Constraint Algorithm for Extremals in Optimal Control Problems

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Abstract

A characterization of different kinds of extremals of optimal control problems is given if we take an open control set. A well known constraint algorithm for implicit differential equations is adapted to the study of such problems. Some necessary conditions of Pontryagin's Maximum Principle determine the primary constraint submanifold for the algorithm. Some examples in the control literature, such as subRiemannian geometry and control-affine systems, are revisited to give, in a clear geometric way, a subset where the abnormal, normal and strict abnormal extremals stand.

Key words: Pontryagin's Maximum Principle, extremals, optimal control problems, abnormality, strict abnormality, presymplectic.

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1 Introduction

In 1958 the International Congress of Mathematicians was held in Edinburgh, Scotland, where for the first time L. S. Pontryagin talked publicly about the Maximum Principle. This Principle is considered as an outstanding achievement of the Optimal Control Theory. It has been used in a wide range of applications, such as medicine, traffic flow, robotics, economy, etc. Nevertheless, it is worth remarking that the Maximum Principle does not give sufficient conditions to compute an optimal trajectory; it only provides necessary conditions. Thus only candidates to be optimal trajectories are found. To study if they are optimal or not, other results related to the existence of solutions for these problems can be useful, see [13, 18] for more details.

First, the necessary conditions of the Maximum Principle gave rise to different kinds of extremals: normal and abnormal. Later on, a new kind of extremal called strict abnormal came up. In the nineties, R. Montgomery, L. S. Liu and H. J. Sussmann [21, 23] proved that there are strict abnormal extremals being optimal in subRiemannian geometry. Then a new interest in abnormality arose as can be seen in [2, 4, 5, 9, 10, 29, 30].

As for the abnormal extremals, it is important to mention that they do not depend on the cost function, but only on the geometry of the control system. So, given the system, the abnormal extremals are characterized without knowing the cost function. On the contrary, the cost function must be previously fixed to study the normal extremals. As for the strict abnormal extremals, it is necessary the cost function because they must be abnormal extremals, although they cannot be normal. The only way to verify when an extremal is not normal is if the cost function is known.

In this paper, under the assumption of the control set being open, we describe an algorithm to characterize the different kinds of extremals that appear in Pontryagin's Maximum Principle. The existence of such an algorithm is directly linked to the natural Hamiltonian framework where this Maximum Principle is usually stated [17, 22, 27]. A weak version of the Maximum Principle admits a presymplectic formalism that induces a constraint algorithm in the sense given in [7, 14, 15, 16]. Such an algorithm has been used and adapted to study singular optimal control problems [11] and to study optimal control problems with nonholonomic constraints [19].

The constraint algorithm will be used to know where the dynamics of normal extremals take place and also the dynamics of abnormal ones. We obtain sufficient conditions to have both kinds of extremals. These conditions elucidate how to determine the strict abnormality. The adaptation of the algorithm, in its presymplectic form, to the study of the extremals is one of the main contributions of this paper and it is mostly developed in §4.

The importance of the theory elaborated is highlighted by the revisit of some known examples in subRiemannian geometry and in single-input control-affine systems to characterize the abnormal extremals. Using the algorithm and under suitable stop conditions, we get the same results as in, for instance, [3, 4, 21], where some conditions must be imposed by hand. Following the described method all the conditions determining the dynamics of the abnormal extremals appear in a natural way.

The organization of the paper is as follows: in Section 2 after a brief review of some notions in control theory, we state the optimal control problem and Pontryagin's Maximum Principle in the suitable framework for this paper, that is, in the presymplectic one. In Section 3 we introduce the presymplectic constraint algorithm in the general context. Section 4 is devoted to adapt and apply the algorithm to the characterization of extremals in optimal control problems. After studying the fixed time problem, we rewrite the algorithm for the free time case in Section 5. Finally, in Section 6, we study some examples in the control literature using the algorithm developed along the paper.

In the sequel, all the manifolds are real, second countable and $\mathcal{C}^\infty$. The maps are assumed to be $\mathcal{C}^\infty$. Sum over repeated indices is understood.

2 Optimal control problems from a presymplectic viewpoint

First, we introduce some concepts on control theory before drifting to optimal control theory, the natural framework of this paper.

A *control system* is defined by a set of differential equations depending on parameters. More precisely, let M be a smooth manifold, $\dim M = m$, U be an open set of $\mathbb{R}^k$ called the *control set*. A vector field X along the projection $\pi\colon M \times U \to M$ is a map $X\colon M \times U \to TM$ such that the following diagram is commutative

$$\begin{array}{ccc} & & TM \\ & \overset{X}{\nearrow} & \downarrow{\scriptstyle \tau_M} \\ M \times U & \xrightarrow{\pi} & M \end{array}$$

where τ_M is the natural projection of the tangent bundle. We denote the set of these vector fields as $\mathfrak{X}(\pi)$. A *control system* is an element of $\mathfrak{X}(\pi)$.

Let $I \subset \mathbb{R}$, a curve $(\gamma, u)\colon I \to M \times U$ is an integral curve of X if

$$\dot\gamma = X \circ (\gamma, u), \quad \text{that} \quad \text{is,} \;\; \dot\gamma(t) = X(\gamma(t), u(t)). \tag{2.1}$$

In the study of control systems, it is greatly interesting to be able to answer questions such as: where can the system go?, from where?, given two endpoint conditions, is there an integral curve of the system connecting them? All these questions are related to the notion of accessibility and reachable sets.

