STRONG REFLEXIVITY OF ABELIAN GROUPS

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Abstract

A reflexive topological group G is called strongly reflexive if each closed subgroup and each Hausdorff quotient of the group G and of its dual group is reflexive.

In this paper we establish the adequate concept of strong reflexivity for convergence groups and we prove that the product of countable many locally compact topological groups and complete metrizable nuclear groups are BB-strongly reflexive.

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Introduction

Along this paper we deal with strong reflexivity of topological groups and convergence groups. All the groups considered will be Abelian. For an Abelian topological group G, the symbol ΓG denotes the set of continuous characters (i.e., continuous homomorphisms from G into $\mathbb{T}$, the multiplicative group of complex numbers with modulus 1). The set ΓG with multiplication defined pointwise and endowed with the compact open topology is a Hausdorff topological Abelian group which is called the *dual group of* G and is denoted by $G^\wedge$. The bidual group of G, $G^{\wedge\wedge}$ is defined as $(G^\wedge)^\wedge$ and $\alpha_G : G \to G^{\wedge\wedge}$ stands for the canonical embedding. A topological Abelian group is said to be *reflexive* if α_G is a topological isomorphism.

The Pontryagin duality theorem states that every locally compact Abelian group is reflexive. This yields, in an obvious way, that also closed subgroups and Hausdorff quotients of locally compact Abelian groups are reflexive. This is not the case for other reflexive groups, which may have non reflexive closed subgroups or non reflexive quotients. For instance, Leptin proved in [11] the existence of a product of discrete groups with a non reflexive closed subgroup. Thus, it is natural to introduce a new class of reflexive groups stable for those operations. This is done in [1], where such groups are called strongly reflexive.

On the other hand, in a set of papers by Beattie, Binz, Butzmann, Müller and several others, a new concept of reflexivity of topological groups is given using the continuous convergence structure to define the dual of a topological Abelian group. Since the continuous convergence structure does not derive in general (unless the departure group is locally compact) from a topology, the dual group is only a “convergence group”. However this incursion into convergence groups is only an auxiliary tool: the bidual is again topological. Thus, if duals are endowed with the continuous convergence structure instead of the compact open topology, a new kind of reflexivity is obtained by requiring that the canonical embedding into the bidual be a bicontinuous isomorphism. In [7] such groups are called BB-reflexive and it is proved that this new notion of reflexivity is independent of the classical notion of Pontryagin reflexivity. It is an open question to determine the class of groups for which they coincide; in [6] it is proved that it contains metrizable groups, and as Corollary 2.5 states, it also contains Čech complete groups and direct sums of locally compact groups.

The importance of this new concept is mainly related to completeness properties of the group. For example, a BB-reflexive topological group must be complete (Proposition 2.2). On the other hand a topological vector space is BB-reflexive if and only if it is locally convex and complete [2]. The class of BB-reflexive groups is more likely to be stable for the operation of taking closed subgroups. So, in this respect BB-reflexivity behaves better than Pontryagin reflexivity. It makes sense to define BB-strongly reflexive groups, as those BB-reflexive groups such that the Hausdorff quotients of them and of their duals are also BB-reflexive. As Theorem 3.4 states, in the class of BB-strongly reflexive groups, the general correspondences between duals of closed subgroups and the whole character groups modulo annihilators characteristic for Pontryagin duality, are also valid.

1 Preliminary background

For the definitions of convergence structure and convergence space we refer the reader to [9] and [2]. Topological notions such as continuity, cluster point, closed, open or compact sets, etc, can be stated in terms of convergence of filters, therefore they have corresponding definitions for convergence spaces. A topology defines in a natural way a convergence structure, namely, the one given by its convergent filters or nets. However, not every convergence structure comes from a topology on the supporting set. A convergence structure Λ on a set X is said to be *topological* if it is given by the convergent filters of some topology.

A set $A \subset X$ is *open* if it belongs to every filter which converges to a point of A. The family of all open sets in the convergence space (X, Λ) fulfills the axioms of a topology τ_Λ, called the *associated topology to the convergence structure* Λ. The convergence in τ_Λ has more convergent filters than Λ and they coincide if the convergence structure Λ is topological. One may prove that a map f defined on a convergence space (X, Λ) with values in a topological space Y is continuous iff $f : (X, \tau_\Lambda) \to Y$ is continuous.

