

# On The Structure of Path-like Trees

F. A. Muntaner-Batle

Facultat de Ciències Polítiques i Jurídiques  
Universitat Internacional de Catalunya, c/ Immaculada 22  
08017 Barcelona, Spain.

M. Rius-Font

\*Departament de Matemàtica Aplicada i Telemàtica  
Universitat Politècnica de Catalunya, Jordi Girona Salgado 1  
08034 Barcelona, Spain.

Mathematics Subject Classifications: 05C05, 05C78

## Abstract

We study the structure of path-like trees. In order to do this, we introduce a set of trees that we call expandable trees. In this paper we also generalize the concept of path-like trees and we call such generalization generalized path-like trees. As in the case of path-like trees, generalized path-like trees, have very nice labeling properties.

*Key words:* Tree, path-like tree,  $T_p$ -tree, expandable tree,  $\alpha$ -valuation, super edge-magic labeling, special super edge-magic labeling, harmonious labeling, super-edge antimagic labeling.

## 1 Introduction

For the undefined concepts and notation, the reader is directed to either [4] or [6].

This paper is mainly devoted to study the structure of a particular family of trees called path-like trees. Path-like trees were first introduced by Barrientos in [3] in his effort to find families of trees with nice labeling properties. They are defined as follows:

---

\*Supported by the Spanish Research Council under project MTM2004-07891-C02-01 and by the Catalan Research Council under project 2001SGR00258.



lemma 2.1 not to be expandable, and in fact it will play a crucial role in the development of this paper.

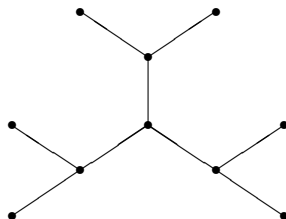


Figure 2: The tree  $T^*$

Next we define a normalized embedding of the path in the grid  $P_k \times P_l$  as follows (see [2]): Let  $\mathcal{L}$  be the 2-dimensional grid. If we fix a crossing point as  $(0, 0)$  then each crossing point in  $\mathcal{L}$  is perfectly determined by an ordered pair  $(i, j)$  where  $i$  denotes the row (level) and  $j$  denotes the column of  $\mathcal{L}$ . Let  $\mathcal{I}$  be an embedding of a path  $P$  in  $\mathcal{L}$  such that:

1. One end vertex of the path  $P$  is  $(0, 0)$ .
2. Each row of the embedding contains at least two vertices of the path  $P$ , and each vertical subpath is in the embedding isomorphic to  $P_2$ .
3. Assume that  $i$  is an even integer and that  $(i, j), (i, j + 1), (i, j + 2), \dots, (i, j + t)$  is a maximal straight horizontal subpath in the embedding of the path  $P$  in  $\mathcal{L}$ . If  $(i + 1, m)$  belongs to the embedding of the path  $P$  in  $\mathcal{L}$ , then  $m \leq j + t$ .
4. Assume that  $i$  is an odd integer and that  $(i, j), (i, j - 1), (i, j - 2), \dots, (i, j - s)$  is a maximal straight horizontal subpath in the embedding of the path  $P$  in  $\mathcal{L}$ . If  $(i + 1, m)$  belongs to the embedding of the path  $P$  in  $\mathcal{L}$ , then  $m \geq j - s$ .

Then the embedding  $\mathcal{I}$  is called a normalized embedding of the path  $P$  in the grid  $\mathcal{L}$ .

The following remark is an immediate consequence of the definition of expandable tree and the fact that every path-like tree can be obtained from a normalized embedding of a path in the 2-dimensional grid (see [2]).

Remark: Every path-like tree is an expandable tree.

In this paper, we take the opportunity to introduce the definition of generalized path-like trees, since they are a natural generalization of path-like trees, and they also have very nice labeling properties.

Let  $P = (V, E)$  be a path with  $V = \{i\}_{i=1}^n$  and with  $E = \{i(i+1)\}_{i=1}^{n-1}$ . Choose a subset  $S \subseteq E$ . For every  $x(x+1) \in S$ , replace  $x(x+1)$  by a new edge  $ij$  where  $i < j$  and

such that  $i + j = 2x + 1$ . Then any graph obtained in this way is called a generalized path-like tree. Notice that the following holds for generalized path-like trees:

1. If  $i + j = 2x + 1$  ( $j > i$ ) then  $d_P(i, x) = d_P(x + 1, j)$ .
2. A generalized path-like tree is not necessarily a tree.
3. Every path-like tree is a generalized path-like tree.
4. Generalized path-like trees are bipartite graphs.
5. Not all generalized path-like trees which are trees, are in fact path-like trees.
6. Generalized path-like trees admit many different types of labelings as for instance,  $\alpha$ -labelings, harmonious labelings, special super-edge magic labelings, etc.

