Eigenvalue distribution in scale free graphs

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Abstract

Scale free graphs can be found very often as models of real networks and are characterized by a power law degree distribution, that is, for a constant $\gamma \geq 1$ the number of vertices of degree d is proportional to $d^{-\gamma}$. Experimental studies show that the eigenvalue distribution also follows a power law for the highest eigenvalues. Hence it has been conjectured that the power law of the degrees determines the power law of the eigenvalues. In this paper we show that we can construct a scale free graph with non highest eigenvalue power law distribution. For $\gamma = 1$ we can construct a scale free graph with small spectrum and a regular graph with eigenvalue power law distribution.

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1 Introduction

In 1999 Faloutsos et al. [5] made an experimental study of a part of the real Internet graph of Oregon at the level of autonomous systems, finding a power law distribution for the degrees respect to their multiplicities, and a power law distribution for the highest eigenvalues of the adjacency matrix respect to their order, such that $\lambda_k = \lambda_1 i^{-\gamma'}$ with $0.45 < \gamma' < 0.5$. As a consequence, they conjectured a power law eigenvalue distribution for scale free graphs. After this, the spectra of some random and deterministic models proposed for scale free graphs have been also studied, obtaining a power law distribution for the highest eigenvalues too.

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Mihail and Papadimitriou [9] trying to explain this phenomenon showed in 2002 that the largest eigenvalues of graphs whose highest degrees are Zipf-like (power laws) distributed with slope γ are distributed according to a power law with slope $\gamma/2$. On year later Gkantsidis, Mihail and Zegura in [10] performed the spectral analysis of a biggest part of the Internet topology at the AS level, by adapting the standard spectral filtering method of examining the eigenvectors corresponding to the largest eigenvalues of matrices related to the adjacency matrix of the topology. They found that the highest eigenvalues are approximately the square roots of the highest degrees.

Chung et al. in [1] proposed a model based in random graphs with given expected degrees and their relations to several key invariants. They proved that for this model the eigenvalues of the (normalized) Laplacian follow the semicircle law, whereas the spectrum of the adjacency matrix obeys the power law. The same authors showed in [2] that the largest eigenvalue of the adjacency matrix of a random graph with expected degree sequence determined by Δ (the maximum degree) and $\tilde{d}$ (weighted average of the squares of the expected degrees) is almost surely $(1+o(1))\, max\{\tilde{d}, \sqrt{\Delta}\}$. And in the case that the k-th largest expected degree Δ_k is significantly larger than $\tilde{d}^2$, then the k-th largest eigenvalue of the adjacency matrix is almost surely $\lambda_k = (1 + o(1))\sqrt{\Delta_k}$. For a random power law graph with exponent $\beta > 2.5$, the largest eigenvalue is almost surely $(1 + o(1))\sqrt{\Delta}$. Moreover, the k-th largest eigenvalues of a random power law graph with exponent β have power law distribution with exponent $2\beta - 1$ if the maximum degree is sufficiently large and k is bounded above by a function depending on β, Δ and d (the average degree). When $2 < \beta < 2.5$, the largest eigenvalue is heavily concentrated at $c\Delta^{3-\beta}$ for some constant c depending on β and d.

The spectra of the Albert-Barábasi model was studied in [8] by simulation, obtaining a power law distribution for the highest eigenvalues. Later in [6], it is shown that at time t the largest k eigenvalues of the adjacency matrix of the graph have $\lambda_k = (1 \pm o(1))\Delta_k^{1/2}$ **whp**, where Δ_k is the k-th largest degree.

The spectra of some deterministic models have been also studied. Comellas and Gago in [3] proposed a simple model based in connections of several star graphs. They showed that the power law of the eigenvalue distribution has exponent $\alpha/2$, where α is the exponent of the degree distribution. For this, they used the fact that in a star graph the highest eigenvalue is the square root of the highest degree. They conjectured that the power law distribution of the eigenvalues must be due to the fact that the vertices of highest degrees may have many leaves.

Therefore, the distribution of the degrees seems to imply the distribution of the eigenvalues. However, in this paper we are going to see that this is not true for certain classes of scale free graphs. In Section 3 it is shown that the Cartesian product of a scale free graph with power law eigenvalue distribution with a regular graph is a scale free graph without eigenvalue power law distribution. Moreover, in Section 4 we construct a scale free graph with $\gamma = 1$ with a small spectrum (only three positive eigenvalues). In the last section, we construct a regular graph with an eigenvalue power law distribution, showing that the converse of the conjecture is also false.

