ON THE k-SYMPLECTIC, k-COSYMPLECTIC AND MULTISYMPLECTIC FORMALISMS OF CLASSICAL FIELD THEORIES

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Abstract

The objective of this work is twofold: First, we analyze the relation between the k-cosymplectic and the k-symplectic Hamiltonian and Lagrangian formalisms in classical field theories. In particular, we prove the equivalence between k-symplectic field theories and the so-called autonomous k-cosymplectic field theories, extending in this way the description of the symplectic formalism of autonomous systems as a particular case of the cosymplectic formalism in non-autonomous mechanics. Furthermore, we clarify some aspects of the geometric character of the solutions to the Hamilton-de Donder-Weyl and the Euler-Lagrange equations in these formalisms. Second, we study the equivalence between k-cosymplectic and a particular kind of multisymplectic Hamiltonian and Lagrangian field theories (those where the configuration bundle of the theory is trivial).

Key words: *k-symplectic manifolds, k-cosymplectic manifolds, multisymplectic manifolds, Hamiltonian and Lagrangian field theories.*

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1 Introduction

The *k-symplectic* and *k-cosymplectic formalisms* are the simplest geometric frameworks for describing classical field theories. The k-symplectic formalism [13, 25] (also called *polysymplectic formalism*) is the generalization to field theories of the standard symplectic formalism in autonomous mechanics, and is used to give a geometric description of certain kinds of field theories: in a local description, those whose Lagrangian and Hamiltonian functions do not depend on the coordinates in the basis (in many of them, the space-time coordinates). The foundations of the k-symplectic formalism are the k-symplectic manifolds intoduced in [2, 3, 4]. The k-cosymplectic formalism is the generalization to field theories of the standard cosymplectic formalism for non-autonomous mechanics, [21, 22], and it describes field theories involving the coordinates in the basis on the Lagrangian and on the Hamiltonian. The foundations of the k-cosymplectic formalism are the k-cosymplectic manifolds introduced in [21, 22]. One of the advantages of these formalisms is that only the tangent and cotangent bundle of a manifold are required for their development. (A brief review of k-symplectic and k-cosymplectic geometry is given in Section 2.2). Other different polysymplectic formalisms for describing field theories have been proposed in [10, 11, 15, 23, 26, 27, 30].

In these formalisms, the field equations (Hamilton-de Donder-Weyl and Euler-Lagrange equations) can be written in a geometrical way using integrable k-vector fields. However, although integral sections of integrable k-vector fields (i.e., integrable distributions) that are solutions to the geometrical field equations are proved to be solutions to the Hamilton-de Donder-Weyl or the Euler-Lagrange equations, the converse is not always true. This also occurs when other geometric descriptions of classical field theories in terms of multivector fields are considered (see [7, 8, 28] for details in the case of multisymplectic field theories). Here we prove that, in the k-cosymplectic formalism, every solution to the Hamilton-de Donder-Weyl equations is, in fact, an integral section of an integrable k-vector field that is a solution to the geometrical field equations in the Hamiltonian formalism. Nevertheless, in the k-symplectic Hamiltonian formalism, this is no longer true, unless some additional conditions on the solutions to the Hamilton-de Donder-Weyl are required. All these features are discussed in Sections 2.3, 2.4, 2.5, 3.2, and 3.3.

After reviewing the k-cosymplectic Hamiltonian formalism in Section 2.4, Section 2.5 contains other relevant results of this work. In particular, the relation between the k-cosymplectic and the k-symplectic Hamiltonian formalism is studied here, proving the equivalence between k-symplectic Hamiltonian systems and a class of k-cosymplectic Hamiltonian systems: the so-called *autonomous k-cosymplectic Hamiltonian systems*. This generalizes the situation in classical mechanics, where the symplectic formalism for describing autonomous Hamiltonian systems can be recovered as a particular case of the cosymplectic Hamiltonian formalism when systems described by time-independent Hamiltonian functions are considered.

A more general geometric framework for describing classical field theories is the *multisymplectic formalism* [5, 12, 24], first introduced in [16, 17, 18], which is based on the use of multisymplectic manifolds. In particular, jet bundles are the appropriate domain for stating the Lagrangian formalism [31], and different kinds of multimomentum bundles are used for developing the Hamiltonian description [9, 14, 19]. (A brief review of multisymplectic Hamiltonian and Lagrangian field theories is given in Sections 4.1, 4.2, and 5.1).

Multisymplectic models allow us to describe a higher variety of field theories than the k-cosymplectic or k-symplectic models, since for the latter the configuration bundle of the theory must be a trivial bundle; however, this restriction does not oocur for the former. Another goal of this paper is to show the equivalence between the multisymplectic and k-cosymplectic descriptions, when theories with trivial configuration bundles are considered, for both the Hamiltonian and Lagrangian formalisms. In this way we complete the results obtained in [20], where an initial analysis about the relation between multisymplectic, k-cosymplectic and k-symplectic structures was carried out. This study is explained in Sections 4.3, and 5.2.

All manifolds are real, paracompact, connected and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood.

2 k-symplectic and k-cosymplectic Hamiltonian formalisms

2.1 k-vector fields and integral sections

(See [21] and [29] for details). If M is a differentiable manifold, let $T^1_kM = TM \oplus \overset{k}{\ldots} \oplus TM$ be the Whitney sum of k copies of TM, and $\tau^1_M \colon T^1_kM \longrightarrow M$ its canonical projection. T^1_kM is usually called the *k-tangent bundle* or *tangent bundle of k^1-velocities of M*.

Definition 1 *A* k-vector field *on* M *is a section* $\mathbf{X}\colon M \longrightarrow T^1_k M$ *of the projection* τ^1_M.

Giving a k-vector field $\mathbf{X}$ is equivalent to giving a family of k vector fields $X_1, \ldots, X_k$ on M obtained by projecting $\mathbf{X}$ onto every factor; that is, $X_A = \tau_A \circ \mathbf{X}$, where $\tau_A\colon T^1_k M \to TM$ is the canonical projection onto the A^{th}-copy TM of $T^1_k M$. For this reason we will denote a k-vector field by $\mathbf{X} = (X_1, \ldots, X_k)$.

Definition 2 *An* integral section *of the k-vector field* $\mathbf{X} = (X_1, \ldots, X_k)$ *passing through a point* $x \in M$ *is a map* $\phi\colon U_0 \subset \mathbb{R}^k \to M$*, defined on some neighborhood* U_0 *of* $0 \in \mathbb{R}^k$*, such that*

$$\phi(0) = x,\ \phi_*(t)\left(\frac{\partial}{\partial t^A}\Big|_t\right) = X_A(\phi(t)) \quad , \quad \text{for every } t \in U_0,\ 1 \leq A \leq k\,.$$

A k-vector field is said to be integrable *if there is an integral section passing through every point of* M*.*

Remark: k-vector fields in a manifold $\mathcal{M}$ can also be defined more generally as sections of the bundle $\Lambda^k(\mathrm{T}\mathcal{M}) \to \mathcal{M}$ (i.e., the contravariant skew-symmetric tensors of order k in $\mathcal{M}$). The k-vector fields defined in Definition 1 are a particular class: the so-called *decomposable* or *homogeneous k-vector fields*, which can be associated with distributions on $\mathcal{M}$. We remark that a k-vector field $\mathbf{X} = (X_1, \ldots, X_k)$ is integrable if, and only if, $\{X_1, \ldots, X_k\}$ define an involutive distribution on $\mathcal{M}$. (See [7] for a detailed exposition on these topics).

2.2 k-symplectic and k-cosymplectic manifolds

(See [21] and [29] for details).

Definition 3 (Awane [2]) *A* k-symplectic structure *on a manifold* M *of dimension* $N = n + kn$ *is a family* $(\omega^A, V; 1 \leq A \leq k)$*, where each* ω^A *is a closed 2-form and* V *is an integrable* nk*-dimensional distribution on* M *such that*

$$(i) \quad \omega^A|_{V\times V} = 0, \qquad (ii) \quad \cap_{A=1}^k \ker \omega^A = \{0\}\ .$$

Then (M, ω^A, V) *is called a* k-symplectic manifold.

Theorem 1 (Awane [2]) *Let* $(\omega^A, V; 1 \leq A \leq k)$ *be a k-symplectic structure on* M*. For every point of* M *there exists a local chart of coordinates* (q^i, p^A_i)*,* $1 \leq i \leq n$*,* $1 \leq A \leq k$*, such that*

$$\omega^A = dq^i \wedge dp^A_i \quad , \quad V = \left\langle \frac{\partial}{\partial p^1_i}, \ldots, \frac{\partial}{\partial p^k_i} \right\rangle_{i=1,\ldots,n} \quad ; \quad 1 \leq A \leq k \quad .$$

The canonical model for this geometrical structure is $((T^1_k)^*Q, \omega^A, V)$, where Q is a n-dimensional differentiable manifold and $(T^1_k)^*Q = T^*Q \oplus \stackrel{k}{\ldots} \oplus T^*Q$ is the Whitney sum of k copies of the cotangent bundle T^*Q, which is usually called the *k-cotangent bundle* or *bundle of* k^1*-covelocities* of Q. We use the following notation for the canonical projections:

$$\pi^A\colon (T^1_k)^*Q \to T^*Q \quad , \quad \pi^1_Q\colon (T^1_k)^*Q \to Q \quad ; \qquad (1 \leq A \leq k)\ ,$$

(here π^A is the canonical projection onto the A^{th}-copy T^*Q of $(T^1_k)^*Q$). So, if $q \in Q$ and $(\alpha^1_q, \ldots, \alpha^k_q) \in (T^1_k)^*Q$, we have

$$\pi^A(\alpha^1_q, \ldots, \alpha^k_q) = \alpha^A_q \quad , \quad \pi^1_Q(\alpha^1_q, \ldots, \alpha^k_q) = q \quad (1 \leq A \leq k)\ .$$

If (q^i), $1 \leq i \leq n$, are local coordinates on $U \subseteq Q$, the induced local coordinates (q^i, p_i^A) on $(\pi_Q^1)^{-1}(U) = (T_k^1)^*U$ are given by

$$q^i(\alpha_q^1, \dots, \alpha_q^k) = q^i(q) \quad , \quad p_i^A(\alpha_q^1, \dots, \alpha_q^k) = \alpha_q^A\left(\frac{\partial}{\partial q^i}\Big|_q\right) .$$

The canonical k-symplectic structure in $(T_k^1)^*Q$ is constructed as follows: we define the differential forms

$$\theta^A = (\pi^A)^*\theta \quad , \quad \omega^A = (\pi^A)^*\omega \quad ; \quad 1 \leq A \leq k \; , \tag{1}$$

where θ is the Liouville 1-form on T^*Q and $\omega = -\mathrm{d}\theta$ is the canonical symplectic form on T^*Q. Obviously $\omega^A = -d\theta^A$. In local coordinates we have

$$\theta^A = p_i^A \mathrm{d}q^i \quad , \quad \omega^A = \mathrm{d}q^i \wedge \mathrm{d}p_i^A \quad ; \quad 1 \leq A \leq k \; . \tag{2}$$

The canonical k-symplectic manifold is $((T_k^1)^*Q, \omega^A, V)$ where $V = \ker(\pi_Q^1)_*$.

Definition 4 *Let M be a differentiable manifold of dimension $k(n+1)+n$. A k–*cosymplectic structure *is a family $(\eta^A, \Omega^A, \mathcal{V})$ $(1 \leq A \leq k)$, where $\eta^A \in \Omega^1(M)$, $\Omega^A \in \Omega^2(M)$, and $\mathcal{V}$ is an nk-dimensional distribution on M, such that*

1. $\eta^1 \wedge \dots \wedge \eta^k \neq 0, \quad \eta^A|_{\mathcal{V}} = 0, \quad \Omega^A|_{\mathcal{V}\times\mathcal{V}} = 0.$
2. $(\cap_{A=1}^k \ker \eta^A) \cap (\cap_{A=1}^k \ker \Omega^A) = \{0\}, \quad dim(\cap_{A=1}^k \ker \Omega^A) = k.$
3. *The forms η^A and Ω^A are closed, and $\mathcal{V}$ is integrable.*

*Then, $(M, \eta^A, \Omega^A, \mathcal{V})$ is said to be a k–*cosymplectic manifold.

