all $y \in X$.

On the other hand, $\Lambda_E(h) \in H_E$ (see Proposition 4.1 later in next section). □

The set $\text{Fix}(\Lambda_E)$ will be characterized as the set of maximal elements of C_E in Section 5.

Theorem 3.7. *Let h be a normal fuzzy subset of X (i.e. $\exists x_0 \in X$ such that $h(x_0) = 1$) $\Lambda_E(h) = h$ if and only if $h = h_x$ ($x \in X$).*

Proof. If $\Lambda_E(h) = h$, then $h \in C_E$ (Theorem 3.6) and being h a normal fuzzy subset, we have $h = h_x$, for some $x \in X$.

Conversely, if $h = h_x$ for some $x \in X$, then using Lemma 2.2$\Lambda_E(h_x)(y) = \text{INF}_{z\in X}\hat{T}(h_x(z)|E(z,y)) = \text{INF}_{z\in X}\hat{T}(E(z,x)|E(z,y)) = E(x,y) = h_x(y)$, for all $y \in X$. □

Theorem 3.7 characterizes only the columns of elements $x \in X$, and it cannot be extended to the whole set C_E, as it is shown in next example.

Example 3.8. *$X = \{x_1, x_2\}$, $E(x_1, x_2) = 0$ T an arbitrary t-norm. We define the following extension of (X, E) : $\overline{X} \neq X \cup \{y\}$, $\overline{E}(x_1, y) = \overline{E}(x_2, y) = 0$.*

The column of y is (restricted to X), the constant fuzzy set $h(x_1) = h(x_2) = 0$. So that $\Lambda_E(h_y) = X$ i.e. $\Lambda_E(h)(x_1) = \Lambda_E(h)(x_2) = 1$.

However, there are also fixed points of Λ_E that are not columns $h_x, x \in X$.

Example 3.9. *For a given $n \in \mathbb{N}$, $n \geq 2$, let us consider $X = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right\} \subseteq [0, 1]$, $T = L$ (the Lukasiewicz t-norm) and E defined by $E(x, y) = 1 - |x - y|$ for all $x, y \in X$.*

Let h be the non-normal fuzzy subset defined by $h(x) = 1 - \left|\frac{3}{2n} - x\right|$, $x \in X$. Obviously $h \neq h_x$ for all $x \in X$, and it is easy to check that $\Lambda_E(h) = h$.

4 Characterizing Im(Λ_E)

This section is devoted to the study of $\text{Im}(\Lambda_E)$. The map Λ_E^2 will play an essential role and the main result of this section will identify its fixed points with the image of Λ_E.

Let us start by noting that $\Lambda_E(h)$ is always a generator, for any $h \in [0, 1]^X$.

Proposition 4.1. *$Im(\Lambda_E) \subseteq H_E$.*

Proof. For any $h \in [0,1]^X$, we have to prove that $\Lambda_E(h) \in H_E$.

$$\begin{aligned}
\hat{T}\left(\Lambda_E(h)(x_1)|\Lambda_E(h)(x_2)\right) &= \hat{T}\left(\operatorname*{INF}_{y\in X}\hat{T}(h(y)|E(y,x))|\operatorname*{INF}_{z\in X}\hat{T}(h(z)|E(z,x_2))\right)\\
&= \operatorname*{INF}_{z\in X}\hat{T}\left(\operatorname*{INF}_{y\in X}\hat{T}(h(y)|E(y,x_1))|\hat{T}(h(z)|E(z,x_2))\right)\\
&\geq \operatorname*{INF}_{z\in X}\hat{T}\left(\hat{T}(h(z)|E(z,x_1))|\hat{T}(h(z)|E(z,x_2))\right)\\
&\geq \operatorname*{INF}_{z\in X}\hat{T}(E(x_1,y)\ |E(x_2,y)) = E(x_1,x_2)
\end{aligned}$$

(applying Lemmas 2.1, 2.2 and the T-transitivity of E).