If H is a subspace of a convergence space (X, Λ), $\tau_\Lambda|_H$ is finer than the associated topology to the convergence structure $\Lambda|H$, and they coincide for compact convergence subspaces, as stated in the following Lemma.

Lemma 1.1 *If (X, Λ) is a convergence space and $H \subset X$ is compact, then $\tau_\Lambda|_H = \tau_{\Lambda|H}$*

Proof. Since the convergence structure Λ is finer than τ_Λ, H is compact in τ_Λ. Then, $\tau_\Lambda|_H$ and $\tau_{\Lambda|H}$ are comparable compact topologies therefore, they coincide. □

A Hausdorff topological space X is a *k-space* if its closed sets are characterized by the following fact: $F \subset X$ is closed in X if and only if $F \cap K$ is closed in K, for every compact subset K of X. This condition means that the topology of a k-space is the finest topology with the same compact sets and it is equivalent to the following one: A function defined on X with values in a topological space Y is continuous iff its restriction to any compact subset is continuous. It is well known that for a topological Abelian group the finest topology with the same compact subsets is not in general a group topology. By this reason Noble introduced in [13] the notion of k-group as the appropriate analogue to k-space for Hausdorff topological groups. A topological group is a *k-group* if its topology is the finest group topology with the same compact subsets (equivalently: each homomorphism from G into another topological group is continuous if its restriction to each compact is continuous). This notion has some better permanence properties than the one of k-space. Quotient groups and products of k-groups are k-groups. Obviously, every k-space is also a k-group.

In the framework of convergence spaces we have the following result.

Proposition 1.2 *Let (X, Λ) a convergence space such that the associated topological space (X, τ_Λ) is Hausdorff. Then the following conditions are equivalent.*

a) $F \subset X$ is closed in X if and only if $F \cap K$ is closed in K, for every compact subset K of X.

b) A function f defined on X with values in a topological space Y is continuous if and only if its restriction to any compact subset of X is continuous.

Proof.

$a) \Rightarrow b)$ Let $f : (X, \Lambda) \to (Y, \tau)$ be such that $f|_K$ is continuous for every Λ-compact K. In order to prove that f is continuous, it is enough to see that $f^{-1}(C) \cap K$ is Λ-closed in K for each closed subset C of Y, but this yields from the equality $f^{-1}(C) \cap K = (f|_K)^{-1}(C)$ and the continuity of $f|_K$.

$b) \Rightarrow a)$ Consider the family
$\mathcal{H} = \{H \subset X$ such that $K \cap H$ is Λ-closed for all Λ-compact, $K\}$.
This family fulfills the axioms of closed sets for a topology $\tau_{\mathcal{H}}$ which is finer than τ_Λ and coincides with it on the Λ-compact subsets of X. Hence the identity map from (X, τ_Λ) to $(X, \tau_{\mathcal{H}})$ is bicontinuous and that means that $\mathcal{H}$ is the family of closed subsets of (X, Λ).

$\square$

Remark. Observe that each one of the above equivalent conditions implies, by Lemma 1.1, that the associated topological space (X, τ_Λ) is a k-space. We will call *k-convergence spaces* those convergence spaces satisfying one of them.

Locally compact convergence spaces are convergence spaces for which every convergent filter has a compact member. They are k-convergence spaces as can be shown in the following Proposition.

Proposition 1.3 *Let (X, Λ) be a locally compact convergence space such that (X, τ_Λ) is Hausdorff. Then, a function f defined on X with values in a convergence space Y is continuous iff its restriction to any compact subset is continuous.*

Proof. Let $f : (X, \Lambda) \to (Y, \Lambda')$ be such that $f|_K$ is continuous for every Λ-compact K. Let $\mathcal{F}$ be a filter in X convergent to x. Since X is locally compact, the filter $\mathcal{F}$ has a compact member K. The trace of $\mathcal{F}$ in K is a filter which converges to x in K, so its image by the continuous function $f|_K$ is a filter in Y which converges to $f(x)$. Therefore, the filter $f(\mathcal{F})$ converges to $f(x)$. $\square$

2 BB-reflexive convergence groups

Fischer defined the convergence groups, as groups endowed with a convergence structure compatible with the group operation. All the convergence groups considered in this paper will be Hausdorff, that is, a filter converges to at most one point.