2 Notation and first results

If $G = (V, E)$ is a simple connected graph, let $\boldsymbol{A}$ denote its adjacency matrix. The spectrum of the graph is the spectrum of its adjacency matrix, and we denote it by $\mathrm{sp}(G) = \mathrm{sp}(\boldsymbol{A}) = \{\lambda_1^{[m_1]}, \ldots, \lambda_d^{[m_d]}\}$, where m_i are the corresponding multiplicities of the eigenvalues λ_i, $1 \leq i \leq d$. We are going to introduce a similar notation for the degree distribution of a graph G, considering the degrees in increasing order and together with their multiplicities:

$$DD(G) = \{d_1^{[n_1]}, \ldots, d_k^{[n_k]}\}.$$

The following results and notation can be found in [4]. The direct product $G \times H$ of the graphs G and H, is a graph with $V(G \times H) = V(G) \times V(H)$, and two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if u' is adjacent to v' and u is adjacent to v. It is also called tensor product, categorial product, cardinal product or Kronecker product. The adjacency matrix of the direct product of two graphs is $\boldsymbol{A}_{G\times H} = \boldsymbol{A}_G \otimes \boldsymbol{A}_H$, and if $\mathrm{sp}(G) = \{\lambda_1, \ldots, \lambda_r\}$ and $\mathrm{sp}(H) = \{\mu_1, \ldots, \mu_s\}$, then its spectrum is

$$\mathrm{sp}(G \times H) = \{\lambda_i \mu_j,\ 1 \leq i \leq r,\ 1 \leq i \leq s\}. \qquad (1)$$

The Cartesian product $G \,\square\, H$ of the graphs G and H is the graph such that $V(G \,\square\, H) = V(G) \times V(H)$ and two vertices (u, u') and (v, v') are adjacent in $G \,\square\, H$ if and only if either $u = v$ and u' is adjacent to v' , or $u' = v'$ and u is adjacent to v. The adjacency matrix of the Cartesian product is the sum of Kronecker of their matrices, $\boldsymbol{A}_{G\,\square\,H} = \boldsymbol{A}_H \otimes \boldsymbol{I}_r + \boldsymbol{I}_s \otimes \boldsymbol{A}_G$, and so if $\mathrm{sp}(G) = \{\lambda_1, \ldots, \lambda_r\}$ and $\mathrm{sp}(H) = \{\mu_1, \ldots, \mu_s\}$, then

$$\mathrm{sp}(G \,\square\, H) = \{\lambda_i + \mu_j,\ 1 \leq i \leq r,\ 1 \leq i \leq s\}. \qquad (2)$$

The degree distributions of the direct and Cartesian products of two graphs are useful for constructing our scale free graphs.

Lemma 2.1. *Let G_1, G_2 be two graphs with $DD(G_1) = \{d_1^{[n_1]}, \dots, d_k^{[n_k]}\}$, $DD(G_2) = \{\delta_1^{[m_1]}, \dots, \delta_l^{[m_l]}\}$ respectively, then the graph $G_1 \times G_2$ has degree distribution*

$$DD(G_1 \times G_2) = \{(d_i\delta_j)^{[n_i m_j]},\ 1 \le i \le k,\ 1 \le j \le l\}.$$

And for the Cartesian product the resulting degree distribution is:

Lemma 2.2. *Let G_1, G_2 be two graphs with $DD(G_1) = \{d_1^{[n_1]}, \dots, d_k^{[n_k]}\}$, $DD(G_2) = \{\delta_1^{[m_1]}, \dots, \delta_k^{[m_k]}\}$ respectively, then the graph $G_1 \Box G_2$ has degree distribution*

$$DD(G_1 \Box G_2) = \{(d_i + \delta_j)^{[n_i m_j]},\ 1 \le i \le k,\ 1 \le j \le l\}.$$

These lemmas can be easily verified by summing the rows or the columns of the corresponding adjacency matrices.

3 Scale free graphs and Cartesian products

If we consider a scale free graph G with eigenvalue power law distribution $\mathrm{sp}(G) = \{\lambda_i = K' i^{-\gamma'},\ 1 \le i \le n\}$, and we take the Cartesian product of G with a regular graph H, the following proposition tell us that in some cases the power law distribution of the degrees is preserved in the resulting graph.