In a similar way, we obtain $\hat{T}(\Lambda_E(h)(x_2)|\Lambda_E(h)(x_1)) \geq E(x_1,x_2)$, and therefore $E_T(\Lambda_E(h)(x_1),\Lambda_E(h)(x_2)) \geq E(x_1,x_2)$, for all $x_1,x_2 \in X$, so that $\Lambda_E(h) \in H_E$. □

At this point, it is not clear whether the set $\mathrm{Im}(\Lambda_E)$ coincides with H_E or, on the contrary, it is strictly contained by H_E.

To answer this, we turn out our attention to the operator Λ_E^2.

Proposition 4.2. *Given $h_1,h_2 \in [0,1]^X$,*

a. If $h_1 \leq h_2$ then $\Lambda_E^2(h_1) \leq \Lambda_E^2(h_2)$

b. $\Lambda_E^2(h_1 \vee h_2) \geq \Lambda_E^2(h_1) \vee \Lambda_E^2(h_2)$

c. $\Lambda_E^2(h_1 \wedge h_2) \leq \Lambda_E^2(h_1) \wedge \Lambda_E^2(h_2)$.

Proof. Trivial. □

Proposition 4.3. $\Lambda_E^2 \geq \phi_E$

Proof. Given $h \in [0,1]^X$, we have:

$$\begin{aligned}
\Lambda_E^2(h)(x) &= \Lambda_E(\Lambda_E(h))(x) = \operatorname*{INF}_{y\in X}\hat{T}(\Lambda_E(h)(y)\ |\ E(y,x))\\
&= \operatorname*{INF}_{y\in X}\hat{T}\left(\operatorname*{INF}_{z\in X}\hat{T}(h(z)|E(z,y))|E(y,x)\right)\\
&\geq \operatorname*{INF}_{y\in X}\operatorname*{SUP}_{z\in X}\hat{T}\left(\hat{T}(h(z)|E(z,y))|E(y,x)\right)\\
&\geq \operatorname*{SUP}_{z\in X}\operatorname*{INF}_{y\in X}\hat{T}\left(\hat{T}(h(z)|E(z,y))|E(y,x)\right)
\end{aligned}$$

$$\begin{aligned} &\geq \operatorname*{SUP}_{z\in X} T(h(z), \operatorname*{INF}_{y\in X} \hat{T}(E(y,z)|E(y,x))) \\ &= \operatorname*{SUP}_{z\in X} T(h(z), E(z,x)) = \phi_E(h)(x), \end{aligned}$$

for all $x \in X$. □

Corollary 4.4. *$\Lambda_E^2(h) \geq h$, for all $h \in [0,1]^X$.*

Proof. $\Lambda_E^2(h) \geq \phi_E(h) \geq h$. □

Lemma 4.5. *Given a column h_x, $x \in X$ and $\alpha \in [0,1]$ we have:*

a) $\Lambda^2(h_x) = h_x$

b) If $g = \hat{T}(\alpha|h_x)$ then $\Lambda_E^2(g) = g$.

Proof. a) Trivial (see Theorem 3.7)

b) According to Proposition 3.5 and Theorem 3.7,

$$\Lambda_E^2(g) = \Lambda_E(\Lambda_E(\hat{T}(\alpha|h_x))) \leq \Lambda_E(T(\alpha, \Lambda_E(h_x))) =$$

$$\Lambda_E(T(\alpha, h_x)) = \hat{T}(\alpha|\Lambda_E(h_x)) = \hat{T}(\alpha|h_x) = g.$$

On the other hand, $\Lambda_E^2(g) \geq g$ (Corollary 4.4), so that $\Lambda_E^2(g) = g$. □

Lemma 4.6. *Let $\{g_i\}_{i\in I}$ be a family of fixed points of Λ_E^2. Then $\bigwedge_{i\in I} g_i$ is also a fixed point of Λ_E^2.*

Proof. It follows from $\Lambda_E^2\left(\bigwedge_{i\in I} g_i\right) = \bigwedge_{i\in I}(\Lambda_E^2(g_i)) = \bigwedge_{i\in I} g_i$ (Proposition 4.2.a), and from $\Lambda_E^2(\bigwedge_{i\in I} g_i) \geq \bigwedge_{i\in I} g_i$ (corollary 4.4). □

Theorem 4.7. *Fix(Λ_E^2) =Im(Λ_E).*

Proof. $\Lambda_E h(x) = \mathrm{INF}_{y\in X}\, \hat{T}(h(y)|E(x,y))) = \mathrm{INF}_{y\in X}\, \hat{T}(h(y)|h_y(x))$.