**Example 3:** The following example is an example of a generalized path-like tree, which is a tree and that is not a path-like tree. It is easy to see it, since it contains a vertex of degree 5.

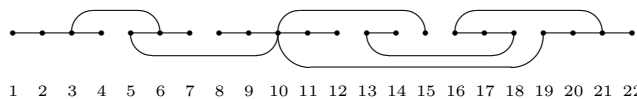


Figure 3: a generalized path-like tree which is not a path-like tree

**Question:** Is it possible to find an efficient algorithm that, given a bipartite graph  $G$ , allows us to decide whether it is a generalized path-like tree?.

We conclude this introduction by defining what we mean by the contraction of a tree  $T$ .

Let  $T$  be any tree. We define the contraction of  $T$ , and we denote it by  $\text{contract}(T)$ , to be the tree with the following two properties:

1. The tree  $\text{contract}(T)$  contains no vertex of degree two.
2. The tree  $T$  can be obtained from  $\text{contract}(T)$  by subdividing edges.

**Example 4:**

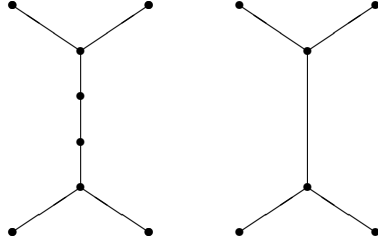


Figure 4:  $T$  and  $\text{contract}(T)$

## 2 Path-like trees and expandable trees

We start this section by providing the following result:

### Lemma 2.1

1. Let  $T$  be an expandable tree. Then any subdivision of  $T$  is expandable.
2. Let  $T$  be an expandable tree. Then  $\text{contract}(T)$  is expandable.
3. Let  $T$  be any tree for which  $\text{contract}(T) \not\cong T^*$ , then  $\text{contract}(T)$  is a caterpillar.
4. The tree  $T^*$  is not expandable.
5. Let  $T$  be any caterpillar with  $\Delta(T) \leq 4$ . Then  $T$  is expandable.

*Proof.*

Since items 1,2 and 5 are clear, the proofs of these items are left to the reader. Next we prove items 3 and 4.

3. Assume to the contrary, that  $T$  is a tree such that  $\text{contract}(T) \not\cong T^*$  and  $\text{contract}(T)$  is not a caterpillar. Let  $S = \{\text{leaves of } \text{contract}(T)\}$  and let  $T'$  denote the new tree  $\text{contract}(T) \setminus S$ . Since  $\text{contract}(T)$  is not a caterpillar, it follows that there exists  $\alpha \in V(T')$  such that  $\deg_{T'}(\alpha) \geq 3$ . Assume that  $\beta_1, \beta_2, \beta_3 \in N(\alpha)$  (where  $N(\alpha)$  denotes the neighborhood of  $\alpha$ ). If  $\deg(\beta_i) \geq 3$  for all  $i \in \{1, 2, 3\}$ , we are done. Therefore there exists  $i \in \{1, 2, 3\}$  such that  $\deg_{T'}(\beta_i) < 3$ . It is clear that  $\beta_i$  is not a leaf in  $\text{contract}(T)$ , since for if not  $\beta_i$  would not be a vertex of  $T'$ . Thus,  $\deg_{\text{contract}(T)}(\beta_i) \geq 2$ . However,  $\text{contract}(T)$  contains no vertex of degree 2, since it is the contraction of some tree. This shows that  $\deg_{\text{contract}(T)}(\beta_i) \geq 3$ . Therefore  $T^* \subseteq \text{contract}(T)$ .
4. Assume to the contrary, that  $T^*$  is expandable and let  $\text{center}(T^*) = \{A\}$ . Then the vertices of degree three of  $T^*$  connect consecutive rows. Hence a vertex of

degree three is adjacent at most to two vertices of degree three. However in our tree, vertex  $A$  is adjacent to three vertices of degree three.  $\square$

Next we state and prove the following theorem:

**Theorem 2.2** *For any given tree  $T$  with  $\Delta(T) \leq 4$ , TFAE*

1.  $T$  is expandable.
2.  $\text{Contract}(T)$  is a caterpillar.
3.  $T$  does not contain any subdivision of  $T^*$  as a subtree.