Now, we introduce some results to determine the accessibility of the systems. See [25, 28] for more details. Given a control system $X \in \mathfrak{X}(\pi)$, we have:

Definition 2.1. *Let $x \in M$.*

1. *The* ***reachable set*** *$R(x, T)$* ***from*** *x* ***at time*** *T* ***in*** *M is the set of points given by evaluating at time T all the curves in M starting at x and satisfying Equation (2.1) for some control u, that is,*

$$R(x, T) = \{\gamma(T) \mid (\gamma, u) \text{ satisfies (2.1)}\;,\; \gamma(0) = x\;,\; \operatorname{Im} u \subset U\}.$$

2. *The* ***reachable set*** *$R(x, \leq T)$* ***from*** *x* ***up to*** *T* ***in*** *M is given by*

$$R(x, \leq T) = \bigcup_{0 \leq t \leq T} R(x, t).$$

Here we have considered as initial condition x at time 0 for solving Equation (2.1). The same reachable sets are defined taking t_0 as the initial time and $t_0 + T$ as the final time, instead of T.

Bearing in mind the reachable sets, we define the notion of accessibility.

Definition 2.2. *Let $x \in M$. The system (2.1) is **accessible from** x if there exists $T > 0$ such that the set of interior points of $R(x, \leq t)$, $\operatorname{int} R(x, \leq t)$, is not empty for every $t \in (0, T]$.*

Thus the system is accessible from $x \in M$ if $R(x) = \cup_{t \geq 0} R(x, t)$ has a nonempty interior.

Consider the following set of vector fields $\mathscr{V} = \{X(\cdot, u) \,|\, u \in U\}$.

Definition 2.3. *The smallest involutive distribution containing the family of vector fields $\mathscr{V}$ is called the **accessibility distribution** $\mathcal{C}$ of the control system $X \in \mathfrak{X}(\pi)$.*

A necessary and sufficient condition of accessibility from a point in M is:

Theorem 2.4. ***([25, 28])*** *The system is accessible from $x \in M$ if and only if the dimension of the accessibility distribution $\mathcal{C}$ containing $\mathscr{V}$ is equal to the dimension of M. In this case, $R(x, \leq T)$ has a nonempty interior for every $T > 0$.*

From now on, we assume the accessibility of the system to guarantee the existence of integral curves between any two close enough given points.

Once the control systems and their accessibility are described, we can introduce a cost function $F\colon M \times U \to \mathbb{R}$ to get in optimal control theory and the functional

$$\mathcal{S}[\gamma, u] = \int_I F(\gamma, u)\, dt$$

defined on curves (γ, u) with a compact interval as domain. We are interested in the following problem:

Problem 2.5. ***(Optimal Control Problem,** OCP**)***
Given the elements M, U, X, F, $I = [a, b]$, x_a, $x_b \in M$. Find (γ, u) such that

(1) $\gamma(a) = x_a$, $\gamma(b) = x_b$,

(2) $\dot{\gamma}(t) = X(\gamma(t), u(t))$, $t \in I$, *and*

(3) $\mathcal{S}[\gamma, u]$ *is minimum over all curves on $M \times U$ satisfying* (1) *and* (2).

Although the mappings $u\colon I \to U$ are usually measurable and bounded, we need to assume that the vector field X along π and the cost function $F\colon M \times U \to \mathbb{R}$ are differentiable enough on $M \times U$ with respect to M and on U.

2.1 Presymplectic formalism

As was said in §1, Pontryagin's Maximum Principle gives conditions to find candidates to be optimal solutions of the above problem. The Maximum Principle can be approached from different viewpoints. Here we use the presymplectic formalism that gives a weaker Maximum Principle than the classical one [3, 17, 18, 20, 26]. For more details in presymplectic formalism see [7, 14, 15, 16, 24].

We consider the manifold $T^*M \times U$ where U is an open set of $\mathbb{R}^k$. Let Ω be the closed 2-form on $T^*M \times U$ given by the pull-back through $\pi_1 \colon T^*M \times U \to T^*M$ of the canonical 2-form on T^*M. The kernel of Ω at $a \in T^*M \times U$ is given by

$$\ker \Omega_a = \{v \in T_a(T^*M \times U) \,|\, \Omega(v,w) = 0 \ , \ \forall\, w \in T_a(T^*M \times U)\}.$$

The kernel contains the π_1-vertical vector fields, that is, π_1-projectable vector fields $Z \in \mathfrak{X}(T^*M \times U)$ such that $(\pi_1)_* Z = 0$. In local natural coordinates (x, p, u) for $T^*M \times U$, we have

$$\Omega = \mathrm{d}p_i \wedge \mathrm{d}x^i \quad \text{and} \quad \ker \Omega = \left\{\partial/\partial u^i\right\} \,.$$

The pair $(T^*M \times U, \Omega)$ is a presymplectic manifold.

In an analogous way to the case of Hamiltonian systems defined in a symplectic manifold [1], associated to a function $H \colon T^*M \times U \to \mathbb{R}$, called *Hamiltonian function*, we have a *presymplectic Hamiltonian system* $(T^*M \times U, \Omega, H)$. The curves of this Hamiltonian system are integral curves of a vector field X_H along π_1 satisfying the following presymplectic equation

$$i_{X_H}\Omega = \mathrm{d}H \tag{2.2}$$

on $T^*M \times U$. See [22] for more details. The vector field X_H is called the *Hamiltonian vector field associated to H*.

As Ω is degenerate, Equation (2.2) does not have solution at every point in $T^*M \times U$ and, in general, at the points where this solution exists it is not unique.

Locally, the Hamiltonian vector field along π_1 is given by

$$X_H = A^i \frac{\partial}{\partial x^i} + B_i \frac{\partial}{\partial p_i} + C^l \frac{\partial}{\partial u^l},$$

the presymplectic equation (2.2) provides us Hamilton's equations

$$A^i = \frac{\partial H}{\partial p_i} \ , \quad B_i = -\frac{\partial H}{\partial x^i},$$

where $i = 1, \ldots, m$ and $l = 1, \ldots, k$, and C^l are free because $\{\partial/\partial u^l\}_{l=1,\ldots,k}$ are in the kernel of Ω.