For every k-cosymplectic structure $(\eta^A, \Omega^A, \mathcal{V})$ on M, there exists a family of k vector fields $\{R_A\}_{1\leq A\leq k}$, which are called *Reeb vector fields*, characterized by the following conditions

$$i(R_A)\eta^B = \delta_A^B \quad , \quad i(R_A)\Omega^B = 0 \quad ; \quad 1 \leq A, B \leq k \; .$$

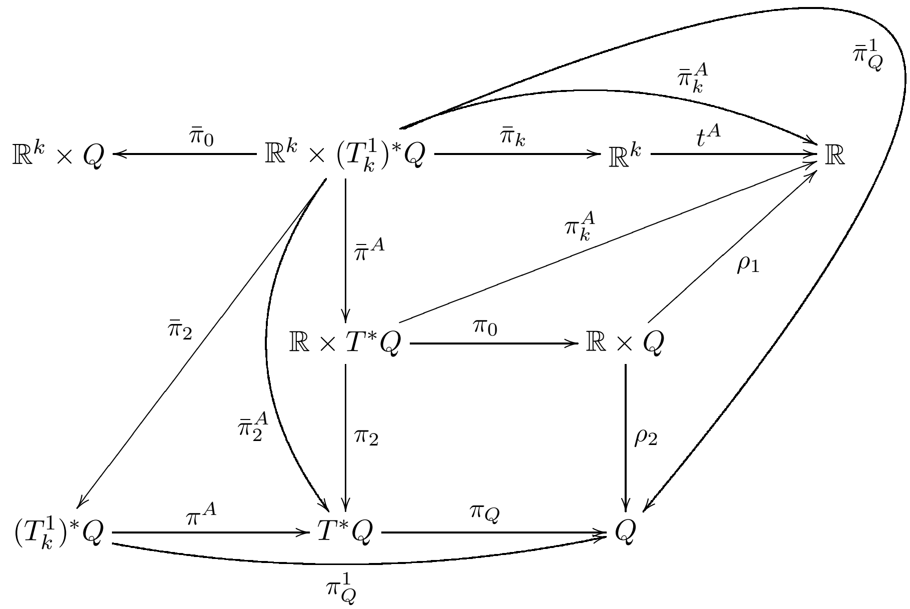
Theorem 2 (Darboux Theorem)*: If M is a k–cosymplectic manifold, then for every point of M there exists a local chart of coordinates (t^A, q^i, p_i^A), $1 \leq A \leq k$, $1 \leq i \leq n$, such that*

$$\eta^A = \mathrm{d}t^A, \quad \Omega^A = \mathrm{d}q^i \wedge \mathrm{d}p_i^A, \quad \mathcal{V} = \left\langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \right\rangle_{i=1,\dots,n} .$$

The canonical model for these geometrical structures is $(\mathbb{R}^k \times (T_k^1)^*Q, \eta^A, \Omega^A, \mathcal{V})$. If (t^A) are coordinates in $\mathbb{R}^k$, and (q^i) are local coordinates on $U \subset Q$, then the induced local coordinates (t^A, q^i, p_i^A) on $\mathbb{R}^k \times (T_k^1)^*U$ are given by

$$t^A(t, \alpha_q^1, \dots, \alpha_q^k) = t^A \quad , \quad q^i(t, \alpha_q^1, \dots, \alpha_q^k) = q^i(q) \quad , \quad p_i^A(t, \alpha_q^1, \dots, \alpha_q^k) = \alpha_q^A\left(\frac{\partial}{\partial q^i}\Big|_q\right) .$$

Considering the canonical projections (submersions), we have the commutative diagram:



(3)

In particular, if $t = (t^1, \ldots, t^k) \in \mathbb{R}^k$, $q \in Q$ and $(t, \alpha^1_q, \ldots, \alpha^k_q) \in \mathbb{R}^k \times (T^1_k)^*Q$, we have

$$\begin{array}{rclcrcl} \bar{\pi}_2(t, \alpha^1_q, \ldots, \alpha^k_q) & = & (\alpha^1_q, \ldots, \alpha^k_q) & , & \bar{\pi}^A_2(t, \alpha^1_q, \ldots, \alpha^k_q) & = & (\alpha^A_q) \\ \bar{\pi}^1_Q(t, \alpha^1_q, \ldots, \alpha^k_q) & = & q & , & \bar{\pi}^A_k(t, \alpha^1_q, \ldots, \alpha^k_q) & = & t^A \\ \bar{\pi}^k(t, \alpha^1_q, \ldots, \alpha^k_q) & = & t & , & \bar{\pi}^A(t, \alpha^1_q, \ldots, \alpha^k_q) & = & (t^A, \alpha^A_q) \end{array}$$

The canonical k-cosymplectic structure in $\mathbb{R}^k \times (T^1_k)^*Q$ is constructed as follows: we define the differential forms

$$\eta^A = (\bar{\pi}^A_k)^* \mathrm{d}t^A \quad , \quad \Theta^A = (\bar{\pi}^A_2)^*\theta \quad , \quad \Omega^A = (\bar{\pi}^A_2)^*\omega \quad ; \quad 1 \leq A \leq k \, . \tag{4}$$

Obviously $\Omega^A = -d\Theta^A$. In local coordinates we have

$$\eta^A = \mathrm{d}t^A \quad , \quad \Theta^A = p^A_i \mathrm{d}q^i \quad , \quad \Omega^A = \mathrm{d}q^i \wedge \mathrm{d}p^A_i \quad ; \quad 1 \leq A \leq k \tag{5}$$

The canonical k-cosymplectic manifold is $(\mathbb{R}^k \times (T^1_k)^*Q, \eta^A, \Omega^A, \mathcal{V})$ where $\mathcal{V} = \ker\,(\bar{\pi}_0)_*$, and locally $\mathcal{V} = \left\langle \dfrac{\partial}{\partial p^A_i} \right\rangle_{1 \leq A \leq k,\ 1 \leq i \leq n}$. Moreover, the Reeb vector fields are $R_A = \dfrac{\partial}{\partial t^A}$, $1 \leq A \leq k$, which are defined intrinsically in $\mathbb{R}^k \times (T^1_k)^*Q$ and span locally the vertical distribution with respect to the projection $\bar{\pi}_2$; i.e., the distribution generated by $\ker\,(\bar{\pi}_2)_*$.

Finally, taking into account (1), (4), and the commutativity of the diagram (3), we have that

$$\Theta^A = \bar{\pi}_2^* \theta^A \quad , \quad \Omega^A = \bar{\pi}_2^* \omega^A \quad ; \quad 1 \leq A \leq k \, . \tag{6}$$

Furthermore, the vector fields spanning the distributions $\mathcal{V}$ on $\mathbb{R}^k \times (T^1_k)^*Q$, and V on $(T^1_k)^*Q$ are also $\bar{\pi}_2$-related.

2.3 k-symplectic Hamiltonian systems

Consider the k-symplectic manifold $((T^1_k)^*Q, \omega^A, V)$, and let $H \in \mathrm{C}^\infty((T^1_k)^*Q)$ be a Hamiltonian function. $((T^1_k)^*Q, H)$ is called a *k-symplectic Hamiltonian system*. The *Hamilton-de Donder-Weyl equations* (HDW-equations for short) for this system are the set of partial diferential equations:

$$\frac{\partial H}{\partial q^i} = -\sum_{A=1}^{k} \frac{\partial \psi^A_i}{\partial t^A} \quad , \quad \frac{\partial H}{\partial p^A_i} = \frac{\partial \psi^i}{\partial t^A}, \quad ; \qquad 1 \leq i \leq n,\ 1 \leq\ A \leq k \, , \tag{7}$$

where $\psi\colon \mathbb{R}^k \to (T^1_k)^*Q$, $\psi(t) = (\psi^i(t), \psi^A_i(t))$, is a solution.

We denote by $\mathfrak{X}^k_H((T^1_k)^*Q)$ the set of k-vector fields $\mathbf{X} = (X_1, \ldots, X_k)$ on $(T^1_k)^*Q$ which are solutions to the equations

$$\sum_{A=1}^{k} i(X_A)\omega^A = \mathrm{d}H \, . \tag{8}$$

In a local system of canonical coordinates, each X_A is locally given by

$$X_A = (X_A)^i \frac{\partial}{\partial q^i} + (X_A)^B_i \frac{\partial}{\partial p^B_i} \quad , \qquad 1 \leq \; A \leq k \; , \tag{9}$$

then, using (2), we obtain that the equation (8) is equivalent to the equations

$$\frac{\partial H}{\partial q^i} = -\sum_{A=1}^{k} (X_A)^A_i \quad , \quad \frac{\partial H}{\partial p^A_i} = (X_A)^i \quad , \quad 1 \leq i \leq n \; . \tag{10}$$

The existence of k-vector fields that are solutions to (8) is assured, and in a local system of coordinates they depend on $n(k^2-1)$ arbitrary functions. Nevertheless, they are not necessarily integrable, and hence the integrability conditions imply that the number of arbitrary functions will in general be less than $n(k^2-1)$.

Proposition 1 *Let $\mathbf{X} = (X_1, \ldots, X_k)$ be an integrable k-vector field in $(T^1_k)^*Q$ and $\psi\colon \mathbb{R}^k \to (T^1_k)^*Q$ an integral section of $\mathbf{X}$. Then $\psi(t) = (\psi^i(t), \psi^A_i(t))$ is a solution to the HDW-equations (7) if, and only if, $\mathbf{X} \in \mathfrak{X}^k_H((T^1_k)^*Q)$.*

(*Proof*): If $\psi(t) = (\psi^i(t), \psi^A_i(t))$ is an integral section of $\mathbf{X}$, then

$$\frac{\partial \psi^i}{\partial t^B} = (X_B)^i \quad , \quad \frac{\partial \psi^A_i}{\partial t^B} = (X_B)^A_i \, . \tag{11}$$

and therefore (10) are the HDW-equations (7). ■

Remark: It is important to point out that the equations (7) and (8) are not equivalent, because there is no way to prove that every solution to the HDW-equations (7) is an integral section of some integrable k-vector field of $\mathfrak{X}^k_H((T^1_k)^*Q)$, unless some additional conditions are required. In particular, we could assume the following condition (which holds for a large class of mathematical applications and physical field theories):

Definition 5 *A map $\psi\colon \mathbb{R}^k \to (T^1_k)^*Q$, solution to the equations (7), is said to be an* admissible solution *to the HDW-equations for a k-symplectic Hamiltonian system $((T^1_k)^*Q, H)$, if $\operatorname{Im}\psi$ is a closed embedded submanifold of $(T^1_k)^*Q$.*

We say that $((T^1_k)^*Q, H)$ is an admissible k-symplectic Hamiltonian system *if all the solutions to its HDW-equations are admissible.*

Proposition 2 *Every admissible solution to the HDW-equations (7) is an integral section of an integrable k-vector field $\mathbf{X} \in \mathfrak{X}^k_H((T^1_k)^*Q)$.*

(*Proof*): Let $\psi\colon \mathbb{R}^k \to (T^1_k)^*Q$ be an admissible solution to the HDW-equations (7). By hypothesis, $\operatorname{Im}\psi$ is a k-dimensional closed submanifold of $(T^1_k)^*Q$. As ψ is an embedding, we can define a k-vector field $\mathbf{X}|_{\operatorname{Im}\psi}$ (at support on $\operatorname{Im}\psi$), and tangent to $\operatorname{Im}\psi$, by

$$X_A(\psi(t)) = (\psi)_*(t) \left(\frac{\partial}{\partial t^A}\Big|_t \right)$$

which is a solution to (8) on the points of $\mathrm{Im}\,\psi$, since (10) holds on these points as a consequence of (7) and (11). Furthermore, by hypothesis, $\mathrm{Im}\,\psi$ is a closed submanifold of $(T^1_k)^*Q$; therefore we can extend this k-vector field $\mathbf{X}|_{\mathrm{Im}\,\psi}$ to an integrable k-vector field $\mathbf{X}\in\mathfrak{X}^k_H((T^1_k)^*Q)$ in such a way that this extension is a solution to the equations (8) (remember that these equations have solutions everywhere on $(T^1_k)^*Q$), and which obviously has ψ as an integral section. This extension is made at least locally, and then the global k-vector field is constructed using partitions of unity. ■

In this way, for admissible k-symplectic Hamiltonian systems, the field equations (8) are a geometric version of the HDW-equations (7).

2.4 k-cosymplectic Hamiltonian systems

Consider the k-cosymplectic manifold $(\mathbb{R}^k\times(T^1_k)^*Q,\eta^A,\Omega^A,\mathcal{V})$, and let $\mathcal{H}\in\mathrm{C}^\infty(\mathbb{R}^k\times(T^1_k)^*Q)$ be a Hamiltonian function. $(\mathbb{R}^k\times(T^1_k)^*Q,\mathcal{H})$ is called a *k-cosymplectic Hamiltonian system*. The *HDW-equations* for this system are the set of partial diferential equations:

$$\frac{\partial\mathcal{H}}{\partial q^i}=-\sum_{A=1}^k\frac{\partial\bar{\psi}^A_i}{\partial t^A}\quad,\quad\frac{\partial\mathcal{H}}{\partial p^A_i}=\frac{\partial\bar{\psi}^i}{\partial t^A}\quad;\quad 1\le A\le k,\,1\le\ i\le n\,. \tag{12}$$

where the solutions $\bar{\psi}(t)=(t,\bar{\psi}^i(t),\bar{\psi}^A_i(t))$ are sections of the projection $\bar{\pi}_k\colon\mathbb{R}^k\times(T^1_k)^*Q\to\mathbb{R}^k$.