If G is a convergence group, we use the symbol ΓG to denote the set of all continuous homomorphisms from G into $\mathbb{T}$. The *continuous convergence structure* Λ_c in ΓG is defined in the following way:
A filter $\mathcal{F}$ in ΓG converges in Λ_c to an element $\xi \in \Gamma G$ if for every $x \in G$ and every filter

$\mathcal{H}$ in G that converges to x, $\omega(\mathcal{F} \times \mathcal{H})$ converges to $\xi(x)$ in $\mathbb{T}$ (here, $\mathcal{F} \times \mathcal{H}$ denotes the filter generated by the products $F \times H$, where $F \in \mathcal{F}$, $H \in \mathcal{H}$ and $\omega(\mathcal{F} \times \mathcal{H})$ denotes the filter generated by $\omega(F \times H) := \{f(x); f \in F, x \in H\}$).

It can be said that Λ_c is the coarsest convergence structure in ΓG for which the evaluation mapping $\omega : \Gamma G \times G \to \mathbb{T}$ is continuous ($\Gamma G \times G$ has the natural product structure). The dual group ΓG of a convergence group (G, Λ), endowed with the convergence structure Λ_c, is a convergence group which is denoted by $\Gamma_c G$ and is called the *convergence dual group of* G.

A convergence group is called *BB-reflexive* if the canonical homomorphism $\kappa_G : G \to \Gamma_c\Gamma_c G$ is a bicontinuous isomorphism (here $\Gamma_c\Gamma_c G$ has the obvious meaning). Observe that, due to the continuity of $\omega : \Gamma G \times G \to \mathbb{T}$, κ_G is always continuous.

For locally compact Abelian topological groups, the compact open topology and the continuous convergence structure in the dual group, have the same convergent filters. This fact characterizes the locally compact groups in the class of topological reflexive groups [12].

Proposition 2.1 *Let G be a locally compact convergence group then:*

a) The continuous convergence structure on the dual group is topological and it coincides with the compact open topology.

b) If compact subsets of G are topological, $\Gamma_c G$ is complete.

Proof.

a) Let $\mathcal{F}$ be a filter τ_{co}-convergent to the neutral element of ΓG and let $\mathcal{H}$ be a convergent filter in G. Since G is locally compact the filter $\mathcal{H}$ has a compact member H. If $W \in \mathcal{B}_{\mathbb{T}}(1)$, the set (H, W) is a neighborhood of the neutral element of ΓG therefore, it contains some $F \in \mathcal{F}$. As $\omega(F \times H) \subset W$, we conclude that $\mathcal{F}$ is Λ_c-convergent. The converse holds without any restrictions.

b) We are going to see that $\Gamma_c G$ is complete in the uniformity of uniform convergence on compact sets. If (f_α) is a Cauchy net in this uniformity, for all $x \in G$, $(f_\alpha(x))$ is also Cauchy in $\mathbb{T}$. Let f be the homomorphism defined by $f(x) = \lim(f_\alpha(x))$. Since for each compact $K \subset G$, $(f_\alpha|_K)$ is in $C(K, T)$ and this topological space is complete, we have that $f|_K$ is continuous for each compact $K \subset G$ and therefore, by Proposition 1.3, it is continuous on G. It is also clear that the convergence of (f_α) to f is uniform on compact sets.

$\square$

We collect in the next Proposition some properties of the continuous convergence structure on the dual of a topological group G.

Proposition 2.2 *Let G be a topological Abelian group, then:*

a) $\Gamma_c G$ is a locally compact convergence group.

b) $(\Gamma G, \tau_{\Lambda_c})$ is a k-space.

c) If $A \subset \Gamma G$ is equicontinuous, the continuous convergence structure on A coincides with the topology of pointwise convergence and with the compact open topology.

d) Compact subsets of $\Gamma_c G$ are equicontinuous.

e) Compact subsets of $\Gamma_c G$ are topological.

f) $\Gamma_c \Gamma_c G$ is topological and complete.

g) If α_G is continuous, $G^\wedge$ and $\Gamma_c G$ have the same compact subsets, and therefore $G^{\wedge\wedge}$ is a topological subgroup of $\Gamma_c\Gamma_c G$.

h) In case α_G continuous, $G^\wedge$ is a k-space if and only if $\tau_{co} = \tau_{\Lambda_c}$.