In § 3, we will study carefully how to solve this kind of systems. Now we state Pontryagin's Maximum Principle from a presymplectic viewpoint.

2.2 Pontryagin's Maximum Principle

It is known that optimal control theory admits a formulation as a presymplectic hamiltonian system, [11, 12, 22]. The corresponding hamiltonian, Pontryagin's Hamiltonian $H\colon T^*M \times U \to \mathbb{R}$, is given by

$$H(\lambda, u) = \langle \lambda, X(x,u) \rangle + p_0 F(x,u),$$

where $\lambda \in T^*_x M$ and $p_0 \in \{-1, 0\}$.

Remark 2.6. Let us introduce some notation to make things easier. Let X be a vector field on a manifold M, we associate it a hamiltonian function as follows

$$\begin{array}{rcl} H_X\colon T^*M & \longrightarrow & \mathbb{R} \\ \lambda & \longmapsto & H_X(\lambda) = \langle \lambda, X(x) \rangle \end{array}$$

where $\lambda \in T^*_x M$. The same is defined if the vector field X is along $\pi\colon M \times U \to M$. Then we can rewrite Pontryagin's Hamiltonian as follows

$$H(\lambda, u) = H_X(\lambda, u) + p_0 F(x,u) \tag{2.3}$$

and we can state Pontryagin's Maximum Principle in a presymplectic viewpoint.

Theorem 2.7. ***(Pontryagin's Maximum Principle, presymplectic form)*** *Let U be an open set in $\mathbb{R}^k$. Let $(\gamma, u)\colon [a,b] \to M \times U$ be a solution of the optimal control problem 2.5 with initial conditions x_a, x_b. Then there exist $\lambda\colon [a,b] \to T^*M$ along γ, and $p_0 \in \{-1, 0\}$ such that:*

1. *(λ, u) is an integral curve of the Hamiltonian vector field X_H that satisfies*

$$i_{X_H} \Omega = \mathrm{d}H; \tag{2.4}$$

2. *$\gamma = \pi_M \circ \lambda$ where $\pi_M\colon T^*M \to M$ is the natural projection of the cotangent bundle;*

3. *γ satisfies the initial conditions, $\gamma(a) = x_a$ and $\gamma(b) = x_b$;*

4. *(a) $\max_{\widetilde{u} \in U} H(\lambda(t), \widetilde{u})$ is constant everywhere in $t \in [a,b]$;*
   (b) $(p_0, \lambda(t)) \neq 0$ for each $t \in [a,b]$.

As was mentioned in § 2.1, the presymplectic equation (2.4) does not have solution in the whole manifold $T^*M \times U$. As we will see in § 3, it has solution if we restrict the equation to the submanifold defined implicitly by

$$S = \{a \in T^*M \times U \,|\, i_v \,\mathrm{d}H = 0, \quad \text{for } v \in \ker \Omega_a\}.$$

Locally, this condition for Pontryagin's Hamiltonian becomes

$$S = \{a \in T^*M \times U \,|\, \frac{\partial H}{\partial u^l}(a) = 0, \quad l = 1, \dots, k\}.$$

Remark 2.8. Observe that this is a necessary condition for the Hamiltonian to have an extremum over the controls as long as U is an open set. In the classic Pontryagin's Maximum Principle [26], the Hamiltonian is equal to the maximum of the Hamiltonian over the controls. Therefore, Theorem 2.7 is weaker than the classic Maximum Principle.

For a general statement of Pontryagin's Maximum Principle see [3, 17, 18, 20, 26]. There are another approaches of the presymplectic formalism of optimal control, see [19, 22], and also other approaches from a symplectic viewpoint, see for instance [27].

From now on, we will refer to Theorem 2.7 as Pontryagin's Maximum Principle without mentioning the fact that it is a weak form of the Maximum Principle.

The necessary conditions 1-4 of Theorem 2.7 determine different kinds of extremals.

Definition 2.9. *A curve* $(\gamma, u)\colon [a,b] \to M \times U$ *is*

1. *an* ***extremal for*** *OCP if there exist* $\lambda\colon [a,b] \to T^*M$ *and* $p_0 \in \{-1, 0\}$ *such that* (λ, u) *satisfies the necessary conditions of Pontryagin's Maximum Principle;*
2. *a* ***normal extremal for*** *OCP if it is an extremal and* $p_0 = -1$*, that is, the Hamiltonian is*
$$H^{[-1]} = H_X - F\ ; \tag{2.5}$$
3. *an* ***abnormal extremal for*** *OCP if it is an extremal and* $p_0 = 0$*, that is, the Hamiltonian is*
$$H^{[0]} = H_X\ ; \tag{2.6}$$
4. *a* ***strictly abnormal extremal for*** *OCP if it is not a normal extremal, but it is an abnormal extremal;*

The curve $(\lambda, u)\colon [a,b] \to T^*M \times U$ *is called* ***biextremal for*** *OCP.*

In §4 we take advantage of the necessary conditions in Theorem 2.7 to try to determine where the different kinds of extremals just defined are contained. We are specially interested in determining strict abnormal extremals and abnormal extremals as a consequence of the results published in [21, 23].

3 Presymplectic constraint algorithm

As was seen in § 2.2, the optimal control theory accepts a geometric formulation from a presymplectic viewpoint. Before proceeding let us introduce more generally the presymplectic manifolds and related concepts. For a more general framework, see [16].

Let M be a smooth manifold and Ω a 2-form on M. Given $x \in M$, the kernel of Ω at x is defined by $\ker \Omega_x = \{v \in T_xM \,|\, i_v\Omega = 0\}$. It is a vector subspace of the tangent space T_xM. We say that Ω is *regular* if the dimension of $\ker \Omega_x$ does not depend on the point $x \in M$.