*Proof.*

1. (1)  $\implies$  (3). If  $T$  is an expandable tree then  $\text{contract}(T)$  is also expandable. Thus  $T^* \not\subseteq \text{contract}(T)$ . Therefore  $T$  does not contain any subdivision of  $T^*$  as a subgraph.
2. (3)  $\implies$  (2). Let  $T$  be a tree such that  $\Delta(T) \leq 4$ , and such that  $T$  does not contain any subdivision of  $T^*$  as a subgraph. Then  $\text{contract}(T)$  does not contain  $T^*$  as a subgraph. Therefore  $\text{contract}(T)$  is a caterpillar.
3. (2)  $\implies$  (1). Let  $T$  be a tree with  $\Delta(T) \leq 4$  such that  $\text{contract}(T)$  is a caterpillar. Then  $\text{contract}(T)$  is expandable. Therefore since  $T$  is a subdivision of  $\text{contract}(T)$  it follows that  $T$  is expandable.  $\square$

Since any path-like tree is an expandable tree, we obtain the following corollary.

**Corollary 2.1**

1. If  $T$  is a path-like tree then  $\text{contract}(T)$  is a caterpillar.
2. If  $T$  is a path-like tree then  $T$  does not contain any subdivision of  $T^*$  as a subgraph.

In [1] Bača et al. proved the next result.

**Theorem 2.3** *Let  $T$  be any path-like tree. Then there are at most two vertices  $u, v \in V(T)$  with:*

1.  $\deg(u)=\deg(v)=3$ .
2. if  $N(u) = \{u_1, u_2, u_3\}$  and  $N(v) = \{v_1, v_2, v_3\}$  then  $\deg(u_i)=\deg(v_i)=1$  for  $i \in \{1, 2\}$  and  $\deg(u_3)=\deg(v_3)=2$  where  $N(u)$  and  $N(v)$  denote the neighborhood of  $u$  and  $v$  respectively.

Next we provide a different proof for this result using the theory developed in this paper.

*Proof.*

We proceed by contradiction. Assume that  $u, v$  are two vertices in  $V(T)$  with the properties described in the statement of the theorem. Then there is exactly one  $u - v$  path in  $T$ . Assume to the contrary, that there is a third vertex in  $T$ , called  $w$ , with the properties of the vertices described in the statement of the theorem. Again, there is a unique  $u - w$  path in  $T$  and also there is a unique  $v - w$  path in  $T$ . Furthermore, these  $u - w$  and  $v - w$  paths are of the form

$$\begin{cases} u = u'_0, u'_1, \dots, u'_l, w_0, w_1, \dots, w_m = w \\ v = v'_0, v'_1, \dots, v'_s, w_0, w_1, \dots, w_m = w \end{cases}$$

where  $\{u'_i\}_{i=0}^l \cap \{v'_i\}_{i=0}^s = \emptyset$ . Then the subtree  $H$  of  $T$  induced by the vertices

$$\{u'_i\}_{i=1}^l \cup \{v'_i\}_{i=1}^s \cup \{w_i\}_{i=1}^{m-1} \cup \{u, u_1, u_2, v, v_1, v_2, w\}$$

is a subdivision of the tree  $T^*$ . □

### 3 Particular families of path-like trees

This section is devoted to obtain necessary and sufficient conditions for different families of trees that guarantee that our trees either are or are not path-like trees. In particular we study the following two families of trees.

- Trees with exactly one vertex of degree three and exactly three vertices of degree one. (The remaining vertices are of degree two.)
- Trees with exactly one vertex of degree four and exactly four vertices of degree one. (The remaining vertices are of degree two.)

Next we introduce the following lemma.