Proof.

a) This is Proposition 1 of [7]. We observe that the requirement that α_G be continuous can be dropped in the proof.

b) follows from a) and Proposition 1.3.

c) it is proved in Lemma 1 of [8].

d) it is proved in Theorem 7 of [8].

e) follows from c) and d).

f) follows from a) and Proposition 2.1. b).

g) it is proved in [7], Remark 1 and Theorem 1.

h) follows from b) and g).

□

Theorem 2.3 *For a topological group G, the following assertions are equivalent:*

a) The topological groups $\Gamma_c\Gamma_c G$ and $G^{\wedge\wedge}$ coincide.

b) α_G is continuous and every homomorphism $\Psi\colon G^\wedge \to \mathbb{T}$ such that $\Psi_{|K}$ is continuous for all compact subsets K, is continuous.

Proof.

a) $\Rightarrow$) b) Suppose $\Gamma_c\Gamma_c G$ and $G^{\wedge\wedge}$ are the same topological group. Since κ_G is continuous it is clear that α_G is continuous. Take now $\Psi\colon G^{\wedge} \to \mathbb{T}$ such that $\Psi_{|K}$ is continuous, for every compact $K \subset G^{\wedge}$. In particular $\Psi_{|K}$ is continuous, for every compact $K \subset \Gamma_c G$. Now Proposition 1.3 and Proposition 2.2 a) imply the continuity of $\Psi\colon \Gamma_c G \to \mathbb{T}$. Thus $\Psi \in \Gamma_c\Gamma_c G = G^{\wedge\wedge}$.

b) $\Leftarrow$) a) From Proposition 2.2 g) we have that $G^{\wedge\wedge}$ is a topological subgroup of $\Gamma_c\Gamma_c G$. In order to see that they coincide, take $\Psi \in \Gamma_c\Gamma_c G$. Again Proposition 2.2 g) and b) imply that $\Psi\colon G^{\wedge} \to \mathbb{T}$ is continuous, so it belongs to $G^{\wedge\wedge}$

$\square$

Corollary 2.4 *Let G be a topological group such that α_G is continuous and $G^{\wedge}$ is a k-group then $G^{\wedge\wedge}$ and $\Gamma_c\Gamma_c G$ coincide as topological groups.*

The following example shows that $G^{\wedge}$ can be a k-group without being k-space and in this case, h) in the above Proposition does not hold. It shows furthermore that, the associated topology to the continuous convergence structure on the dual of a topological group is not in general a group topology.

Example. Let G be the topological group $\omega\mathbb{R} \times \mathbb{R}^{\omega}$ where $\omega\mathbb{R}$ and $\mathbb{R}^{\omega}$ denote the countable direct sum and the product of real lines respectively. We have that $G^{\wedge} = \mathbb{R}^{\omega} \times \omega\mathbb{R}$, is a k-group but not a k-space (see [1](17.9)). Thus, $(G^{\wedge}, \tau_{\Lambda_c})$ cannot be a topological group; for otherwise, being τ_{Λ_c} a k-space topology, it should be also k-group topology but there is already a k-group topology on $G^{\wedge}$, namely τ_{co}. On the other hand, $\Gamma_c G$ and $G^{\wedge}$ have the same compact subsets, therefore $\tau_{co} \neq \tau_{\Lambda_c}$.

The notion of Čech-completeness has interesting implications in the context of topological groups as shown in [14]. Čech complete groups are k-spaces (in particular k-groups). It is also interesting to note that the class of Čech complete topological groups contains locally compact groups, metrizable complete groups, and is closed with respect to the operations of taking closed subgroups, Hausdorff quotients and countable products.