Under the assumption of regularity of Ω, $\ker \Omega = \cup_{x\in M} \ker \Omega_x$ is a vector subbundle of the tangent bundle TM. The set of all the vector fields $X \in \mathfrak{X}(M)$ such that $X(x) \in \ker \Omega_x$ for all $x \in M$ is also denoted by $\ker \Omega$. The vector subbundle $\ker \Omega$ is involutive if and only if Ω is a closed form.

A *presymplectic form* on M is a closed and regular 2-form. A *presymplectic manifold* is a manifold M with a presymplectic form $\Omega \in \Omega^2(M)$. It is obvious that a symplectic manifold is presymplectic with $\ker \Omega = \{0\}$.

If (M, Ω) is a presymplectic manifold, some usual notions of symplectic manifolds also appear here. So if $H \in \mathcal{C}^\infty(M)$, we may consider the equation

$$i_X\Omega = \mathrm{d}H \tag{3.7}$$

where the unknown, the vector field $X \in \mathfrak{X}(M)$, is called the Hamiltonian vector field associated with the Hamiltonian function H.

If $\ker \Omega \neq \{0\}$, the mapping $\Omega^\sharp \colon TM \to T^*M$ given by $\Omega^\sharp(v_x) = i_{v_x}\Omega$ is not onto. Thus Equation (3.7) does not always have a solution. It is indispensable to claim for $\mathrm{d}H \in \mathrm{Im}\,\Omega^\sharp$. This condition can depend on the point $x \in M$ where we compute $\Omega^\sharp_x$. With this in mind, we define a *presymplectic system* (M, Ω, H) as a presymplectic manifold (M, Ω) and a function $H \in \mathcal{C}^\infty(M)$.

Problem 3.1. ***(Presymplectic)*** *Given a presymplectic system (M, Ω, H), find (N, X) such that*

(a) N is a submanifold of M,

(b) $X \in \mathfrak{X}(M)$ is tangent to N on N,

(c) N is maximal among all the submanifolds satisfying (a) and (b).

The solution to this problem gives rise to the so-called presymplectic algorithm described as follows, see [7, 14, 15, 16] for more details. The condition (c) cannot be assured in general.

Step zero: Let $N_0 = \{x \in M \,|\, \exists\, v_x \in T_xM\,, \quad i_{v_x}\Omega = \mathrm{d}_xH\}$ be the *primary constraint submanifold*.

Proposition 3.2. *$N_0 = \{x \in M \,|\, (\mathrm{L}_Z H)_x = 0\,,\ Z \in \ker \Omega\}$, where L_Z is the Lie derivative with respect to Z.*

The proof is a straightforward consequence of the fact that if $\alpha_x \in T^*_xM$, we have $\alpha_x \in \mathrm{Im}\Omega^\sharp_x$ if and only if $\ker \Omega_x \subset \ker \alpha_x$.

On the points of N_0 there exists a solution of the presymplectic equation, but the solution is not unique. Indeed, if X_0 is a solution, then $X_0 + \ker\Omega$ is the set of all the solutions. We may consider X_0 as a vector field defined on the whole M because N_0 is closed and we assume that N_0 is a submanifold of M.

Take the pair $(N_0, X_0 + \ker\Omega)$, rewritten as (N_0, X^{N_0}) where X^{N_0} denotes the set of all the vector fields solving the problem in the step zero. Observe that we need an element in X^{N_0} tangent to N_0.

Step one: Let now

$$N_1 = \{x \in N_0 \,|\, \exists\, X \in X^{N_0}\,, \quad X(x) \in T_xN_0\}$$

providing a new pair (N_1, X^{N_1}) where X^{N_1} is the set of the vector fields solution and we assume again that N_1 is a submanifold. This step stabilizes the constraints in N_0.

Once again the vector fields X^{N_1} are tangent to N_0, but not necessarily to N_1. Hence, inductively, we arrive at (N_i, X^{N_i}) where we assume that N_i is a submanifold of M and we define $N_{i+1} = \{x \in N_i \,|\, \exists\, X \in X^{N_i}\,, \quad X(x) \in T_xN_i\}$, obtaining the sequence

$$M \supseteq N_0 \supseteq N_1 \supseteq \ldots \supseteq N_i \supseteq N_{i+1} \supseteq \ldots$$

and the corresponding $X^{N_{i+1}}$. Let

$$N_f = \bigcap_{i\geq 0} N_i, \quad X^{N_f} = \bigcap_{i\geq 0} X^{N_i},$$

if N_f is a nontrivial submanifold of M, (N_f, X^{N_f}) is the solution to Problem 3.1.

Note that each step of the algorithm can reduce the set of points of M where there exists solution, that is $N_i \subseteq N_{i-1}$, and can also reduce the degrees of freedom of the set of vector fields solution, $X^{N_i} \subseteq X^{N_{i-1}}$.

If at one step $N_i = N_{i+1}$, we have already finished.

This presymplectic algorithm comes from the Dirac-Bergmann theory of constraints developed in the fifties for quantuum field theory. In [15] the algorithm is studied from a geometric viewpoint what has been reviewed in this section.

In the next sections, we adapt this algorithm in the context of the weak Pontryagin's Maximum Principle.

4 Characterization of extremals

Let us make profit of the presymplectic constraint algorithm explained in §3 to know where the different kinds of extremals are.

As stated in § 2.1, 2.2, we consider the presymplectic Hamiltonian system $(T^*M \times U, \Omega, H)$ with the Hamiltonian function $H = H_X + p_0F$ given in Equation (2.3).