Proposition 3.1. *Let G be a scale free graph of order n and let H be a δ-regular graph of order m, then if δ is small enough the graph $G \Box H$ is also scale free.*

Proof. If $DD(G) = \{d_i^{[n_{d_i}]} : n_{d_i} = K d_i^{-\gamma}, 1 \le i \le k\}$ and $DD(H) = \{\delta^{[m]}\}$, we can apply Lemma 2.2 to show that the degree distribution of the resulting graph is

$$DD(G \Box H) = \{(d_1 + \delta)^{[mn_{d_1}]}, \dots, (d_k + \delta)^{[mn_{d_k}]} :\ n_{d_i} = K d_i^{-\gamma},\ 1 \le i \le k\}$$

Suppose that $DD(G \Box H)$ has a power law degree distribution. The multiplicities $n_{d_i+\delta} = K'(d_i + \delta)^{-\gamma'},\ 1 \le i \le k$ for suitable constants K' and γ'. As $n_{d_i+\delta} = mn_{d_i} = mKd_i^{-\gamma}$, we see that $K' = mK\big(1 + \delta/d_i\big)^{\gamma}$ and $\gamma = \gamma'$. Therefore, if δ is small with respect to the largest degrees, the graph is scale free. $\Box$

On the other hand, the power law distribution of the highest eigenvalues is not always preserved by the Cartesian product. If the distribution of eigenvalues of the regular graph is $\mathrm{sp}(H) = \{\mu_1, \dots, \mu_s\}$, then applying (2) yields

$$\mathrm{sp}(G \Box H) = \{K' i^{-\gamma'} + \mu_j,\ 1 \le i \le r,\ 1 \le j \le s\}.$$

The distribution of the Cartesian product depends on the distance between the eigenvalues of the regular graph, and thus the highest eigenvalues of the resulting graph might not to follow a power law. For instance, we consider a scale free graph G with $n = 10000$ vertices, power law eigenvalue distribution with $\gamma' = 2.3/2 = 1.15$ (which might happens in real scale free network) and $K' = 3000$, and we take the Cartesian product of G with a complete graph K_{10} (with $\mathrm{sp}(K_{10}) = \{9^{[1]}, -1^{[9]}\}$). The resulting graph has 100000 vertices. The power law distribution of the degrees has not change very much, so it is still a scale free graph. But in Figure 3 we can observe that the distribution of the highest eigenvalues does not follow a power law.

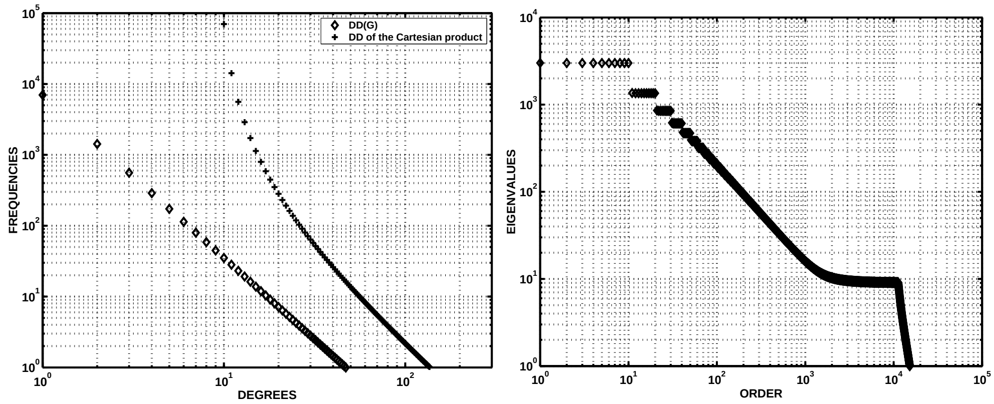

Figure 1: On the left we see that the power-law distribution of the degrees is preserved for the highest degrees. On the right, we see that the highest eigenvalues of the Cartesian product have not power law distribution.

4 Scale free graphs with small spectrum

The construction of a scale free graph with small spectrum is based on the description of its adjacency matrix. First consider the set of prime numbers $\{1 = p_1, p_2, \ldots, p_n, \ldots\}$ we call $\pi_n = \prod_{i=1}^{n} p_i$ and $\pi'_n = \prod_{i=1}^{n}(p_i + 1)$. We make the Cartesian product of star graphs $SP_n = S_{p_n+1}\Box \ldots \Box S_{p_1+1}$. The adjacency matrix of this graph $\boldsymbol{SP}_n = \boldsymbol{S}_{p_n+1} \otimes \ldots \otimes \boldsymbol{S}_{p_1+1}$ has order $\pi'_n \times \pi'_n$. The degree distribution follows exactly a power law with $\gamma = 1$.