For every $y \in X$, we can define a fuzzy subset g_y in the following way: $g_y(x) = \hat{T}(h(y)|h_y(x))$ that is of the form of Lemma 4.5.b and therefore a fixed point of Fix(Λ_E^2). $\Lambda_E h = \mathrm{INF}_{y\in X}\, g_y$ which thanks to Lemma 4.6 belongs to Fix(Λ_E^2) as well. So, Im$(\Lambda_E) \subset$ Fix (Λ_E^2).

On the other hand, given $g \in [0,1]^X$ such that $\Lambda_E^2(g) = g$, then $g \in$Im(Λ_E) because

$$g = \Lambda_E^2(g) = \Lambda_E(\Lambda_E(g)).$$

□

As a consequence of Theorem 4.7 we can easily check that $\mathrm{Im}(\Lambda_E) \subsetneq H_E$, as it is shown in the next example:

Example 4.8. *Let be $X = \{x_1, x_2, x_3\}$, $T = L$ (the Luckasiewicz t-norm), E the T-indistinguishability operator defined by $E(x_i, x_j) = 0$ if $i \neq j$, and $h \in H_E$ defined by $h(x_1) = 1$, $h(x_2) = 0.5$, $h(x_3) = 0$.*

We have that $\Lambda_E(h) = \{0.5, 0, 0\}$ and $\Lambda_E^2(h) = \{1, 0.5, 0.5\} \neq h = \{1, 0.5, 0\}$ and we can apply Theorem 4.6 to conclude that $h \notin Im(\Lambda_E)$.

Corollary 4.9. $\Lambda_E^3 = \Lambda_E$.

Proof. Consequence of Theorem 4.7. □

Corollary 4.10. $\Lambda_E^{2n} = \Lambda_E^2$, $\Lambda_E^{2n+1} = \Lambda_E$ *with* $n \in \mathbb{N}$.

In particular, Λ_E^2 is a fuzzy closure operator and $\mathrm{Im}\Lambda_E$ is the set of closed sets of a fuzzy topology.

5 Fix(Λ_E)

In Proposition 3.6 we have proved that the set $\mathrm{Fix}(\Lambda_E)$ of fixed points of Λ_E is contained in the set P_X of fuzzy points of E. In this section we will characterize the fixed points of Λ_E as exactly the maximal fuzzy points of E. Moreover, given a fuzzy point h, we can find a fixed point h' of Λ_E with $h \leq h'$.

Considering the natural T-indistinguishability operator E_X associated to E restricted to $\mathrm{Fix}(\Lambda_E)$, we have an isometric embedding of (X, E) into $(\mathrm{Fix}(\Lambda_E), E_X)$. Some of its properties will be shown.

Lemma 5.1. *Let E be a T-indistinguishability operator on X and $h \in H_E$. $\Lambda_E(h) \geq h$ if and only if $h \in P_X$.*

Proof.

$$\Lambda_E(h)(x) = \underset{y \in X}{\mathrm{INF}}\, \hat{T}(h(y)|E(x,y)) \geq h(x)$$

$$\Leftrightarrow \hat{T}(h(y)|E(x,y)) \geq h(x)\ \forall x, y \in X \Leftrightarrow T(h(x), h(y)) \leq E(x,y)$$

□

Next Theorem characterizes the set of fixed points of Λ_E.