From § 2.2, 3 the presymplectic equation (2.4), $i_{X_H}\Omega = \mathrm{d}H$, has solution in the primary constraint submanifold

$$N_0 = \{(\lambda, u) \in T^*M \times U \mid \frac{\partial H}{\partial u^l} = \lambda_j \frac{\partial X^j}{\partial u^l} + p_0 \frac{\partial F}{\partial u^l} = 0 \ , \quad l = 1, \dots, k\}. \tag{4.8}$$

So the Optimal Control Problem 2.5 could have solutions if we restrict the given equation to N_0. The tangency condition of the vector field X_H to N_0 defines

$$N_1 = \{(\lambda, u) \in N_0 \mid X_H \left(\frac{\partial H}{\partial u^l}\right) = 0, \quad l = 1, \dots, k\}, \tag{4.9}$$

which corresponds with step one in §3. Recall that locally, $X_H = A^i\partial/\partial x^i + B_i\partial/\partial p_i + C^l\partial/\partial u^l$ where $A^i = \partial H/\partial p_i$ and $B_i = -\partial H/\partial x^i$.

At this point, some specific questions about our control system must be included. One desirable objective is to determine the input controls and another objective is to restrict the problem to a smaller submanifold of $T^*M \times U$. Observe that, generally, a step of the algorithm can provide us new constraints and the determination of some controls at the same time. If the new constraints are independent of the previous constraints, they must be stabilized as was explained in Step 1 in § 3. Hence the algorithm continues. At each step of the algorithm, we are in one the following cases:

1. All the controls are determined and there are no more conditions to be stabilized, then the algorithm stops and provides us a unique vector field whose integral curves are biextremals.

2. All the controls are determined, but there are more conditions to be stabilized. Then the algorithm goes on, but the vector field is completely determined.

3. Not all the controls are determined and there are no more conditions to be stabilized. Then the algorithm stops, but there is not a unique vector field whose integral curves are biextremals.

4. Not all the controls are determined, but there are more conditions to be stabilized, then the algorithm goes on. So there may not be a unique vector field whose integral curves are biextremals.

5. The conditions to be stabilized could not define a submanifold. In this case a careful detailed study must be done. More precisely, when a subset N_i is not a submanifold, we have to restrict to each submanifold contained in N_i and stabilize each condition. For instance for $xy = 0$ we have to stabilize separately when $x = 0$ and when $y = 0$.

6. The final constraint submanifold is discrete, the biextremals are constant. They are usually not interesting because it is supposed that we move from one point to another different point in the state manifold.

7. The final constraint submanifold is empty, then there are no biextremals.

Remark 4.1. It remains to discuss if we miss some extremals using the constraint algorithm as it happens in subRiemannian geometry in [21], where using a less geometric approach they miss the constant extremals. There, this is not a big flaw since usually we want to move from a point in the state manifold to another point.

Remark 4.2. The case 5 comes from singular sets of the vector fields distributions that appear along the algorithm, that is, distributions without constant rank.

4.1 Characterization of abnormality

First, we characterize a subset of $T^*M \times U$ where the abnormal biextremals are if they exist. In this situation we take $p_0 = 0$ and the corresponding Pontryagin's Hamiltonian is (2.6)

$$H^{[0]} = H_X.$$

Then, in this case, the primary constraint submanifold (4.8) becomes

$$N_0^{[0]} = \{(\lambda, u) \in T^*M \times U \mid \lambda_j \frac{\partial X^j}{\partial u^l} = 0, \quad l = 1, \ldots, k\}, \tag{4.10}$$

the submanifold (4.9) is

$$N_1^{[0]} = \{(\lambda, u) \in N_0^{[0]} \mid \lambda_j \, (X^i \frac{\partial^2 X^j}{\partial x^i \partial u^l} - \frac{\partial X^j}{\partial x^i}\frac{\partial X^i}{\partial u^l} + C^r \frac{\partial^2 X^j}{\partial u^r \partial u^l}) = 0, \quad l = 1, \ldots, k\},$$

and the algorithm continues.

Once we have the final constraint submanifold $N_f^{[0]}$, we have to delete the biextremals in the zero fiber because these extremals do not satisfy the necessary condition (4.b) of Pontryagin's Maximum Principle 2.7. For the sake of simplicity and clarity, we rename this actual final constraint submanifold with the same name $N_f^{[0]}$.

Proposition 4.3. *If there exist $(\lambda, u) \in N_f^{[0]}$ with $\lambda \neq 0$, then $(\gamma, u) = (\pi_M \times \mathrm{Id})(\lambda, u)$ is an abnormal extremal.*

4.2 Characterization of normality

The normal and abnormal extremals in Definition 2.9 do not constitute a disjoint partition of the set of extremals. We have already described abnormality in § 4.1. Let us do the same with normality. For $p_0 = -1$, Pontryagin's Hamiltonian (2.5) is

$$H^{[-1]} = H_X - F.$$

Then the primary constraint submanifold (4.8) becomes

$$N_0^{[-1]} = \{(\lambda, u) \in T^*Q \times U \mid \lambda_j \frac{\partial X^j}{\partial u^l} - \frac{\partial F}{\partial u^l} = 0, \quad l = 1, \ldots, k\}, \tag{4.11}$$

the submanifold (4.9) is

$$N_1^{[-1]} = \{(\lambda, u) \in N_0^{[-1]} \mid \quad \lambda_j\, (X^i \frac{\partial^2 X^j}{\partial x^i \partial u^l} - \frac{\partial X^j}{\partial x^i}\frac{\partial X^i}{\partial u^l} + C^r \frac{\partial^2 X^j}{\partial u^r \partial u^l})$$

$$-X^i \frac{\partial^2 F}{\partial x^i \partial u^l} - C^r \frac{\partial^2 F}{\partial u^r \partial u^l} = 0, \quad l = 1, \ldots, k\},$$

Note the significant role that the cost function plays for normal extremals: the possibility for the controls to be determined essentially depends on the given cost function. To a better understanding of all this process we address the reader to the examples in § 5, 6.