We denote by $\mathfrak{X}^k_{\mathcal{H}}(\mathbb{R}^k\times(T^1_k)^*Q)$ the set of k-vector fields $\bar{\mathbf{X}}=(\bar{X}_1,\dots,\bar{X}_k)$ on $\mathbb{R}^k\times(T^1_k)^*Q$ wich are solutions to the equations

$$\sum_{A=1}^k i(\bar{X}_A)\Omega^A=\mathrm{d}\mathcal{H}-\sum_{A=1}^k R_A(\mathcal{H})\eta^A\quad,\quad\eta^A(\bar{X}_B)=\delta^A_B\;;\quad 1\le A,B\le k\,. \tag{13}$$

Since $R_A=\partial/\partial t^A$ and $\eta^A=\mathrm{d}t^A$, then we can write locally the above equations as follows

$$\sum_{A=1}^k i(\bar{X}_A)\Omega^A=\mathrm{d}\mathcal{H}-\sum_{A=1}^k\frac{\partial\mathcal{H}}{\partial t^A}\mathrm{d}t^A\quad,\quad\mathrm{d}t^A(\bar{X}_B)=\delta^A_B\quad;\quad 1\le A,B\le k\,. \tag{14}$$

In a local system of coordinates, $\bar{X}_A$ are locally given by

$$\bar{X}_A=(\bar{X}_A)^B\frac{\partial}{\partial t^B}+(\bar{X}_A)^i\frac{\partial}{\partial q^i}+(\bar{X}_A)^B_i\frac{\partial}{\partial p^B_i}\,. \tag{15}$$

and, using (2), we obtain that the equations (13) are equivalent to the equations

$$\frac{\partial\mathcal{H}}{\partial p^A_i}=(\bar{X}_A)^i\quad,\quad\frac{\partial\mathcal{H}}{\partial q^i}=-\sum_{A=1}^k(\bar{X}_A)^A_i\quad,\quad(\bar{X}_A)^B=\delta^B_A\,, \tag{16}$$

The existence of k-vector fields that are solutions to (14) is assured, and in a local system of coordinates they depend on $n(k^2-1)$ arbitrary functions, but for integrable solutions the number of arbitrary functions is, in general, less than $n(k^2-1)$.

Proposition 3 *Let $\bar{\mathbf{X}}=(\bar{X}_1,\dots,\bar{X}_k)$ be an integrable k-vector field in $\mathbb{R}^k\times(T^1_k)^*Q$ and $\bar{\psi}\colon\mathbb{R}^k\to\mathbb{R}^k\times(T^1_k)^*Q$ an integral section of $\bar{\mathbf{X}}$. Then $\bar{\psi}(t)=(t,\bar{\psi}^i(t),\bar{\psi}^A_i(t))$ is a solution to the HDW-equations (12) if, and only if, $\bar{\mathbf{X}}\in\mathfrak{X}^k_{\mathcal{H}}(\mathbb{R}^k\times(T^1_k)^*Q)$.*

(*Proof*): If $\bar{\psi}(t) = (t, \bar{\psi}^i(t), \bar{\psi}^A_i(t))$ is an integral section of $\bar{\mathbf{X}}$, we have that

$$\frac{\partial \bar{\psi}^i}{\partial t^B} = (\bar{X}_B)^i \quad , \quad \frac{\partial \bar{\psi}^A_i}{\partial t^B} = (\bar{X}_B)^A_i \, , \tag{17}$$

and therefore we obtain that (16) are the HDW-equations (7). ■

Furthermore we have:

Proposition 4 *Every section $\bar{\psi} \colon \mathbb{R}^k \to \mathbb{R}^k \times (T^1_k)^* Q$ of the projection $\bar{\pi}_k$ that is a solution to the HDW-equations (12) is an integral section of an integrable k-vector field $\bar{\mathbf{X}} \in \mathfrak{X}^k_{\mathcal{H}}(\mathbb{R}^k \times (T^1_k)^* Q)$.*

(*Proof*): Let $\bar{\psi} \colon U_0 \subset \mathbb{R}^k \to \mathbb{R}^k \times (T^1_k)^* Q$ be a section of the projection $\bar{\pi}_k$ that is a solution to the HDW-equations (12). We have that $\bar{\psi}$ is an injective immersion and $\mathrm{Im}\,\bar{\psi}$ is a closed submanifold of $\mathbb{R}^k \times (T^1_k)^* Q$, since $\mathrm{Im}\,\bar{\psi} = \mathrm{graph}\,\psi$, for $\psi = \bar{\pi}_2 \circ \bar{\psi} \colon \mathbb{R}^k \to (T^1_k)^* Q$. Then the construction of the integrable k-vector field in $\mathbb{R}^k \times (T^1_k)^* Q$, which has $\bar{\psi}$ as integral section and is a solution to (13), follows the same pattern as in proposition 2. ■

So the equations (13) are a geometric version of the HDW-equations(12).

2.5 Autonomous k-cosymplectic Hamiltonian systems

Following a terminology analogous to that in mechanics, we define:

Definition 6 *A k-cosymplectic Hamiltonian system $(\mathbb{R}^k \times (T^1_k)^* Q, \mathcal{H})$ is said to be* autonomous *if* $\mathrm{L}(R_A)\mathcal{H} = \dfrac{\partial \mathcal{H}}{\partial t^A} = 0$, *for* $1 \leq A \leq k$.

Observe that the condition in definition 6 means that $\mathcal{H}$ does not depend on the variables t^A, and thus $\mathcal{H} = \bar{\pi}_2^* H$ for some $H \in \mathrm{C}^\infty((T^1_k)^* Q)$.

For an autonomous k-cosymplectic Hamiltonian system, the equations (13) become

$$\sum_{A=1}^k i(\bar{X}_A)\Omega^A = \mathrm{d}\mathcal{H} \quad , \quad \eta^A(\bar{X}_B) = \delta^A_B \; ; \quad 1 \leq A, B \leq k \, . \tag{18}$$

Therefore:

Proposition 5 *Every autonomous k-cosymplectic Hamiltonian system $(\mathbb{R}^k \times (T^1_k)^* Q, \mathcal{H})$ defines a k-symplectic Hamiltonian system $((T^1_k)^* Q, H)$, where $\mathcal{H} = \bar{\pi}_2^* H$, and conversely.*

We have the following result for solutions to the Hamilton-de Donder-Weyl equations:

Theorem 3 *Let $(\mathbb{R}^k \times (T^1_k)^* Q, \mathcal{H})$ be an autonomous k-cosymplectic Hamiltonian system and let $((T^1_k)^* Q, H)$ be its associated k-symplectic Hamiltonian system . Then, every section $\bar{\psi} \colon \mathbb{R}^k \to \mathbb{R}^k \times (T^1_k)^* Q$, that is, a solution to the HDW-equations (12) for the system $(\mathbb{R}^k \times (T^1_k)^* Q, \mathcal{H})$ defines a map $\psi \colon \mathbb{R}^k \to (T^1_k)^* Q$ that is a solution to the HDW-equations (7) for the system $((T^1_k)^* Q, H)$; and conversely.*

(*Proof*): Since $\mathcal{H} = \bar{\pi}_2^* H$ we have

$$\frac{\partial \mathcal{H}}{\partial q^i} = \frac{\partial H}{\partial q^i} \quad , \quad \frac{\partial \mathcal{H}}{\partial p_i^A} = \frac{\partial H}{\partial p_i^A} \, . \tag{19}$$

Let $\bar{\psi} \colon \mathbb{R}^k \to \mathbb{R}^k \times (T_k^1)^* Q$ be a section of the projection $\bar{\pi}_k$, which in coordinates is expressed as $\bar{\psi}(t) = (t, \bar{\psi}^i(t), \bar{\psi}_i^A(t))$. Then we construct the map $\psi = \bar{\pi}_2 \circ \bar{\psi} \colon \mathbb{R}^k \to (T_k^1)^* Q$, which in coordinates is expressed as $\psi(t) = (\psi^i(t), \psi_i^A(t)) = (\bar{\psi}^i(t), \bar{\psi}_i^A(t))$. Then, if $\bar{\psi}$ is a solution to the HDW-equations (12), from (19) we obtain that ψ is a solution to the HDW-equations (7).

Conversely, consider a map $\psi \colon \mathbb{R}^k \to (T_k^1)^* Q$. We define $\bar{\psi} = (Id_{\mathbb{R}^k}, \psi) : \mathbb{R}^k \to \mathbb{R}^k \times (T_k^1)^* Q$. Furthermore, if $\psi(t) = (\psi^i(t), \psi_i^A(t))$, then $\bar{\psi}(t) = (t, \bar{\psi}^i(t), \bar{\psi}_i^A(t))$, with $\bar{\psi}^i(t) = \psi^i(t)$ and $\bar{\psi}_i^A(t) = \psi_i^A(t)$ (observe that, in fact, $\mathrm{Im}\,\bar{\psi} = \mathrm{graph}\,\psi$). Hence, if ψ is a solution to the HDW-equations (7), from (19) we obtain that $\bar{\psi}$ is a solution to the HDW-equations (12). ■

For k-vector fields that are solutions to the geometric field equations (8) and (18) we have:

Proposition 6 *Let $(\mathbb{R}^k \times (T_k^1)^* Q, \mathcal{H})$ be an autonomous k-cosymplectic Hamiltonian system and let $((T_k^1)^* Q, H)$ be its associated k-symplectic Hamiltonian system. Then every k-vector field $\mathbf{X} \in \mathfrak{X}_H^k (T_k^1)^* Q)$ defines a k-vector field $\bar{\mathbf{X}} \in \mathfrak{X}_{\mathcal{H}}^k (\mathbb{R}^k \times (T_k^1)^* Q)$.*

Furthermore, $\mathbf{X}$ is integrable if, and only if, its associated $\bar{\mathbf{X}}$ is integrable too.

(*Proof*): Let $\mathbf{X} = (X_1, \ldots, X_k) \in \mathfrak{X}_H^k ((T_k^1)^* Q)$. For every $A = 1, \ldots, k$, let $\bar{X}_A \in \mathfrak{X}(\mathbb{R}^k \times (T_k^1)^* Q)$ be the *suspension* of the corresponding vector field $X^A \in \mathfrak{X}((T_k^1)^* Q)$, which is defined as follows (see [1], p. 374, for this construction in mechanics): for every $\mathrm{p} \in (T_k^1)^* Q$, let $\gamma_{\mathrm{p}}^A \colon \mathbb{R} \to (T_k^1)^* Q$ be the integral curve of X_A passing through p; then, if $t_0 = (t_0^1, \ldots, t_0^k) \in \mathbb{R}^k$, we can construct the curve $\bar{\gamma}_{\bar{\mathrm{p}}}^A \colon \mathbb{R} \to \mathbb{R}^k \times (T_k^1)^* Q$, passing through the point $\bar{\mathrm{p}} \equiv (t_0, \mathrm{p}) \in \mathbb{R}^k \times (T_k^1)^* Q$, given by $\bar{\gamma}_{\bar{\mathrm{p}}}^A (t^A) = (t_0^1, \ldots, t^A + t_0^A, \ldots, t_0^k; \gamma_{\mathrm{p}}(t^A))$. Therefore, $\bar{X}_A$ is the vector field tangent to $\bar{\gamma}_{\bar{\mathrm{p}}}^A$ at (t_0, p). In natural coordinates, if X_A is locally given by (9), then $\bar{X}_A$ is locally given by

$$\bar{X}_A = \frac{\partial}{\partial t^A} + (\bar{X}_A)^i \frac{\partial}{\partial q^i} + (\bar{X}_A)_i^B \frac{\partial}{\partial p_i^B} = \frac{\partial}{\partial t^A} + \bar{\pi}_2^* (X_A)^i \frac{\partial}{\partial q^i} + \bar{\pi}_2^* (X_A)_i^B \frac{\partial}{\partial p_i^B} \, .$$

Observe that $\bar{X}_A$ are $\bar{\pi}_2$-projectable vector fields, and $(\bar{\pi}_2)_* \bar{X}_A = X_A$. In this way we have defined a k-vector field $\bar{\mathbf{X}} = (\bar{X}_1, \ldots, \bar{X}_k)$ in $\mathbb{R}^k \times (T_k^1)^* Q$. Therefore, taking (6) into account,

$$\sum_{A=1}^k i(\bar{X}_A) \Omega^A - \mathrm{d}\mathcal{H} = \sum_{A=1}^k i(\bar{X}_A) \bar{\pi}_2^* \omega^A - \mathrm{d}(\bar{\pi}_2^* H) = \bar{\pi}_2^* (\sum_{A=1}^k i((\pi_2)_* \bar{X}_A) \omega^A - \mathrm{d}H) = 0 \, ,$$

since $\mathbf{X} = (X_1, \ldots, X_k) \in \mathfrak{X}_H^k (T_k^1)^* Q)$, and therefore $\bar{X} = (\bar{X}_1, \ldots, \bar{X}_k) \in \mathfrak{X}_{\mathcal{H}}^k (\mathbb{R}^k \times (T_k^1)^* Q)$.

Furthermore, if $\psi \colon \mathbb{R}^k \to (T_k^1)^* Q$ is an integral section of $\mathbf{X}$, then $\bar{\psi} \colon \mathbb{R}^k \to \mathbb{R}^k \times (T_k^1)^* Q$ such that $\bar{\psi} = (Id_{\mathbb{R}^k}, \psi)$ (see Theorem 3) is an integral section of $\bar{\mathbf{X}}$.