Corollary 2.5 *a) Čech complete groups are BB-reflexive if and only if they are Pontryagin reflexive.*

b) Arbitrary direct sums of locally compact topological groups are BB-reflexive.

Proof.

a) Let G be a Čech complete group. It is a k-group, so the canonical mapping α_G is continuous (see [13]). On the other hand, one of the authors proved in [6] that for a metrizable group, the dual group $G^{\wedge}$ is a k-space. The same can be proved for Čech complete groups in a quite analogous way. Therefore, for this class of groups it also holds that reflexivity is equivalent to BB-reflexivity.

b) A well known theorem of Kaplan [10], states that arbitrary products and direct sums of locally compact groups are Pontryagin reflexive, being the dual of an arbitrary product of topological groups, topologically isomorphic to the direct sum of the dual groups and conversely, the dual of the direct sum of groups, topologically isomorphic to the product of the dual groups. On the other hand, arbitrary products of locally compact groups are k-groups. So, from Kaplan's result and the above Theorem we conclude that arbitrary direct sums of locally compact groups are BB-reflexive.

□

3 BB-strongly reflexive convergence groups

The definition of strongly reflexive topological groups appeared for the first time in [3]. According to [1](17.1), it can be simplified and stated in the following way: A reflexive topological group G is *strongly reflexive* if every closed subgroup and every Hausdorff quotient of G and of $G^\wedge$ is reflexive. We will show that the analogical notion of BB-strongly reflexive groups admits a further simplification.

A BB-reflexive convergence group G is said to be *BB-strongly reflexive* if for arbitrary closed subgroups H and L of G and of $\Gamma_c G$ respectively, the quotients G/H and $\Gamma_c G/L$ are BB-reflexive. In order to justify our definition we will prove in 3.4 that those requirements about quotients, imply that closed subgroups of G and of $\Gamma_c G$ are BB-reflexive.

A subgroup H of a convergence group (G, Λ) is said to be *dually closed* if, for every element x of $G \setminus H$, there is a continuous character φ in ΓG such that $\varphi(H) = 1$ and $\varphi(x) \neq 1$. It is said to be *dually embedded* if every continuous character defined on H can be extended to a continuous character on G. The *annihilator of* H is defined as the subgroup $H^o := \{\varphi \in \Gamma G : \varphi(H) = 1\}$. It is easy to prove that, a closed subgroup H of a topological or convergence group G is dually closed if and only if the quotient group G/H has sufficiently many continuous characters. For our purposes we also need the following result, whose proof is straightforward.

Lemma 3.1 *Let G be a convergence group and H a subgroup of G. Then H is dually closed if and only if $\kappa_G(H) = H^{oo} \cap \kappa_G(G)$, where H^{oo} denotes the subgroup $(H^o)^o$ of $\Gamma\Gamma_c G$.*

Let $f : G \to H$ be a continuous homomorphism of convergence groups. The *dual mapping* $\Gamma f : \Gamma_c H \to \Gamma_c G$ defined by $(\Gamma f(\chi))(g) := (\chi \circ f)(g)$ is a continuous homomorphism ([4]). If f is onto, then Γf is injective. Let H be a closed subgroup of a convergence group G ; denote by $p : G \to G/H$ the canonical projection and by $i : H \to G$ the inclusion. By means of the dual mappings, Γp and Γi we obtain the natural continuous homomorphisms $\varphi : \Gamma_c(G/H) \to H^o$ and $\psi : \Gamma_c G/H^o \to \Gamma_c H$. Observe that if H is dually embedded, ψ is a continuous isomorphism. We prove now that φ is always a bicontinuous isomorphism.