To make evident the differences and similarities between abnormal and normal biextremals in terms of this algorithm, let us compare locally the primary constraint submanifolds and the dynamical equations for abnormality in § 4.1 and for normality. If (x, p, u) are natural coordinates in $T^*M \times U$, we have

	Abnormal	Normal
Dynamical equations	$\dot{x}^i = \frac{\partial H^{[0]}}{\partial p_i}$ $\dot{p}_i = -\frac{\partial H^{[0]}}{\partial x^i}$	$\dot{x}^i = \frac{\partial H^{[-1]}}{\partial p_i} = \frac{\partial H^{[0]}}{\partial p_i}$ $\dot{p}_i = -\frac{\partial H^{[-1]}}{\partial x^i} = -\frac{\partial H^{[0]}}{\partial x^i} + \frac{\partial F}{\partial x^i}$
Primary constraints	$\frac{\partial H^{[0]}}{\partial u^l} = 0$	$\frac{\partial H^{[-1]}}{\partial u^l} = \frac{\partial H^{[0]}}{\partial u^l} - \frac{\partial F}{\partial u^l} = 0$

where $i = 1, \ldots, m$, $l = 1, \ldots, k$.

Observe that Hamilton's equations for x^i are the same for both Hamiltonian functions since the cost function does not depend on the momenta p's. Hamilton's equations for p_i are equal for cost functions not depending on x^i. For instance, if the cost function is constant, as in the case of time-optimal.

The final constraint submanifolds $N_f^{[0]}$ and $N_f^{[-1]}$, if they exist, restrict the set of points where the biextremals of the Optimal Control Problem 2.5 are. But, even if Hamilton's equations are the same, $N_f^{[0]}$ and $N_f^{[-1]}$ could be different. Then the integral curve in $T^*M \times U$ along the same extremal in M may be different depending on where the initial conditions for the momenta are taken.

As stated in Definition 2.9, there may exist abnormal extremals being normal and viceversa. To study how the extremals are we need to project the biextremals on the base manifold $M \times U$ using $\rho_1 = \pi_M \times \mathrm{Id}\colon T^*M \times U \to M \times U$.

Summarizing all the above comments, we have the following propositions whose proofs are immediate.

Proposition 4.4. *Let (γ, u) be an abnormal extremal. If there exists a covector λ along γ such that $(\lambda, u) \in N_f^{[-1]}$, then (γ, u) is also a normal extremal.*

Let (γ, u) be a normal extremal. If there exists a covector λ along γ such that $(\lambda, u) \in N_f^{[0]}$, then (γ, u) is also an abnormal extremal.

Proposition 4.5. *If there exist $(\lambda^{[0]}, u^{[0]}) \in N_f^{[0]}$ with $\lambda^{[0]} \neq 0$ and $(\lambda^{[-1]}, u^{[-1]}) \in N_f^{[-1]}$ such that $\pi_M(\lambda^{[0]}) = \pi_M(\lambda^{[-1]})$, then $\gamma = \pi_M(\lambda^{[0]})$ is an abnormal extremal and also a normal extremal.*

Remark 4.6. In this second proposition we do not consider the control as a part of the extremal, because it may happen that different controls give the same extremals in M depending on the control system. So we project onto M the biextremals to compare them. Under some assumptions about the control systems, such as control-linearity, different controls give different extremals. If so happens, we will project the biextremals onto $M \times U$ through ρ_1 to compare them.

4.3 Characterization of strict abnormality

The way to find the strict abnormal extremals is to project all the biextremals in $N_f^{[0]}$ and $N_f^{[-1]}$ through $\rho = \pi_M \circ \pi_1 \colon T^*M \times U \to M$ due to Remark 4.6. Denoting by $P = \rho(N_f^{[0]}) \cap \rho(N_f^{[-1]})$, it may happen that

(i) $P = \emptyset$ and $\rho(N_f^{[0]}) \neq \emptyset$, then all the abnormal extremals are strict.

(ii) $P = \emptyset$ and $\rho(N_f^{[-1]}) \neq \emptyset$, in this case all the normal extremals are strict normal.

(iii) $P \neq \emptyset$ and $\rho(N_f^{[0]}) = P$, then there are no strict abnormal extremals.

(iv) $P \neq \emptyset$ and $\rho(N_f^{[0]}) \neq P$, in this case there are strict abnormal extremals, but only locally since the extremal could have pieces in P. So at some points the extremal can be locally normal.

(v) $P \neq \emptyset$ and $\rho(N_f^{[0]}) = \rho(N_f^{[-1]}) = P$, then all the abnormal extremals are also normal and viceversa.

So far we know how to search for abnormal extremals and also for normal extremals. While in § 4.1 we do not care about the cost function, in § 4.2 a cost function is essential for the process. To characterize strict abnormal extremals the cost function is fundamental because these extremals are abnormal, but not normal. The only way to guarantee that an extremal is not normal is to use the cost function.

Summarizing this section, we have the following characterization of strict abnormality whose proof is immediate.

Proposition 4.7. *Let (γ, u) be an abnormal extremal. If there does not exist λ along γ such that $(\lambda, u) \in N_f^{[-1]}$, then (γ, u) is a strict abnormal extremal.*

If there exists $(\lambda^{[0]}, u^{[0]}) \in N_f^{[0]}$ with $\lambda^{[0]} \neq 0$ such that there are not any $(\lambda^{[-1]}, u^{[-1]}) \in N_f^{[-1]}$ satisfying $\pi_M(\lambda^{[0]}) = \pi_M(\lambda^{[-1]})$, then $\gamma = \rho(\lambda^{[0]}, u^{[0]})$ is a strict abnormal extremal.

5 Free time optimal control problem

Once all the theory has been introduced let us deal with the particular case of the free time optimal control problem. Recall that in this case the interval of definition of the extremals is another unknown of the problem. Consider the following problem:

Problem 5.1. ***(Free Time Optimal Control Problem,** FOCP)*
Given the elements M, U, X, F, x_a, $x_b \in M$ (as in § 2). Find (γ, u) and $I = [a, b] \subset \mathbb{R}$ such that

(1) $\gamma(a) = x_a$, $\gamma(b) = x_b$,

(2) $\dot\gamma(t) = X(\gamma(t), u(t))$, $t \in I$, *and*

(3) $\mathcal{S}[\gamma, u]$ *is minimum over all curves on* $M \times U$ *satisfying* (1) *and* (2).