Theorem 5.2. *Let E be a T-indistinguishability operator on X. $Fix(\Lambda_E)$ is the set of all fuzzy points $h \in P_X$ which are maximal in P_X.*

Proof. a)

Let h be a fixed point of Λ_E and $h' \in P_X$ with $h \leq h'$.

$$h(x) = \operatorname*{INF}_{y \in Y} \hat{T}(h(y)|E(x,y)) \geq \operatorname*{INF}_{y \in Y} \hat{T}(h'(y)|E(x,y)) \geq h'(x)$$

So, $h = h'$.

b)

Let h be a fuzzy point not in $\mathrm{Fix}(\Lambda_E)$. There exists $x_0 \in X$ with $h(x_0) < \mathrm{INF}_{y \in Y} \hat{T}(h(y), E(x_0, y))$.

We can define a new fuzzy subset h' by

$$h'(x) = \begin{cases} h(x_0) & \text{if } x \neq x_0 \\ \mathrm{INF}_{y \in Y} \hat{T}(h(y), E(x_0, y)) & \text{otherwise.} \end{cases}$$

h' is a fuzzy point and $h' > h$ which means that h is not maximal in P_X. □

Using Zorn's Lemma, we can see that every fuzzy point is contained in fixed point of Λ_E.

Corollary 5.3. *Given a fuzzy point h, there exists a fixed point h' of Λ_E with $h \leq h'$.*

Theorem 5.4. *Let (X, E) be a T-indistinguishability operator, $(|X| < \infty)$, and $h \in [0,1]^X$ such that $h(x) < 1$ for all $x \in X_0$. $\Lambda_E(h) = h$ if and only if, $h = h_a$ $(a \notin X)$ satisfying $\forall x \in X$ $\exists u_x \in X$ such that $\hat{T}\left(\overline{E}(a, u_x)|E(x, u_x)\right) = \overline{E}(x, a)$.*

Proof. Let suppose that $\Lambda_E(h) = h$. In this case, $h \in C_E$ (Theorem 3.6) and being $h(x) < 1$ for all $x \in X$, we have that $h = h_a$, $a \notin X$. Further, $\Lambda_E(h_a)(x) = \mathrm{INF}_{y \in X} \hat{T}(h_a(y)|E(y, x)) = h_a(x)$ which, being X finite, implies that for all $x \in X$ there exists u_x such that $\hat{T}(h_a(u_x)|E(u_x, x)) = \hat{T}(\overline{E}(a, u_x)|E(x, u_x)) = h_a(x) = \overline{E}(x, a)$.

Conversely, let $h = h_a$ $(a \notin X)$ be a fuzzy subset satisfying that for all x there exists u_x such that $\hat{T}(\overline{E}(a, u_x)|E(x, u_x)) = \overline{E}(x, a)$. In this case, $\Lambda_E(h)(x) = \mathrm{INF}_{y \in X} \hat{T}(h(y)|E(y, x)) \geq \mathrm{INF}_{y \in X \cup \{a\}} \hat{T}(h(y)|\overline{E}(y, x)) \geq h(x)$, for all $x \in X$. On the other hand, $\mathrm{INF}_{y \in X} \hat{T}(h(y)|E(y, x)) \leq \hat{T}(h(u_x)|E(x, u_x)) = \hat{T}(\overline{E}(a, u_x)|E(x, u_x)) = E(x, a) = h(x)$, so that $\Lambda_E(h)(x) = h(x)$ for all $x \in X$. □

This theorem can be easily extended to non-finite set X by replacing the condition $\forall x \in X\ \exists u_x \in X$ s.t. $\hat{T}(\overline{E}(a, u_x)|E(x, u_x)) = E(x, u_x)$ by the more technical one $\forall x \in X$, $\forall \epsilon \in [0,1]\ \exists u_{x,\epsilon}$ s.t. $\hat{T}(\overline{E}(a, u_{x,\epsilon})|E(x, u_{x,\epsilon})) < E(x, a) + \epsilon$. The proof is similar to that of Theorem 3.7.