Now, if $\bar{\psi}$ is an integral section of $\bar{\mathbf{X}}$, the equations (17) hold for $\bar{\psi}(t) = (t, \bar{\psi}^i(t), \bar{\psi}_i^A(t))$ and, as $(\bar{X}_A)^i = \bar{\pi}_2^* (X_A)^i$ and $(\bar{X}_A)_i^B = \bar{\pi}_2^* (X_A)_i^B$, this is equivalent to saying that the equations (11) hold for $\psi(t) = (\psi^i(t), \psi_i^A(t))$; that is, ψ is an integral section of $\mathbf{X}$. ■

Remark: The converse statement is not true. In fact, the k-vector fields that are solution to the geometric field equations (18) are not completely determined, as the equations (16) show, and then there are k-vector fields in $\mathfrak{X}_{\mathcal{H}}^k (\mathbb{R}^k \times (T_k^1)^* Q)$ that are not $\bar{\pi}_2$-projectable (in fact, it suffices to take their undetermined component functions to be not $\bar{\pi}_2$-projectable). However, we have the following particular result:

Proposition 7 *Let $((T^1_k)^*Q, H)$ be an admissible k-symplectic Hamiltonian system, and $(\mathbb{R}^k \times (T^1_k)^*Q, \mathcal{H})$ its associated autonomous k-cosymplectic Hamiltonian system. Then, every integrable k-vector field $\bar{\mathbf{X}} \in \mathfrak{X}^k_{\mathcal{H}}(\mathbb{R}^k \times (T^1_k)^*Q)$ defines an integrable k-vector field $\mathbf{X} \in \mathfrak{X}^k_H((T^1_k)^*Q)$.*

(*Proof*): If $\bar{\mathbf{X}} \in \mathfrak{X}^k_{\mathcal{H}}(\mathbb{R}^k \times (T^1_k)^*Q)$ is an integrable k-vector field, denote by $\bar{\mathcal{S}}$ the set of its integral sections (i.e., solutions to the the HDW-equations (12)). Let $\mathcal{S}$ be the set of maps $\psi\colon \mathbb{R}^k \to (T^1_k)^*Q$ associated with these sections by Theorem 3, which are admissible solutions to the HDW-equations (7), by the hypothesis that $((T^1_k)^*Q, \omega^A, H)$ is an admissible k-symplectic Hamiltonian system. Then, by proposition 2 we can construct an integrable k-vector field $\mathbf{X} \in \mathfrak{X}^k_H((T^1_k)^*Q)$ for which $\mathcal{S}$ is its set of integral sections (which are admissible solutions to the HDW-equations (7)). ■

3 k-symplectic and k-cosymplectic Lagrangian formalisms

(See [25, 29] for details on the construction of this formalism).

3.1 Canonical structures in the bundles T^1_kQ and $\mathbb{R}^k \times T^1_kQ$

Consider the bundle $\tau^1_Q\colon T^1_kQ \to Q$ (see Section 2.1). If (q^i) are local coordinates on $U \subseteq Q$ then the induced local coordinates (q^i, v^i), $1 \leq i \leq n$, in $TU = (\tau^1_Q)^{-1}(U)$ are given by $q^i(v_q) = q^i(q)$, $v^i(v_q) = v_q(q^i)$, and the induced local coordinates (q^i, v^i_A), $1 \leq i \leq n$, $1 \leq A \leq k$, in $T^1_kU = (\tau^1_Q)^{-1}(U)$ are given by

$$q^i(v_{1q}, \dots, v_{kq}) = q^i(q), \qquad v^i_A(v_{1q}, \dots, v_{kq}) = v_{Aq}(q^i)\,.$$

For a vector $Z_q \in T_qQ$, and for $A = 1, \dots, k$, we define its *vertical A-lift*, $(Z_q)^{V_A}$, at the point $(v_{1q}, \dots, v_{kq}) \in T^1_kQ$, as the vector tangent to the fiber $(\tau^1_Q)^{-1}(q) \subset T^1_kQ$, which is given by

$$(Z_q)^{V_A}(v_{1q}, \dots, v_A) = \frac{d}{ds}(v_{1q}, \dots, v_{A-1q}, v_{A_q} + sZ_q, v_{A+1q}, \dots, v_{kq})|_{s=0}\,.$$

In local coordinates, if $X_q = a^i \left.\dfrac{\partial}{\partial q^i}\right|_q$, we have $(Z_q)^{V_A}(v_{1q}, \dots, v_{kq}) = a^i \left.\dfrac{\partial}{\partial v^i_A}\right|_{(v_{1q}, \dots, v_{kq})}$. Then, the *canonical k-tangent structure* on T^1_kQ is the set $(S^1, \dots, S^k)$ of tensor fields of type $(1,1)$ defined by

$$S^A(w_q)(Z_{w_q}) = ((\tau^1_Q)_*(w_q)(Z_{w_q}))^{V_A}(w_q) \quad , \quad \text{for } w_q \in T^1_kQ,\ Z_{w_q} \in T_{w_q}(T^1_kQ);\ A = 1, \dots, k\,.$$

In local coordinates we have

$$S^A = \frac{\partial}{\partial v^i_A} \otimes \mathrm{d}q^i\,. \tag{20}$$

The *Liouville vector field* $\Delta \in \mathfrak{X}(T^1_kQ)$ is the infinitesimal generator of the following flow

$$\psi\colon \mathbb{R} \times T^1_kQ \longrightarrow T^1_kQ \quad , \quad \psi(s, v_{1_q}, \dots, v_{k_q}) = (e^s v_{1_q}, \dots, e^s v_{k_q})\,,$$

and in local coordinates it has the form

$$\Delta = \sum_{A=1}^k v^i_A \frac{\partial}{\partial v^i_A}\,.$$

Now, consider the manifold $J^1\pi_{\mathbb{R}^k}$ of 1-jets of sections of the trivial bundle $\pi_{\mathbb{R}^k}\colon \mathbb{R}^k\times Q \to \mathbb{R}^k$, which is diffeomorphic to $\mathbb{R}^k \times T^1_kQ$, via the diffeomorphism given by

$$\begin{array}{rcl} J^1\pi_{\mathbb{R}^k} & \to & \mathbb{R}^k \times T^1_kQ \\ j^1_t\phi = j^1_t(Id_{\mathbb{R}^k}, \phi_Q) & \to & (t, v_1, \dots, v_k) \ , \end{array} \tag{21}$$

where $\phi_Q\colon \mathbb{R}^k \stackrel{\phi}{\to} \mathbb{R}^k \times Q \stackrel{\pi_Q}{\to} Q$, and $v_A = (\phi_Q)_*(t)(\dfrac{\partial}{\partial t^A}\Big|_t)$, for $1 \leq A \leq k$. We denote by $\bar\tau^1_Q\colon \mathbb{R}^k \times T^1_kQ \to Q$ the canonical projection. If (q^i) are local coordinates on $U \subseteq Q$, then the induced local coordinates (t^A, q^i, v^i_A) on $(\bar\tau^1_Q)^{-1}(U) = \mathbb{R}^k \times T^1_kU$ are

$$t^A(t, v_{1q}, \dots, v_{kq}) = t^A; \quad q^i(t, v_{1q}, \dots, v_{kq}) = q^i(q); \quad v^i_A(t, v_{1q}, \dots, v_{kq}) = v_{Aq}(q^i)\,.$$

We consider the extension of S^A to $\mathbb{R}^k \times T^1_kQ$, which we denote by $\bar S^A$, and they have the same local expressions (20). Finally, we introduce the *Liouville vector field* $\bar\Delta \in \mathfrak{X}(\mathbb{R}^k \times T^1_kQ)$, which is the infinitesimal generator of the following flow

$$\begin{array}{rcl} \mathbb{R} \times (\mathbb{R}^k \times T^1_kQ) & \longrightarrow & \mathbb{R}^k \times T^1_kQ \\ (s, (t, v_{1q}, \dots, v_{kq})) & \longrightarrow & (t, e^s v_{1q}, \dots, e^s v_{kq})\,, \end{array}$$

and in local coordinates it has the form

$$\bar\Delta = \sum_{i,A} v^i_A \frac{\partial}{\partial v^i_A}\,, \tag{22}$$

3.2 k-symplectic Lagrangian formalism

Let $L \in \mathrm{C}^\infty(T^1_kQ)$ be a Lagrangian function.

A family of forms $\theta^A_L \in \Omega^1(T^1_kQ)$, $1 \leq A \leq k$, is introduced by using the k-tangent structure of T^1_kQ, as follows

$$\theta^A_L = \mathrm{d}L \circ S^A \quad 1 \leq A \leq k \quad ,$$

and hence we define $\omega^A_L = -\mathrm{d}\theta^A_L$. In coordinates

$$\theta^A_L = \frac{\partial L}{\partial v^i_A}\mathrm{d}q^i \quad , \quad \omega^A_L = \mathrm{d}q^i \wedge \mathrm{d}\left(\frac{\partial L}{\partial v^i_A}\right) = \frac{\partial^2 L}{\partial q^j \partial v^i_A}\mathrm{d}q^i \wedge \mathrm{d}q^j + \frac{\partial^2 L}{\partial v^j_B \partial v^i_A}\mathrm{d}q^i \wedge \mathrm{d}v^j_B\ .$$

We can also define the *Energy Lagrangian function* associated to L, $E_L \in \mathrm{C}^\infty(T^1_kQ)$, as $E_L = \Delta(L) - L$. Its local expression is

$$E_L = v^i_A \frac{\partial L}{\partial v^i_A} - L\ .$$

Finally, the Legendre map $FL\colon T^1_kQ \longrightarrow (T^1_k)^*Q$ was introduced by Günther [13], and we rewrite it as follows: if $(v_{1_q}, \dots, v_{k_q}) \in (T^1_k)_qQ$

$$[FL(v_{1_q}, \dots, v_{k_q})]^A(w_q) = \frac{d}{ds}L(v_{1_q}, \dots, v_{A_q} + s w_q, \dots, v_{k_q})|_{s=0},$$

for each $A = 1, \dots, k$. We have that FL is locally given by

$$(q^i, v^i_A) \longrightarrow \left(q^i, \frac{\partial L}{\partial v^i_A}\right). \tag{23}$$

Furthermore, from (2) and (23) we obtain that

$$\theta_L^A = FL^*\theta^A \quad , \quad \omega_L^A = FL^*\omega^A \tag{24}$$

The Lagrangian L is said to be *regular* if $(\frac{\partial^2 L}{\partial v_A^i \partial v_B^j})$ is a non-singular matrix at every point of $T_k^1 Q$. Then, from (23) and (24) we get:

Proposition 8 *Let $L \in \mathrm{C}^\infty(T_k^1 Q)$ be a Lagrangian. The following conditions are equivalent:*

1) L is regular. 2) FL is a local diffeomorphism. 3) $(T_k^1 Q, \omega_L^A, V)$, where $V = Ker(\tau_Q^1)_*$, is a k-symplectic manifold. ■

A Lagrangian function L is said to be *hyperregular* if the corresponding Legendre map FL is a global diffeomorphism. If L is regular, $(T_k^1 Q, L)$ is said to be a *k-symplectic Lagrangian system*. If L is not regular $(T_k^1 Q, L)$ is a *k-presymplectic Lagrangian system*.

The *Euler-Lagrange equations* for L are:

$$\sum_{A=1}^k \frac{\partial}{\partial t^A}\Big|_t \left(\frac{\partial L}{\partial v_A^i}\Big|_{\varphi(t)} \right) = \frac{\partial L}{\partial q^i}\Big|_{\varphi(t)} \quad , \quad v_A^i(\varphi(t)) = \frac{\partial \varphi^i}{\partial t^A} \quad , \quad 1 \le i \le n,\ 1 \le A \le k \tag{25}$$

whose solutions are maps $\varphi\colon \mathbb{R}^k \to T_k^1 Q$ that, as a consequence of the last group of equations (25), are first prolongations to $T_k^1 Q$ of maps $\phi = \tau_Q^1 \circ \varphi\colon \mathbb{R}^k \to Q$; that is, φ are *holonomic*. This means that $\varphi = \phi^{(1)}$ where

$$\begin{array}{rccl} \phi^{(1)}: & \mathbb{R}^k & \to & T_k^1 Q \\ & t & \mapsto & \phi^{(1)}(t) = (\phi_*(t)(\frac{\partial}{\partial t^1}\Big|_t), \ldots, \phi_*(t)(\frac{\partial}{\partial t^k}\Big|_t)) \end{array} .$$

Let $\mathfrak{X}_L^k(T_k^1 Q)$ be the set of k-vector fields $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_k)$ in $T_k^1 Q$, wich are solutions to

$$\sum_{A=1}^k i(\Gamma_A)\omega_L^A = \mathrm{d}E_L \,. \tag{26}$$

If $\Gamma_A = (\Gamma_A)^i \dfrac{\partial}{\partial q^i} + (\Gamma_A)_B^i \dfrac{\partial}{\partial v_B^i}$ locally, then $\mathbf{\Gamma}$ is a solution to (26) if, and only if, $(\Gamma_A)^i$ and $(\Gamma_A)_B^i$ satisfy

$$\begin{aligned} \left(\frac{\partial^2 L}{\partial q^i \partial v_A^j} + \frac{\partial^2 L}{\partial q^j \partial v_A^i} \right) (\Gamma_A)^j - \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} (\Gamma_A)_B^j &= v_A^j \frac{\partial^2 L}{\partial q^i \partial v_A^j} - \frac{\partial L}{\partial q^i} \\ \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} (\Gamma_A)^i &= \frac{\partial^2 L}{\partial v_B^j \partial v_A^i} v_A^i \,. \end{aligned}$$

If the Lagrangian is regular, the above equations are equivalent to

$$\frac{\partial^2 L}{\partial q^j \partial v_A^i} v_A^j + \frac{\partial^2 L}{\partial v_A^i \partial v_B^j} (\Gamma_A)_B^j = \frac{\partial L}{\partial q^i} \quad , \quad (\Gamma_A)^i = v_A^i \,.$$

The last group of these equations is the local expression of the condition that $\mathbf{\Gamma}$ is a SOPDE (see [25]), and hence, if it is integrable, its integral sections are first prolongations $\phi^{(1)}\colon \mathbb{R}^k \to T_k^1 Q$ of maps $\phi\colon \mathbb{R}^k \to Q$, and using the first group of equations, we deduce that $\phi^{(1)}$ are solutions to the Euler-Lagrange equations (25). If L is not regular then, in general, the equations (25) or (26) have no solutions anywhere in T^1_kQ, but they do in a submanifold S of T^1_kQ (in the most favourable situations). Moreover, solutions to (26) are not SOPDE necessarily.