Pontryagin's Maximum Principle is the same as Theorem 2.7, but replacing $(4.a)$ by

$$(4.a')\ \max_{\widetilde{u}\in U} H(\lambda(t), \widetilde{u})\, \textit{is zero everywhere}\, t \in I.$$

Similar to Remark 2.8, condition $(4.a')$ and the presymplectic equation imply that Pontryagin's Hamiltonian is zero. Thus presymplectic equation (2.4) must be restricted to the submanifold defined by the condition

$$H = H_X + p_0 F = 0.$$

Hence, it must also be stabilized in the algorithm.

The primary constraint submanifold is

$$N_0 = \{(\lambda, u) \in T^*M \times U \mid \frac{\partial H}{\partial u^l} = \lambda_j \frac{\partial X^j}{\partial u^l} + p_0 \frac{\partial F}{\partial u^l} = 0,\ H = 0, \quad l = 1, \dots, k\}.$$

For abnormality, the Hamiltonian is $H^{[0]} = H_X$ and the primary constraint submanifold becomes

$$N_0^{[0]} = \{(\lambda, u) \in T^*M \times U \mid \lambda_j \frac{\partial X^j}{\partial u^l} = 0,\ H_X = 0, \quad l = 1, \dots, k\}.$$

But, for normality, the Hamiltonian is $H^{[-1]} = H_X - F$ and the primary constraint submanifold is

$$N_0^{[-1]} = \{(\lambda, u) \in T^*M \times U \mid \lambda_j \frac{\partial X^j}{\partial u^l} - \frac{\partial F}{\partial u^l} = 0,\ H_X - F = 0,\ l = 1, \dots, k\}.$$

In both cases, the algorithm must continue and the characterization of the extremals is exactly the same as is explained in §4.

6 Examples

6.1 Geodesics in Riemannian geometry

Let M be an m-dimensional Riemannian manifold and $\{Y_1, \dots, Y_m\}$ be linear independent vector fields on M. Consider the following control-linear system

$$X = u^1 Y_1 + \dots + u^m Y_m.$$

The problem of finding the geodesic curves in M can be addressed as an optimal control problem for the previous system with cost function $F(x, u) = \|X\|$, being $\|\cdot\|$ the Riemannian norm.

For abnormality $p_0 = 0$, the primary constraint submanifold (4.10) is

$$N_0^{[0]} = \{(\lambda, u) \in T^*M \times U \mid \langle \lambda, Y_l \rangle = 0, \quad l = 1, \dots, m\}.$$

So the controls do not appear in the primary constraint submanifold. As the number of controls coincides with the dimension of the state space, the annihilator of all the control vector fields is the zero covector. But Theorem 2.7 says that $(p_0, \lambda) \neq 0$. So in Riemannian geometry there are no abnormal extremals neither strict abnormal extremals, as is stated in [21].

For normality, $p_0 = -1$. The primary constraint submanifold (4.11) is

$$N_0^{[-1]} = \{(\lambda, u) \in T^*M \times U \mid \langle \lambda, Y_l \rangle - \frac{\partial F}{\partial u^l} = 0, \quad l = 1, \dots, m\}.$$

For instance, if the vector fields $\{Y_1, \dots, Y_m\}$ are orthonormal, the cost function is

$$F(x, u) = \frac{1}{2}((u^1)^2 + \dots + (u^m)^2),$$

where $1/2$ is written by convention, and we have

$$N_0^{[-1]} = \{(\lambda, u) \in T^*M \times U \mid \langle \lambda, Y_l \rangle - u^l = 0, \quad l = 1, \dots, m\}.$$

Hence, all the controls are known and the Hamiltonian vector field X_H is uniquely determined, then $N_0^{[-1]} = N_f^{[-1]}$ from § 4. We are in the case *(ii)* in § 4.3.

The projections on M of the integral curves of X_H satisfy the well-known geodesic equations on M as can be easily proved.

6.2 SubRiemannian geometry

As before, let M be an m-dimensional Riemannian manifold and $\{Y_1, \ldots, Y_k\}$ be linear independent vector fields on M, but with $k < m$. Now the corresponding control-linear system is

$$X = u^1 Y_1 + \ldots + u^k Y_k$$

and we state the same problem as in the previous section: to find the geodesic curves in M satisfying the previous system, that is, with $F(x, u) = \|X\|$ as the cost function. This optimal control system describes subRiemannian geometry.

For abnormality $p_0 = 0$, the primary constraint submanifold (4.10) is

$$N_0^{[0]} = \{(\lambda, u) \in T^*M \times U \mid \langle \lambda, Y_l \rangle = 0, \quad l = 1, \ldots, k\}$$

and the Hamiltonian vector field on $N_0^{[0]}$ is $X_{H^{[0]}} = u^j X_{Y_j}$, where X_{Y_j} denotes $X_{H_{Y_j}}$.

The tangency condition is

$$X_{H^{[0]}}(H_{Y_l}) = u^j X_{Y_j}(H_{Y_l}) = u^j dH_{Y_l}(X_{Y_j}) = -u^j \{H_{Y_j}, H_{Y_l}\} = \sum_{j=1, j\neq l}^{k} u^j H_{[Y_j, Y_l]} = 0$$

for $l = 1, \ldots, k$. See [1] for the properties of Poisson brackets used above. Then

$$N_1^{[0]} = \{(\lambda, u) \in N_0^{[0]} \mid \sum_{j=1, j\neq l}^{k} u^j H_{[Y_j, Y_l]} = 0, \quad l = 1, \ldots, k\}.$$

See [21] for a characterization of the abnormal extremals when there are only two control vector fields, a particular simple case. In [2] the abnormal extremals are studied more generally, without any assumption about the number of control vector fields.