Proposition 4.1. *The degree distribution of SP_n is*

$$DD(SP_n) = \{d_\beta^{[n_{d_\beta}]} : d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n},\ \beta = (\beta_1, \ldots, \beta_n),\ \beta_i \in \{0,1\},\\ n_{d_\beta} = 2\pi_n / d_\beta\}.$$

Proof. The degree distribution of a star graph is $DD(S_{p_n+1}) = \{1^{[n]}, p_n^{[1]}\}$. Thus applying Lemma 2.1 we obtain

$$DD(SP_2) = DD(S_{p_2+1} \,\Box\, S_{p_1+1}) = \{1^{[2\cdot 2]}, 2^{[1\cdot 2]}\}.$$

Each element $d^{[n_d]}$ of $DD(SP_n)$ gives rise to two elements $(p_{n+1} \cdot d)^{[n_d]}$ and $d^{[p_{n+1} \cdot n_d]}$ in $DD(SP_{n+1})$. Observe that the product of each degree by its multiplicity is constant. Moreover, as there are $2\pi_n$ leaves in SP_n, this product is $2\pi_n$. Hence we get the result. Note that the distribution follows a power law with $\gamma = 1$. $\square$

The spectrum of SP_n is easy to compute as it is the product of all the eigenvalues of the stars (Equation (1)), hence we get

$$\mathrm{sp}(SP_n) = \{\sqrt{\pi_n}^{[2^{n-1}]}, -\sqrt{\pi_n}^{[2^{n-1}]}, 0^{[\pi'_n - 2^n]}\}.$$

The degree distribution of this graph follows a perfect power law, but the problem is that this graph is not connected. To solve this problem, consider that at the $n+1$-step in the graph SP_{n+1} we connect the second $\pi'_n - 1$ vertices of the graph to the first one forming a star. In this way, we obtain a connected graph SF_{n+1} with adjacency matrix

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{S}_{\pi'_n} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{O} \end{pmatrix}, \quad \boldsymbol{B} = \overbrace{(\boldsymbol{SP}_n \mid \ldots \mid \boldsymbol{SP}_n)}^{p_{n+1}}.$$

When we calculate the degree distribution of the graph SF_{n+1} to see if it preserves the power law, we get

$$DD(SF_{n+1}) = \begin{cases} (\pi_{n+1} + \pi'_n - 1)^{[1]} & \text{if } \beta_i = 1\ \forall\, i, \\ (p_2^{\beta_2} \cdots p_n^{\beta_n})^{[n'_d]},\ n'_d = \frac{2\pi_n}{d} + \beta_n & \text{otherwise.} \end{cases}$$

Observe that the distribution of the new graph is very similar to the previous one, and for large values of n the multiplicities $n'_d \approx n_d$. Therefore we can assure that SF_{n+1} is a scale free graph.
The spectrum of SF_{n+1} is characterized in the following theorem.

Theorem 4.2. *The characteristic polynomial of the graph* SF_{n+1} *is*

$$\Phi_{SF_{n+1}}(\lambda) = [\lambda^2 \pm \sqrt{\pi'_n - 1}\lambda - p_{n+1}\pi_n][\lambda^2 - p_{n+1}\pi_n]^{2^n-4}\lambda^{\pi'_{n+1}-2^n}.$$

Proof. After $n+1$ steps of the inductive construction the adjacency matrix of the graph has order $\pi'_{n+1} \times \pi'_{n+1}$, and the system to solve is

$$\boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} \boldsymbol{S}_{\pi'_n} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{O} \end{pmatrix} \boldsymbol{x} = \lambda \boldsymbol{x}, \tag{3}$$

where $\boldsymbol{B} = \overbrace{(\boldsymbol{SP}_n, \ldots, \boldsymbol{SP}_n)}^{p_{n+1}}$ and $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{p_{n+1}+1})$.
First of all we are going to prove that $\mathrm{Ker}(\boldsymbol{SP}_n) \subset \mathrm{Ker}(\boldsymbol{S}_{\pi'_n})$. The spectrum of a star graph on $a+1$ vertices is $\mathrm{sp}(\boldsymbol{S}_{a+1}) = \{\sqrt{a}^{[1]}, -\sqrt{a}^{[1]}, 0^{[a-2]}\}$, and the kernel is formed by vectors characterized by having the first coordinate and the sum of the others equal to 0. A base of the kernel could be

$$\mathrm{Ker}(\boldsymbol{S}_{a+1}) = \{\boldsymbol{\phi}_i^{a+1} = [0, 1, 0, \ldots, \overbrace{-1}^{i}, \ldots, 0], \quad 3 \le i \le a+1\}.$$