There is a nice relation between the couples of fuzzy subsets h, h' of $\mathrm{Im}\Lambda_E$ which are one image of the other one that will be studied next. We shall call h and h' dual fuzzy subsets.

Proposition 5.5. *Let h be a fixed point of Λ_E and $\alpha \in [0,1]$. If $T(\alpha, h)$ and $\hat{T}(\alpha|h)$ are in ImΛ_E, then they are dual fuzzy subsets.*

Proof. It is a consequence of Proposition 3.5.:
3.5.a) states that

$$\Lambda_E(T(\alpha, h)) = \hat{T}(\alpha|h)).$$

On the other hand,

$$\Lambda_E(\hat{T}(\alpha|h) = \Lambda_E(\hat{T}(\alpha|\Lambda_E(h)) = \Lambda_E^2(T(\alpha, h)) = T(\alpha, h),$$

where the last equality follows from Theorem 4.7. □

If T is an archimedean t-norm with additive generator t, then we can associate to T the quasi-arithmetic mean m_t generated by t; i.e.: $m(x,y) = t^{-1}\left(\frac{t(x)+t(y)}{2}\right)$ for all $x, y \in [0,1]$. It can be proved that in this way we have a bijection between continuous archimedean t-norms and continuous quasi-arithmetic means [11]. Then a fixed point h of Λ happens to be the quasi-arithmetic mean of the dual fuzzy subsets $T(\alpha, h)$ and $\hat{T}(\alpha|h)$.

Proposition 5.6. *Let h be a fixed point of Λ_E, $\alpha \in [0,1]$ and $T(\alpha, h)$ and $\hat{T}(\alpha|h)$ dual non-normalized fuzzy subsets in ImΛ_E with T an archimedan t-norm with additive generator t. Then h is the quasi-arithmetic mean of these dual fuzzy subsets.*

Proof.

$$m_t(T(\alpha, h), \hat{T}(\alpha|h)) = t^{-1}\left(\frac{T(\alpha, h), \hat{T}(\alpha|h)}{2}\right) =$$

$$t^{-1}\left(\frac{t\left(t^{[-1]}(t(\alpha) + t(h))\right) + t\left(t^{[-1]}(t(h) - t(\alpha))\right)}{2}\right) =$$

$$t^{-1}\left(\frac{(t(\alpha)+t(h)+t(h)-t(\alpha)}{2}\right)=h.$$

□

This means in particular that these dual fuzzy subsets and h generate the same T-indistinguishability operator [12] and the same T-preorder [8].

Proposition 5.7. [2] *Let E be a T-indistinguishability operator on a set X. The fuzzy relation E on $[0,1]^X$ defined for all $h, h' \in [0,1]^X$ by*

$$E_X(h,h') = \underset{x\in X}{INF} \overset{\leftrightarrow}{T} (h(x), h'(x))$$

is a T-indistinguishability operator.

E_X is called the natural T-indistinguishability operator on $[0,1]^X$.

Restricting E_X to the set P_X of fuzzy points of X, we have the following result.

Proposition 5.8. *Let E be a T-indistinguishability operator on X. If h is a fixed point of Λ and h_x is the column corresponding to the element x of X,*

$$E_X(h,h_x) = h(x).$$

Proof.

$$h(y) = \underset{z\in X}{\mathrm{INF}}\, \hat{T}(h(z)|E(y,z)) \geq \underset{z\in X}{\mathrm{INF}}\, \hat{T}(h(z)|T(E(y,x),E(x,z))).$$

By Lemma 2.3, this last expression is greater or equal than

$$T(E(y,x), \underset{z\in X}{\mathrm{INF}}\, \hat{T}(h(z)|E(x,z)) = T(h_x(y), h(x)).$$