We define *admissible solutions* to the Euler-Lagrange equations and *admissible k-symplectic Lagrangian systems* in the same way as in the Hamiltonian case (definition 5). Then the statement of Proposition 2 can be proved analogously for these admissible solutions. This proof holds for regular k-symplectic Lagrangian systems, and for the non-regular case the proof is still valid considering the submanifold S of $(T^1_k)^*Q$ where the Lagrangian field equations have solutions.

3.3 k-cosymplectic Lagrangian formalism and autonomous k-cosymplectic Lagrangian systems

Let $\mathcal{L} \in \mathrm{C}^\infty(\mathbb{R}^k \times T^1_kQ)$ be a Lagrangian .

A family of forms $\Theta^A_{\mathcal{L}} \in \Omega^1(\mathbb{R}^k \times T^1_kQ)$, $1 \leq A \leq k$, is introduced by using the k-tangent structure of $\mathbb{R}^k \times T^1_kQ$, as follows

$$\Theta^A_{\mathcal{L}} = \mathrm{d}\mathcal{L} \circ \bar{S}^A \quad 1 \leq A \leq k \quad ,$$

and hence we define $\Omega^A_{\mathcal{L}} = -\mathrm{d}\Theta^A_{\mathcal{L}}$. In coordinates

$$\Theta^A_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial v^i_A}\mathrm{d}q^i \quad , \quad \Omega^A_{\mathcal{L}} = \frac{\partial^2 \mathcal{L}}{\partial q^j \partial v^i_A}\mathrm{d}q^i \wedge \mathrm{d}q^j + \frac{\partial^2 \mathcal{L}}{\partial v^j_B \partial v^i_A}\mathrm{d}q^i \wedge \mathrm{d}v^j_B + \frac{\partial^2 \mathcal{L}}{\partial t^B \partial v^i_A}\mathrm{d}q^i \wedge \mathrm{d}t^B \ . \tag{27}$$

We can also define the *Energy Lagrangian function* associated to $\mathcal{L}$, $\mathcal{E}_{\mathcal{L}} \in \mathrm{C}^\infty(\mathbb{R}^k \times T^1_kQ)$ as $\mathcal{E}_{\mathcal{L}} = \bar{\Delta}(\mathcal{L}) - \mathcal{L}$, whose local expression is

$$\mathcal{E}_{\mathcal{L}} = v^i_A \frac{\partial \mathcal{L}}{\partial v^i_A} - \mathcal{L} \ .$$

Finally, the Legendre map $F\mathcal{L} \colon \mathbb{R}^k \times T^1_kQ \longrightarrow \mathbb{R}^k \times (T^1_k)^*Q$, is defined as follows:

$$F\mathcal{L}(t, v_{1q}, \dots, v_{kq}) = (t, \dots, [F\mathcal{L}(t, v_{1q}, \dots, v_{kq})]^A, \dots)$$

where

$$[F\mathcal{L}(t, v_{1q}, \dots, v_{kq})]^A(w_q) = \frac{d}{ds}\mathcal{L}\left(t, v_{1q}, \dots, v_{Aq} + sw_q, \dots, v_{kq}\right)|_{s=0},$$

for each $A = 1, \dots, k$; and it is locally given by

$$F\mathcal{L} : (t^A, q^i, v^i_A) \longrightarrow \left(t^A, q^i, \frac{\partial \mathcal{L}}{\partial v^i_A}\right) . \tag{28}$$

It is obvious that

$$\Theta^A_{\mathcal{L}} = F\mathcal{L}^*\Theta^A \, , \quad \Omega^A_{\mathcal{L}} = F\mathcal{L}^*\Omega^A, \quad 1 \leq A \leq k \quad . \tag{29}$$

Observe that $F\mathcal{L} = \mathrm{Id}_{\mathbb{R}^k} \times FL \colon \mathbb{R}^k \times T^1_kQ \to \mathbb{R}^k \times (T^1_k)^*Q$, (see (6), (24) and (29)).

The Lagrangian $\mathcal{L} = \mathcal{L}(t^B, q^j, v^j_B)$ is *regular* if the matrix $(\frac{\partial^2 \mathcal{L}}{\partial v^i_A \partial v^j_B})$ is not singular at every point of $\mathbb{R}^k \times T^1_kQ$. Then, from (5), (28) and (29) we deduce the following proposition (See [22]):

Proposition 9 *Let $\mathcal{L} \in \mathrm{C}^\infty(\mathbb{R}^k \times T^1_kQ)$ be a Lagrangian. The following conditions are equivalent:*

1) $\mathcal{L}$ is regular. 2) $F\mathcal{L}$ is a local diffeomorphism. 3) $(\mathbb{R}^k \times T^1_kQ, \mathrm{d}t^A, \Omega^A_{\mathcal{L}}, \mathcal{V})$, where $\mathcal{V} = \ker(\bar{\tau}_0)_*$, is a k-cosymplectic manifold.

■

A Lagrangian function $\mathcal{L}$ is said to be *hyperregular* if the corresponding Legendre map $F\mathcal{L}$ is a global diffeomorphism. If $\mathcal{L}$ is regular, $(\mathbb{R}^k \times T^1_kQ, \mathcal{L})$ is said to be a *k-cosymplectic Lagrangian system*. If $\mathcal{L}$ is not regular, $(\mathbb{R}^k \times T^1_kQ, \mathcal{L})$ is a *k-precosymplectic Lagrangian system*.

The *Euler-Lagrange equations* are (25), but now the Lagrangian is $\mathcal{L} = \mathcal{L}(t^B, q^j, v^j_B)$, and their solutions are sections $\bar\varphi\colon \mathbb{R}^k \to \mathbb{R}^k \times T^1_kQ$ of the natural projection $\mathbb{R}^k \times T^1_kQ \to \mathbb{R}^k$, which are first prolongations to $\mathbb{R}^k \times T^1_kQ$ of sections $\phi\colon \mathbb{R}^k \to Q$ of the natural projection $\mathbb{R}^k \times Q \to \mathbb{R}^k$; that is, $\bar\varphi$ are *holonomic*. This means that $\bar\varphi = \phi^{[1]}$ where

$$\begin{array}{rcl} \phi^{[1]} : \mathbb{R}^k & \longrightarrow & \mathbb{R}^k \times T^1_kQ \\ t & \longrightarrow & \phi^{[1]}(t) = \left(t, \phi_*(t)(\dfrac{\partial}{\partial t^1}), \ldots, \phi_*(t)(\dfrac{\partial}{\partial t^k})\right) \end{array}$$

Furthermore, we denote by $\mathfrak{X}^k_{\mathcal{L}}(\mathbb{R}^k \times T^1_kQ)$ the set of k-vector fields $\bar{\mathbf{\Gamma}} = (\bar\Gamma_1, \ldots, \bar\Gamma_k)$ in $\mathbb{R}^k \times T^1_kQ$, that are solutions to the equations

$$\sum_{A=1}^k i(\bar\Gamma_A)\Omega^A_{\mathcal{L}} = \mathrm{d}\mathcal{E}_{\mathcal{L}} - \sum_{A=1}^k \frac{\partial \mathcal{L}}{\partial t^A}\mathrm{d}t^A \quad , \quad \mathrm{d}t^A(\bar\Gamma_B) = \delta^A_B \quad ; \quad 1 \leq A, B \leq k \; . \tag{30}$$

In a local system of natural coordinates, if

$$\bar\Gamma_A = (\bar\Gamma_A)^B \frac{\partial}{\partial t^B} + (\bar\Gamma_A)^i \frac{\partial}{\partial q^i} + (\bar\Gamma_A)^i_B \frac{\partial}{\partial v^i_B} \tag{31}$$

then $\bar{\mathbf{\Gamma}}$ is a solution to (30) if, and only if, $(\bar\Gamma_A)^i$ and $(\bar\Gamma_A)^i_B$ satisfy

$$\begin{gathered} (\bar\Gamma_A)^B = \delta^B_A \quad , \quad (\bar\Gamma_A)^i \frac{\partial^2 L}{\partial t^B \partial v^i_A} = v^i_A \frac{\partial^2 \mathcal{L}}{\partial t^B \partial v^i_A} \quad , \quad (\bar\Gamma_A)^i \frac{\partial^2 \mathcal{L}}{\partial v^j_B \partial v^i_A} = v^i_A \frac{\partial^2 \mathcal{L}}{\partial v^j_B \partial v^i_A} \\ \frac{\partial^2 \mathcal{L}}{\partial q^j \partial v^i_A}\left(v^i_A - (\bar\Gamma_A)^i\right) + \frac{\partial^2 \mathcal{L}}{\partial t^A \partial v^i_A} + v^k_A \frac{\partial^2 \mathcal{L}}{\partial q^k \partial v^i_A} + (\bar\Gamma_A)^k_B \frac{\partial^2 \mathcal{L}}{\partial v^k_B \partial v^i_A} = \frac{\partial \mathcal{L}}{\partial q^i} \end{gathered} \tag{32}$$

When $\mathcal{L}$ is regular, we obtain that $(\bar\Gamma_A)^i = v^i_A$, and the last equation can be written as follows

$$\frac{\partial^2 \mathcal{L}}{\partial t^A \partial v^i_A} + v^k_A \frac{\partial^2 \mathcal{L}}{\partial q^k \partial v^i_A} + (\bar\Gamma_A)^k_B \frac{\partial^2 \mathcal{L}}{\partial v^k_B \partial v^i_A} = \frac{\partial \mathcal{L}}{\partial q^i} \, , \tag{33}$$

then $\bar{\mathbf{\Gamma}}$ is a SOPDE (see [22]), and hence, if it is integrable, its integral sections are holonomic and they are solutions to the Euler-Lagrange equations for $\mathcal{L}$. If $\mathcal{L}$ is not regular, the existence of solutions to the equations (25) for $\mathcal{L}$ or to (30) is not assured, in general, except in a submanifold of T^1_kQ (in the most favourable situations). Moreover, solutions to (30) are not SOPDE necessarily.

Definition 7 *A k-cosymplectic (or k-precosymplectic) Lagrangian system is said to be* autonomous *if* $\dfrac{\partial \mathcal{L}}{\partial t^A} = 0$ *or, what is equivalent,* $\dfrac{\partial \mathcal{E}_{\mathcal{L}}}{\partial t^A} = 0$, $1 \leq A \leq k$.

Now, all the results obtained in Section 2.5 can be stated and proved in the same way, considering the systems $(\mathbb{R}^k \times T^1_kQ, \mathcal{L})$ and (T^1_kQ, L) instead of $(\mathbb{R}^k \times (T^1_k)^*Q, \mathcal{H})$ and $((T^1_k)^*Q, H)$.

Finally, the k-symplectic and k-cosymplectic Lagrangian and Hamiltonian systems are related by means of the *Legendre maps* FL and $F\mathcal{L}$.

4 Multisymplectic Hamiltonian formalism

4.1 Multisymplectic manifolds and multimomentum bundles

(See, for instance, [9]).

Definition 8 *The couple* $(\mathcal{M},\Omega)$*, with* $\Omega\in\Omega^{k+1}(\mathcal{M})$ *(*$2\leq k+1\leq \dim\mathcal{M}$*), is a* multisymplectic manifold *if* Ω *is closed and* 1*-nondegenerate; that is, for every* $p\in\mathcal{M}$*, and* $X_p\in T_p\mathcal{M}$*, we have that* $i(X_p)\Omega_p=0$ *if, and only if,* $X_p=0$*.*

A very important example of multisymplectic manifold is the *multicotangent bundle* $\Lambda^k\mathrm{T}^*Q$ of a manifold Q, which is the bundle of k-forms in Q, and is endowed with a canonical multisymplectic $(k+1)$-form. Other examples of multisymplectic manifolds which are relevant in field theory are the so-called *multimomentum bundles*: let $\pi\colon E\to M$ be a fiber bundle, ($\dim M=k$, $\dim E=n+k$), where M is an oriented manifold with volume form $\omega\in\Omega^k(M)$, and denote by (t^A,q^i) ($1\leq A\leq k$, $1\leq n$) the natural coordinates in E adapted to the bundle, such that $\omega=\mathrm{d}t^1\wedge\ldots\wedge\mathrm{d}t^k\equiv\mathrm{d}^kt$. First we have $\Lambda^k_2T^*E\equiv\mathcal{M}\pi$, which is the bundle of k-forms on E vanishing by the action of two π-vertical vector fields. This is called the *extended multimomentum bundle*, and its canonical submersions are denoted by

$$\kappa\colon\mathcal{M}\pi\to E\quad;\quad\bar\kappa=\pi\circ\kappa\colon\mathcal{M}\pi\to M$$

We can introduce natural coordinates in $\mathcal{M}\pi$ adapted to the bundle $\pi\colon E\to M$, which are denoted by (t^A,q^i,p^A_i,p), and such that $\omega=\mathrm{d}^kt$. Then, denoting $\mathrm{d}^{k-1}t^A=i\left(\dfrac{\partial}{\partial t^A}\right)\mathrm{d}^kt$, the elements of $\mathcal{M}\pi$ can be written as $p_i^A\,\mathrm{d}q^i\wedge d^{k-1}t_A\,+\,p\,\mathrm{d}^kt$.