The matrix $\boldsymbol{S}^2_{a+1}$ has the same eigenvectors as $\boldsymbol{S}_{a+1}$, and the eigenvalues of $\boldsymbol{S}^2_{a+1}$ are the square of the eigenvalues of $\boldsymbol{S}_{a+1}$. Therefore the kernels are equal but $\boldsymbol{S}^2_{a+1}$ has only one non-zero eigenvalue a of multiplicity 2. On the other hand, the eigenvalues of a Kronecker product of n matrices $\boldsymbol{SP}^2_n = \boldsymbol{S}^2_{p_n+1} \otimes \cdots \otimes \boldsymbol{S}^2_{p_1+1}$ are all the possible products of the eigenvalues of the stars, and thus

$$\mathrm{sp}(\boldsymbol{SP}_n) = \{\pi_n^{[2^n]}, 0^{[\pi'_n - 2^n]}\}.$$

The corresponding eigenvectors are also the Kronecker product of the corresponding eigenvectors, so that

$$\mathrm{Ker}(\boldsymbol{SP}^2_n) = \{\boldsymbol{\psi}_j^n = \boldsymbol{\phi}_{i_n}^{p_n+1} \otimes \cdots \otimes \boldsymbol{\phi}_{i_1}^{p_1+1},\ 1 \le i_k \le k+1,\ 2^n \le j \le \pi'_n\}.$$

These vectors can also be considered as

$$\boldsymbol{\psi}_j^n = \boldsymbol{\phi}_{i_n}^{p_n+1} \otimes \boldsymbol{\psi}_j^{n-1} = [\boldsymbol{O}_{\pi'_{n-1}}, \boldsymbol{\psi}_j^{n-1}, \boldsymbol{O}_{\pi'_{n-1}}, \ldots, \underbrace{-\boldsymbol{\psi}_j^{n-1}}_{i_n}, \ldots, \boldsymbol{O}_{\pi'_{n-1}}].$$

Observe that the first coordinate of $\boldsymbol{\psi}_j^n$ is zero and the sum of the orders too, therefore $\boldsymbol{\psi}_j^n \in \mathrm{Ker}(\boldsymbol{S}_{\pi'_n})$. So if $\boldsymbol{x} \in \mathrm{Ker}(\boldsymbol{SP}_n) = \mathrm{Ker}(\boldsymbol{SP}^2_n)$, then $\boldsymbol{x}$ is a linear combination of $\boldsymbol{\psi}_j^n$. Therefore $\boldsymbol{x} \in \mathrm{Ker}(\boldsymbol{S}_{\pi'_n})$.

Using this result is easy to verify that $\boldsymbol{x} \in \mathrm{Ker}(\boldsymbol{A}) \Leftrightarrow \boldsymbol{x}_1 \in \mathrm{Ker}(\boldsymbol{SP}_n)$. Now consider the general equation (3) in the form

$$\begin{aligned} \boldsymbol{S}_{\pi'_n}\boldsymbol{x}_1 + \boldsymbol{SP}_n\Big(\sum_{i=2}^{p_{n+1}+1} \boldsymbol{x}_i\Big) &= \lambda\boldsymbol{x}_1 && (4)\\ \boldsymbol{SP}_n\boldsymbol{x}_1 &= \lambda\boldsymbol{x}_i, \qquad 2 \leq i \leq p_{n+1}+1. && (5) \end{aligned}$$

If $\boldsymbol{x} \in \mathrm{Ker}(\boldsymbol{A})$, both equations [4] and [5] are zero, and from the second we get that $\boldsymbol{x}_1 \in \mathrm{Ker}(\boldsymbol{SP}_n)$. If $\boldsymbol{x}_1 \in \mathrm{Ker}(\boldsymbol{SP}_n) \subset \mathrm{Ker}(\boldsymbol{S}_{\pi'_n})$, then

$$\begin{aligned} \boldsymbol{SP}_n\Big(\sum_{i=2}^{p_{n+1}+1} \boldsymbol{x}_i\Big) &= \lambda\boldsymbol{x}_1 && (6)\\ 0 &= \lambda\boldsymbol{x}_i, \qquad 2 \leq i \leq p_{n+1}+1. && (7) \end{aligned}$$