**Lemma 3.1** *Let  $P$  be a path embedded in the two dimensional grid. Assume that there is an elementary transformation on a unitary path (without loss of generality we may assume that it is a horizontal path), that (possible together with other elementary transformations) allows us to obtain another path. The following holds:*

1. *All horizontal subpaths are unitary.*
2. *All horizontal subpaths have to be relocated.*
3. *All vertical subpaths have equal length.*

*Proof.*

Let  $P$  be an embedding of the path in the grid. Without loss of generality, we may assume that we replace a unitary horizontal edge  $e$ . Assume that this movement (possible together with other movements) allows us to obtain the path. Then  $e$  has to be replaced between the two end vertices of the vertical subpaths that edge  $e$  communicates. For otherwise, the vertex or vertices of degree 3 that will appear after this replacement cannot disappear with other replacements. This shows that the two vertical subpaths that edge  $e$  communicates must have the same length. Inductively, we have to replace all horizontal subpaths, since every time that we relocate a unitary path, we create at least one vertex of degree 3, except for the last movements on each side of the embedding. Therefore,

1. All vertical subpaths have the same length.
2. All horizontal subpaths must be relocated.
3. As a consequence of (2), all horizontal subpaths must have unitary length, since for otherwise they could not be relocated.  $\square$

**Theorem 3.2** *Let  $T$  be a tree with exactly one vertex of degree three, namely  $v$ . Assume that  $v$  has attached three paths of lengths  $n_1, n_2$  and  $n_3$  (so that  $|V(T)| = n_1 + n_2 + n_3 + 1$ ). Then  $T$  is a path-like tree if and only if for some permutation  $\sigma$  of  $\{1, 2, 3\}$  there exist two natural numbers  $\alpha, \beta$  such that  $\alpha | n_{\sigma(1)}, \beta | (n_{\sigma(2)} + n_{\sigma(3)} + 1)$  and  $\alpha + n_{\sigma(3)} = \beta$ .*

*Proof.*

( $\implies$ ) Assume that  $L_i$  is a unitary subpath of the path-like tree. We denote by  $L'_i$  the unitary subpath from which  $L_i$  is the elementary transformation.

Without loss of generality assume that  $L_i$  is a unitary horizontal subpath, with a vertex (without loss of generality, the right one) of degree 3. Replacing  $L_i$  by  $L'_i$  we are left with two cases:



1. Case 1: the subpath  $L'_i$  contains a vertex of degree 2 and a vertex of degree 3. In this case the part of the path-like tree that ends in the "right side" of  $L_i$  satisfies the conditions of Lemma 3.1, and therefore, its length is  $n_1 = \alpha \cdot h$  where  $\alpha$  is the length of the segment of the path-like tree that comes before  $L_i$  and  $\alpha - 1$  is the length of the one of the subpaths attached to  $v$ . (Without loss of generality  $\alpha = n_2 + 1$ ). Therefore, if we take  $\beta = n_2 + n_3 + 1$  we get that  $\alpha + n_3 = \beta$ .
2. Case 2: the subpath  $L'_i$  contains two vertices of degree 3. In this case, like in case 1, the length of the part of the path-like tree that ends at  $L_i$  is  $n_1 = \alpha \cdot h$  and using a similar argument to the one used in case 1, the back part has length  $\beta \cdot h' = n_2 + n_3 + 1$  with  $\alpha + n_2 = \beta$ .

( $\Leftarrow$ ) For a proof it suffices to see the figure below, where  $n_1$  is the number of edges of the path  $f - v$  and  $n_2$  is the number of edges of the path  $v - l$ . The horizontal dots, represent paths of lengths  $2k\alpha$  and  $2h\beta$  ( $k, h \in \mathbf{N}$ ) respectively.  $\square$

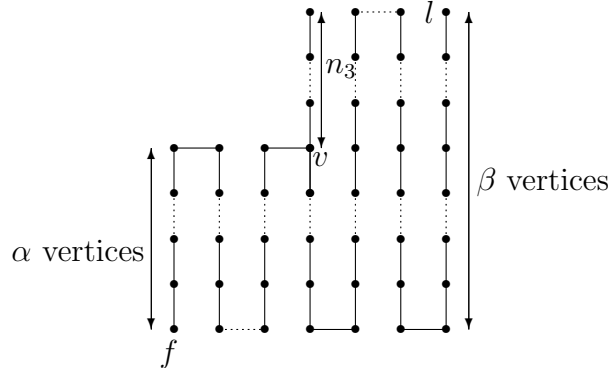


Figure 5: proof of theorem 3.2

**Theorem 3.3** *Let  $T$  be a tree with exactly one vertex, namely  $v$ , of the degree 4, exactly 4 end vertices and the remaining vertices being of degree 2. Let  $n_1, n_2, n_3$  and  $n_4$  be the lengths of the paths from  $v$  to each one of the end vertices. Then  $T$  is a path-like tree if and only if there exists a permutation  $\sigma$  of the set  $\{1, 2, 3, 4\}$  such that*