$\mathcal{M}\pi$ is a subbundle of Λ^kT^*E, and hence $\mathcal{M}\pi$ is also endowed with canonical forms. First we have the "tautological form" $\Theta\in\Omega^k(\mathcal{M}\pi)$, which is defined as follows: let $(x,\alpha)\in\Lambda^k_2T^*E$, with $x\in E$ and $\alpha\in\Lambda^k_2T^*_xE$; then, for every $X_1,\ldots,X_m\in T_{(x,\alpha)}(\mathcal{M}\pi)$, we have

$$\Theta((x,\alpha))(X_1,\ldots,X_m):=\alpha(x)(T_{(x,\alpha)}\kappa(X_1),\ldots,T_{(x,\alpha)}\kappa(X_m))\tag{34}$$

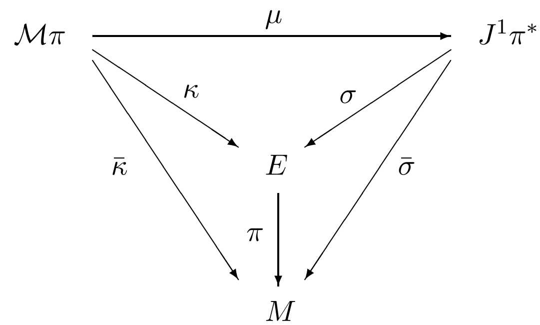
Thus we define the multisymplectic form

$$\Omega:=-\mathrm{d}\Theta\in\Omega^{k+1}(\mathcal{M}\pi)\tag{35}$$

and the local expressions of the above forms are

$$\Theta=p_i^A\mathrm{d}q^i\wedge\mathrm{d}^{k-1}t_A+p\,\mathrm{d}^kt\ ,\ \Omega=-\mathrm{d}p_i^A\wedge\mathrm{d}q^i\wedge\mathrm{d}^{k-1}t_A-\mathrm{d}p\wedge\mathrm{d}^kt\tag{36}$$

Consider $\pi^*\Lambda^kT^*M$, which is another bundle over E, whose sections are the π-semibasic k-forms on E, and denote by $J^1\pi^*$ the quotient $\Lambda^k_2T^*E/\pi^*\Lambda^kT^*M$. $J^1\pi^*$ is usually called the *restricted multimomentum bundle* associated with the bundle $\pi\colon E\to M$. Natural coordinates in $J^1\pi^*$ (adapted to the bundle $\pi\colon E\to M$) are denoted by (t^A,q^i,p^A_i). We have the natural submersions specified in the following diagram



4.2 Multisymplectic Hamiltonian formalism

The Hamiltonian formalism in $J^1\pi^*$ presented here is based on the construction made in [5] (see also [6] and [9]).

Definition 9 *A section* $h\colon J^1\pi^* \to \mathcal{M}\pi$ *of the projection* μ *is called a Hamiltonian section. The differentiable forms* $\Theta_h := h^*\Theta$ *and* $\Omega_h := -\mathrm{d}\Theta_h = h^*\Omega$ *are called the* Hamilton-Cartan k and $(k+1)$ forms *of* $J^1\pi^*$ *associated with the Hamiltonian section* h*.* $(J^1\pi^*, h)$ *is said to be a Hamiltonian system in* $J^1\pi^*$*.*

In natural coordinates we have that $h(t^A, q^i, p_i^A) = (t^A, q^i, p_i^A, p = -\mathcal{H}(t^A, q^i, p_i^A))$, and $\mathcal{H} \in C^\infty(U)$, $U \subset J^1\pi^*$, is a *local Hamiltonian function*. Then we have

$$\Theta_h = p_i^A \mathrm{d}q^i \wedge \mathrm{d}^{k-1}t_A - \mathcal{H}\mathrm{d}^k t \ , \ \Omega_h = -\mathrm{d}p_i^A \wedge \mathrm{d}q^i \wedge \mathrm{d}^{k-1}t_A + \mathrm{d}\mathcal{H} \wedge \mathrm{d}^k t \ .$$

The field equations for these multisymplectic Hamiltonian systems can be stated as

$$\psi^* \, i(X)\Omega_h = 0 \quad , \quad \text{for every } X \in \mathfrak{X}(J^1\pi^*) \ , \tag{37}$$

where $\bar\psi\colon M \to J^1\pi^*$ are sections of the projection $\bar\sigma$ that are solutions to these equations. In natural coordinates, writing $\bar\psi(t) = (t, \bar\psi^i(t), \bar\psi_i^A(t))$, we have that this equation is equivalent to the *Hamilton-de Donder-Weyl equations* for the multisymplectic Hamiltonian system $(J^1\pi^*, h)$

$$\frac{\partial \mathcal{H}}{\partial q^i} = -\sum_{A=1}^{k} \frac{\partial \bar\psi_i^A}{\partial t^A} \quad , \quad \frac{\partial \mathcal{H}}{\partial p_i^A} = \frac{\partial \bar\psi^i}{\partial t^A} \quad ; \quad 1 \leq A \leq k, \, 1 \leq \ i \leq n \, . \tag{38}$$

We denote by $\mathfrak{X}_h^k(J^1\pi^*)$ the set of k-vector fields $\bar{\mathbf{X}} = (\bar X_1, \ldots, \bar X_k)$ in $J^1\pi^*$ which are solution to the equations

$$i(\bar{\mathbf{X}})\Omega_h = i(\bar X_1) \ldots i(\bar X_k)\Omega_h = 0 \quad , \quad i(\bar{\mathbf{X}})\omega = i(\bar X_1) \ldots i(\bar X_k)\omega = 1 \ , \tag{39}$$

(we denote by $\omega = \mathrm{d}^k t$ the volume form in M and its pull-backs to all the manifolds. The contraction of k-vector fields and forms is the usual one between tensorial objects).

In a system of natural coordinates, the components of $\bar{\mathbf{X}}$ are given by (15), then $i(\bar{\mathbf{X}})\omega = 1$ leads to $(\bar X_A)^B = 1$, for every $A, B = 1, \ldots, k$, and hence the other equation (39) gives

$$\frac{\partial \mathcal{H}}{\partial q^i} = -\sum_{A=1}^{k} (\bar X_A)_i^A \quad , \quad \frac{\partial \mathcal{H}}{\partial p_i^A} = (\bar X_A)^i \ . \tag{40}$$

The existence of k-vector fields that are solutions to (39) is assured, and in a local system of coordinates they depend on $n(k^2-1)$ arbitrary functions, but the number of arbitrary functions for integrable solutions is, in general, less than $n(k^2-1)$.

Proposition 10 *Let* $\bar{\mathbf{X}} = (\bar X_1, \ldots, \bar X_k)$ *be an integrable* k*-vector field in* $J^1\pi^*$ *and* $\bar\psi\colon M \to J^1\pi^*$ *an integral section of* $\bar{\mathbf{X}}$*. Then* $\bar\psi(t) = (t, \bar\psi^i(t), \bar\psi_i^A(t))$ *is a solution to the equations (38), and hence to (37), if, and only if,* $\bar{\mathbf{X}} \in \mathfrak{X}_h^k(J^1\pi^*)$*.*

(*Proof*): If $\bar\psi(t) = (t, \bar\psi^i(t), \bar\psi_i^A(t))$ is an integral section of $\bar{\mathbf{X}}$, we have that

$$\frac{\partial \bar\psi^i}{\partial t^B} = (\bar X_B)^i \quad , \quad \frac{\partial \bar\psi_i^A}{\partial t^B} = (\bar X_B)_i^A \, , \tag{41}$$

and therefore we obtain that (40) are the HDW-equations (38). ∎

4.3 Relation with the k-cosymplectic Hamiltonian formalism

In order to compare the multisymplectic and the k-cosymplectic formalisms of field theory, from now on we consider the case when $\pi\colon E \to M$ is the trivial bundle $\mathbb{R}^k \times Q \to \mathbb{R}^k$. Then we can establish relations among the canonical multisymplectic form on $\mathcal{M}\pi \equiv \Lambda^k_2 T^*(\mathbb{R}^k \times Q)$, the canonical k-symplectic structure on $(T^1_k)^*Q$, and the canonical k-cosymplectic structure on $\mathbb{R}^k \times (T^1_k)^*Q$ (see also [20]). First recall that in $M = \mathbb{R}^k$ we have the canonical volume form $\omega = \mathrm{d}t^1 \wedge \ldots \wedge \mathrm{d}t^k \equiv \mathrm{d}^k t$. Then:

Proposition 11 *1. $\mathcal{M}\pi \equiv \Lambda^k_2 T^*(\mathbb{R}^k \times Q)$ is diffeomorphic to $\mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q$.*

2. $J^1\pi^*$ is diffeomorphic to $\mathbb{R}^k \times (T^1_k)^*Q$.

(*Proof*):

1. Consider the canonical embedding $\imath_t\colon Q \hookrightarrow \mathbb{R}^k \times Q$ given by $i_t(q) = (t, q)$, and the canonical submersion $\rho_2\colon \mathbb{R}^k \times Q \to Q$. We can define the map
$$\begin{array}{rccc} \bar\Psi\colon & \Lambda^k_2 T^*(\mathbb{R}^k \times Q) & \longrightarrow & \mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q \\ & \alpha_{(t,q)} & \mapsto & (t, p, \alpha^1_q, \ldots, \alpha^k_q) \end{array}$$
where
$$\begin{aligned} p &= \alpha_{(t,q)}\left(\frac{\partial}{\partial t^1}\Big|_{(t,q)}, \ldots, \frac{\partial}{\partial t^k}\Big|_{(t,q)}\right) \\ \alpha^A_q(X) &= \alpha_{(t,q)}\left(\frac{\partial}{\partial t^1}\Big|_{(t,q)}, \ldots, \frac{\partial}{\partial t^{A-1}}\Big|_{(t,q)}, (\imath_t)_*X, \frac{\partial}{\partial t^{A+1}}\Big|_{(t,q)}, \ldots, \frac{\partial}{\partial t^k}\Big|_{(t,q)}\right) \quad , \quad X \in \mathfrak{X}(Q) \end{aligned}$$
(note that t^A and p are now global coordinates in the corresponding fibres). The inverse of $\bar\Psi$ is given by
$$\alpha_{(t,q)} = p\,\mathrm{d}^k t|_{(t,x)} + (\rho_2)^*_{(t,q)}\alpha^A_q \wedge \mathrm{d}^{k-1}t_A|_{(t,q)} \ .$$
Thus, $\bar\Psi$ is a diffeomorphism. Locally $\bar\Psi$ is written as the identity.

2. It is a straighforward consequence of the above item because
$$J^1\pi^* = \Lambda^k_2 T^*E/\pi^*\Lambda^k T^*M \simeq \mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q/\mathbb{R} \simeq \mathbb{R}^k \times (T^1_k)^*Q$$

■

Next, using a procedure analogous to that in the above proof, we can give the

Relationship between the canonical geometric structures in $\mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q$ and in $(T^1_k)^*Q$.

Let $\jmath\colon (T^1_k)^*Q \hookrightarrow \mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q$ be the natural embedding of $(T^1_k)^*Q$ into $\mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q$ as the zero-section of the bundle $\mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q \to (T^1_k)^*Q$. Starting from the canonical forms Θ and Ω in $\mathcal{M}\pi \simeq \mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q$ we can define the forms θ^A on $(T^1_k)^*Q$, $1 \leq A \leq k$, by
$$\begin{aligned} \theta^A(X) &= \jmath^*\left[\Theta\left(\frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^{A-1}}, \jmath_*X, \frac{\partial}{\partial t^{A+1}}, \ldots, \frac{\partial}{\partial t^k}\right)\right] \\ &= -\left(\jmath^*\left[i\left(\frac{\partial}{\partial t^k}\right)\ldots i\left(\frac{\partial}{\partial t^1}\right)(\Theta \wedge \mathrm{d}t^A)\right]\right)(X) \quad , \quad X \in \mathfrak{X}((T^1_k)^*Q) \ . \end{aligned}$$

Then for $X,Y\in\mathfrak{X}((T^1_k)^*Q)$, we get the 2-forms ω^A on $(T^1_k)^*Q$ given as

$$\begin{aligned}\omega^A(X,Y) &= -\mathrm{d}\theta^A(X,Y)=\jmath^*\left[\Omega\left(\jmath_*X,\frac{\partial}{\partial t^1},\ldots,\frac{\partial}{\partial t^{A-1}},\jmath_*Y,\frac{\partial}{\partial t^{A+1}},\ldots,\frac{\partial}{\partial t^k}\right)\right]\\ &= (-1)^{k+1}\left(\jmath^*\left[i\left(\frac{\partial}{\partial t^k}\right)\ldots i\left(\frac{\partial}{\partial t^1}\right)(\Omega\wedge\mathrm{d}t^A)\right]\right)(X,Y)\,. \end{aligned}\tag{42}$$

From (36) we obtain the local expressions

$$\theta^A=p^A_i\mathrm{d}q^i\quad,\quad\omega^A=\mathrm{d}q^i\wedge\mathrm{d}p^A_i\,.$$

Furthermore, we have the involutive distribution $V=\ker(\pi^1_Q)_*$, and hence $(\omega^A,V;1\leq A\leq k)$ is the canonical k-symplectic structure in $(T^1_k)^*Q$.