For two-control input vector fields, the submanifold $N_1^{[0]}$ is defined implicitly by the constraints $\{u^1 H_{[Y_1, Y_2]} = 0, \ u^2 H_{[Y_2, Y_1]} = 0\}$. As both controls cannot be identically zero, otherwise there is no motion, then the only constraint is $H_{[Y_1, Y_2]} = 0$. Following the algorithm we obtain

$$N_2^{[0]} = \{(\lambda, u) \in N_1^{[0]} \mid u^1 H_{[Y_1, [Y_1, Y_2]]} + u^2 H_{[Y_2, [Y_1, Y_2]]} = 0\}.$$

In order not to contradict the assumption of accessibility taken in the whole paper, at least one of $H_{[Y_1, [Y_1, Y_2]]}$ and $H_{[Y_2, [Y_1, Y_2]]}$ must be nonzero. Hence, as we have a linear dependence between the controls, the motion is determined up to reparametrization.

As recapitulation, the abnormal extremals, if they exist, are in

$$N_2^{[0]} - \{(\lambda, u) \in N_2^{[0]} \mid H_{[Y_1, [Y_1, Y_2]]} = 0, \ H_{[Y_2, [Y_1, Y_2]]} = 0\},$$

see [21] for similar results with another approach.

For normality, $p_0 = -1$ and the primary constraint submanifold (4.11) is

$$N_0^{[-1]} = \{(\lambda, u) \in T^*M \times U \mid \langle \lambda, Y_l \rangle - \frac{\partial F}{\partial u^l} = 0, \quad l = 1, \dots, k\}.$$

As in § 6.1, if the vector fields $\{Y_1, \dots, Y_k\}$ are orthonormal, the cost function is

$$F(x, u) = \frac{1}{2}((u^1)^2 + \ldots + (u^k)^2)$$

and we have $N_0^{[-1]} = \{(\lambda, u) \in T^*M \times U \mid \langle \lambda, Y_l \rangle - u^k = 0, \quad l = 1, \dots, k\}$. Hence, for normality the reasoning follows as in Riemannian geometry.

6.3 Control-affine systems

Now we consider a m-dimensional manifold M and the control-affine system

$$X = Y + u^1 Y_1 + \ldots + u^k Y_k,$$

where $\{Y_1, \dots, Y_k\}$ are linear independent vector fields and Y is the drift vector field. Let F be the cost function for an optimal control problem.

For abnormality $p_0 = 0$, the primary constraint submanifold (4.10) is

$$N_0^{[0]} = \{(\lambda, u) \in T^*M \times U \mid \langle \lambda, Y_l \rangle = 0, \quad l = 1, \dots, k\}$$

and the Hamiltonian vector field is $X_{H^{[0]}} = X_Y + u^j X_{Y_j}$ on $N_0^{[0]}$, with the same notation as in § 6.2.

The tangency condition is

$$X_{H^{[0]}}(H_{Y_l}) = H_{[Y, Y_l]} + \sum_{j=1, j \neq l}^{k} u^j H_{[Y_j, Y_l]} = 0, \quad l = 1, \dots, k.$$

Then $N_1^{[0]} = \{(\lambda, u) \in N_0^{[0]} \mid H_{[Y,Y_l]} + \sum_{j=1, j\neq l}^{k} u^j H_{[Y_j, Y_l]} = 0, \quad l = 1, \dots, k\}$.

In [3, 4, 30] this situation is studied when there are at most two controls and in [29] more general results related to control-affine systems are given.

Depending of the rank of the matrices $A = (H_{[Y_j, Y_l]})$ and $B = (A \,|\, H_{[Y, Y_l]})$, we have the following situations, compare with [3, 4, 29, 30]:

(i) The rank of A is maximum and then all the controls are determined. Hence, given the initial conditions, the abnormal extremals are known.

(ii) The rank of A is not maximum and is equal to the rank of B. Then some controls are determined and others are free. There are no new constraints and the algorithm ends.

(iii) The rank of A is not maximum and different from the rank of B. Then some controls are determined and others are free. But there are also new constraints and the algorithm continues. At every step, a similar analysis must be done.

For normality, $p_0 = -1$ and the primary constraint submanifold (4.11) is

$$N_0^{[-1]} = \{(\lambda, u) \in T^*M \times U \mid \langle \lambda, Y_l \rangle - \frac{\partial F}{\partial u^l} = 0, \quad l = 1, \ldots, k\}.$$

For instance, if the cost function is

$$F(x, u) = \frac{1}{2}((u^1)^2 + \ldots + (u^k)^2),$$

we have $N_0^{[-1]} = \{(\lambda, u) \in T^*M \times U \mid \langle \lambda, Y_l \rangle - u^k = 0, \quad l = 1, \ldots, k\}$. Hence, for normality the reasoning follows as in the previous examples.

7 Conclusion and outlook

In this paper we have given a method to study different kinds of extremals in optimal control problems with an open control set. This method is based on the suitable reinterpretation of the so-called *presymplectic algorithm* in other fields. The dependence on the cost function makes difficult to give general characterizations of normal and strict abnormal extremals since each problem must be studied by itself. However, the abnormal extremals only depend on the geometry of the control system, so some general results can be deduced.

One line of future research is to apply this general algorithm in the study of optimal control problems with affine connection control systems, which model the motion of different types of mechanical systems such as rigid bodies, nonholonomic systems and robotic arms [6].

Furthermore, we are interested in the characterization of extremals for particular optimal control problems for mechanical systems, as for instance time-optimal problems and control-quadratic cost function.

Apart from having sufficient conditions to determine where the extremals are, it may be interesting to prove the density and the optimality of them, similar to the work done in [21]. Moreover, under what assumptions we can obtain necessary conditions to determine the different extremals.

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