From

$$h(y) \geq T(h_x(y), h(x))$$

it follows

$$h(x) \leq \hat{T}(h_x(y)|h(y)).$$

on the other hand, since h is a fuzzy point,

$$h_x(y) = E(x,y) \geq T(h(x), h(y))$$

or equivalently,

$$h(x) \leq \hat{T}(h(y)|h_x(y)).$$

$$h(x) \leq Min(\hat{T}(h_x(y)|h(y)), \hat{T}(h(y)|h_x(y))) \ \forall x, y \in X$$

and therefore

$$h(x) \leq \operatorname*{INF}_{y \in X} Min(\hat{T}(h_x(y)|h(y)), \hat{T}(h(y)|h_x(y))) = E_X(h_x, h).$$

But since

$$Min(\hat{T}(h_x(x)|h(x)), \hat{T}(h(x)|h_x(x))) = h(x),$$

we finally get our result. □

Corollary 5.9. *Let E be a T-indistinguishability operator on X. The map $\tau : X \to \mathrm{Fix}(\Lambda_E)$ defined by $\tau(x) = h_x$ is an isometric embedding.*

Proof. Trivial: $E_X(h_x, h_y) = h_y(x) = h_x(y) = E(x, y)$. □

Corollary 5.10. *Let E be a T-indistinguishability operator on X and h, h' fixed points of Λ_E. Then*

$$E_X(h, h') \geq T(h(x), h'(x)) \ \forall x \in X$$

Proof.

$$E_X(h, h') \geq T(E_X(h, h_x), E_X(h_x, h')) = T(h(x), h'(x)).$$

□

Proposition 5.11. *Let E be a T-indistinguishability operator on X and $h, h' \in P_X$. Then*

$$E_X(h, h') \leq E_X(\Lambda_E(h), \Lambda_E(h')).$$

Proof.

$$\Lambda_E(h)(x) = \operatorname*{INF}_{y \in X} \hat{T}(h(y)|E(x, y)) \geq \operatorname*{INF}_{y \in X} T(\hat{T}(h(y)|h'(y), \hat{T}(h'(y)|E(x, y)) \geq$$

$$T(\operatorname*{INF}_{y \in X} T(\hat{T}(h(y)|h'(y)), \operatorname*{INF}_{y \in X} \hat{T}(h'(y)|E(x, y)) \geq$$

$$T(\operatorname*{INF}_{y \in X} T(\hat{T}(h(y)|h'(y)), \Lambda_E(h')(x))$$

and therefore,

$$\hat{T}(\Lambda_E(h')(x)|\Lambda_E(h)(x) \geq \underset{y\in X}{\mathrm{INF}}\, \hat{T}(h(y)|h'(y)) \geq E_X(h,h').$$

Similarly,

$$\hat{T}(\Lambda_E(h)(x)|\Lambda_E(h')(x) \geq \underset{y\in X}{\mathrm{INF}}\, \hat{T}(h'(y)|h(y)) \geq E_X(h,h')$$

and

$$E_X(\Lambda_E(h),\Lambda_E(h')) \geq E_X(h,h').$$

□

6 Concluding remarks

A new map $\Lambda_E : [0,1]^X \to [0,1]^X$ associated to a T-indistinguishability operator E on a set X has been introduced. It allows us to characterize the columns of E as its fixed points. The set $\mathrm{Im}(\Lambda_E)$ has also been characterized as the set of fixed points of Λ_E^2. In this way, $\mathrm{Im}(\Lambda_E)$ appears as a well differentiated subset of H_E. $\mathrm{Fix}(\Lambda_E)$ has been characterized as the set of maximal fuzzy points of E.

Let us conclude with a very simple example that gives a geometrical interpretation of the maps and sets studied in the paper.

Example 6.1. *Let $X = \{a,b\}$ and consider the T-indistinguishability operator E with $E(a,b) = m$. Every fuzzy subset h of X can be identified with the point $(h(a),h(b))$ of $[0,1]^2$.*

H_E, the set of generators of E is then the region of $[0,1]^2$ defined by the inequation

$$\overset{\leftrightarrow}{T}(x,y) \geq m$$

and P_X is the part of H_E limited by the inequation

$$T(x,y) \leq m.$$

If $h = (p,q)$, then $\Lambda_E(h) = (\hat{T}(q|m),\hat{T}(p|m))$ and
$\Lambda_E^2(h) = (\hat{T}(\hat{T}(p|m)|m),\hat{T}(\hat{T}(q|m)|m))$.