Conversely, starting from this k-symplectic structure in $(T^1_k)^*Q$ we can obtain the canonical forms in $\mathcal{M}\pi\simeq\mathbb{R}^k\times\mathbb{R}\times(T^1_k)^*Q$, by doing

$$\Theta=p\mathrm{d}^kt+\sigma_2^*\theta^A\wedge\mathrm{d}^{k-1}t_A\quad,\quad\Omega=-\mathrm{d}\Theta=-\mathrm{d}p\wedge\mathrm{d}^kt+\sigma_2^*\omega^A\wedge\mathrm{d}^{k-1}t_A\tag{43}$$

where $\sigma_2\colon\mathbb{R}^k\times\mathbb{R}\times(T^1_k)^*Q\to(T^1_k)^*Q$ is the canonical submersion.

Summarizing, we have proved that:

Theorem 4 *The canonical multisymplectic form on $\mathcal{M}\pi\simeq\mathbb{R}^k\times\mathbb{R}\times(T^1_k)^*Q$ and the 2-forms of the canonical k-symplectic structure on $(T^1_k)^*Q$ are related by (42), and (43).*

Relationship between the canonical geometric structures in $\mathbb{R}^k\times\mathbb{R}\times(T^1_k)^*Q$ and in $\mathbb{R}^k\times(T^1_k)^*Q$.

In an analogous way, we can also relate the canonical geometric structures in $\mathbb{R}^k\times\mathbb{R}\times(T^1_k)^*Q$ and in $\mathbb{R}^k\times(T^1_k)^*Q$. In fact, denoting by $\mathfrak{i}\colon\mathbb{R}^k\times(T^1_k)^*Q\hookrightarrow\mathbb{R}^k\times\mathbb{R}\times(T^1_k)^*Q$ the natural embedding of $\mathbb{R}^k\times(T^1_k)^*Q$ into $\mathbb{R}^k\times\mathbb{R}\times(T^1_k)^*Q$ as the zero-section of the bundle $\mu\colon\mathbb{R}^k\times\mathbb{R}\times(T^1_k)^*Q\to\mathbb{R}^k\times(T^1_k)^*Q$; then from the canonical forms Θ and Ω in $\mathcal{M}\pi\simeq\mathbb{R}^k\times\mathbb{R}\times(T^1_k)^*Q$ we can define the forms Θ^A on $\mathbb{R}^k\times(T^1_k)^*Q$ as follows: for $\bar X\in\mathfrak{X}(\mathbb{R}^k\times(T^1_k)^*Q)$, and $1\leq A\leq k$,

$$\begin{aligned}\Theta^A(\bar X) &= \mathfrak{i}^*\left[\Theta\left(\frac{\partial}{\partial t^1},\ldots,\frac{\partial}{\partial t^{A-1}},\mathfrak{i}_*\bar X,\frac{\partial}{\partial t^{A+1}},\ldots,\frac{\partial}{\partial t^k}\right)\right]\\ &= -\left(\mathfrak{i}^*\left[i\left(\frac{\partial}{\partial t^k}\right)\ldots i\left(\frac{\partial}{\partial t^1}\right)(\Theta\wedge\mathrm{d}t^A)\right]\right)(\bar X)\end{aligned}$$

Then, for $\bar X,\bar Y\in\mathfrak{X}(\mathbb{R}^k\times(T^1_k)^*Q)$, we obtain the 2-forms Ω^A on $\mathbb{R}^k\times(T^1_k)^*Q$,

$$\begin{aligned}\Omega^A(\bar X,\bar Y) &= -\mathrm{d}\Theta^A(\bar X,\bar Y)=\mathfrak{i}^*\left[\Omega\left(\mathfrak{i}_*\bar X,\frac{\partial}{\partial t^1},\ldots,\frac{\partial}{\partial t^{A-1}},\mathfrak{i}_*\bar Y,\frac{\partial}{\partial t^{A+1}},\ldots,\frac{\partial}{\partial t^k}\right)\right]\\ &= (-1)^{k+1}\left(\mathfrak{i}^*\left[i\Big(\frac{\partial}{\partial t^k}\Big)\ldots i\Big(\frac{\partial}{\partial t^1}\Big)(\Omega\wedge\mathrm{d}t^A)\right]\right)(\bar X,\bar Y)\,.\end{aligned}\tag{44}$$

(These forms have the same coordinate expressions as θ^A and ω^A). Furthermore, although the 1-forms η^A are canonically defined on $\mathbb{R}^k\times(T^1_k)^*Q$, we can recover them from the multisymplectic form Ω as follows: for $\bar X\in\mathfrak{X}(\mathbb{R}^k\times(T^1_k)^*Q)$,

$$\eta^A(\bar X)=(-1)^{k-A}\mathfrak{i}^*\left[\Omega\left(\frac{\partial}{\partial p},\frac{\partial}{\partial t^1},\ldots,\frac{\partial}{\partial t^{A-1}},\mathfrak{i}_*\bar X,\frac{\partial}{\partial t^{A+1}},\ldots,\frac{\partial}{\partial t^k}\right)\right]\,.\tag{45}$$

whose coordinate expressions are $\eta^A = \mathrm{d}t^A$. These forms can also be defined by introducing the canonical embedding

$$\begin{array}{rccc} \jmath_0\colon & \mathbb{R}^k \times (T^1_k)^*Q & \hookrightarrow & \mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q \\ & (t, \alpha^1_x, \ldots, \alpha^k_x) & \to & (t, 1, 0_x, \ldots, 0_x) \end{array}$$

and then making

$$\eta^A(\bar{X}) = \jmath_0^*\left[\Theta\left(\frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^{A-1}}, (\jmath_0)_*\bar{X}, \frac{\partial}{\partial t^{A+1}}, \ldots, \frac{\partial}{\partial t^k}\right)\right] \quad , \quad \bar{X} \in \mathfrak{X}(\mathbb{R}^k \times (T^1_k)^*Q) \ . \tag{46}$$

Furthermore, we have the involutive distribution $\mathcal{V} = \ker(\bar{\pi}_2)_* = \left\langle \dfrac{\partial}{\partial t^A} \right\rangle$, and hence $(\eta^A, \Omega^A, \mathcal{V}; 1 \leq A \leq k)$ is the canonical k-cosymplectic structure in $\mathbb{R}^k \times (T^1_k)^*Q$.

Conversely, starting from this k-cosymplectic structure in $\mathbb{R}^k \times (T^1_k)^*Q$ we can obtain the canonical forms in $\mathcal{M}\pi \simeq \mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q$, by doing

$$\Theta = p\mathrm{d}^k t + \bar{\sigma}_2^*\Theta^A \wedge \mathrm{d}^{k-1}t_A \quad , \quad \Omega = -\mathrm{d}\Theta = -\mathrm{d}p \wedge \mathrm{d}^k t + \bar{\sigma}_2^*\Omega^A \wedge \mathrm{d}^{k-1}t_A \tag{47}$$

where $\bar{\sigma}_2\colon \mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q \to (T^1_k)^*Q$ is the canonical submersion.

Summarizing, we have proved that:

Theorem 5 *The canonical multisymplectic form on $\mathcal{M}\pi \simeq \mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q$ and the* 1 *and* 2*-forms of the canonical k-cosymplectic structure on $\mathbb{R}^k \times (T^1_k)^*Q$ are related by (44), (45) (or (46)), and (47).*

Relationship between the canonical geometric structures in $J^1\pi^* \simeq \mathbb{R}^k \times (T^1_k)^*Q$.

It is important to point out that, as the bundle $\mu\colon \mathcal{M}\pi \simeq \mathbb{R}^k \times \mathbb{R} \times (T^1_k)^*Q \to J^1\pi^* \simeq \mathbb{R}^k \times (T^1_k)^*Q$ is trivial, then Hamiltonian sections can be taken to be global sections of the projection μ by giving a global Hamiltonian function $\mathrm{H} \in C^\infty(\mathbb{R}^k \times (T^1_k)^*Q)$. Then we can also relate the non-canonical multisymplectic form with the k-cosymplectic structure in $\mathbb{R}^k \times (T^1_k)^*Q$ as follows: starting from the forms Θ_h and Ω_h in $\mathbb{R}^k \times (T^1_k)^*Q$, we can define the forms Θ^A and Ω^A on $\mathbb{R}^k \times (T^1_k)^*Q$ as follows: for $\bar{X}, \bar{Y} \in \mathfrak{X}(\mathbb{R}^k \times (T^1_k)^*Q)$, and $1 \leq A \leq k$,

$$\begin{aligned} \Theta^A(\bar{X}) &= -\left(i\left(\frac{\partial}{\partial t^k}\right)\ldots i\left(\frac{\partial}{\partial t^1}\right)(\Theta_h \wedge \mathrm{d}t^A)\right)(\bar{X}) \\ \Omega^A(\bar{X}, \bar{Y}) &= -\mathrm{d}\Theta^A(\bar{X}, \bar{Y}) = (-1)^{k+1}\left(i\left(\frac{\partial}{\partial t^k}\right)\ldots i\left(\frac{\partial}{\partial t^1}\right)(\Omega_h \wedge \mathrm{d}t^A)\right)(\bar{X}, \bar{Y}) \ , \end{aligned} \tag{48}$$

and the 1-forms $\eta^A = \mathrm{d}t^A$ are canonically defined.

Conversely, starting from the canonical k-cosymplectic structure on $\mathbb{R}^k \times (T^1_k)^*Q$, and from $\mathcal{H}$, we can construct

$$\Theta_h = -\mathcal{H}\mathrm{d}^k t + \Theta^A \wedge \mathrm{d}^{k-1}t_A \quad , \quad \Omega = -\mathrm{d}\Theta = \mathrm{d}\mathcal{H} \wedge \mathrm{d}^k t + \Omega^A \wedge \mathrm{d}^{k-1}t_A \tag{49}$$

So we have:

Theorem 6 *The multisymplectic form and the* 2*-forms of the canonical k-cosymplectic structure on $J^1\pi^* \simeq \mathbb{R}^k \times (T^1_k)^*Q$ are related by (48) and (49).*

Finally, the following result about the solutions to the Hamiltonian equations establishes the complete equivalence between both formalisms:

Theorem 7 *A k-vector field $\bar{\mathbf{X}} = (\bar{X}_1, \ldots, \bar{X}_k)$ in $J^1\pi^* \simeq \mathbb{R}^k \times (T^1_k)^*Q$ is a solution to the equations (13) if, and only if, it is also a solution to the equations (39); that is, $\mathfrak{X}^k_h(\mathbb{R}^k \times (T^1_k)^*Q) = \mathfrak{X}^k_H(\mathbb{R}^k \times (T^1_k)^*Q)$.*

(*Proof*): The proof is immediate, bearing in mind that in natural coordinates the solutions to the equations (13) and (39) are partially determined by the equations (16) and (40) respectively, and these are equivalent. ■

5 Multisymplectic Lagrangian formalism

5.1 Multisymplectic Lagrangian systems

(For details, see [7] and the references quoted therein). Consider the first-order jet bundle $\pi_E\colon J^1\pi \to E$, which is also a bundle over M with projection $\bar{\pi}\colon J^1\pi \longrightarrow M$, and is endowed with natural coordinates (t^A, q^i, v^i_A), adapted to the bundle structure. A *Lagrangian density* is a $\bar{\pi}$-semibasic k-form on $J^1\pi$, and hence it can be expressed as $\mathbb{L} = \mathcal{L}\omega$, where $\mathcal{L} \in C^\infty(J^1\pi)$ is the *Lagrangian function* associated with $\mathbb{L}$ and ω. Using the canonical structures of $J^1\pi$, we can define the *Poincaré-Cartan k and $(k+1)$-forms*, which have the following local expressions:

$$\begin{aligned}
\Theta_{\mathbb{L}} &= \frac{\partial\mathcal{L}}{\partial v^i_A}\mathrm{d}q^i \wedge \mathrm{d}^{k-1}t_A - \left(\frac{\partial\mathcal{L}}{\partial v^i_A}v^i_A - \mathcal{L}\right)\mathrm{d}^k t \\
\Omega_{\mathbb{L}} &= -\frac{\partial^2\mathcal{L}}{\partial v^j_B \partial v^i_A}\mathrm{d}v^j_B \wedge \mathrm{d}q^i \wedge \mathrm{d}^{k-1}t_A - \frac{\partial^2\mathcal{L}}{\partial q^j \partial v^i_A}\mathrm{d}q^j \wedge \mathrm{d}q^i \wedge \mathrm{d}^{k-1}t_A + \\
&\quad \frac{\partial^2\mathcal{L}}{\partial v^j_B \partial v^i_A}v^i_A \mathrm{d}v^j_B \wedge \mathrm{d}^k t + \left(\frac{\partial^2\mathcal{L}}{\partial q^i \partial v^j_B}v^j_B - \frac{\partial\mathcal{L}}{\partial q^j} + \frac{\partial^2\mathcal{L}}{\partial t^A \partial v^i_A}\right)\mathrm{d}q^i \wedge \mathrm{d}^k t
\end{aligned}$$

$(J^1\pi, \mathbb{L})$ is said to be a Lagrangian system. The Lagrangian system and the Lagrangian function are *regular* if $\Omega_{\mathbb{L}}$ is a multisymplectic $(k+1)$-form. Elsewhere they are *singular* (or *non-regular*), and $\Omega_{\mathbb{L}}$ is a pre-multisymplectic form. The regularity condition is locally equivalent to $det(\frac{\partial^2\mathcal{L}}{\partial v^A_\alpha \partial v^B_\nu}) \neq 0$, at every point in $J^1\pi$.