Proposition 3.2 *If G is a convergence group, the natural homomorphism $\varphi : \Gamma_c(G/H) \to H^o$ is a bicontinuous isomorphism.*

Proof. It is clear that $\varphi : \Gamma_c(G/H) \to H^o$ is a continuous isomorphism. In order to prove that φ^{-1} is continuous, take a convergent filter in H^o, say $\mathcal{F} \to 0$. We must check that $\varphi^{-1}(\mathcal{F}) \to 0$ in $\Gamma_c(G/H)$ i.e. that $\omega_{G/H}(\varphi^{-1}(\mathcal{F}) \times \mathcal{H}) \to 0$ in $\mathbb{T}$ for every filter $\mathcal{H} \to [x]$ in G/H. This is satisfied because, by the definition of quotient structure, $\mathcal{H} \supset p(\mathcal{L})$ for some filter $\mathcal{L} \to z$ with $z \in p^{-1}([x])$ and thereof, $\omega_{G/H}(\varphi^{-1}(\mathcal{F}) \times \mathcal{H}) \supset \omega_{G/H}(\varphi^{-1}(\mathcal{F}) \times p(\mathcal{L})) = \omega_G(\mathcal{F} \times \mathcal{L}) \to 0$ in $\mathbb{T}$. □

Remark. We have obtained this bicontinuous isomorphism without any assumptions on the convergence group G. However, this is not the case for the Pontryagin duality; if G is a topological group, the natural mapping $\varphi : (G/H)^\wedge \to H^o$ is a continuous isomorphism, and further requirements are needed in order that it be a topological isomorphism.

Proposition 3.3 *If G is a BB-reflexive convergence group, every dually closed and dually embedded subgroup of G, is BB-reflexive.*

Proof. The homomorphism $\kappa_H : H \to \Gamma_c\Gamma_c H$ is injective because so is κ_G. From the commutativity of the following diagram and taking into account that $\Gamma\psi$ is continuous monomorphism and φ^{H^o} and $\kappa_{G|H}$ are bicontinuous isomorphisms we obtain that κ_H is surjective and that κ_H^{-1} is continuous.

$$\begin{array}{ccc} \Gamma_c\Gamma_c H & \xrightarrow{\Gamma\psi} & \Gamma_c(\Gamma_c G/H^o) \\ {\scriptstyle\kappa_H}\uparrow & & \downarrow{\scriptstyle\varphi^{H^o}} \\ H & \xrightarrow{\kappa_{G|H}} & H^{oo} \end{array}$$

□

The already mentioned example of Leptin of a closed non reflexive subgroup of a product of discrete groups [11], shows that there are dually closed and embedded subgroups of Pontryagin reflexive groups which are not Pontryagin reflexive. Hence, the analogue to the last proposition does not hold in the Pontryagin setting.

Theorem 3.4 *If G is a BB-strongly reflexive convergence group, then:*

a) *Closed subgroups of G are dually closed.*

b) *For every closed subgroup H of G, the homomorphisms $\varphi^H : \Gamma_c(G/H) \to H^o$, $\phi : \Gamma_c(\Gamma_c G/H^o) \to H$ and $\psi : \Gamma_c G/H^o \to \Gamma_c H$ are bicontinuous isomorphisms.*

c) *Closed subgroups of G are dually embedded.*

d) *Closed subgroups of G are BB-reflexive.*

e) *$\Gamma_c G$ is BB-strongly reflexive. Therefore it satisfies a), b), c) and d).*

Proof.

a) For every closed subgroup H of G, the group G/H is BB-reflexive. Thus, it has sufficiently many continuous characters, and so, H is dually closed.

b) $\varphi^H : \Gamma_c(G/H) \to H^o$ and $\varphi^{H^o} : \Gamma_c(\Gamma_c G/H^o) \to H^{oo}$ are bicontinuous isomorphisms by Proposition 3.2.

Since H is dually closed and G is BB-reflexive, by Lemma 3.1, $\kappa_{G|H} : H \to H^{oo}$ is a bicontinuous isomorphism, and so is also $\phi = {\kappa_{G|H}}^{-1} \circ \varphi^{H^o}$.

Now, the commutativity of the following diagram

$$\begin{array}{ccc} \Gamma_c G/H^o & \xrightarrow{\kappa_{\Gamma_c G/H^o}} & \Gamma_c\Gamma_c(\Gamma_c G/H^o) \\ \psi\big\downarrow & & \big\uparrow \Gamma\varphi^{H^o} \\ \Gamma_c H & \xleftarrow[\Gamma\kappa_{G|H}]{} & \Gamma_c H^{oo} \end{array}$$

together with the fact that $\kappa_{\Gamma_c G/H^o}$, $\Gamma\varphi^{H^o}$ and $\Gamma\kappa_{G|H}$ are bicontinuous isomorphisms, imply that ψ is a bicontinuous isomorphism.