- $(n_{\sigma(1)} + 1) | n_{\sigma(3)}$
- $(n_{\sigma(2)} + 1) | n_{\sigma(4)}$

*Proof.*

( $\implies$ ) We can view the tree  $T$  as two paths with exactly one vertex  $v$  in common. Let  $P$  be one of these two paths and let  $n_i < n_j$  be the lengths of the segments of  $P$

which come before and after  $v$ . Assume that  $L$  is a unitary segment of  $P$  with an end at  $v$  that comes from applying an elementary transformation to a given segment  $L'$ . When we substitute again  $L$  by  $L'$ , the path  $P$  meets the conditions of the previous proposition. Therefore it is formed by segments of the same length that are joined by unitary segments, where  $n_j = (n_i + 1)$ . With the remaining of the tree, that is to say, the other path, we proceed in a similar way.

( $\Leftarrow$ ) For a proof it suffices to see the figure below, where  $n_3$  is the number of edges of the path  $f - v$ ,  $n_4$  is the number of edges of the path  $v - l$ ,  $n_1$  is the number of vertices of the path  $g - v$  and  $n_2$  is the number of vertices of the path  $v - h$ . The horizontal dots, represent paths of lengths  $2kn_1$  and  $2hn_2$  ( $k, h \in \mathbf{N}$ ) respectively.  $\square$

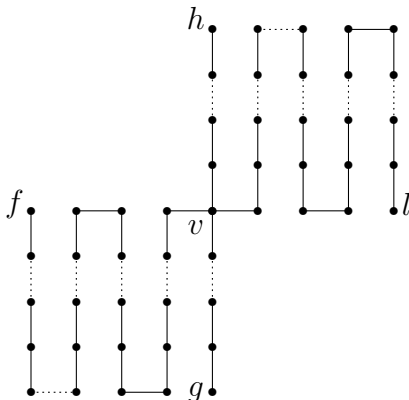


Figure 6: proof of theorem 3.3

## 4 Path-like trees and $T_p$ -trees

In [8] Hedge and Shetty defined a tree to be a  $T_p$ -tree as follows:

Let  $T$  be a tree and  $u_0, v_0$  be two adjacent vertices in  $T$ . Let there be two pendant vertices  $u, v$  in  $T$  such that the length of the  $u_0 - u$  path and the length of the  $v_0 - v$  path are equal. If the edge  $u_0v_0$  is deleted from  $T$ , and vertices  $u$  and  $v$  are joined by an edge  $uv$ , such a transformation is called an elementary parallel transformation (or an ept)<sup>1</sup> and the edge  $u_0v_0$  is called a transformable edge. If by a sequence of ept's  $T$  can be reduced to a path,  $T$  is called a  $T_p$ -tree (transformed tree) and any such sequence regarded as a composition of mappings (ept's) denoted by  $P$ , is called a parallel transformation of  $T$ . The path, the image of  $T$  under  $P$ , is denoted by  $T(P)$ .

<sup>1</sup>The concept of ept was first defined by Acharya, however we do not have any reference for it. In [8] Hedge and Shetty claimed that they found out about ept's by personal communication with Acharya.

$T_p$ -trees were introduced due to their nice labeling properties, and some of such properties were explored in [8].

The reason why we introduce this section is because the definitions of path-like trees and  $T_p$ -trees look very similar at a first glance, and hence we thought it would be interesting to explore the relationship existing among these two concepts. Although, one may be tempted to think that the definitions of path-like trees and  $T_p$ -trees are equivalent, this is not true, and the relationship existing among the two concepts is as shown in the next Venn diagram.