If $h = (p,q)$ is not in H_E and $p > q$, then $\phi_E(h) = (p,T(m,p))$ and $\psi_E(h) = (\hat{T}(q|m),\hat{T}(p|m))$; if $p < q$, then $\phi_E(h) = (T(m,q),q)$ and $\psi_E(h) = (\hat{T}(q|m),\hat{T}(p|m))$.

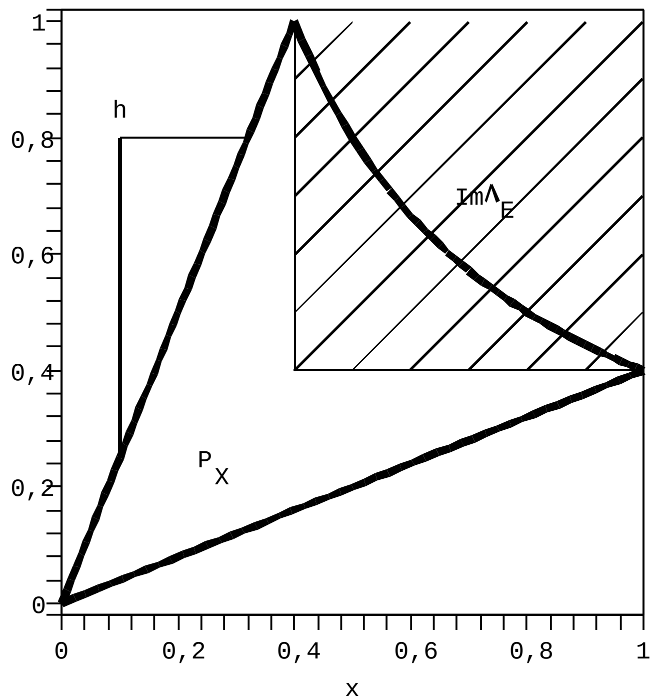

Figure 1:

Taking $m = 0.4$ *and* T *the product t-norm,* H_E *is the region of* $[0,1]^2$ *defined by the inequations*

$$\begin{cases} x - 0.4y \geq 0 \\ 0,4x - y \geq 0 \end{cases}$$

and P_X *is the part of this region below the hyperbola*

$$xy = 0.4.$$

The fixed points of Λ_E *are the maximal elements of* P_X *and therefore are the points in this hyperbola.*

If $h = (p, q)$, *then* $\Lambda_E(h) = (MIN(1, \frac{0.4}{q}), MIN(1, \frac{0.4}{p}))$ *and* $\Lambda_E^2(h) = (MAX(p, 0.4), MAX(q, 0.4))$. $Fix(\Lambda_E^2) = \text{Im}(\Lambda_E)$ *is the square*

$$\begin{cases} 0.4 \leq x \leq 1 \\ 0.4 \leq y \leq 1 \end{cases}$$

In this set, the image under Λ_E *of a fuzzy subset bellow the hyperbola* $xy = 0.4$ *(i.e.: bellow Fix*(Λ_E)*) is a point above it and vice versa, which gives a clear picture of Corollary 4.10.*

Finally, if $h = (p, q)$ *is not in* H_E *and* $p > q$ *,then* $\phi_E(h) = (p, mp)$ *and* $\psi_E(h) = (\frac{m}{q}, q)$*; if* $p < q$*, then* $\phi_E(h) = (mq, q)$ *and* $\psi_E(h) = (p, \frac{m}{p})$*. For example, if* $h = (0.1, 0.8)$*,* $\phi_E(h) = (0.32, 0.8)$ *and* $\psi_E(h) = (0.1, 0.25)$ *are obtained by projecting* h *to its closest edge of* H_E *horizontally and vertically respectively. See Figure 1.*

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