Assuming that $\lambda \neq 0$, we can multiply the Equation (6) for λ, and use the second equation to get $0 = \lambda^2\boldsymbol{x}_1$. Hence λ must be zero, and therefore $\boldsymbol{x} \in \mathrm{Ker}(\boldsymbol{A})$. Note also that $dim(\mathrm{Ker}(\boldsymbol{A})) = \pi'_{n+1} - 2^n$ as there are 2^n eigenvectors of $\boldsymbol{SP}_n$ that not are in the $\mathrm{Ker}(\boldsymbol{SP}_n)$.
If $\boldsymbol{x} \notin \mathrm{Ker}(\boldsymbol{SP}_n)$, $\boldsymbol{x}$ must be an eigenvector associated to the eigenvalue π_n and from Equation (5) we can isolate each x_i and substitute it in Equation (4) to get

$$\lambda\boldsymbol{S}_{\pi'_n}\boldsymbol{x}_1 + p_{n+1}\pi_n\boldsymbol{x}_1 = \lambda^2\boldsymbol{x}_1. \qquad (8)$$

This equation can be arranged as

$$\boldsymbol{S}_{\pi'_n}\boldsymbol{x}_1 = \lambda^{-1}(\lambda^2 - p_{n+1}\pi_n)\boldsymbol{x}_1.$$

Therefore $\boldsymbol{x}_1$ must be an eigenvector of $\boldsymbol{S}_{\pi'_n}$. Now we consider two cases. First suppose that $\boldsymbol{x}_1 \notin \mathrm{Ker}(\boldsymbol{S}_{\pi'_n})$, then $\boldsymbol{x}_1$ must be an eigenvector associated to either of the eigenvalues $\pm\sqrt{\pi'_n - 1}$, and thus (8) can be arranged as

$$\lambda^2 \pm \sqrt{\pi'_n - 1}\lambda - p_{n+1}\pi_n = 0.$$

Note that from this equation we get four different eigenvalues. Second, suppose that $\boldsymbol{x}_1 \in \mathrm{Ker}(\boldsymbol{S}_{\pi'_n})$, then the equation to solve is $p_{n+1}\pi_n\,\boldsymbol{x}_1 = \lambda^2\boldsymbol{x}_1$, which leads to the two last equations

$$\lambda \pm \sqrt{p_{n+1}\pi_n} = 0.$$

The multiplicity of each of these eigenvalues is $2^{n-1} - 2$. □

5 Regular graphs with eigenvalues power law distribution

As we have seen in the previous section, the power law distribution of the degrees does not determine the power law distribution of the highest eigenvalues of the graph. As we will see now, the converse is also not true. To this end, we construct a regular graph with an eigenvalue power law distribution. The construction of this graph is very simple. We just consider direct products of complete graphs K_{p_n}. The degree distribution of a complete graph is $DD(K_a) = \{(a-1)^{[a]}\}$, and the spectrum is

$$\mathrm{sp}(K_a) = \{(a-1)^{[1]}, -1^{[a-1]}\}.$$

Proposition 5.1. *The graph $B_n = K_{p_n+1} \times \cdots \times K_{p_1+1}$ is π_n-regular and its spectrum follows a power law with $\gamma = 1$. $(n \geq 2)$*

Proof. The proof is very similar to the one of Lemma 4.1. For the first graph $B_2 = K_{2+1} \times K_{1+1} = K_3 \times K_2$, by applying Lemma(2.1) we get

$$DD(B_2) = \{(1 \cdot 2)^{[2\cdot 3]}\}, \qquad \mathrm{sp}(B_2) = \{2^{[1]}, 1^{[2]}, -1^{[2]}, -2^{[1]}\}.$$

In each step we multiply the degree distribution by $DD(K_{p_n+1}) = \{p_n^{[p_n+1]}\}$, so thus we obtain

$$DD(B_n) = \{\pi_n^{[\pi'_n]}\}.$$

For the spectra, in each step we multiply by $\mathrm{sp}(K_{p_n+1}) = \{p_n^{[1]}, -1^{[p_n]}\}$, which means that either we multiply the eigenvalues or the multiplicities by p_n as in the proof of Proposition 4.1, hence

$$\mathrm{sp}(B_n) = \{\pm\lambda_i^{[m_i]} : \lambda_i m_i = \pi_n, \quad 1 \leq i \leq n\}.$$

Which means that the distribution of the eigenvalues follows a power law with $\gamma = 1$. □

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