The Lagrangian field equations can be stated as

$$(\phi^1)^*\, i(X)\Omega_{\mathbb{L}} = 0 \quad , \quad \text{for every } X \in \mathfrak{X}(J^1\pi) \ ,$$

where $\phi\colon M \to E$ are sections of the projection π, and $\phi^1\colon M \to J^1\pi$ are their canonical liftings, which are solutions to these equations. In natural coordinates, writing $\phi(t) = (t, \phi^i(t))$, we have that this equation is equivalent to the *Euler-Lagrange equations* (25) for the Lagrangian $\mathcal{L}$. Furthermore, we denote by $\mathfrak{X}^k_{\mathbb{L}}(J^1\pi)$ the set of k-vector fields $\bar{\mathbf{\Gamma}} = (\bar{\Gamma}_1, \ldots, \bar{\Gamma}_k)$ in $J^1\pi$, that are solutions to the equations

$$i(\bar{\mathbf{\Gamma}})\Omega_{\mathbb{L}} = 0 \quad , \quad i(\bar{\mathbf{\Gamma}})\omega = 1 \tag{50}$$

In a system of natural coordinates the components of $\bar{\mathbf{\Gamma}}$ are given by (31), then $\bar{\mathbf{\Gamma}}$ is a solution to (50) if, and only if, $(\bar{\Gamma}_A)^B = 1$, for every $A, B = 1, \ldots, k$, and $(\bar{\Gamma}_A)^i$ and $(\bar{\Gamma}_A)^i_B$ satisfy the equations (32). When $\mathcal{L}$ is regular, we obtain that $(\bar{\Gamma}_A)^i = v^i_A$, and the equations (33 hold;

then $\bar{\mathbf{\Gamma}}$ is a SOPDE, and hence, if it is integrable, its integral sections are holonomic and they are solutions to the Euler-Lagrange equations for $\mathcal{L}$. If $\mathcal{L}$ is not regular, the existence of solutions to the equations (25) for $\mathcal{L}$ or to (50) is not assured, in general, except in a submanifold of $J^1\pi$ (in the most favourable situations). Moreover, solutions to (50) are not SOPDE necessarily.

Finally, $\Theta_{\mathbb{L}} \in \Omega^1(J^1\pi)$ being π_E-semibasic, we have a natural map $\widetilde{F\mathcal{L}}\colon J^1\pi \to \mathcal{M}\pi$, given by

$$\widetilde{F\mathcal{L}}(\bar{y}) = \Theta_{\mathbb{L}}(\bar{y}) \quad ; \quad \bar{y} \in J^1\pi$$

which is called the *extended Legendre map* associated to the Lagrangian $\mathcal{L}$. The *restricted Legendre map* is $F\mathcal{L} = \mu \circ \widetilde{F\mathcal{L}}\colon J^1\pi \to J^1\pi^*$. Their local expressions are

$$\begin{aligned} \widetilde{F\mathcal{L}} &: (t^A, q^i, v^i_A) \mapsto \left(t^A, q^i, \frac{\partial \mathcal{L}}{\partial v^i_A}, \mathcal{L} - v^i_A \frac{\partial \mathcal{L}}{\partial v^i_A}\right) \\ F\mathcal{L} &: (t^A, q^i, v^i_A) \mapsto \left(t^A, q^i, \frac{\partial \mathcal{L}}{\partial v^i_A}\right) \end{aligned} \tag{51}$$

Moreover, we have $\widetilde{F\mathcal{L}}^*\Theta = \Theta_{\mathbb{L}}$, and $\widetilde{F\mathcal{L}}^*\Omega = \Omega_{\mathbb{L}}$. Observe that the Legendre transformations $F\mathcal{L}$ defined for the k-cosymplectic and the multisymplectic formalisms are the same, as their local expressions (28) and (51) show.

5.2 Relation between multisymplectic and k-cosymplectic Lagrangian systems

In the particular case $E = \mathbb{R}^k \times Q$, we have $J^1\pi \simeq \mathbb{R}^k \times T^1_kQ$ and we can define the Energy Lagrangian function $\mathcal{E}_{\mathcal{L}}$ as

$$\mathcal{E}_{\mathcal{L}} = \Theta_{\mathbb{L}}\left(\frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^k}\right)$$

whose local expression is $\mathcal{E}_{\mathcal{L}} = v^i_A \dfrac{\partial \mathcal{L}}{\partial v^i_A} - \mathcal{L}$. Then we can write

$$\Theta_{\mathbb{L}} = \frac{\partial \mathcal{L}}{\partial v^i_A} \mathrm{d}q^i \wedge \mathrm{d}^{k-1}t_A - \mathcal{E}_{\mathcal{L}} \mathrm{d}^k t \quad , \quad \Omega_{\mathbb{L}} = -\mathrm{d}\left(\frac{\partial \mathcal{L}}{\partial v^i_A}\right) \wedge \mathrm{d}q^i \wedge \mathrm{d}^{k-1}t_A + \mathrm{d}\mathcal{E}_{\mathcal{L}} \wedge \mathrm{d}^k t$$

In this particular case, as in the Hamiltonian case, we can relate the non-canonical Lagrangian multisymplectic (or pre-multisymplectic) form $\Omega_{\mathbb{L}}$ with the non-canonical Lagrangian k-cosymplectic (or k-precosymplectic) structure in $\mathbb{R}^k \times T^1_kQ$ constructed in Section 3.3 as follows: starting from the forms $\Theta_{\mathbb{L}}$ and $\Omega_{\mathbb{L}}$ in $J^1\pi \simeq \mathbb{R}^k \times T^1_kQ$, we can define the forms $\Theta^A_{\mathcal{L}}$ and $\Omega^A_{\mathcal{L}}$ on $\mathbb{R}^k \times T^1_kQ$, as follows: for $X, Y \in \mathfrak{X}(\mathbb{R}^k \times T^1_kQ)$, and $1 \leq A \leq k$,

$$\begin{aligned} \Theta^A_{\mathcal{L}}(X) &= -\left(i\left(\frac{\partial}{\partial t^k}\right) \dots i\left(\frac{\partial}{\partial t^1}\right)(\Theta_{\mathbb{L}} \wedge \mathrm{d}t^A)\right)(X) \\ \Omega^A_{\mathcal{L}}(X, Y) &= -\mathrm{d}\Theta^A_{\mathcal{L}} = (-1)^{k+1}\left(i\left(\frac{\partial}{\partial t^k}\right) \dots i\left(\frac{\partial}{\partial t^1}\right)(\Omega_{\mathbb{L}} \wedge \mathrm{d}t^A)\right)(X, Y) \,. \end{aligned} \tag{52}$$

and the 1-forms $\eta^A = \mathrm{d}t^A$ are canonically defined.

Conversely, starting from the Lagrangian k-cosymplectic (or k-precosymplectic) structure on $\mathbb{R}^k \times T^1_kQ$, and from $\mathcal{E}_{\mathcal{L}}$, we can construct on $\mathbb{R}^k \times T^1_kQ \simeq J^1\pi$

$$\Theta_{\mathbb{L}} = -\mathcal{E}_{\mathcal{L}} \mathrm{d}^k t + \Theta^A_{\mathcal{L}} \wedge \mathrm{d}^{k-1}t_A \quad , \quad \Omega_{\mathbb{L}} = -\mathrm{d}\Theta_{\mathbb{L}} = \mathrm{d}\mathcal{E}_{\mathcal{L}} \wedge \mathrm{d}^k t + \Omega^A_{\mathcal{L}} \wedge \mathrm{d}^{k-1}t_A \tag{53}$$

So we have proved that:

Theorem 8 *The Lagrangian multisymplectic (or pre-multisymplectic) form and the Lagrangian 2-forms of the k-cosymplectic (or k-precosymplectic) structure on* $J^1\pi \equiv \mathbb{R}^k \times T^1_kQ$ *are related by (52) and (53).*

The discussion in the above section about the Lagrangian equations proves the following result, which establishes the complete equivalence between both formalisms:

Theorem 9 *A k-vector field* $\bar{\mathbf{\Gamma}} = (\bar{\Gamma}_1, \dots, \bar{\Gamma}_k)$ *in* $J^1\pi \simeq \mathbb{R}^k \times T^1_kQ$ *is a solution to the equations (50) if, and only if, it is also a solution to the equations (30); that is, we have that* $\mathfrak{X}^k_{\mathcal{L}}(\mathbb{R}^k \times T^1_kQ) = \mathfrak{X}^k_{\mathcal{L}}(\mathbb{R}^k \times T^1_kQ)$.

Appendix: Correspondences between the formalisms

Hamiltonian formalism

	k**-symplectic**	k**-cosymplectic**	**Multisymplectic**
Phase space	$(T^1_k)^*Q$	$\mathbb{R}^k \times (T^1_k)^*Q$	$\mathcal{M}\pi \to J^1\pi^*$
Canonical forms	$\theta^A \in \Lambda^1((T^1_k)^*Q)$	$\Theta^A \in \Lambda^1(\mathbb{R}^k \times (T^1_k)^*Q)$	$\Theta \in \Lambda^k(\mathcal{M}\pi)$
	$\omega^A = -d\theta^A$	$\Omega^A = -d\Theta^A$	$\Omega = -d\Theta$
Hamiltonians	$H : (T^1_k)^*Q \to \mathbb{R}$	$\mathcal{H} : \mathbb{R}^k \times (T^1_k)^*Q \to \mathbb{R}$	$h : J^1\pi^* \to \mathcal{M}\pi$
			$\Theta_h = h^*\Theta\,,\ \Omega_h = h^*\Omega$
Geometric equations	$\displaystyle\sum_{A=1}^k i(X_A)\omega^A = dH$	$\displaystyle\sum_{A=1}^k i(\bar{X}_A)\Omega^A = dH - \frac{\partial H}{\partial t^A}dt^A$ $dt^A(\bar{X}_B) = \delta^A_B$	$i(\bar{\mathbf{X}})\Omega_h = 0$ $i(\bar{\mathbf{X}})\omega = 1$
	$(X_1, \dots, X_k)$ k-vector field on $(T^1_k)^*Q$	$(\bar{X}_1, \dots, \bar{X}_k)$ k-vector field on $\mathbb{R}^k \times (T^1_k)^*Q$	$\bar{\mathbf{X}}$ k-vector field on $J^1\pi^*$

Lagrangian formalism

	k**-symplectic**	k**-cosymplectic**	**Multisymplectic**
Phase space	T^1_kQ	$\mathbb{R}^k \times T^1_kQ$	$J^1\pi$
Lagrangians	$L : T^1_kQ \to \mathbb{R}$	$\mathcal{L} : \mathbb{R}^k \times T^1_kQ \to \mathbb{R}$	$\mathcal{L} : J^1\pi \to \mathbb{R}\,,\ \mathbb{L} = \mathcal{L}\omega$
Lagrangian forms	$\theta^A_L \in \Lambda^1(T^1_kQ)$	$\Theta^A_{\mathcal{L}} \in \Lambda^1(\mathbb{R}^k \times T^1_kQ)$	$\Theta_{\mathbb{L}} \in \Lambda^k(J^1\pi)$
	$\omega^A_L = -d\theta^A$	$\Omega^A_{\mathcal{L}} = -d\Theta^A_{\mathcal{L}}$	$\Omega_{\mathbb{L}} = -d\Theta_{\mathbb{L}}$
Geometric equations	$\displaystyle\sum_{A=1}^k i(\Gamma_A)\omega^A_L = E_L$	$\displaystyle\sum_{A=1}^k i(\bar{\Gamma}_A)\Omega^A_{\mathcal{L}} = d\mathcal{E}_{\mathcal{L}} - \frac{\partial \mathcal{L}}{\partial t^A}dt^A$ $dt^A(\bar{\Gamma}_B) = \delta^A_B$	$i(\bar{\mathbf{\Gamma}})\Omega_{\mathbb{L}} = 0$ $i(\bar{\mathbf{\Gamma}})\omega = 1$
	$(\Gamma_1, \dots, \Gamma_k)$ k-vector field on T^1_kQ	$(\bar{\Gamma}_1, \dots, \bar{\Gamma}_k)$ k-vector field on $\mathbb{R}^k \times T^1_kQ$	$\bar{\mathbf{\Gamma}}$ k-vector field on $J^1\pi$

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