c) Every character χ in $\Gamma_c H$ is the image of a character in $\Gamma_c G/H^o$ through the previous isomorphism ψ. The latter comes from a character in $\Gamma_c G$ that extends χ.

d) All closed subgroups are dually closed and dually embedded by a) and c) respectively. Therefore, by Proposition 3.3, they are BB-reflexive.

e) $\Gamma_c G$ and its quotients are BB-reflexive by the definition of BB-strongly reflexive group. The same happens with $\Gamma_c\Gamma_c G$, which is bicontinuously isomorphic to G.

$\square$

The following theorems confirm that we have actually introduced a new class of topological groups; more precisely the class of BB-strongly reflexive groups is strictly larger than that of locally compact Abelian groups.

We will need the following Lemma

Lemma 3.5 *If (G, Λ) is a convergence group and $H \subset G$ a closed subgroup, then τ_Λ/H is the associated topology to Λ/H.*

Proof. We must show that $\tau_\Lambda/H = \tau_{\Lambda/H}$. Let $p : G \to G/H$ be the canonical projection.

$\subset$) Take $O \in \tau_\Lambda/H$ and let $\mathcal{F}$ be a filter in G/H, Λ/H-convergent to $[z] \in O$. If $\mathcal{L}$ is a filter in G such that $\mathcal{L} \xrightarrow{\Lambda} x \in p^{-1}[z]$ and $p(\mathcal{L}) \subset \mathcal{F}$, then $p^{-1}(O) \in \mathcal{L}$ since $p^{-1}(O)$ is τ_Λ-open and $x \in p^{-1}(O)$. Thus $O = p(p^{-1}(O)) \in p(\mathcal{L}) \subset \mathcal{F}$ and so, $O \in \tau_{\Lambda/H}$.

$\supset$) Let $U \in \tau_{\Lambda/H}$ and let $\mathcal{L}$ be a filter in G such that $\mathcal{L} \xrightarrow{\Lambda} t \in p^{-1}(U)$. Then $p(\mathcal{L}) \to [t] \in U$ in G/H and, since U is $\tau_{\Lambda/H}$-open, $U \in p(\mathcal{L})$. Thus $p^{-1}(U) \in \mathcal{L}$, and so $U \in \tau_\Lambda/H$.

□

Theorem 3.6 *Let $\{G_n\}_{n\in\mathbb{N}}$ be a sequence of locally compact Abelian groups. Then, $G = \prod G_n$ is BB-strongly reflexive.*

Proof. In [1](17.3), it is proved that G is Pontryagin strongly reflexive, then for each closed subgroup H of G, the quotient G/H is Čech complete and Pontryagin reflexive, thus it is BB-reflexive.

Let L be a closed subgroup of $\Gamma_c G$. Since $G^\wedge$ is a k-space, τ_{co} is the associated topology to the continuous convergence structure and L is a closed subgroup of $G^\wedge$. Being G Pontryagin strongly reflexive, L is dually closed; so, there exists a closed subgroup H of G such that $H^o = L$ ([1](14.2)). We are going to see that $\Gamma_c G/H^o$ is reflexive.

The convergence group $\Gamma_c G$ is locally compact and the same happens with the quotient $\Gamma_c G/H^o$. Thus, $\Gamma_c(\Gamma_c G/H^o)$ is topological and it carries the compact open topology. On the other hand, the topological group associated to $\Gamma_c G/H^o$ is, by Lemma 3.5, $G^\wedge/H^o$. Therefore $\Gamma(\Gamma_c G/H^o) = \Gamma(G^\wedge/H^o)$.
The group $\Gamma_c(\Gamma_c G/H^o)$ is bicontinuously isomorphic to $H^{oo} = \kappa_G(H)$ which is isomorphic to H and $\psi : \Gamma_c G/H^o \to \Gamma_c H$ is a continuous isomorphism, since H is dually embedded. So, taking into account the commutativity of the diagram

$$\begin{array}{ccc} \Gamma_c G/H^o & \xrightarrow{\kappa_{\Gamma_c G/H^o}} & \Gamma_c\Gamma_c(\Gamma_c G/H^o) \\ \psi\downarrow & & \uparrow\Gamma\varphi^{H^o} \\ \Gamma_c H & \underset{\Gamma\kappa_{G|H}}{\longleftarrow} & \Gamma_c H^{oo} \end{array}$$

and due to the fact that $\Gamma_c H$ is locally compact, using Proposition 1.3, we only need to prove that the restrictions of ψ^{-1} to the compact subsets of $\Gamma_c H$ are continuous.