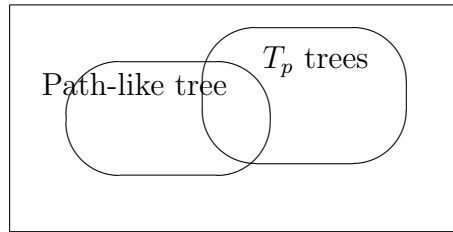


Figure 7: Venn diagram

It is clear that

$$\{\text{path-like trees}\} \cap \{T_p \text{ trees}\} \neq \emptyset.$$

Therefore, we will show the following two facts:

1.  $|\{\text{path-like trees}\} \setminus \{T_p\text{-trees}\}| = \infty$
2.  $|\{T_p\text{-trees}\} \setminus \{\text{path-like trees}\}| = \infty$

1. Proof of 1: to prove 1 it suffices to notice that by the definition of  $T_p$ -tree its maximum degree is upper bounded by 3. Thus any path-like tree with maximum degree 4 is not a  $T_p$ -tree.  $\square$
2. Proof of 2: it suffices to find  $T_p$ -trees that contain  $T^*$  as a subdivision. Such  $T_p$ -trees are not path-like trees. It is clear that there are infinitely many of such  $T_p$ -trees.  $\square$

Next, we provide an example of a  $T_p$ -tree which is not a path-like tree. Consider the tree  $T$  in Figure 8:

It is clear that  $T$  is not a path-like tree, since it contains a subdivision of  $T^*$ . Next, if we perform the indicated ept's in the given order, we obtain a path.

$$\begin{aligned}
 ab &\longrightarrow a_0b_0 \\
 cd &\longrightarrow c_0d_0 \\
 ef &\longrightarrow e_0f_0
 \end{aligned}$$

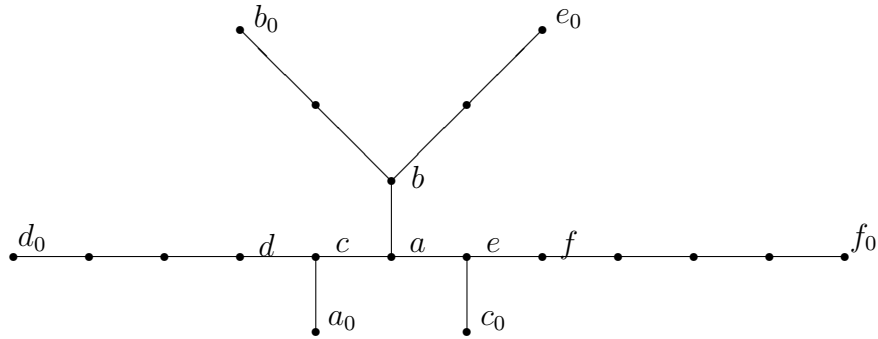


Figure 8: the tree  $T$

Therefore the tree  $T$  is a  $T_p$ -tree.

## 5 Acknowledgements

The authors want to acknowledge Martin Bača, Joan Gimbert and Yuquin Lin, for the useful discussions and encouragement.

## References

- [1] M. Bača, Y. Lin, F. A. Muntaner-Batle, Super edge antimagic labelings of the path-like trees, *Utilitas Math.* To appear.
- [2] M. Bača, Y. Lin, F. A. Muntaner-Batle, Normalized embeddings of path-like trees. *Utilitas Math.* To appear.
- [3] C. Barrientos, Difference Vertex Labelings, Ph. D. Thesis. Universitat Politècnica de Catalunya (2004).
- [4] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, second edition. Wadsworth & Brooks/Cole Advanced Books and Software, Monterey (1986).

- [5] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, On super edge-magic graphs, *Ars Combin.* **64** (2002), 81-95.
- [6] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of combinatorics*, **DS6** (2000).
- [7] R.L. Graham and N.J. Sloane, On additive bases and harmonious graphs, *S.I.A.M. J. Alg. Discrete Math.*, **1** (1980) 382-404.
- [8] S.M. Hedge and S. Shetty, On graceful trees. *Appl. Math. E-Notes* **2** (2002), 192-197.
- [9] F.A. Muntaner-Batle, Special super edge-magic labelings of bipartite graphs, *J. Combin. Math. Combin. Comput.* **39** (2001) 107-120.
- [10] A. Rosa, On certain valuations of the vertices of a graph, *Theory of graphs (Internat. Symposium, Rome, July 1966)*, Gordon and Breach, N.Y. and Dunot Paris (1967) 349-355.