Let C be a compact subset of $\Gamma_c H$, $i : H \to G$ the inclusion and $p : \Gamma_c G \to \Gamma_c G/H^o$ the canonical projection; C is topological and equicontinuous. As it is proved in [1](8.2), there exists an equicontinuous set E in ΓG such that $\Gamma i(E) = C$. Let $\bar{E}$ the τ_{co}-clausure of E; $\bar{E}$ is closed and equicontinuous and therefore compact in $G^\wedge$. Being α_G continuous, $\bar{E}$ is also compact in $\Gamma_c G$. Consequently $p(\bar{E})$ is compact in $\Gamma_c G/H^o$. Since $\psi^{-1}(C) \subset \psi^{-1}(\Gamma i(\bar{E})) = p(\bar{E})$ and $\psi^{-1}(C)$ is closed in $\Gamma_c G/H^o$, we have that $\psi^{-1}(C)$ is compact in $\Gamma_c G/H^o$.

We are going to see now that $\psi^{-1}(C)$ is topological:
The set $K = \bar{E}$ is topological and compact in $G^\wedge = \sum G_n^\wedge$; so, there exists some $n \in \mathbb{N}$ such that $K \subset G_1^\wedge + G_2^\wedge + \ldots + G_n^\wedge =: G^n$ and $p(K) \subset p(G^n)$ which is topologically isomorphic to $G^n/G^n \cap H^o$. Let us see that $G^n/G^n \cap H^o$ inherits from $\Gamma_c G/H^o$ the natural topology. Let $\mathcal{F}$ be a filter in $G^n/G^n \cap H^o$ convergent to $[x]$ in the natural topology, $q : G^n \to G^n/G^n \cap H^o$ the canonical projection and $\mathcal{H}$ a filter in G^n convergent

to $x \in q^{-1}([x])$ such that $q(\mathcal{H}) \subset \mathcal{F}$. If $\mathcal{L}$ converges to y in $G = \prod G_n$, since $\mathcal{H}$ is in $G^n = G_1^\wedge + G_2^\wedge + \ldots + G_n^\wedge$, $\mathcal{H}(\mathcal{L}) \to x(y)$; therefore $\mathcal{H} \to x$ in $\Gamma_c G$ and then $\mathcal{F} \to [x]$ in $\Gamma_c G/H^o$.

For each compact C of $\Gamma_c H$, we have seen that $\psi^{-1}(C)$ is compact and topological. The map $\psi : \psi^{-1}(C) \to C$, surjective and continuous, is in fact a topological isomorphism, and consequently the restriction of ψ^{-1} to the compact set C is continuous.

□

For the class of nuclear groups introduced by Banaszczyk in [1] we have the following result.

Theorem 3.7 *Every complete metrizable nuclear group is BB-strongly reflexive.*

Proof. By [1](17.3) every complete metrizable nuclear group G is Pontryagin strongly reflexive. Then, for every closed subgroup H of G, H and G/H are Pontryagin reflexive. So, being H and G/H metrizable, are also BB-reflexive ([6]).

Dual convergence groups of BB-reflexive groups are also BB-reflexive, therefore $\Gamma_c G$ and $\Gamma_c H$ are BB-reflexive.

Let L be a closed subgroup of $\Gamma_c G$. Being τ_{co} the associated topology to the continuous convergence structure, L is a closed subgroup of $G^\wedge$. As in the proof of Theorem 3.6, there exists a closed subgroup H of G such that $H^o = L$. Using now that $\Gamma_c G/H^o$ is bicontinuosly isomorphic to $\Gamma_c H$ ([5]), we obtain $\Gamma_c G/H^o$ is BB-reflexive.

□

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